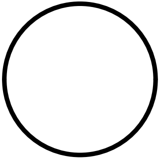
1-loop massive vacuum diagram

We are going to live in $d = 4 - 2\varepsilon$ dimensional space-time



$$\int \frac{d^d k}{D^n} = i\pi^{d/2} m^{d-2n} V(n)$$

$$D = m^2 - k^2 - i0 \qquad m = 1$$

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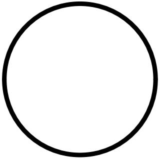

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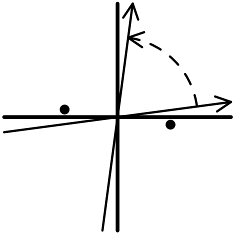

Poles $k_0 = \pm\left(\sqrt{\vec{k}^2 + 1} - i0\right)$

Wick rotation $k_0 = ik_0$, $k^2 = -k^2$

$$\int \frac{d^d k}{(k^2 + 1)^n} = \pi^{d/2} V(n)$$

α parametrization

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty e^{-a\alpha} \alpha^{n-1} d\alpha$$

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$$V(n) = \frac{\Gamma(-d/2+n)}{\Gamma(n)}$$

Integer n

Proportional to

$$V_1 = \frac{4}{(d-2)(d-4)}\Gamma(1+\varepsilon)$$

For example,

$$V(2) = -\frac{d-2}{2}V_1 = \Gamma(\varepsilon)\,.$$

$V(n)$ is UV divergent at $d \to 4$ if $n \leq 2$: $1/\varepsilon$ pole

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$$\Gamma(1+\varepsilon) = \exp\left[-\gamma\varepsilon + \sum_{n=2}^{\infty}\frac{(-1)^n\zeta_n}{n}\varepsilon^n\right]$$

$$\zeta_n = \sum_{k=1}^{\infty}\frac{1}{k^n}$$

$$\zeta_2 = \frac{\pi^2}{6} \qquad \zeta_3 \approx 1.202 \qquad \zeta_4 = \frac{\pi^4}{90} \qquad \cdots$$

Integrals in d dimensions

$$\int cf(k)\, d^d k = c \int f(k)\, d^d k$$

$$\int [f(k) + g(k)]\, d^d k = \int f(k)\, d^d k + \int g(k)\, d^d k$$

$$\int f(k+q)\, d^d k = \int f(k)\, d^d k$$

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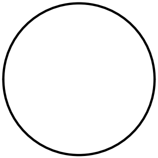
In particular

$$\int \frac{\partial f(k)}{\partial k^\mu}\, d^dk = 0$$

Massless vacuum diagrams



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by dimensionality (argument fails at $n = d/2$)

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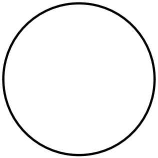

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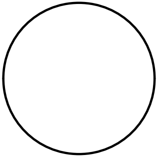

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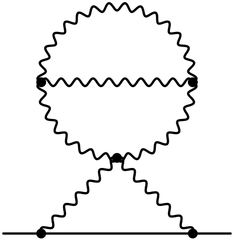

$$= 0$$

d-dimensional solid angle

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$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

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$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$\Omega_1 = 2 \qquad \Omega_2 = 2\pi \qquad \Omega_3 = 4\pi \qquad \Omega_4 = 2\pi^2 \qquad \cdots$$

UV divergence

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2 - i0)^2}$$

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UV divergence $(1/\varepsilon)$ $k \to \infty$ — IR regularization

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2)^2}\bigg|_{UV} = \frac{i}{8\pi^2} \int_\lambda^\infty k^{-1-2\varepsilon} dk = \frac{i\lambda^{-2\varepsilon}}{(4\pi)^2 \varepsilon} = \frac{i}{(4\pi)^2} \frac{1}{\varepsilon}$$

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Any IR regularization is OK

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2)^2}\bigg|_{UV} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(m^2 - k^2)^2}$$

$$= \frac{i m^{-2\varepsilon}}{(4\pi)^2} \Gamma(\varepsilon) = \frac{i}{(4\pi)^2} \frac{1}{\varepsilon}$$

Feynman parametrization

$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int e^{-a_1\alpha_1 - a_2\alpha_2} \alpha_1^{n_1-1} \alpha_2^{n_2-1} d\alpha_1 \, d\alpha_2$$

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Substitution $\alpha_1 = \eta x$, $\alpha_2 = \eta(1-x)$

$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 \frac{x^{n_1-1}(1-x)^{n_2-1} dx}{[a_1 x + a_2(1-x)]^{n_1+n_2}}$$

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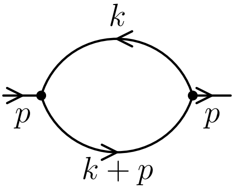
$\alpha_i = \eta x_i$

$$\frac{1}{a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}} = \frac{\Gamma(n_1 + n_2 \cdots + n_k)}{\Gamma(n_1)\Gamma(n_2)\cdots\Gamma(n_k)}$$
$$\int \frac{\delta(x_1 + x_2 \cdots + x_l - 1) x_1^{n_1-1} x_2^{n_2-1} \cdots x_k^{n_k-1} dx_1\, dx_2 \cdots dx_k}{[a_1 x_1 + a_2 x_2 \cdots + a_k x_k]^{n_1 + n_2 \cdots + n_k}}$$

1-loop massless propagator diagram



$$\int \frac{d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2}(-p^2)^{d/2-n_1-n_2} G(n_1, n_2)$$

$$D_1 = -(k+p)^2 \qquad D_2 = -k^2$$

$(p^2 = -1)$ Symmetric $1 \leftrightarrow 2$

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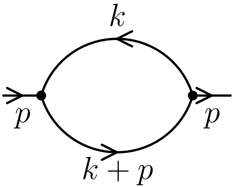

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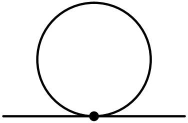
Vanishes for integer $n_1 \leq 0$ or $n_2 \leq 0$



1-loop massless propagator diagram

Wick rotation, α parametrization

$$G(n_1, n_2) = \frac{\pi^{-d/2}}{\Gamma(n_1)\Gamma(n_2)}$$

$$\times \int e^{-\alpha_1 (k+p)^2 - \alpha_2 k^2} \alpha_1^{n_1 - 1} \alpha_2^{n_2 - 1} d\alpha_1 \, d\alpha_2 \, d^d k$$

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Shift $k' = k + \frac{\alpha_1}{\alpha_1+\alpha_2} p$

$$G(n_1,n_2) = \frac{\pi^{-d/2}}{\Gamma(n_1)\Gamma(n_2)} \int \exp\left[-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right] \alpha_1^{n_1-1}\alpha_2^{n_2-1} d\alpha_1\, d\alpha_2 \times \int e^{-(\alpha_1+\alpha_2)k^2} d^d k$$

$$= \frac{1}{\Gamma(n_1)\Gamma(n_2)} \times \int \exp\left[-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right] (\alpha_1+\alpha_2)^{-d/2} \alpha_1^{n_1-1}\alpha_2^{n_2-1} d\alpha_1\, d\alpha_2$$

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Substitution $\alpha_1 = \eta x$, $\alpha_2 = \eta(1-x)$

$$G(n_1, n_2) = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 x^{n_1-1}(1-x)^{n_2-1} dx$$

$$\times \int_0^\infty e^{-\eta x(1-x)} \eta^{-d/2+n_1+n_2-1} d\eta$$

$$= \frac{\Gamma(-d/2+n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 x^{d/2-n_2-1}(1-x)^{d/2-n_1-1} dx$$

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$$G(n_1, n_2) = \frac{\Gamma(-d/2+n_1+n_2)\Gamma(d/2-n_1)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2)}$$

1-loop massless propagator diagram

$$\frac{G(n_1, n_2 + 1)}{G(n_1, n_2)} = -\frac{(d - 2n_1 - 2n_2)(d - n_1 - n_2 - 1)}{n_2(d - 2n_2 - 2)}$$

1-loop massless propagator diagram

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For integer $n_{1,2}$, proportional to

$$G_1 = -\frac{2g_1}{(d-3)(d-4)}$$

$$g_1 = \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

Divergences

$k \to \infty$: the denominator $(k^2)^{n_1+n_2}$
UV-divergent if $d \geq 2(n_1 + n_2)$ $(d \to 4$: $n_1 + n_2 \leq 2)$
$1/\varepsilon$ pole of $\Gamma(-d/2 + n_1 + n_2)$ for $n_1 = n_2 = 1$

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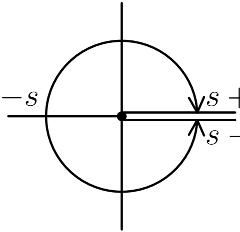
$k \to 0$: the denominator $(k^2)^{n_2}$
IR-divergent if $d \leq 2n_2$ ($d \to 4$: $n_2 \geq 2$)
$1/\varepsilon$ pole of $\Gamma(d/2-n_2)$ for $n_2 \geq 2$
Similarly $k+p \to 0$

Analytical properties



$$I(s \pm i0) = G_1 s^{-\varepsilon} e^{\pm i\pi\varepsilon}$$

$$I(s+i0) - I(s-i0)$$

$$= G_1 s^{-\varepsilon} 2\varepsilon \sin(\pi\varepsilon) \to 2\pi i$$

$$I(p^2) = -\frac{i}{\pi^{d/2}} \int \frac{d^d k}{(-k^2 - i0)(-(k+p)^2 - i0)} = G_1 (-p^2)^{-\varepsilon}$$

Tensors in d dimensions

$$\delta^\mu_\mu = d$$

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Projector onto completely antisymmetric tensors

$$\delta^{[\mu_1}_{\nu_1} \delta^{\mu_2}_{\nu_2} \cdots \delta^{\mu_n]}_{\nu_n}$$

For example

$$\delta^{[\mu_1}_{\nu_1} \delta^{\mu_2]}_{\nu_2} = \frac{1}{2!} \left(\delta^{\mu_1}_{\nu_1} \delta^{\mu_2}_{\nu_2} - \delta^{\mu_2}_{\nu_1} \delta^{\mu_1}_{\nu_2} \right)$$

Its trace — the number of independent components

$$\delta^{[\mu_1}_{\mu_1} \delta^{\mu_2}_{\mu_2} \cdots \delta^{\mu_n]}_{\mu_n} = \binom{d}{n} = \frac{1}{n!} d(d-1) \cdots (d-n+1)$$

Integer d: any tensor antisymmetric in $n > d$ indices is zero
Non-integer d: the traces are non-zero for all n,
the projectors are non-zero

γ-matrices in d dimensions

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

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How many different products of the matrices γ^μ are there for an integer d? Each of d matrices γ^μ occurs either 0 or 1 times. The number of independent products is 2^d. For any even integer d, products of γ^μ span the whole space of matrices. The number of independent $N \times N$ matrices is N^2. This means that γ^μ must be $2^{d/2} \times 2^{d/2}$ matrices:

$$\operatorname{Tr} 1 = 2^{d/2}$$

Any γ-matrix expression can be expanded in

$$\Gamma^{\mu_1 \ldots \mu_n} = \gamma^{[\mu_1} \ldots \gamma^{\mu_n]}$$

For a non-integer d, this basis is infinite.

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But conventionally

$$\operatorname{Tr} 1 = 4$$

γ-matrices in d dimensions

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γ_5 in d dimensions

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$$= -(d-8)\operatorname{Tr}\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\ \Rightarrow\ (d-4)\operatorname{Tr}\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta = 0$$

$\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^\mu = (d-8)\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta +$ terms with fewer γ-matrices