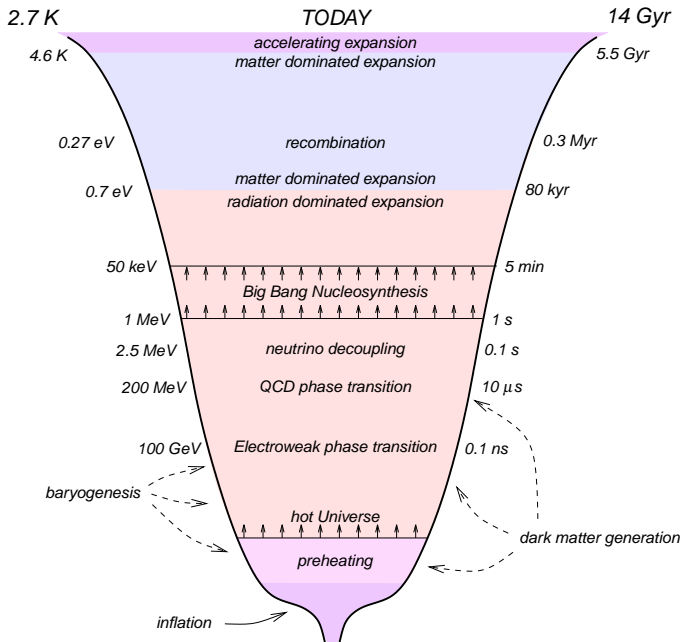


The Big Bang Cosmology:  
Lecture #2  
Friedmann–Robertson–Walker Cosmology

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# Outline

- 1 The FRW metric, reddening and the Hubble law
- 2 Gas of free particles in the expanding Universe
- 3 The dynamics of the expanding Universe
  - Einstein equations
  - Examples of cosmological solutions
  - Particles horizon
  - Events horizon

# Convention

## metric

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu, \quad \mu, \nu = 0, 1, 2, 3 \quad \text{signature } (+, -, -, -)$$

$$x^\mu \rightarrow x'^\mu(x^\mu) : \quad B_{\nu\lambda}^{\mu}{}'(x') = \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x^\tau}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\lambda} B_{\tau\rho}^\sigma(x),$$

## Christoffel symbols:

$$\tilde{A}^\mu(\tilde{x}) = A^\mu(x) - \Gamma_{\nu\lambda}^\mu(x) A^\nu(x) dx^\lambda$$

$$A^\mu(\tilde{x}) - \tilde{A}^\mu(\tilde{x}) = \nabla_\nu A^\mu \cdot dx^\nu \quad \Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\rho} (\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda}).$$

## Riemann tensor

$$\nabla_\mu \nabla_\nu A^\lambda - \nabla_\nu \nabla_\mu A^\lambda = A^\sigma R_{\sigma\mu\nu}^\lambda, \quad R_{\nu\lambda\rho}^\mu = \partial_\lambda \Gamma_{\nu\rho}^\mu - \partial_\rho \Gamma_{\nu\lambda}^\mu + \Gamma_{\sigma\lambda}^\mu \Gamma_{\nu\rho}^\sigma - \Gamma_{\sigma\rho}^\mu \Gamma_{\nu\lambda}^\sigma.$$

## Ricci tensor and scalar

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}, \quad R \equiv g^{\mu\nu} R_{\mu\nu} = R^{\lambda\mu}{}_{\lambda\mu}$$

# Homogeneous and isotropic 3d manifolds

$$dl^2 = d\rho^2 + r^2(\rho)(d\theta^2 + \sin^2 \theta d\phi^2)$$

$$r(\rho) = \begin{cases} R \sin(\rho/R), & \text{3-sphere} \\ \rho, & \text{3-plane} \\ R \sinh(\rho/R), & \text{3-hyperboloid} \end{cases}$$

$\rho$  is a geodesic distance;

$$S = 4\pi r^2(\rho);$$

$$\Delta\theta = \frac{l}{r(\rho)}$$

$$d\rho^2 = \frac{dr^2}{\cosh^2 \frac{\rho}{R}} = \frac{dr^2}{1 + \frac{r^2}{R^2}}$$

$$d\rho^2 = \frac{dr^2}{\cos^2 \frac{\rho}{R}} = \frac{dr^2}{1 - \frac{r^2}{R^2}}$$

$$dl^2 = \frac{dr^2}{1 - \kappa \frac{r^2}{R^2}} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

# Homogeneous and isotropic 3d manifolds

$$dl^2 = \gamma_{ji} dx^i dx^j = \frac{dr^2}{1 - \varkappa \frac{r^2}{R^2}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$\varkappa = \begin{cases} +1, & \text{3-sphere} \\ 0, & \text{3-plane} \\ -1, & \text{3-hyperboloid} \end{cases}$$

$${}^{(3)}R_{ijkl} = \frac{\varkappa}{R^2} (\gamma_{ik} \gamma_{jl} - \gamma_{il} \gamma_{jk})$$

$${}^{(3)}R_{ij} = 2 \frac{\varkappa}{R^2} \gamma_{ij},$$

$${}^{(3)}R = 6\varkappa R^{-2}$$

# FRW(L) metric

$$ds^2 = dt^2 - a^2(t) \gamma_{ij} dx^i dx^j ,$$

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

Special frame: different parts look similar

Also this is comoving frame: world lines of particles at rest are geodesics,

$$\frac{du^\mu}{ds} + \Gamma_{\nu\lambda}^\mu u^\nu u^\lambda = 0 .$$

$$\gamma_{ij} \approx \delta_{ij}$$

# Photons in the expanding Universe

$$S = -\frac{1}{4} \int d^4x \sqrt{-g} g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho}$$

$$dt = a d\eta$$

conformally flat metric

$$ds^2 = dt^2 - a^2(t) \delta_{ij} dx^i dx^j \longrightarrow ds^2 = a^2(\eta) [d\eta^2 - \delta_{ij} dx^i dx^j]$$

$$S = -\frac{1}{4} \int d^4x \eta^{\mu\nu} \eta^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho},$$

$$A_\mu^{(\alpha)} = e_\mu^{(\alpha)} e^{ik\eta - i\mathbf{k}\mathbf{x}}, \quad k = |\mathbf{k}|$$

$$\Delta x = 2\pi/k, \quad \Delta\eta = 2\pi/k$$

$$\lambda(t) = a(t) \Delta x = 2\pi \frac{a(t)}{k}, \quad T = a(t) \Delta\eta = 2\pi \frac{a(t)}{k}$$



Redshift and the Hubble law  $\lambda_0 = \lambda_i \frac{a_0}{a(t_i)} \equiv \lambda_i(1 + z(t_i))$

$$\mathbf{p}(t) = \frac{\mathbf{k}}{a(t)}, \quad \omega(t) = \frac{k}{a(t)}$$

for not very distant objects

$$a(t_i) = a_0 - \dot{a}(t_0)(t_0 - t_i) \longrightarrow a(t_i) = a_0[1 - H_0(t_0 - t_i)]$$

$$z(t_i) = H_0(t_0 - t_i) = H_0 r, \quad z \ll 1$$

$$H_0 = h \cdot 100 \frac{\text{km}}{\text{s} \cdot \text{Mpc}}, \quad h = 0.73^{+0.04}_{-0.03}$$

similar reddening for other relativistic particles (small  $H$ ,  $\dot{H}$ , etc.)

$$\mathbf{p} = \frac{\mathbf{k}}{a(t)}$$

is true for massive particles as well

# Gas of free particles in the expanding Universe

homogeneous gas  
in comoving coordinates:

$$dN = f(t, \mathbf{p}) d^3\mathbf{X} d^3\mathbf{p}$$

$$d^3\mathbf{x} = \text{const}, \quad d^3\mathbf{k} = \text{const}, \quad f(k) = \text{const}$$

$$f(k) d^3\mathbf{x} d^3\mathbf{k} = \text{const}$$

comoving volume equals physical volume

$$d^3\mathbf{x} d^3\mathbf{k} = d^3(a\mathbf{x}) d^3\left(\frac{\mathbf{k}}{a}\right) = d^3\mathbf{X} d^3\mathbf{p}$$

$$f(\mathbf{p}, t) = f(\mathbf{k}) = f[a(t) \cdot \mathbf{p}].$$

$$t = t_i : f_i(\mathbf{p}) \longrightarrow f(\mathbf{p}, t) = f_i\left(\frac{a(t)}{a(t_i)} \mathbf{p}\right)$$

Massless bosons (photons)

fermions

$$f_i(\mathbf{p}) = f_{Pl} \left( \frac{|\mathbf{p}|}{T_i} \right) = \frac{1}{(2\pi)^3} \frac{1}{e^{|\mathbf{p}|/T_i} - 1}$$

$$f(\mathbf{p}, t) = f \left( \frac{a(t)|\mathbf{p}|}{a_i T_i} \right) = f \left( \frac{|\mathbf{p}|}{T_{eff}(t)} \right)$$

$$T_{eff}(t) = \frac{a_i}{a(t)} T_i$$

decoupling at  $T \gg m$  :

neutrinos, hot(warm) dark matter

$$\text{decoupling at } T \ll m : f(\mathbf{p}) = \frac{1}{(2\pi)^3} \exp \left( -\frac{m - \mu_i}{T_i} \right) \exp \left( -\frac{a^2(t)\mathbf{p}^2}{2ma_i^2 T_i} \right)$$

$$f(\mathbf{p}, t) = \frac{1}{(2\pi)^3} \exp \left( -\frac{m - \mu_{eff}}{T_{eff}} \right) \exp \left( -\frac{\mathbf{p}^2}{2mT_{eff}} \right)$$

$$T_{eff}(t) = \left( \frac{a_i}{a(t)} \right)^2 T_i, \quad \frac{m - \mu_{eff}(t)}{T_{eff}} = \frac{m - \mu_i}{T_i}$$

## Einstein equations

$$ds^2 = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j ,$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} .$$

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{0i} = 0, \quad R_{ij} = (\ddot{a}a + 2\dot{a}^2 + 2\kappa)\gamma_{ij},$$

$$R = g^{\mu\nu}R_{\mu\nu} = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2}\right)$$

$T_{\mu\nu}$ : macroscopic description

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - g_{\mu\nu}p$$

liquid with  $\rho(t)$  and  $p(t)$

in the comoving frame  $u^0 = 1, \mathbf{u} = 0$

$$T_{\mu\nu} = \text{diag}(\rho, a^2(t)\gamma_{ij}p)$$

Friedman equation (00) : 
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{\kappa}{a^2}$$

$$\nabla_{\mu} T^{\mu 0} = 0 \longrightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

the equation of state

$$p = p(\rho)$$

many-component liquid,  
in case of thermal equilibrium

other equations

$$-3d(\ln a) = \frac{d\rho}{\rho + p} = d(\ln s)$$

entropy is conserved in a comoving frame

$$sa^3 = \text{const}$$

# Examples of cosmological solutions

$$\kappa = 0 \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho$$

dust:

$$p = 0$$

singular at  $t = t_s$ 

$$\rho = \frac{\text{const}}{a^3}, \quad a(t) = \text{const} \cdot (t - t_s)^{2/3}, \quad \rho(t) = \frac{\text{const}}{(t - t_s)^2}$$

$$t_s = 0, \quad H(t) = \frac{\dot{a}}{a}(t) = \frac{2}{3t}, \quad \rho = \frac{3}{8\pi G} H^2 = \frac{1}{6\pi G} \frac{1}{t^2}$$

the Universe is too young

$$t_0 = \frac{2}{3H_0} = 0.93 \cdot 10^{10} \text{ yr} \quad (h = 0.7)$$

# Cosmological (particle) horizon $l_H(t)$

distance covered by photons emitted at  $t = 0$

the size of causally-connected region — the size of the visible part of the Universe

in conformal coordinates:  $ds^2 = 0 \rightarrow |d\mathbf{x}| = d\eta$   
coordinate size of the horizon equals  $\eta(t)$

$$l_H(t) = a(t)\eta(t) = a(t) \int_0^t \frac{dt'}{a(t')}$$

dust

$$l_H(t) = 3t = \frac{2}{H(t)}, \quad l_{H,0} = 2.7 \cdot 10^{28} \text{ cm} \quad (h = 0.7)$$

# Examples of cosmological solutions

radiation:

$$\rho = \frac{1}{3}\rho$$

singular at  $t = t_s$

$$\rho = \frac{\text{const}}{a^4}, \quad a(t) = \text{const} \cdot (t - t_s)^{1/2}, \quad \rho(t) = \frac{\text{const}}{(t - t_s)^2}$$

$$t_s = 0, \quad H(t) = \frac{\dot{a}}{a}(t) = \frac{1}{2t}, \quad \rho = \frac{3}{8\pi G} H^2 = \frac{3}{32\pi G} \frac{1}{t^2}$$

$$l_H(t) = a(t) \int_0^t \frac{dt'}{a(t')} = 2t = \frac{1}{H(t)}.$$

In case of thermal equilibrium

$$\rho_b = \frac{\pi^2}{30} g_b T^4, \quad \rho_f = \frac{7}{8} \frac{\pi^2}{30} g_f T^4$$

$$\rho = \frac{\pi^2}{30} g_* T^4, \quad g_* = \sum_b g_b + \frac{7}{8} \sum_f g_f$$



$$\rho = \frac{\pi^2}{30} g_* T^4, \quad g_* = \sum_b g_b + \frac{7}{8} \sum_f g_f$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho \longrightarrow H = \frac{T^2}{M_{Pl}^*}$$

$$M_{Pl}^* = \sqrt{\frac{90}{8\pi^3 g_*}} M_{Pl} = \frac{1}{1.66\sqrt{g_*}} M_{Pl}.$$

$$g_* = g_*(T)$$

$$T(t) \approx \frac{\text{const}}{a(t)}$$

# Examples of cosmological solutions

vacuum:

$$T_{\mu\nu} = \rho_{vac} \eta_{\mu\nu}$$

$$p = -\rho$$

$$S_G = -\frac{1}{16\pi G} \int R \sqrt{-g} d^4x, \quad S_\Lambda = -\Lambda \int \sqrt{-g} d^4x.$$

$$a = \text{const} \cdot e^{H_{ds} t}, \quad H_{ds} = \sqrt{\frac{8\pi}{3} G \rho_{vac}}$$

de Sitter space: space-time of constant curvature

$$ds^2 = dt^2 - e^{2H_{ds}t} d\mathbf{x}^2$$

$\ddot{a} > 0$ ,

no initial singularity

$$ds^2 = dt^2 - e^{2H_{dS}t} d\mathbf{x}^2$$

no cosmological horizon:  $l_H(t) = e^{H_{dS}t} \int_{-\infty}^t dt' e^{-H_{dS}t'} = \infty$

de Sitter (events) horizon ( $\mathbf{x} = 0, t$ ):

from which distance  $l(t)$  one can detect light emitted at  $t'$ ?

in conformal coordinates:  $ds^2 = 0 \longrightarrow |d\mathbf{x}| = d\eta$

coordinate size:  $\eta(t \rightarrow \infty) - \eta(t) = \int_t^\infty \frac{dt'}{a(t')}$

physical size:  $l_{dS} = a(t) \int_t^\infty \frac{dt'}{a(t')} = \frac{1}{H_{dS}}$

observer will never be informed what happens at distances larger

$$l_{dS} = H_{dS}^{-1}$$

Constant expansion:

$$\ddot{a} = 0$$

negative curvature

marginal matter:

$$p = w\rho,$$

gas of straight strings

$$w = -\frac{1}{3}, \quad p = -\frac{1}{3}\rho$$

Big Rip...

$$w < -1$$