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# Introduction to Rigid Supersymmetric Theories ‡

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*Abstract*

In these lectures we discuss the supersymmetry algebra and its irreducible representations. We construct the theories of rigid supersymmetry and gave their superspace formulations. The perturbative quantum properties of the extended supersymmetric theories are derived, including the superconformal invariance of a large class of these theories as well as the chiral effective action for $N = 2$ Yang-Mills theory. The superconformal transformations in four dimensional superspace are derived and encoded into one superconformal Killing superfield. It is also shown that the anomalous dimensions of chiral operators in a superconformal quantum field are related to their $R$ weight.

Some of this material follows the book of reference [0] by the author. Certain chapters of this book are reproduced, however, in other sections the reader is referred to the relevant parts of reference [0]. In this review, chapter 6 on superconformal theories and three sections of chapter 5 on flat directions, non-holomorphicity and the chiral effective action for $N = 2$ Yang-Mills theory are new material.

The aim of the lectures is to provide the reader with the material required to understand more recent developments in the non-perturbative properties of quantum extended supersymmetric theories.

‡ This material is based on lectures presented at the Nato conference on "Confinement, Duality, and Nonperturbative Aspects of QCD", the Issac Newton Institute, Cambridge, UK and at the TASI 1998 Summer School, Boulder, Colorado, USA.

## Contents



## Lecture 1. The Supersymmetry Algebra

This section is identical to chapter 2 of reference [0]. The equation numbers are keep the same as in this book. I thank World Scientific Publishing for their permission to reproduced this material.

In the 1960's, with the growing awareness of the significance of internal symmetries such as $SU(2)$ and larger groups, physicists attempted to find a symmetry which would combine in a non-trivial way the space-time Poincaré group with an internal symmetry group. After much effort it was shown that such an attempt was impossible within the context of a Lie group. Coleman and Mandula[4] showed on very general assumptions that any Lie group which contained the Poincaré group $P$, whose generators $P_a$ and $J_{ab}$ satisfy the relations

$$\begin{aligned}
[P_a, P_b] &= 0 \\
[P_a, J_{bc}] &= (\eta_{ab} P_c - \eta_{ac} P_b) \\
[J_{ab}, J_{cd}] &= -(\eta_{ac} J_{bd} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc} - \eta_{bc} J_{ad})
\end{aligned} \tag{2.1}$$

and an internal symmetry group $G$ with generators $T_s$ such that

$$[T_r, T_s] = f_{rst} T_t \tag{2.2}$$

must be a direct product of $P$ and $G$; or in other words

$$[P_a, T_s] = 0 = [J_{ab}, T_s] \tag{2.3}$$

They also showed that $G$ must be of the form of a semisimple group with additional $U(1)$ groups.

It is worthwhile to make some remarks concerning the status of this no-go theorem. Clearly there are Lie groups that contain the Poincaré group and internal symmetry groups in a non-trivial manner; however, the theorem states that these groups lead to trivial physics. Consider, for example, two-body scattering; once we have imposed conservation of angular momentum and momentum the scattering angle is the only unknown quantity. If there were a Lie group that had a non-trivial mixing with the Poincaré group then there would be further generators associated with space-time. The resulting conservation laws will further constrain, for example, two-body scattering, and so the scattering angle can only take on discrete values. However, the scattering process is expected to be analytic in the scattering angle, $\theta$, and hence we must conclude that the process does not depend on $\theta$ at all.

Essentially the theorem shows that if one used a Lie group that contained an internal group which mixed in a non-trivial manner with the Poincaré group then the S-matrix for all processes would be zero. The theorem assumes among other things, that the S-matrix exists and is non-trivial, the vacuum is non-degenerate and that there are no massless particles. It is important to realise that the theorem only applies to symmetries that act on S-matrix elements and not on all the other many symmetries that occur in quantum field theory. Indeed it is not uncommon to find examples of the latter symmetries. Of course, no-go theorems are only as strong as the assumptions required to prove them.

In a remarkable paper Gelfand and Likhtman [1] showed that provided one generalised the concept of a Lie group one could indeed find a symmetry that included the Poincaré group and an internal symmetry group in a non-trivial way. In this section we will discuss this approach to the supersymmetry group; having adopted a more general notion of a group, we will show that one is led, with the aid of the Coleman-Mandula theorem, and a few assumptions, to the known supersymmetry group. Since the structure of a Lie group, at least in some local region of the identity, is determined entirely by its Lie algebra it is necessary to adopt a more general notion than a Lie algebra. The vital step in discovering the supersymmetry algebra is to introduce generators $Q^i_\alpha$, which satisfy anti-commutation relations, i.e.

$$\begin{aligned}\{Q^i_\alpha, Q^j_\beta\} &= Q^i_\alpha Q^j_\beta + Q^j_\beta Q^i_\alpha \\ &= \text{ some other generator}\end{aligned} \tag{2.4}$$

The significance of the $i$ and $\alpha$ indices will become apparent shortly. Let us therefore assume that the supersymmetry group involves generators $P_a$, $J_{ab}$, $T_s$ and possibly some other generators which satisfy commutation relations, as well as the generators $Q^i_\alpha$ ($i = 1, 2, \ldots, N$). We will call the former generators which satisfy Eqs. (2.1), (2.2) and (2.3) to be even and those satisfying Eq. (2.4) to be odd generators.

Having let the genie out of the bottle we promptly replace the stopper and demand that the supersymmetry algebras have a $Z_2$ graded structure. This simply means that the even and odd generators must satisfy the rules:

$$\begin{aligned} [\text{even}, \text{even}] &= \text{even} \\ \{\text{odd}, \text{odd}\} &= \text{even} \\ [\text{even}, \text{odd}] &= \text{odd} \end{aligned} \tag{2.5}$$

We must still have the relations

$$[P_a, T_s] = 0 = [J_{ab}, T_s] \tag{2.6}$$

since the even (bosonic) subgroup must obey the Coleman-Mandula theorem.

Let us now investigate the commutator between $J_{ab}$ and $Q^i_\alpha$. As a result of Eq.(2.5) it must be of the form

$$[Q^i_\alpha, J_{ab}] = (b_{ab})_\alpha^\beta Q^i_\beta \tag{2.7}$$

since by definition the $Q^i_\alpha$ are the only odd generators. We take the $\alpha$ indices to be those rotated by $J_{ab}$. As in a Lie algebra we have some generalised Jacobi identities. If we denote an even generator by $B$ and an odd generator by $F$ we find that

$$\begin{aligned} \big[[B_1, B_2], B_3\big] + \big[[B_3, B_1], B_2\big] + \big[[B_2, B_3], B_1\big] &= 0 \\ \big[[B_1, B_2], F_3\big] + \big[[F_3, B_1], B_2\big] + \big[[B_2, F_3], B_1\big] &= 0 \\ \{[B_1, F_2], F_3\} + \{[B_1, F_3], F_2\} + [\{F_2, F_3\}, B_1] &= 0 \\ [\{F_1, F_2\}, F_3] + [\{F_1, F_3\}, F_2] + [\{F_2, F_3\}, F_1] &= 0 \end{aligned} \tag{2.8}$$

The reader may verify, by expanding each bracket, that these relations are indeed identically true.

The identity

$$\big[[J_{ab}, J_{cd}], Q^i_\alpha\big] + \big[[Q^i_\alpha, J_{ab}], J_{cd}\big] + \big[[J_{cd}, Q^i_\alpha], J_{ab}\big] = 0 \tag{2.9}$$

upon use of Eq. (2.7) implies that

$$[b_{ab}, b_{cd}]_\alpha^\beta = -\eta_{ac}(b_{bd})_\alpha^\beta - \eta_{bd}(b_{ac})_\alpha^\beta + \eta_{ad}(b_{bc})_\alpha^\beta + \eta_{bc}(b_{ad})_\alpha^\beta \tag{2.10}$$

This means that the $(b_{cd})_\alpha^\beta$ form a representation of the Lorentz algebra or in other words the $Q^i_\alpha$ carry a representation of the Lorentz group. We will select $Q^i_\alpha$ to be in the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation of the Lorentz group, i.e.

$$[Q^i_\alpha, J_{ab}] = \frac{1}{2}(\sigma_{ab})_\alpha^\beta Q^i_\beta \tag{2.11}$$

We can choose $Q^i_\alpha$ to be a Majorana spinor, i.e.

$$Q^i_\alpha = C_{\alpha\beta}\bar{Q}^{\beta i} \tag{2.12}$$

where $C_{\alpha\beta} = -C_{\beta\alpha}$ is the charge conjugation matrix (see appendix A of [0]). This does not represent a loss of generality since, if the algebra admits complex conjugation as an involution we can always redefine the supercharges so as to satisfy (2.12) (see Note 1 at the end of this chapter).

The above calculation reflects the more general result that the $Q^i_\alpha$ must belong to a realization of the even (bosonic) subalgebras of the supersymmetry group. This is a simple consequence of demanding that the algebra be $Z_2$ graded. The commutator of any even generator $B_1$, with $Q^i_\alpha$ is of the form

$$[Q^i_\alpha, B_1] = (h_1)^{i\beta}_{\alpha j} Q^j_\beta \tag{2.13}$$

The generalised Jacobi identity

$$\big[[Q^i_\alpha, B_1], B_2\big] + \big[[B_1, B_2], Q^i_\alpha\big] + \big[[B_2, Q^i_\alpha], B_1\big] = 0 \tag{2.14}$$

implies that

$$[h_1, h_2]^{i\beta}_{\alpha j} Q^j_\beta = \big[Q^i_\alpha [B_1, B_2]\big] \tag{2.15}$$

or in other words the matrices $h$ represent the Lie algebra of the even generators.

The above remarks imply that

$$[Q^i_\alpha, T_r] = (l_r)^i_j Q^j_\alpha + (t_r)^i_j (i\gamma_5)^\beta_\alpha Q^j_\beta \tag{2.16}$$

where $(l_r)^i_j + i\gamma_5 (t_r)^i_j$ represent the Lie algebra of the internal symmetry group. This results from the fact that $\delta^\alpha_\beta$ and $(\gamma_5)^\beta_\alpha$ are the only invariant tensors which are scalar and pseudoscalar.

The remaining odd-even commutator is $[Q^i_\alpha, P_a]$. A possibility that is allowed by the generalised Jacobi identities that involve the internal symmetry group and the Lorentz group is

$$[Q^i_\alpha, P_a] = c(\gamma_a)^\beta_\alpha Q^i_\beta \tag{2.17}$$

However, the $\big[[Q^i_\alpha, P_a], P_b\big] + \ldots$ identity implies that the constant $c = 0$, i.e.

$$[Q^i_\alpha, P_a] = 0 \tag{2.18}$$

More generally we could have considered $(c\gamma_a + d\gamma_a\gamma_5)Q$, on the right-hand side of (2.17), however, then the above Jacobi identity and the Majorana condition imply that $c = d = 0$. (See Note 2 at the end of this chapter). Let us finally consider the $\{Q^i_\alpha, Q^j_\beta\}$ anticommutator. This object must be composed of even generators and must be symmetric under interchange of $\alpha \leftrightarrow \beta$ and $i \leftrightarrow j$. The even generators are those of the Poincaré group, the internal symmetry group and other even generators which, from the Coleman-Mandula theorem, commute with the Poincaré group, i.e. they are scalar and pseudoscalar. Hence the most general possibility is of the form

$$\{Q^i_\alpha, Q^j_\beta\} = r(\gamma^a C)_{\alpha\beta} P_a \delta^{ij} + s(\sigma^{ab} C)_{\alpha\beta} J_{ab} \delta^{ij} + C_{\alpha\beta} U^{ij} + (\gamma_5 C)_{\alpha\beta} V^{ij} \tag{2.19}$$

In this equation $U^{ij}$ and $V^{ij}$ are new generators which we will discuss further below. We have not included a $(\gamma^b\gamma_5 C)_{\alpha\beta} L_b^{ij}$ term as the $(Q, Q, J_{ab})$ Jacobi identity implies that $L_b^{ij}$ mixes nontrivially with the Poincaré group and so is excluded by the no-go theorem.

The fact that we have only used numerically invariant tensors under the Poincaré group is a consequence of the generalised Jacobi identities between two odd and one even generators.

To illustrate the argument more clearly, let us temporarily specialise to the case $N = 1$ where there is only one supercharge $Q_\alpha$. Equation (2.19) then reads

$$\{Q_\alpha, Q_\beta\} = r(\gamma^a C)_{\alpha\beta} P_a + s(\sigma^{ab} C)_{\alpha\beta} J_{ab}.$$

Using the Jacobi identity

$$\{[P_a, Q_\alpha], Q_\beta\} + \{[P_a, Q_\beta], Q_\alpha\} + [\{Q_\alpha, Q_\beta\}, P_a] = 0,$$

we find that

$$0 = s(\sigma^{cd} C)_{\alpha\beta}[J_{cd}, P_a] = s(\sigma^{cd} C)_{\alpha\beta}(-\eta_{ac} P_d + \eta_{ad} P_c),$$

and, consequently, $s = 0$. We are free to scale the generator $P_a$ in order to bring $r = 2$.

Let us now consider the commutator of the generator of the internal group and the supercharge. For only one supercharge, Eq. (2.16) reduces to

$$[Q_\alpha, T_r] = l_r Q_\alpha + i(\gamma_5)_\alpha^{\ \beta} t_r Q_\beta.$$

Taking the adjoint of this equation, multiplying by $(i\gamma^0)$ and using the definition of the Dirac conjugate given in appendix A of reference [0], we find that

$$[\bar{Q}^\alpha, T_r] = l_r^* \bar{Q}^\alpha + \bar{Q}^\beta (i t_r^*)(\gamma_5)_\beta^{\ \alpha}.$$

Multiplying by $C_{\gamma\alpha}$ and using Eq. (2.12), we arrive at the equation

$$[Q_\alpha, T_r] = l_r^* Q_\alpha + i t_r^* (\gamma_5)_\alpha^{\ \beta} Q_\beta.$$

Comparing this equation with the one we started from, we therefore conclude that

$$l_r^* = l_r, \qquad t_r^* = t_r.$$

The Jacobi identity

$$[\{Q_\alpha, Q_\beta\}, T_r] + \big[[T_r, Q_\alpha], Q_\beta\big] + \big[[T_r, Q_\beta], Q_\alpha\big] = 0$$

results in the equation

$$\begin{aligned}&[0 + \big(l_r \delta_\alpha^\gamma + i t_r (\gamma_5)\alpha^\gamma\big) 2(\gamma_a C)_{\gamma\beta} P_a] + (\alpha \leftrightarrow \beta) = \\ &\quad 2P_a\{l_r(\gamma_a C)_{\alpha\beta} + i t_r(\gamma_5\gamma_a C)_{\alpha\beta}\} + (\alpha \leftrightarrow \beta) = 0.\end{aligned}$$

Since $(\gamma_a C)_{\alpha\beta}$ and $(\gamma_5 \gamma_a C)_{\alpha\beta}$ are symmetric and antisymmetric in $\alpha\beta$ respectively, we conclude that $l_r = 0$ but $t_r$ has no constraint placed on it. Consequently, we find that we have only one internal generator $R$ and we may scale it such that

$$[Q_\alpha, R] = i(\gamma_5)_\alpha^\beta Q_\beta.$$

The $N = 1$ supersymmetry algebra is summarised in Eq. (2.27).

Let us now return to the extend supersymmetry algebra. The even generators $U^{ij} = -U^{ji}$ and $V^{ij} = -V^{ji}$ are called central charges [5] and are often also denoted by $Z^{ij}$. It is a consequence of the generalised Jacobi identities $\big((Q,Q,Q)$ and $(Q,Q,Z)\big)$ that they commute with all other generators including themselves, i.e.

$$[U^{ij}, \text{anything}] = 0 = [V^{ij}, \text{anything}] \tag{2.20}$$

We note that the Coleman-Mandula theorem allowed a semi-simple group plus $U(1)$ factors. The details of the calculation are given in note 5 at the end of the chapter. Their role in supersymmetric theories will emerge in later chapters.

In general, we should write, on the right-hand side of (2.19), $(\gamma^a C)_{\alpha\beta}\omega^{ij} P_a + \ldots$, where $\omega^{ij}$ is an arbitrary real symmetric matrix. However, one can show that it is possible to redefine (rotate and rescale) the supercharges, whilst preserving the Majorana condition, in such a way as to bring $\omega^{ij}$ to the form $\omega^{ij} = r\delta^{ij}$ (see Note 3 at the end of this chapter). The $[P_a, \{Q_\alpha^i, Q_\beta^j\}] + \ldots = 0$ identity implies that $s = 0$ and we can normalise $P_a$ by setting $r = 2$ yielding the final result

$$\{Q_\alpha^i, Q_\beta^j\} = 2(\gamma_a C)_{\alpha\beta}\delta^{ij} P^a + C_{\alpha\beta} U^{ij} + (\gamma_5 C)_{\alpha\beta} V^{ij} \tag{2.21}$$

In any case $r$ and $s$ have different dimensions and so it would require the introduction of a dimensional parameter in order that they were both non-zero.

Had we chosen another irreducible Lorentz representation for $Q_\alpha^i$ other than $(j + \frac{1}{2}, j) \oplus (j, j + \frac{1}{2})$ we would not have been able to put $P_a$, i.e. a $(\frac{1}{2}, \frac{1}{2})$ representation, on the right-hand side of Eq. (2.21). The simplest choice is $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$. In fact this is the only possible choice (see Note 4).

Finally, we must discuss the constraints placed on the internal symmetry group by the generalised Jacobi identity. This discussion is complicated by the particular way the Majorana constraint of Eq. (2.12) is written. A two-component version of this constraint is

$$\bar{Q}_{\dot{A}i} = (Q_A^i)^*; \quad A, \dot{A} = 1, 2 \tag{2.22}$$

(see A of [0] for two-component notation). Equation (2.19) and (2.16) then become

$$\begin{aligned}
\{Q_A^i, \bar{Q}_{\dot{B}j}\} &= -2i(\sigma^a)_{A\dot{B}}\delta_j^i P_a \\
\{Q_A^i, Q_B^j\} &= \varepsilon_{AB}(U^{ij} + iV^{ij}) \\
[Q_A^i, J_{ab}] &= +\frac{1}{2}(\sigma_{ab})_A^B Q_B^i
\end{aligned} \tag{2.23}$$

and

$$[Q^i_A, T_r] = (l_r + it_r)^i_j Q^j_A \tag{2.24}$$

Taking the complex conjugate of the last equation and using the Majorana condition we find that

$$[Q_{\dot{A}i}, T_r] = Q_{\dot{A}k}(U^\dagger_r)^k_i \tag{2.25}$$

where $(U_r)^i_j = (l_r + it_r)^i_j$. The $(Q, \bar{Q}, T)$ Jacobi identity then implies that $\delta^i_j$ be an invariant tensor of $G$, i.e.

$$U_r + U^\dagger_r = 0 \tag{2.26}$$

Hence $U_r$ is an antihermitian matrix and so represents the generators of the unitary group $U(N)$. However, taking account of the central charge terms in the $(Q, Q, T)$ Jacobi identity one finds that there is for every central charge an invariant antisymmetric tensor of the internal group and so the possible internal symmetry group is further reduced. If there is only one central charge, the internal group is $Sp(N)$ while if there are no central charges it is $U(N)$.

To summarise, once we have adopted the rule that the algebra be $Z_2$ graded and contain the Poincaré group and an internal symmetry group then the generalised Jacobi identities place very strong constraints on any possible algebra. In fact, once one makes the further assumption that $Q^i_\alpha$ are spinors under the Lorentz group then the algebra is determined to be of the form of equations (2.1), (2.6), (2.11), (2.16), (2.18) and (2.21).

The simplest algebra is for $N = 1$ and takes the form

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= 2(\gamma_a C)_{\alpha\beta} P^a \\
[Q_\alpha, P_a] &= 0 \\
[Q_\alpha, J_{cd}] &= \frac{1}{2}(\sigma_{cd})^\beta_\alpha Q_\beta \\
[Q_\alpha, R] &= i(\gamma_5)^\beta_\alpha Q_\beta
\end{aligned} \tag{2.27}$$

as well as the commutation relations of the Poincaré group. We note that there are no central charges (i.e. $U^{11} = V^{11} = 0$), and the internal symmetry group becomes just a chiral rotation with generator $R$.

We now wish to prove five of the statements above. This is done here rather than in the above text, in order that the main line of argument should not become obscured by technical points. These points are best clarified in two-component notation.

*Note 1:* Suppose we have an algebra that admits a complex conjugation as an involution; for the supercharges this means that

$$(Q^i_A)^* = b_i{}^j Q_{\dot{A}j}; \quad (Q_{\dot{A}j})^* = d^j{}_k Q^k_A$$

There is no mixing of the Lorentz indices since $(Q^i_A)^*$ transforms like $Q_{\dot{A}i}$, namely in the $(0, \frac{1}{2})$ representation of the Lorentz group, and not like $Q^i_A$ which is in the $(\frac{1}{2}, 0)$ representation. The lowering of the $i$ index under $*$ is at this point purely a notational device. Two successive $*$ operations yield the unit operation and this implies that

$$(b_i{}^j)^* d^j{}_k = \delta^i_k \tag{2.28}$$

and in particular that ${b_i}^j$ is an invertible matrix. We now make the redefinitions

$$\begin{aligned} {Q'}^i_A &= Q^i_A \\ {Q'}_{\dot{A}i} &= {b_i}^j Q_{\dot{A}j} \end{aligned} \tag{2.29}$$

Taking the complex conjugate of ${Q'}_{Ai}$, we find

$$({Q'}^i_A)^* = (Q^i_A)^* = {b_i}^j Q_{\dot{A}j} = {Q'}_{\dot{A}j}$$

while

$$({Q'}_{\dot{A}i})^* = ({b_i}^j)^* (Q_{\dot{A}j})^* = ({b_i}^j)^* {d^j}_k Q^k_A = Q^i_A \tag{2.30}$$

using Eq. (2.28).

Thus the ${Q'}^i_A$ satisfy the Majorana condition, as required. If the $Q$'s do not initially satisfy the Majorana condition, we may simply redefine them so that they do.

*Note 2:* Suppose the $[Q_A, P_a]$ commutator were of the form

$$[Q_A, P_a] = e(\sigma_a)_{A\dot{B}} Q^{\dot{B}} \tag{2.31}$$

where $e$ is a complex number and for simplicity we have suppressed the $i$ index. Taking the complex conjugate (see A of reference [0]), we find that

$$[Q_{\dot{A}}, P_a] = -e^* (\sigma_a)_{B\dot{A}} Q^B \tag{2.32}$$

Consideration of the $\big[[Q_A, P_a], P_b\big] + \ldots = 0$ Jacobi identity yields the result

$$-|e|^2 (\sigma_a)_{A\dot{B}} (\sigma^b)^{C\dot{B}} - (a \leftrightarrow b) = 0 \tag{2.33}$$

Consequently $e = 0$ and we recover the result

$$[Q_A, P_a] = 0 \tag{2.34}$$

*Note 3:* The most general form of the $Q^{Ai}, Q^{\dot{B}}_j$ anticommutator is

$$\{Q^{Ai}, Q^{\dot{B}}_j\} = -2i U^i_j (\sigma^m)^{A\dot{B}} P_m + \text{terms involving other Dirac matrices} \tag{2.35}$$

Taking the complex conjugate of this equation and comparing it with itself, we find that $U$ is a Hermitian matrix

$$(U^i_j)^* = U^j_i \tag{2.36}$$

We now make a field redefinition of the supercharge

$${Q'}^{Ai} = B^i_j Q^{Aj} \tag{2.37}$$

and its complex conjugate

$${Q'}^{\dot{A}}_i = (B^i_j)^* Q^{\dot{A}}_j \tag{2.38}$$

Upon making this redefinition in Eq. (2.35), the $U$ matrix becomes replaced by

$$U'^i_j = B^i_k U^k_l (B^j_l)^* \quad \text{or} \quad U' = BUB^\dagger \tag{2.39}$$

Since $U$ is a Hermitian matrix, we may diagonalise it in the form $c_i \delta^i_j$ using a unitarity matrix $B$. We note that this preserves the Majorana condition on $Q^{Ai}$. Finally, we may scale $Q^i \to (1/\sqrt{c}^i) Q^i$ to bring $U$ to the form $U = d_i \delta^i_j$, where $d_i = \pm 1$. In fact, taking $A = B = 1$ and $i = j = k$, we realise that the right-hand side of Eq. (2.35) is a positive definite operator and since the energy $-iP_0$ is assumed positive definite, we can only find $d_i = +1$. The final result is

$$\{Q^{Ai}, Q^{\dot{B}}_j\} = -2i\delta^i_j (\sigma^m)^{A\dot{B}} P_m \tag{2.40}$$

*Note 4:* Let us suppose that the supercharge $Q$ contains an irreducible representation of the Lorentz group other than $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$, say, the representation $Q_{A_1 \ldots A_n, \dot{B}_1 \ldots \dot{B}_m}$ where the $A$ and $B$ indices are understood to be separately symmetrised and $n + m$ is odd in order that $Q$ is odd and $n + m > 1$. By projecting the $\{Q, Q^\dagger\}$ anticommutator we may find the anti-commutator involving $Q_{A_1 \ldots A_n, \dot{B}_1 \ldots \dot{B}_m}$ and its hermitian conjugate. Let us consider in particular the anticommutator involving $Q = Q_{11 \ldots 1, \mathrm{ii} \ldots \mathrm{i}}$, this must result in an object of spin $n + m > 1$. However, by the Coleman-Mandula no-go theorem no such generator can occur in the algebra and so the anticommutator must vanish, i.e. $QQ^\dagger + Q^\dagger Q = 0$.

Assuming the space on which $Q$ acts has a positive definite norm, one such example being the space of on-shell states, we must conclude that $Q$ vanishes. However if $Q_{11 \ldots 1, \mathrm{ii} \ldots \mathrm{i}}$ vanishes, so must $Q_{A_1 \ldots A_n, \dot{B}_1 \ldots \dot{B}_m}$ by its Lorentz properties, and we are left only with the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation.

*Note 5:* We now return to the proof of equation (2.20). Using the (Q,Q,Z) Jacobi identity it is straightforward to show that the supercharges $Q$ commute with the central charges $Z$. The (Q,Q,U) Jacobi identity then implies that the central charges commute with themselves. Finally, one considers the $(Q, Q, T_r)$ Jacobi identity. This relation shows that the commutator of $T_r$ and $Z$ takes the generic form $[T_r,\ Z] = \ldots Z$. However, the generators $T_r$ and $Z$ form the internal symmetry group of the supersymmetry algebra and from the no-go theorem we know that this group must be a semisimple Lie group times $U(1)$ factors. We recall that a semisimple Lie group is one that has no normal Abelian subgoups other that the group itself and the identity element. As such, we must conclude that $T_r$ and $Z$ commute, and hence our final result that the central charges commute with all generators, that is they really are central.

Although the above discussion started with the Poincaré group, one could equally well have started with the conformal or (anti-) de Sitter groups and obtained the superconformal and super (anti)de Sitter algebras. For completeness, we now list these algebras. The superconformal algebra which has the generators $P_n, J_{mn}, D, K_n, A, Q^{\alpha i}$, $S^{\alpha i}$ and the internal symmetry generators $T_r$ and $A$ is given by the Lorentz group plus:

$$[J_{mn}, P_k] = \eta_{nk} P_m - \eta_{mk} P_n$$

$$
\begin{aligned}
[J_{mn}, K_k] &= \eta_{nk} K_m - \eta_{mk} K_n \\
[D, P_K] &= -P_K \quad [D, K_K] = +K_K \\
[P_m, K_n] &= -2J_{mn} + 2\eta_{mn} D \quad [K_n, K_m] = 0 \quad [P_n, P_m] = 0 \\
[Q^i_\alpha, J_{mn}] &= \frac{1}{2}(\gamma_{mn})_\alpha^\beta Q^i_\beta \quad [S^i_\alpha, J_{mn}] = \frac{1}{2}(\gamma_{mn})_\alpha^\beta S^{\beta i} \\
\{Q^i_\alpha, Q^j_\beta\} &= -2(\gamma^n C^{-1})_{\alpha\beta} P_n \delta^{ij} \\
\{S^i_\alpha, S^j_\beta\} &= +2(\gamma^n C^{-1})_{\alpha\beta} K_n \delta^{ij} \\
[Q^i_\alpha, D] &= \frac{1}{2} Q^i_\alpha \quad [S^i_\alpha, D] = -\frac{1}{2} S^i_\alpha \\
[Q^i_\alpha, K_n] &= -(\gamma_n)_\alpha^\beta S^i_\beta \quad [S^i_\alpha, P_n] = (\gamma_n)_\alpha^\beta Q^i_\beta \\
[Q^i_\alpha, T_r] &= \big(\delta_\alpha^\beta (\tau_{r_1})^i_j + (\gamma_5)_\alpha^\beta (\tau_{r_2})^i_j\big) Q^j_\beta \\
[S^i_\alpha, T_r] &= \big(\delta_\alpha^\beta (\tau_{r_1})^i_j - (\gamma_5)_\alpha^\beta (\tau_{r_2})^i_j\big) Q^j_\beta \\
[Q^i_\alpha, A] &= -i(\gamma_5)_\alpha^\beta Q^i_\beta \left(\frac{4-N}{4N}\right) \\
[S^i_\alpha, A] &= \frac{4-N}{4N} i(\gamma_5)_\alpha^\beta S^i_\beta \\
\{Q^i_\alpha, S^j_\beta\} &= -2(C^{-1}_{\alpha\beta}) D\delta^{ij} + (\gamma^{mn} C^{-1})_{\alpha\beta} J_{mn} \delta^{ij} + 4i(\gamma_5 C^{-1}_{\alpha\beta}) A\delta^{ij} \\
&\quad - 2(\tau_{r_1})^{ij} (C^{-1})_{\alpha\beta} + \big((\tau_{r_2})^{ij} (\gamma_5 C^{-1})_{\alpha\beta}\big) T_r
\end{aligned} \tag{2.41}
$$

The $T_r$ and $A$ generate $U(N)$ and $\tau_1 + \gamma_5 \tau_2$ are in the fundamental representation of $SU(N)$.

The case of $N = 4$ is singular and one can have either

$$
[Q^i_\alpha, A] = 0 \quad \text{or} \quad [Q^i_\alpha, A] = -i(\gamma_5)_\alpha^\beta Q^i_\beta
$$

and similarly for $S^i_\alpha$ and $A$. One may verify that both possibilities are allowed by the $N = 4$ Jacobi identities and so form acceptable superalgebras.

The anti-de Sitter superalgebra has generators $M_{mn}$, $T_{ij} = -T_{ji}$ and $Q^{\alpha i}$, and is given by

$$
\begin{aligned}
[M_{mn}, M_{pq}] &= \eta_{np} M_{mq} + 3 \text{ terms} \\
[M_{mn}, T_{ij}] &= 0 \quad [Q^i_\alpha, M_{mn}] = \frac{1}{2}(\gamma_{mn})_\alpha{}^\beta Q^i_\beta \\
[Q^i_\alpha, T^{jk}] &= -2i(\delta^{ij} Q^k_\alpha - \delta^{ik} Q^j_\alpha) \\
\{Q^i_\alpha, Q^j_\beta\} &= \delta^{ij} (\gamma_{mn} C^{-1})_{\alpha\beta} i M_{mn} + (C^{-1})_{\alpha\beta} T^{ij} \\
[T^{ij}, T^{kl}] &= -2i(\delta^{jk} T^{il} + 3 \text{ terms})
\end{aligned} \tag{2.42}
$$

## Lecture 2. Models of Rigid Supersymmetry

### 2.1 The Wess-Zumino Model

This section is identical to chapter 5 of reference [0]. The equation numbers are keep the same as in this book. I thank World Scientific Publishing for their permission to reproduced this material.

The first four-dimensional model in which supersymmetry was linearly realised was found by Wess and Zumino[3] by studying two-dimensional dual models[7]. In this chapter we rediscover supersymmetry along the lines given in Chapter 4 and discuss the Wess-Zumino model which is the simplest model of $N = 1$ supersymmetry.

Let us assume that the simplest model possesses one fermion $\chi_\alpha$ which is a Majorana spinor, i.e.

$$\chi_\alpha = C_{\alpha\beta}\bar{\chi}^\beta \tag{5.1}$$

On shell, that is, when

$$\partial\!\!\!/\chi = \text{ interaction} \tag{5.2}$$

$\chi_\alpha$ has two degrees of freedom or two helicity states. Applying our rule concerning equal numbers of fermionic and bosonic degrees of freedom of the previous chapter to the on-shell states we find that we must add two bosonic degrees of freedom to $\chi_\alpha$ in order to form a realization of supersymmetry. These could either be two spin-zero particles or one massless vector particle which also has two helicity states on-shell. We will consider the former possibility in this section and the latter possibility, which is the $N = 1$ Yang-Mills theory, in the next chapter.

In Chapter 8 we will show that these considerations are indeed correct. An irreducible representation of $N = 1$ supersymmetry can be carried either by one parity even spin-zero state, one parity odd spin-zero state and one Majorana spin-$\frac{1}{2}$, or by one massless spin-one and one Majorana spin-$\frac{1}{2}$. Taking the former possibility we have a Majorana spinor $\chi_\alpha$ and two spin-zero states which we will assume to be represented by a scalar field $A$ and pseudoscalar field $B$. For simplicity we will begin by constructing the free theory; the fields $A,\ B,\ \chi_\alpha$ are then subject to

$$\partial^2 A = \partial^2 B = \partial\!\!\!/\chi = 0 \tag{5.3}$$

We now wish to construct the supersymmetry transformations that are carried by this irreducible realization of supersymmetry. Since $\bar{\varepsilon}^\alpha Q_\alpha$ is dimensionless and $Q_\alpha$ has mass dimension $+\frac{1}{2}$, the parameter $\bar{\varepsilon}^\alpha$ must have dimension $-\frac{1}{2}$. On grounds of linearity, dimension, Lorentz invariance and parity we may write down the following set of transformations:

$$\begin{aligned} &\delta A = \bar{\varepsilon} Q A = \bar{\varepsilon}\chi \qquad \delta B = i\bar{\varepsilon}\gamma_5\chi \\ &\delta\chi = \partial\!\!\!/(\alpha A + \beta i\gamma_5 B)\varepsilon \end{aligned} \tag{5.4}$$

where $\alpha$ and $\beta$ are undetermined parameters.

The variation of $A$ is straightforward; however, the appearance of a derivative in $\delta\chi$ is the only way to match dimensions once the transformations are assumed to be linear. The reader will find no trouble verifying that these transformations do leave the set of field equations of Eq.(5.3) intact.

We can now test whether the $N = 1$ supersymmetry algebra of Chapter 2 is represented by these transformations. The commutator of two supersymmetries on $A$ is given by

$$[\delta_1, \delta_2]A = [\bar{\varepsilon}_1 Q, \bar{\varepsilon}_2 Q]A \tag{5.5}$$

which, using Eq.(2.27), becomes

$$\begin{aligned} [\delta_1, \delta_2]A &= 2\bar{\varepsilon}_2 \gamma^a \varepsilon_1 P_a A \\ &= 2\bar{\varepsilon}_2 \gamma^a \varepsilon_1 \partial_a A \\ \text{since} \qquad P_a &= \partial_a \end{aligned} \tag{5.6}$$

On the other hand the transformation laws of Eq.(5.4) imply that

$$\begin{aligned} [\delta_1, \delta_2]A &= \bar{\varepsilon}_2 \not{\partial}(\alpha A + i\gamma_5 \beta B)\varepsilon_1 - (1 \leftrightarrow 2) \\ &= 2\alpha \bar{\varepsilon}_2 \not{\partial} \varepsilon_1 A \end{aligned} \tag{5.7}$$

The term involving $B$ drops out because of the properties of Majorana spinors (see appendix A of reference [0]). Provided $\alpha = +1$ this is indeed the 4-translation required by the algebra. We therefore set $\alpha = +1$. The calculation for $B$ is similar and yields $\beta = +1$. For the field $\chi_\alpha$ the commutator of two supersymmetries gives the result

$$\begin{aligned} [\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\chi &= \not{\partial}[\bar{\varepsilon}_1 \chi + i\gamma_5 \bar{\varepsilon}_1 i\gamma_5 \chi]\varepsilon_2 - (1 \leftrightarrow 2) \\ &= -\frac{1}{4} \bar{\varepsilon}_1 \gamma^R \varepsilon_2 \not{\partial}[\gamma_R + i\gamma_5 \gamma_R i\gamma_5]\chi - (1 \leftrightarrow 2) \\ &= +\frac{1}{2} \bar{\varepsilon}_2 \gamma^a \varepsilon_1 2 \not{\partial} \gamma_a \chi \\ &= 2\bar{\varepsilon}_2 \not{\partial} \varepsilon_1 \chi - \bar{\varepsilon}_2 \gamma^a \varepsilon_1 \gamma_a \not{\partial} \chi \end{aligned} \tag{5.8}$$

The above calculation makes use of a Fierz rearrangement (see Appendix A) as well as the properties of Majorana spinors. However, $\chi_\alpha$ is subject to its equation of motion, i.e. $\not{\partial}\chi = 0$, implying the final result

$$[\delta_1, \delta_2]\chi = 2\bar{\varepsilon}_2 \not{\partial} \varepsilon_1 \chi \tag{5.9}$$

which is the consequence dictated by the supersymmetry algebra. The reader will have no difficulty verifying that the fields $A$, $B$ and $\chi_\alpha$ and the transformations

$$\begin{aligned} &\delta A = \bar{\varepsilon}\chi, \quad \delta B = i\bar{\varepsilon}\gamma_5 \chi \\ &\delta \chi = \not{\partial}(A + i\gamma_5 B)\varepsilon \end{aligned} \tag{5.10}$$

form a representation of the whole of the supersymmetry algebra *provided* $A$, $B$ and $\chi_\alpha$ are on-shell (i.e. $\partial^2 A = \partial^2 B = \not{\partial}\chi = 0$).

We now wish to consider the fields $A$, $B$ and $\chi_\alpha$ when they are no longer subject to their field equations. The Lagrangian from which the above field equations follow is

$$L = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\chi}\partial\!\!\!/\chi \tag{5.11}$$

It is easy to prove that the action $\int d^4xL$ is indeed invariant under the transformation of Eq.(5.10). This invariance is achieved *without the use of the field equations*. The trouble with this formulation is that the fields $A$, $B$ and $\chi_\alpha$ do not form a realization of the supersymmetry algebra when they are no longer subject to their field equations, as the last term in Eq.(5.8) demonstrates. It will prove useful to introduce the following terminology. We shall refer to an irreducible representation of supersymmetry carried by fields which are subject to their equations of motion as an *on-shell representation*. We shall also refer to a Lagrangian as being *algebraically on-shell* when it is formed from fields which carry an on-shell representation, that is, do not carry a representation of supersymmetry off-shell, and the Lagrangian is invariant under these on-shell transformations. The Lagrangian of Eq.(5.11) is then an algebraically on-shell Lagrangian.

That $A$, $B$ and $\chi_\alpha$ cannot carry a representation of supersymmetry off-shell can be seen without any calculation, since these fields do not satisfy the rule of equal numbers of fermions and bosons which was given earlier. Off-shell, $A$ and $B$ have two degrees of freedom, but $\chi_\alpha$ has four degrees of freedom. Clearly, the representations of supersymmetry must change radically when enlarged from on-shell to off-shell.

A possible way out of this dilemma would be to add two bosonic fields $F$ and $G$ which would restore the fermion-boson balance. However, these additional fields would have to occur in the Lagrangian so as to give rise to no on-shell states. As such, they must occur in the Lagrangian in the form $+\frac{1}{2}F^2 + \frac{1}{2}G^2$ assuming the free action to be only bilinear in the fields and consequently be of mass dimension two. On dimensional grounds their supersymmetry transformations must be of the form

$$\delta F = \bar{\varepsilon}\partial\!\!\!/\chi \quad \delta G = i\bar{\varepsilon}\gamma_5\partial\!\!\!/\chi \tag{5.12}$$

where we have tacitly assumed that $F$ and $G$ are scalar and pseudoscalar respectively. The fields $F$ and $G$ cannot occur in $\delta A$ on dimensional grounds, but can occur in $\delta\chi_\alpha$ in the form

$$\delta\chi = [(\mu F + i\tau\gamma_5 G) + \partial\!\!\!/(A + i\gamma_5 B)]\varepsilon \tag{5.13}$$

where $\mu$ and $\tau$ are undetermined parameters.

We note that we can only modify transformation laws in such a way that on-shell (i.e., when $F = G = \partial\!\!\!/\chi = \partial^2 A = \partial^2 B = 0$) we regain the on-shell transformation laws of Eq.(5.10).

We must now test if these new transformations do form a realization of the supersymmetry algebra. In fact, straightforward calculation shows they do, provided $\mu = \tau = +1$. This representation of supersymmetry involving the fields $A$, $B$, $\chi_\alpha$, $F$ and $G$ was found by Wess and Zumino[3] and we now summarize their result:

$$\begin{aligned} &\delta A = \bar{\varepsilon}\chi \qquad \delta B = i\bar{\varepsilon}\gamma_5\chi \\ &\delta\chi = [F + i\gamma_5 G + \partial\!\!\!/(A + i\gamma_5 B)]\varepsilon \\ &\delta F = \bar{\varepsilon}\partial\!\!\!/\chi \qquad \delta G = i\bar{\varepsilon}\gamma_5\partial\!\!\!/\chi \end{aligned} \tag{5.14}$$

The action which is invariant under these transformations, is given by the Lagrangian

$$A = \int d^4x \left\{ -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\chi}\partial\!\!\!/\chi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \right\} \; eqno(5.15)$$

As expected the $F$ and $G$ fields occur as squares without derivatives and so lead to no on-shell states.

The above construction of the Wess-Zumino model is typical of that for a general free supersymmetric theory. We begin with the on-shell states, given for any model in Chapter 8, and construct the on-shell transformation laws. We can then find the Lagrangian which is invariant without use of the equations of motion, but contains no auxiliary fields. One then tries to find a set of auxiliary fields that give an off-shell algebra. Once this is done one can find a corresponding off-shell action. How one finds the nonlinear theory from the free theory is discussed in the later chapters.

The first of these two steps is always possible; however, there is no sure way of finding auxiliary fields that are required in all models, except with a few rare exceptions. This fact is easily seen to be a consequence of our rule for equal numbers of fermi and bose degrees of freedom in any representation of supersymmetry. It is only spin 0's, when represented by scalars, that have the same number of field components off-shell as they have on-shell states. For example, a Majorana spin-$\frac{1}{2}$ when represented by a spinor $\chi_\alpha$ has a jump of 2 degrees of freedom between on and off-shell and a massless spin-1 boson when represented by a vector $A_\mu$ has a jump of 1 degree of freedom. In the latter case it is important to subtract the one gauge degree of freedom from $A_\mu$ thus leaving 3 field components off-shell (see next chapter). Since the increase in the number of degrees of freedom from an on-shell state to the off-shell field representing it changes by different amounts for fermions and bosons, the fermionic-boson balance which holds on-shell will not hold off-shell if we only introduce the fields that describe the on-shell states. The discrepancy must be made up by fields, like $F$ and $G$, that lead to no on-shell states. These latter type of fields are called *auxiliary fields*. The whole problem of finding representations of supersymmetry amounts to finding the auxiliary fields.

Unfortunately, it is not at all easy to find the auxiliary fields. Although the fermi-bose counting rule gives a guide to the number of auxiliary fields it does not actually tell you what they are, or how they transform. In fact, the auxiliary fields are only known for almost all $N = 1$ and 2 supersymmetry theories and for a very few $N = 4$ theories and not for the higher $N$ theories. In particular, they are not known for the $N = 8$ supergravity theory.

Theories for which the auxiliary fields are not known can still be described by a Lagrangian in the same way as the Wess-Zumino theory can be described without the use of $F$ and $G$, namely, by the so called algebraically on-shell Lagrangian formulation, which for the Wess-Zumino theory was given in Eq. (5.11). Such 'algebraically on-shell Lagrangians' are not too difficult to find at least at the linearized level. As explained in Chapter 8 we can easily find the relevant on-shell states of the theory. The algebraically on-shell Lagrangian then consists of writing down the known kinetic terms for each spin.

Of course, we are really interested in the interacting theories. The form of the interactions is however often governed by symmetry principles such as gauge invariance in the above example or general coordinate invariance in the case of gravity theories. When the form of the interactions is dictated by a local symmetry there is a straightforward, although maybe very lengthy way of finding the nonlinear theory from the linear theory. This method, called Noether coupling, is described in Chapter 7. In one guise or another this technique has been used to construct nonlinear 'algebraically on-shell Lagrangians' for all supersymmetric theories.

The reader will now ask himself whether algebraically on-shell Lagrangians may be good enough. Do we really need the auxiliary fields? This question will be addressed in the next chapter, but the following example is a warning against over-estimating the importance of a Lagrangian that is invariant under a set of transformations that mix fermi-bose fields, but do not obey any particular algebra.

Consider the Lagrangian

$$L = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}\bar{\chi}\partial\!\!\!/\chi \tag{5.16}$$

whose corresponding action is invariant under the transformations

$$\partial A = \bar{\varepsilon}\chi \quad \partial\chi = \partial\!\!\!/ A\varepsilon \tag{5.17}$$

However, this theory has nothing to do with supersymmetry. The algebra of transformations of Eq. (5.17) does not close on or off-shell without generating transformations which, although invariances of the free theory, can never be generalized to be invariances of an interacting theory. In fact, the on-shell states do not even have the correct fermi-bose balance required to form an irreducible representation of supersymmetry. This example illustrates the fact that the 'algebraically on-shell Lagrangians' rely for their validity, as supersymmetric theories, on their on-shell algebra.

As a final remark in this section it is worth pointing out that the problem of finding the representations of any group is a mathematical question not dependent on any dynamical considerations for its resolution. Thus the questions of which are physical fields and which are auxiliary fields is a model-dependent statement.

### 2.2 The $N = 1$ Yang-Mills Theory

This account of the construction of the $N = 1$ Yang-Mills theory in $x$-space follows closely chapter 6 of reference [0].

### 2.3 The Extended Theories

The $N = 2$ Yang-Mills theory and $N = 2$ matter are constructed as well as their most general renormalizable coupling. This account closely follows chapter 12 of reference [0].

## Lecture 3. The Irreducible Representations of Supersymmetry

The first part of this section is taken from reference [0] and we have kept the equation numbers the same as in that reference.

In this chapter we wish to find the irreducible representations of supersymmetry [11], or, put another way, we want to know what is the possible particle content of supersymmetric theories. As is well known the irreducible representations of the Poincaré group are found by the Wigner method of induced representations [12]. This method consists of finding a representation of a subgroup of the Poincaré group and boosting it up to a representation of the full group. In practice, one adopts the following recipe: we choose a given momentum $q^\mu$ which satisfies $q^\mu q_\mu = 0$ or $q^\mu q_\mu = -m^2$ depending which case we are considering. We find the subgroup $H$ which leaves $q^\mu$ intact and find a representation of $H$ on the $|q^\mu\rangle$ states. We then induce this representation to the whole of the Poincaré group $P$, in the usual way. In this construction there is a one-to-one correspondence between points of $P/H$ and four-momentum which satisfies $P_\mu P^\mu = 0$ or $P_\mu P^\mu = -m^2$. One can show that the result is independent of the choice of momentum $q^\mu$ one starts with.

In what follows we will not discuss the irreducible representations in general, but only that part applicable to the rest frame, i.e. the representations of $H$ in the states at rest. We can do this safely in the knowledge that once the representation of $H$ on the rest-frame states in known then the representation of $P$ is uniquely given and that every irreducible representation of the Poincaré group can be obtained by considering every irreducible representation of $H$.

In terms of physics the procedure has a simple interpretation, namely, the properties of a particle are determined entirely by its behaviour in a given frame (i.e. for given $q^\mu$). The general behaviour is obtained from the given $q^\mu$ by boosting either the observer or the frame with momentum $q^\mu$ to one with arbitrary momentum.

The procedure outlined above for the Poincaré group can be generalised to any group of the form $S \otimes_s T$ where the symbol $\otimes_s$ denotes the semi-direct product of the groups $S$ and $T$ where $T$ is Abelian. It also applies to the supersymmetry group and we shall take it for granted that the above recipe is the correct procedure and does in fact yield all irreducible representations of the supersymmetry group.

Let us first consider the massless case $q_\mu q^\mu = 0$, for which we choose the standard momentum $q_s^\mu = (m, 0, 0, m)$ for our "rest frame". We must now find $H$ whose group elements leave $q_s^\mu = (m, 0, 0, m)$ intact. Clearly this contains $Q_\alpha^i$, $P_\mu$ and $T_s$, since these generators all commute with $P_\mu$ and so rotate the states with $q_s^\mu$ into themselves. As we will see in the last section one can not have non-vanishing central charges for the massless case.

Under the Lorentz group the action of the generator $\frac{1}{2}\Lambda^{\mu\nu}J_{\mu\nu}$ creates an infinitesimal transformation $q^\mu \to \Lambda^\mu_\nu q^\nu + q^\mu$. Hence $q_s^\mu$ is left invariant provided the parameters obey the relations

$$\Lambda_{30} = 0, \quad \Lambda_{10} + \Lambda_{13} = 0, \quad \Lambda_{20} + \Lambda_{23} = 0 \tag{8.1}$$

Thus the Lorentz generators in $H$ are

$$T_1 = J_{10} + J_{13}, \quad T_2 = J_{20} + J_{23}, \quad J = J_{12} \tag{8.2}$$

These generators form the algebra

$$\begin{aligned} [T_1, J] &= -T_2 \\ [T_2, J] &= +T_1 \\ [T_1, T_2] &= 0 \end{aligned} \tag{8.3}$$

The reader will recognise this to be the Lie algebra of $E_2$, the group of translations and rotations in a two-dimensional plane.

Now the only unitary representations of $E_2$ which are finite dimensional have $T_1$ and $T_2$ trivially realised, i.e.

$$T_1|q_s^\mu\rangle = T_2|q_s^\mu\rangle = 0 \tag{8.4}$$

This results from the theorem that all non-trivial unitary representations of noncompact groups are infinite dimensional. We will assume we require finite-dimensional representations of $H$.

Hence for the Poincaré group, in the case of massless particles, finding representations of $H$ results in finding representations of $E_2$ and consequently for the generator $J$ alone. We choose our states so that

$$J|\lambda\rangle = i\lambda|\lambda\rangle \tag{8.5}$$

Our generators are antihermitian. In fact, $J$ is the helicity operator and we select $\lambda$ to be integer or half-integer (i.e. $J = \underline{q}\cdot\underline{J}/|\underline{q}|$ evaluated at $\underline{q} = (0,0,m)$ where $J_i = \varepsilon_{ijk}J_{jk}$; $i,j = 1,2,3$).

Let us now consider the action of the supercharges $Q_\alpha^i$ on the rest-frame states, $|q_s^\mu\rangle$. The calculation is easiest when performed using the two-component formulation of the supersymmetry algebra of Eq. (2.23). On rest-frame states we find that

$$\begin{aligned}\{Q^{Ai}, Q_j^{\dot B}\} &= -2\delta_j^i(\sigma_\mu)^{A\dot B}q_s^\mu \\ &= -2\delta_j^i(\sigma_0+\sigma_3)^{A\dot B}m = +4m\delta_j^i\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}^{A\dot B}\end{aligned} \tag{8.6}$$

In particular these imply the relations

$$\begin{aligned}\{Q^{1i}, Q_j^{\dot 1}\} &= 0 \\ \{Q^{2i}, Q_j^{\dot 2}\} &= 4m\delta_j^i \\ \{Q_i^{\dot i}, Q^{2j}\} = \{Q_i^1, Q^{\dot 2j}\} &= 0 (8.7)\end{aligned}$$

The first relation implies that

$$\langle q_s^\mu|\big(Q^{1i}(Q^{1i})^* + (Q^{1i})^*Q^{1i}\big)|q_s^\mu\rangle = 0 \tag{8.8}$$

Demanding that the norm on physical states be positive definite and vanishes only if the state vanishes yields

$$Q_2^i|q_s^\mu\rangle = Q_{\dot 2i}|q_s^\mu\rangle = 0 \tag{8.9}$$

Hence, all generators in $H$ have zero action on rest-frame states except $J$, $T_s$, $P_\mu$, $Q_1^i$ and $Q_{\dot 1 i}$. Using Eq. (2.23) we find that

$$\begin{aligned}[Q_1^i, J] &= \frac{1}{2}(\sigma_{12})_1^1 Q_1^i \\ &= -\frac{i}{2}Q_1^i\end{aligned} \tag{8.10}$$

Similarly, we find that complex conjugation implies

$$[(Q_1^i)^*, J] = +\frac{i}{2}(Q_1^i)^* \tag{8.11}$$

The relations between the remaining generators summarised in Eqs. (8.7), (8.10), (8.11) and (2.24) can be summarised by the statement that $Q_1^i$ and $(Q_1^i)^*$ form a Clifford algebra, act as raising and lowering operators for the helicity operator $J$ and transform under the $N$ and $\bar{N}$ representation of $SU(N)$.

We find the representations of this algebra in the usual way; we choose a state of given helicity, say $\lambda$, and let it be the vacuum state for the operator $(Q_1^i)^*$, i.e.

$$\begin{aligned} Q_1^i|\lambda\rangle &= 0 \\ J|\lambda\rangle &= i\lambda|\lambda\rangle \end{aligned} \tag{8.12}$$

The states of this representation are then

$$\begin{aligned} |\lambda\rangle &= |\lambda\rangle \\ |\lambda - \frac{1}{2}, i\rangle &= (Q_1^i)^*|\lambda\rangle \\ |\lambda - 1, [ij]\rangle &= (Q_1^i)^*(Q_1^j)^*|\lambda\rangle \end{aligned} \tag{8.13}$$

etc. These states have the helicities indicated and belong to the $[ijk\ldots]$ anti-symmetric representation of $SU(N)$. The series will terminate after the helicity $\lambda - (N/2)$, as the next state will be an object antisymmetric in $N+1$ indices. Since there are only $N$ labels this object vanishes identically. The states have helicities from $\lambda$ to $\lambda - (N/2)$, there being $N!/\big(m!(N-m)!\big)$ states with helicity $\lambda - (m/2)$.

To obtain a set of states which represent particles of both helicities we must add to the above set the representations with helicities from $-\lambda$ to $-\lambda + (N/2)$. The exception is the so-called CPT self-conjugate sets of states which automatically contain both helicity states.

The representations of the full supersymmetry group are obtained by boosting the above states in accordance with the Wigner method of induced representations.

Hence the massless irreducible representation of $N = 1$ supersymmetry comprises only the two states

$$\begin{aligned} &|\lambda\rangle \\ &|\lambda - \frac{1}{2}\rangle = (Q_1)|\lambda\rangle \end{aligned} \tag{8.14}$$

with helicities $\lambda$ and $\lambda - \frac{1}{2}$ and since

$$Q_1 Q_1|\lambda\rangle = 0 \tag{8.15}$$

there are no more states.

To obtain a CPT invariant theory we must add states of the opposite helicities, i.e. $-\lambda$ and $-\lambda+\frac{1}{2}$. For example, if $\lambda=\frac{1}{2}$ we get on-shell helicity states 0 and $\frac{1}{2}$ and their CPT conjugates with helicities $-\frac{1}{2}, 0$, giving a theory with two spin 0's and one Majorana spin-$\frac{1}{2}$. Alternatively, if $\lambda=2$ then we get on-shell helicity states $3/2$ and 2 and their CPT self conjugates with helicity $-3/2$ and $-2$; this results in a theory with one spin 2 and one spin $3/2$ particles. These on-shell states are those of the Wess-Zumino model and $N=1$ supergravity respectively. Later in this discussion we will give a complete account of these theories.

For $N=4$ with $\lambda=1$ we get the massless states

$$|1\rangle,\ |\frac{1}{2}, i\rangle,\ |0,[ij]\rangle,\ |-\frac{1}{2},[ijk]\rangle,\ |-1,[ijkl]\rangle \tag{8.16}$$

This is a CPT self-conjugate theory with one spin, four spin-$\frac{1}{2}$ and six spin-0 particles.

Table 8.1 below gives the multiplicity for massless irreducible representations which have maximal helicity 1 or less.

Table 8.1 Multiplicities for massless irreducible representations with maximal helicity 1 or less

| $N$ Spin | 1 | 1 | 2 | 2 | 4 |
|---|---|---|---|---|---|
| Spin 1 | – | 1 | 1 | – | 1 |
| Spin $\frac{1}{2}$ | 1 | 1 | 2 | 2 | 4 |
| Spin 0 | 2 | – | 2 | 4 | 6 |

We see that as $N$ increases, the multiplicities of each spin and the number of different types of spin increases. The simplest theories are those for $N = 1$. The one in the first column in the Wess-Zumino model and the one in the second column is the $N = 1$ supersymmetric Yang-Mills theory. The latter contains one spin 1 and one spin $\frac{1}{2}$, consistent with the formula for the lowest helicity $\lambda - (N/2)$, which in this case gives $1 - \frac{1}{2} = \frac{1}{2}$. The $N = 4$ multiplet is CPT self conjugate, since in this case we have $\lambda - (N/2) = 1 - 4/2 = -1$. The Table stops at $N$ equal to 4 since when $N$ is greater than 4 we must have particles of spin greater than 1. Clearly, $N > 4$ implies that $\lambda - (N/2) = 1 - (N/2) < -1$. This leads us to the well-known statement that the $N = 4$ supersymmetric theory is the maximally extended Yang-Mills theory.

The content for massless on-shell representations with a maximum helicity 2 is given in Table 8.2.

Table 8.2 Multiplicity for massless on-shell representations with maximal helicity 2.

| $N$ Spin | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Spin 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Spin $\frac{3}{2}$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 8 |
| Spin 1 | | 1 | 3 | 6 | 10 | 16 | 28 | 28 |
| Spin $\frac{1}{2}$ | | | 1 | 4 | 11 | 26 | 56 | 56 |
| Spin 0 | | | | 2 | 10 | 30 | 70 | 70 |

The $N = 1$ supergravity theory contains only one spin-2 graviton and one spin-3/2 gravitino. It is often referred to as simple supergravity theory. For the $N = 8$ supergravity theory, $\lambda - (N/2) = 2 - \frac{8}{2} = -2$. Consequently it is CPT self conjugate and contains all particles from spin 2 to spin 0. Clearly, for theories in which $N$ is greater than 8, particles of higher than spin 2 will occur. Thus, the $N = 8$ theory is the maximally extended supergravity theory.

It has sometimes been claimed that this theory is in fact the largest possible consistent supersymmetric theory. This contention rests on the widely-held belief that it is impossible to consistently couple massless particles of spin $\frac{5}{2}$ to other particles.In fact superstring theories do include spin 5/2 particles, but these are massive.

We now consider the massive irreducible representations of supersymmetry. We take our rest-frame momentum to be

$$q_s^\mu = (m, 0, 0, 0) \tag{8.17}$$

The corresponding little group is then generated by

$$P_m, Q^{\alpha i}, T^r, Z_1^{ij}, Z_2^{ij}, J_m \equiv \frac{1}{2}\varepsilon_{mnr}J^{nr} \tag{8.18}$$

where $m, n, r = 1, 2, 3$ for the present discussion. The $J_m$ generate the group $SU(2)$. Let us first consider the case where the central charges are trivially realised.

When acting on the rest-frame states the supercharges obey the algebra

$$\begin{aligned} \{Q^{Ai}, (Q^{Bj})^*\} &= 2\delta^A_B\delta^i_j m \\ \{Q^{Ai}, Q^{Bj}\} &= 0 \end{aligned} \tag{8.19}$$

The action of the $T^r$ is that of $U(N)$ with the $SU(2)$ rotation generators satisfy

$$\begin{aligned} [J_m, J_n] &= \varepsilon_{mnr}J_r \\ [Q^{Ai}, J_m] &= i(\sigma_m)^A{}_B Q^{Bi} \end{aligned} \tag{8.20}$$

where $(\sigma_m)$ are the Pauli matrices. We note that as far as $SU(2)$ is concerned the dotted spinor $Q^{\dot{A}i}$ behaves like the undotted spinor $Q_{Ai}$.

We observe that unlike the massless case none of the supercharges are trivially realised and so the Clifford algebra they form has $4N$ elements, that is, twice as many as those for the massless case. The unique irreducible representation of the Clifford algebra is found in the usual way. We define a Clifford vacuum

$$Q^i_A|q^\mu_s\rangle = 0, \quad A = 1, 2\ , i = 1, \ldots, N \tag{8.21}$$

and the representation is carried by the states

$$|q^\mu_s\rangle,\ (Q^i_A)^*|q^\mu_s\rangle,\ (Q^i_A)^*(Q^j_B)^*|q^\mu_s\rangle, \ldots \tag{8.22}$$

Due to the anticommuting nature of the $(Q^j_A)^*$ this series terminates when one applies $(2N+1)Q^*$'s.

The structure of the above representation is not particularly apparent since it is not clear how many particles of a given spin it contains. The properties of the Clifford algebra are more easily displayed by defining the real generators

$$\begin{aligned} \Gamma^i_{2A-1} &= \frac{1}{2m}\big(Q^{Ai} + (Q^{Ai})^*\big) \\ \Gamma^i_{2A} &= \frac{i}{2m}\big(Q^{Ai} - (Q^{Ai})^*\big) \end{aligned} \tag{8.23}$$

where the

$$\Gamma^i_p = (\Gamma^i_1, \Gamma^i_2, \Gamma^i_3, \Gamma^i_4) \tag{8.24}$$

are hermitian. The Clifford algebra of Eq. (8.19) now becomes

$$\{\Gamma^i_p, \Gamma^j_q\} = \delta^{ij}\delta_{pq} \tag{8.25}$$

The $4N$ elements of the Clifford algebra carry the group $SO(4N)$ in the standard manner; the $4N(4N-1)/2$ generators of $SO(4N)$ being

$$O^{ij}_{mn} = \frac{1}{2}[\Gamma^i_m, \Gamma^j_n] \tag{8.26}$$

As there are an even number of elements in the basis of the Clifford algebra, we may define a "parity" ($\gamma_5$) operator

$$\Gamma_{4N+1} = \prod_{p=1}^{4}\prod_{i=1}^{N}\Gamma^i_p \tag{8.27}$$

which obeys the relations

$$\begin{aligned}(\Gamma_{4N+1})^2 &= +1\\ \{\Gamma_{4N+1}, \Gamma^i_p\} &= 0\end{aligned} \tag{8.28}$$

Indeed, the irreducible representation of Eq. (8.22) is of dimension $2^{2N}$ and transforms according to an irreducible representation of $SO(4N)$ of dimension $2^{2N-1}$ with $\Gamma_{4N+1} = -1$ and another of dimension $2^{2N-1}$ with $\Gamma_{4N+1} = +1$. Now any linear transformation of the $Q$'s, $Q^*$'s (for example $\delta Q = rQ$) can be represented by a generator formed from the commutator of the $Q$'s and $Q^*$'s (for example, $r[Q, Q^*]$). In particular the $SU(2)$ rotation generators are given by

$$s_k = -\frac{i}{4m}(\sigma_k)^A{}_B[Q^{jB}, (Q^{jA})^*] \tag{8.29}$$

One may easily verify that

$$[Q^{jA}, s_k] = i(\sigma_k)^A_B Q^{Bj} \tag{8.30}$$

The states of a given spin will be classified by that subgroup of $SO(4N)$ which commutes with the appropriate $SU(2)$ rotation subgroup of $SO(4N)$. This will be the group generated by all generators bilinear in $Q, Q^*$ that have their two-component index contracted, i.e.

$$\begin{aligned}\Lambda^i_j &= \frac{i}{2m}[Q^{Ai}, (Q^j_A)^*]\\ k^{ij} &= \frac{i}{2m}[Q^{Ai}, Q^j_A]\end{aligned} \tag{8.31}$$

and $(k^{ij})^\dagger = k_{ij}$. It is easy to verify that the $\Lambda^i_j$, $k^{ij}$ and $k_{ij}$ generate the group $USp(2N)$ and so the states of a given spin are labelled by representations of $USp(2N)$. That the group is $USp(2N)$ is most easily seen by defining

$$Q^a_A = \begin{cases} Q^i_A\delta^a_i & a = 1, \ldots, N\\ \varepsilon_{AB}(Q^{Bi})^* & a = N+1, \ldots, 2N\end{cases} \tag{8.32}$$

for then the generators $\Lambda^i_j$, $k^{ij}$ and $k_{ij}$ are given by

$$s^{ab} = \frac{i}{2m}[Q^{Aa}, Q^b_A]. \tag{8.33}$$

Using the fact that

$$\{Q_A^a, Q_B^b\} = \varepsilon_{AB}\Omega^{ab} \tag{8.34}$$

where

$$\Omega^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we can verify that

$$[s^{ab}, s^{cd}] = \Omega^{ac}s^{bd} + \Omega^{ad}s^{bc} + \Omega^{bc}s^{ad} + \Omega^{bd}s^{ac} \tag{8.35}$$

which is the algebra of $USp(2N)$.

The particle content of a massive irreducible representation is given by the following

**Theorem** [21]: If our Clifford vacuum is a scalar under the $SU(2)$ spin group and the internal symmetry group, then the irreducible massive representation of supersymmetry has the following content

$$2^{2N} = \left[\frac{N}{2}, (0)\right] + \left[\frac{N-1}{2}, (1)\right] + \ldots + \left[\frac{N-\kappa}{2}, (\kappa)\right] + \ldots + [0, (N)] \tag{8.36}$$

where the first entry in the bracket denotes the spin and the last entry, say $(k)$ denotes which $k$th fold antisymmetric traceless irreducible representation of $USp(2N)$ that this spin belongs to.

Table 8.3 Some massive representations (without central charges) labelled in terms of the $USp(2N)$ representations.

| $N$ / Spin | 1 | | | | 2 | | | 3 | | 4 |
|---|---|---|---|---|---|---|---|---|---|---|
| Spin 2 | | | | 1 | | | 1 | | 1 | 1 |
| Spin $\frac{3}{2}$ | | | 1 | 2 | | 1 | 4 | 1 | 6 | 8 |
| Spin 1 | | 1 | 2 | 1 | 1 | 4 | $5+1$ | 6 | $14+1$ | 27 |
| Spin $\frac{1}{2}$ | 1 | 2 | 1 | | 4 | $5+1$ | 4 | 14 | $14'+6$ | 48 |
| Spin 0 | 2 | 1 | | | 5 | 4 | 1 | $14'$ | 14 | 42 |

Consider an example with two supercharges. The classifying group is $USp(4)$ and the $2^4$ states are one spin 1, four spin 1/2, and five spin 0 corresponding to the $\underline{1}-$, $\underline{4}-$ and $\underline{5}-$dimensional representations of $USp(4)$. For more examples see Table 8.3.

Should the Clifford vacuum carry spin and belong to a non-trivial representation of the internal group $U(N)$, then the irreducible representation is found by taking the tensor product of the vacuum and the representation given in the above theorem.

### Massive Representations with a Central Charge

We now consider the case of particles that are massive, but which also possess a central charge. We take the particles to be in their rest frame with momentum $q^\mu \equiv (M, 0, 0, 0)$.

The isotropy group, H contains ($P^a, Q^i_A, Q^{\dot{A}}_i, \underline{J}, T_r$ and $Z^{ij}$). In the rest frame of the particles, that is for the momentum $q^\mu$, the algebra of the supercharges is given by

$$\{Q^{Ai},\ (Q^{Bj})^*\} = 2\delta^A_B \delta^i_j M \tag{1}$$

and

$$\{Q^i_A,\ Q^j_B\} = \epsilon_{AB} Z^{ij} \tag{2}$$

To discover what is the particle content in a supermultiplet we would like to rewrite the above algebra as a Clifford algebra. The first step in this proceedure is to carry out a unitary transformation on the internal symmetry index of the supercharges i.e. $Q^i_A \to U^i_j Q^j_A$ or $Q_A \to U Q^i_A$ with $U^\dagger U = 1$. Such a transformation preserves the form of the first relation of equation (1). However, the unitary transformation can be chossen [104] such that the central charge ,which transforms as $Z \to UZU^T$, can be brought to the form of a matrix which has all its entries zero except for the 2 by 2 matices down its diagonal. These 2 by 2 matrices are anti-symmetric as a consequence of the anti-symmetry nature of $Z^{ij}$ which is preserved by the unitary transformation. This is the closest one can come to diagonalising an anti-symmetric matrix. Let us for simplicity restrict our attention to $N$ even. To best write down this matrix we replace the $i, j = 1, 2, \ldots N$ internal indices by $i = (a, m),\ j = (b, n),\ a, b = 1, 2,\ m, n = 1, \ldots, \frac{N}{2}$ whereupon

$$Z^{(a,m)(b,n)} = 2\epsilon^{ab}\delta^{mn} Z_n \tag{3}$$

In fact one also show that $Z_n \geq 0$. The supercharges in the rest frame satisfy the relations

$$\{Q^{A(am)},\ (Q^{B(bn)})^*\} = 2\delta^A_B \delta^a_b \delta^m_n M \tag{4}$$

and

$$\{Q^{(am)}_A,\ Q^{(bn)}_B\} = 2\epsilon_{AB}\epsilon^{ab}\delta^{mn} Z_n \tag{5}$$

We now define the supercharges

$$S^{Am}_1 = \frac{1}{\sqrt{2}}(Q^{A1m} + (Q^{B2m}\epsilon_{BA})^*) \tag{6}$$

$$S^{Am}_2 = \frac{1}{\sqrt{2}}(Q^{A1m} - (Q^{B2m}\epsilon_{BA})^*) \tag{7}$$

in terms of which all the anti-commutators vanish except for

$$\{S^{Am}_1,\ (S^{Bn}_1)^*\} = 2\delta^{AB}\delta^{mn}(M - Z_n) \tag{8}$$

$$\{S^{Am}_2,\ (S^{Bn}_2)^*\} = 2\delta^{AB}\delta^{mn}(M + Z_n) \tag{9}$$

This algebra is a Clifford algebra formed from the $2N$ operators $S^{Am}_1$ and $S^{Am}_2$ and their complex conjugates. It follows from equation (9) that if we take the same indices on each supercharge that the right-hand side is positive definite and hence $Z_n \leq M$.

To find the irreducible representation of supersymmetry we follow a similar procedure to that which se followed for massive and massless particles. The result crucially depends on whether $Z_n < M,\ \forall\ n$ or if one or more values of n we saturate the bound $Z_n = M$.

Let us first consider $Z_n < M,\ \forall\ n$. In this case, the right-hand sides of both equations (8) and (9) are non-zero. Taking $S_1^{Am}$ and $S_2^{Am}$ to annihilate the vaccum the physical states are given by the creation opperators $(S_1^{Am})^*$ and $(S_2^{Am})^*$ acting on the vacuum. The resulting representation has $2^{2N}$ states and has the same structure as for the massive case in the absence of a central charge. The states are classified by $USP(2N)$ as for the massive case.

Let us now suppose that $q$ of the $Z_n$'s saturate the bound i.e $Z_n = M$. For these values of $n$ the right-hand side of equation (8) vanishes; taking the expectation value of this relation for any physical state we find that

$$< phys|S_1^{An}(S_1^{An})^*|phys > + < phys|(S_1^{An})^*(S_1^{An})|phys >= 0 \tag{10}$$

The scalar product on the space of physical states satisfies all the axioms of a scalar product and hence we conclude that both of the above terms vanish and as a result

$$(S_1^{Bn})^*|phys >= 0 = (S_1^{An})|phys > \tag{11}$$

This argument is the same as that used to eliminate half of the supercharges and their complex conjugates in the massless case, however in case under consideration here it only eliminates $q$ of the supercharges and their complex conjugates. There remain the $\frac{N}{2}$ supercharges $(S_2^{Bm})$ and the $\frac{N}{2} - q$ supercharges $(S_1^{Bm})$ for the values of $m$ for which $Z_m$ do not saturate the bound as well as their complex conjugates. These supercharges form a Clifford algebra and we can take the $\frac{N}{2}$ supercharges $(S_2^{Bm})$ and the $\frac{N}{2} - q$ supercharges $(S_1^{Bm})$ to annihilate the vaccum and their complex conjugates to be creation operators. The resulting massive irreducible representation of supersymmetry has $2^{2(N-q)}$ states and it has the same form as a massive representation of $N - q$ extended supersymmetry. The states will be classified by $USp(2N - 2q)$.

Clearly, a representation in which some or all of the central charges are equal to their mass has fewer states that the massive representation formed when none of the central charges saturate the mass or a massive representation for which all the central charges vanish. This is a consequence of the fact that the latter Clifford algebra has more of its supercharges active in the irreducible representation. In almost all cases, the representation with some of its central charges saturated contains a smaller range of spins than the massive representation with no central charges. This feature plays a very important role in discussions of duality in supersymmetric theories.

Let us consider the irreducible representations of $N = 4$ supersymmetry which has both of its two possible central charges saturated. These representations are like the corresponding $N = 2$ massive representations. An important example has a $\underline{1}$ of spin one, a $\underline{4}$ of spin one-half and $\underline{5}$ of spin zero. The underlined numbers are their $USp(4)$ representations. This representation arises when the $N = 4$ Yang-Mills theory is spontaneously broken by one of its scalars aquiring a vaccum expectation value. The theory before being spontaneously broken has a massless representation with one spin one, 2 spin 1/2 , and six spin zero's. Examining the massive representations for $N = 4$ in the absence of a central charge one finds that the representation with the smallest spins has all spins from spin 2 to spin 0. Hence the spontaneously broken theory can only be supersymmetric if the representation has a central charge. Another way to get the count in the above representation is to take the massless representation and recall that when the theory is spontaneously broken one of the scalars has been eaten by the vector as a result of the Higgs mechanism.

We close this section by answering a question which may have arisen in the mind of the reader. For the $N$ extended supersymmetry algebra the supersymmetry algebra in the rest frame of equation (3) representation has $\frac{N}{2}$ possible central charges. This makes one central charge for the case of $N = 2$. However, this number conflicts with our understanding that a particle in $N = 2$ supersymmetric Yang-Mills theory can have two central charges corresponding to its electric and magnetic fields. The resolution of this conundrum is that althought one can use a unitary transformation to bring the central charge of a given irreducible representation, i.e. particle, to have only one independent component one can not do this simultaneously for all irreducible multiplets or particles.

Some examples of massive representations with central charges are given in the table below.

Table 8.4 Some massive representations with one central charge ($|Z| = m$). All states are complex.

| $N$ / Spin | 2 | | 4 | | 6 | | 8 |
|---|---|---|---|---|---|---|---|
| Spin 2 | | | | | | 1 | 1 |
| Spin $\frac{3}{2}$ | | | | 1 | 1 | 6 | 8 |
| Spin 1 | | 1 | 1 | 4 | 6 | $14+1$ | 27 |
| Spin $\frac{1}{2}$ | 1 | 2 | 4 | $5+1$ | 14 | $14'+6$ | 48 |
| Spin 0 | 2 | 1 | 5 | 4 | $14'$ | 14 | 42 |

The account of the massive irreducible representations of supersymmetry given here is along similar lines to the review by Ferrara and Savoy given in [21].

## Lecture 4. Superspace

### 4.1 Construction of Superspace

This constructed superspace as the coset space of the super-Poincare group divided by the Lorentz group. It follows closely chapter 14 of reference [0].

### 4.2 Superspace Formulations of Rigid Supersymmetric Theories

The Wess-Zumino model and $N = 1, 2$ Yang-Mills theories are formulated in superspace. This closely follows chapter 15 of reference [0].

## Lecture 5. Quantum Properties of Supersymmetric Models

### 5.1 Super-Feynman Rules and the Non-renormalisation Theorem

The super-Feynman rules of the Wess-Zumino model and $N = 1$ Yang-Mills theory are derived and the non-renormalisation theory is proved. This closely followed chapter 17 of reference [0].

### 5.2 Flat Directions

The potential in a supersymmetric theory is given by the squares of the auxiliary fields. In this section we consider an $N = 1$ supersymmetric model which contains Wess-Zumino multiplets coupled to the $N = 1$ Yang-Mills multiplet with gauge group $G$. Let us denote the auxiliary fields of the Wess-Zumino multiplets by the complex field $\mathcal{F}^i$ where the index $i$ labels the Wess-Zumino multiplets and those of the $N = 1$ Yang-Mills multiplet by $D^a$ where $a = 1, \ldots,$ dimension of $G$. Then the classical potential is given by

$$V = |\mathcal{F}^i|^2 + \frac{1}{2}\sum_a (D^a)^2 \tag{5.2.1}$$

For a general $N = 1$ renormalizable theory the auxiliary fields are given by

$$\mathcal{F}^i = \frac{\partial W(z^j)}{\partial z^i} \tag{5.2.2}$$

and

$$D^a = -g\bar{z}_i (T^a)^i_j z^j + \zeta^a \tag{5.2.3}$$

In equation (5.2.2) $W$ is the superpotential which we recall occurs in the superspace formulation of the theory as $(\int d^4x d^2\theta W + c.c)$ and $z^i$ are the scalars of the Wess-Zumino multiplet. For a renormlizable theory, the superpotential has the form $W(z) = \frac{1}{3!} d_{ijk} z^i z^j z^k + \frac{1}{2!} m_{ij} z^i z^j + e_i z^i$. In equation (5.2.3) $g$ is the gauge coupling constant and $(T^a)^i_j$ are the generators of the group $G$ to which these scalars $z^i$ belong. The terms in the auxiliary fields which are independent of $z^i$ can only occur when we have $U(1)$ factors for $D^a$ and auxiliary fields $\mathcal{F}^i$ that transform trivial under $G$. The resulting $\zeta^a$ and $e_i$ are constants.

Clearly, the potential is positive definite. Another remarkable feature of the potential is that it generically has flat directions. This means that minimizing the potential does not specify a unique field configuration. In other words there exists a vacuum degeneracy. The simplest example is for a Wess-Zumino model in the adjoint representation coupled to a $N = 1$ Yang-Mills multiplet. Taking the superpotential for this theory to vanish the potential is given by

$$V = \frac{1}{2}\sum_a (-g f_{abc} \bar{z}^b z^c)^2 \tag{5.2.4}$$

Clearly, the minimum is given by field configurations whose only non-zero vacuum expectation values are $< z^a > H^a$ where $H^a$ are the Cartan generators of the algebra. This theory is precisely the $N = 2$ supersymmetric Yang-Mills theory when written in terms of $N = 1$ supermuliplets.

In a general quantum field theory such a vacuum degeneracy would be removed by quantum corrections to the potential. However, things are different in supersymmetric theories. In fact, if supersymmetry is not broken the potential does not receive any perturbative quantum corrections [315]. It obviously follows that if supersymmetry is not broken then the vacuum degeneracy is not removed by perturbative quantum corrections [315]. This result was first proved before the advent of the non-renormalisation theorem as formulated in reference [316], but it is particularly obvious given this theorem. For the effective potential we are interested in field configurations where the spinors vanish and the space-time derivatives of all fields are set to zero. For such configurations, the gauge invariant superfields do not contain any $\theta$ dependence as only their first component is non-zero. Quantum corrections, however, contain an integral over all of superspace and to be non-zero requires a $\theta^2\bar{\theta}^2$ factor in the integrand. For the field configurations of interest to us such an integral over the full superspace must vanish and as a result we find that there are no quantum corrections to the effective potential if supersymmetry is not broken.

Finally, we recall why the expectation values of the auxiliary fields vanish if supersymmetry is preserved. In this case the expectation value of the supersymmetry transformations of the spinors must vanish. The transformation of the spinors contain auxiliary fields which occur without space-time derivatives and the bosonic fields which correspond to the dynamical degrees of freedom of the theory. The latter occur with space-time derivative, as they have mass dimension one and $\epsilon$ has dimension $-\frac{1}{2}$. Consequently, if the expectation values of supersymmetry transformations of the spinors vanish so do the expectation values of all the auxiliary fields. By examining the supersymmetry transformations of the spinors given earlier the reader may verify that there are no loop holes in this argument.

Clearly, the rigid $N = 2$ and $N = 4$ theories can be written in terms of $N = 1$ supermultiplets and so the flat directions that occur in these theories are also not removed by quantum corrections. Although this might be viewed as a problem in these theories it has been turned to advantage in the work of Seiberg and Witten. These authors realized that the dependence of these theories on the expectation values of the scalar fields, or the moduli, obeyed interesting properties that can be exploited to solve for part of the effective action of these theories.

### 5.3 Non-holomorphicity

The non-renormalisation theorem states that perturbative quantum corrections to the effective action are of the form

$$\int d^4x_1 \ldots \int d^4x_n \int d^4\theta G(x_1, \ldots x_n) f(\varphi(x_1, \theta), \ldots, V(x_1, \theta), \ldots, D^A\varphi(x_1, \theta), \ldots) \tag{5.3.1}$$

where $\varphi$ and $V$ are the superfields that contain the Wess-Zumino and $N = 1$ Yang-Mills fields respectively.

The most significant aspect of this result is that the corrections arise from a single superspace integral over **all** of superspace, that is they contain a $d^4\theta = d^2\theta d^2\bar{\theta}$ integral and not a sub-integral of the form $d^2\theta$ or $d^2\bar{\theta}$. Such sub-integrals play an important role in supersymmetric theories. For example, the superpotential in the superspace formulation of the Wess-Zumino model has the form $\int d^4x d^2\theta W + c.c.$

While their is no question that this formulation of the non-renormalisation theorem is correct, with the passing of time, it was taken by many workers to mean that their could never be any quantum corrections which were sub-integrals i.e. that is of the form

$$\int d^4xd^2\theta\hat{W}(\varphi) \tag{5.3.2}$$

In particular, it was often said that there could be no quantum corrections to the superpotential.

Consider, however, the expression

$$\int d^4xd^4\theta[(-\frac{D^2}{4\partial^2})\varphi^n] = \int d^4xd^2\theta(-\frac{\bar{D}^2}{4})(-\frac{D^2}{4\partial^2})\varphi^n = \int d^4xd^2\theta\varphi^n \tag{5.3.3}$$

where we have used the relation $\bar{D}^2D^2\psi = 16\partial^2\psi$ where $\psi$ is any chiral superfield. This maneuver illustrates the important point that although an expression can be written as a full superspace integral, it can also be expressible as a local integral over only a subspace of superspace. The above expression when written in terms of the full superspace integral is non-local, however, any effective action contains many non-local contributions. The occurrence of the $\frac{1}{\partial^2}$ is the signal of a massless particle. For a massive particle one would instead find a factor of $\frac{1}{(\partial^2+m^2)}$ which can not be rewritten as a sub-integral. Hence, only when massless particles circulate in the quantum loops can we find a contribution to the effective action which can be written as a sub-superspace integral.

The first example of such a correction to the superpotential was found in reference [313]. In reference [301]. it was shown that all the proofs of the non-renormalisation theorem allowed contributions to the effective action which were integrals over a subspace of superspace if massless particles were present. It was also shown [301] that such corrections were not some pathological exception, but that they generically occurred whenever massless particles were present. This lecture follows the first part of reference [301] and the reader is referred there for a much more complete discussion and several examples. In the Wess-Zumino model such corrections first occur at two loops and were calculated in [302], while in the Wess-Zumino model coupled to $N = 1$ Yang-Mills theory the corrections occur even at one loop [303]. An alternative way of looking at such corrections was given in references [304] and [305].

The reader may wonder what such corrections have to do with non-holomorphicity. The answer is that the corrections we have been considering are non-holomorphic in the coupling constants. The situation is most easily illustrated in the context of the massless Wess-Zumino model where the superpotential is of the form $\int d^4xd^2\theta\lambda\varphi^3 + c.c.$ Since the propagator connects $\varphi$ to $\bar{\varphi}$ we get no corrections at all if we do not include terms that contain both $\lambda$ and $\bar{\lambda}$. Consequently, the corrections we find to the superpotential must contain $\lambda$ and $\bar{\lambda}$ and so is non-holomorphic in $\lambda$.

We can of course prevent the occurrence of such terms if we give masses to all the particles or we do not integrate over the infra-red region of the loop momentum integration for the massless particles. Such is the case if we calculate the Wilsonian effective action. However, if the terms considered here affect the physics in an important way one will necessarily miss such effects and they will only become apparent when one carries out the integrations that one had previously excluded.

### 5.4 Perturbative Quantum Properties of Extended Theories of Supersymmetry

Many of the perturbative properties of the extended theories of supersymmetry are derived. These include the finiteness, or superconformal invariance, of the $N=4$ Yang-Mills theory, the demonstration that $N=2$ Yang-Mills theory coupled to $N=2$ matter has a perturbative beta-function that only has one-loop contributions and the existence of a large class of superconformally invariant quantum $N=2$ theories. This section closely follows chapter 18 of reference [0].

### 5.5 The $N=2$ Chiral Effective Action

The $N=2$ Yang-Mills theory is described [58] by a superfield $A$ which is chiral

$$\bar{D}^{i}_{\dot{B}}\mathcal{A}=0 \tag{5.5.1}$$

and also satisfies the constraint

$$D^{ij}\mathcal{A}=\bar{D}^{ij}\bar{\mathcal{A}} \tag{5.5.2}$$

where $D^{ij}=D^{Ai}D_{A}^{\ j}$, $\quad \bar{D}^{ij}=\bar{D}^{\dot{A}i}\bar{D}_{\dot{A}}^{\ j}$. This last constraint imposes the Bianchi identity on the Yang-Mills field strength and makes the triplet auxiliary field real.

Let us decompose the $N=2$ chiral superfield $\mathcal{A}$ in terms of $N=1$ superfields. Let us label the two superspace Grassmann odd coordinates $\theta^{Ai}$, $i=1,2$ which occur in the $N=2$ superspace as $\theta^{A}_{\ 1}=\theta^{A}$ and $\theta^{A}_{\ 2}=\eta^{A}$ and $\bar{\theta}^{\dot{A}1}=\bar{\theta}^{\dot{A}}$ and $\bar{\theta}^{\dot{A}2}=\eta^{\dot{A}}$. We associate the coordinates $\theta^{A}$ and $\bar{\theta}^{\dot{A}}$ with those of $N=1$ superspace which we will keep manifest. Similarly, we denote the spinorial covariant derivatives of $N=2$ superspace as $D_{A}^{\ 1}=D_{A}$, $D_{A}^{\ 2}=\nabla_{A}$ and $\bar{D}_{\dot{A}1}=\bar{D}_{\dot{A}}$, $\bar{D}_{\dot{A}2}=\bar{\nabla}_{\dot{A}}$. We use also the notation

$$\bar{D}^{2}=\bar{D}^{\dot{B}}\bar{D}_{\dot{B}},\quad D^{2}=D^{B}D_{B},\quad \bar{\nabla}^{2}=\bar{\nabla}^{\dot{B}}\bar{\nabla}_{\dot{B}},\quad \nabla^{2}=\nabla^{B}\nabla_{B} \tag{5.5.3}$$

We lower the $i,j$ indices with $\varepsilon^{ij}=\varepsilon_{ji}=-\varepsilon^{ij}$, $\varepsilon^{12}=1$, in the usual way, that is $T^{i}=\varepsilon^{ij}T_{j}$ and $T_{k}=T^{j}\varepsilon_{jk}$.

To find the $N=1$ superfields contained in $\mathcal{A}$ we must write $\mathcal{A}$ in terms of $\eta_{A},\eta_{\dot{A}}$ and $N=1$ superfields and solve the $N=2$ superspace constraints of equations (1) and (2). The chirality constraint of equation (5.5.1) for $i=2$ implies that $\mathcal{A}$ can be written in the form

$$\mathcal{A}=a+\eta^{A}W_{A}+\eta^{2}K+\ldots \tag{5.5.4}$$

where

$$\eta^{2}=\eta^{A}\eta_{A}\quad \bar{\eta}^{2}=\bar{\eta}^{\dot{A}}\bar{\eta}_{\dot{A}}\quad \theta^{2}=\theta^{A}\theta_{A}\quad \bar{\theta}^{2}=\bar{\theta}^{\dot{A}}\bar{\theta}_{\dot{A}} \tag{5.5.5}$$

and $+\ldots$ denotes terms involving $\bar{\eta}^{\dot{A}}$ which must contain space-time derivatives of the superfields that are already written. The superfields $A$, $W_{A}$ and $K$ depend on $x^{\mu}$, $\theta^{A}$ and $\bar{\theta}^{\dot{A}}$ and so are $N=1$ superfields. We will see that $A$ and $W_{C}$ are the $N=1$ chiral superfields that contain the Wess-Zumino multiplet and the $N = 1$ Yang-Mills mulitplet respectively.

We could continue with this approach, however a more efficient method is to use the covariant derivatives to define the components of the superfield $\mathcal{A}$. Acting with $\nabla_B$ on $\mathcal{A}$, equation (5.5.1) for $i = 2$ implies that the only independent $N = 1$ superfields which depend on $x^\mu, \theta_A$ and $\bar{\theta}_{\dot{A}}$ are are given by

$$\mathcal{A}|_{\eta=0} = A, \quad \nabla_B \mathcal{A}|_{\eta=0} = W_B, \quad -\frac{1}{2}\nabla^2 \mathcal{A}|_{\eta=0} = K. \tag{5.5.6}$$

Acting with $\bar{D}_{\dot{C}}$, equation (5.5.1) for $i = 1$ implies that these $N = 1$ superfields are chiral superfields;

$$\bar{D}_{\dot{B}} A = 0 = \bar{D}_{\dot{B}} W_C = \bar{D}_{\dot{B}} K \tag{5.5.7}$$

It remains to solve the constraint of equation (5.5.2). Taking $i = 1,\ j = 2$ we find it becomes $D^B \nabla_B \mathcal{A} = -\nabla^{\dot{B}} \bar{D}_{\dot{B}} \bar{\mathcal{A}}$. Swopping the last two covariant derivatives on the right hand side and taking $\eta = 0$ we find the constraint $D_B W^B = \bar{D}_{\dot{B}} \bar{W}^{\dot{B}}$. Taking $i = 2,\ j = 2$ in equation (5.5.2) we find the result

$$\nabla^2 \mathcal{A} = \bar{D}^2 \bar{\mathcal{A}} \tag{5.5.8}$$

Evaluating this equation at $\eta = 0$ we find that

$$-\frac{1}{2}\nabla^2 \mathcal{A}|_{\eta=0} = K = -\frac{1}{2}\bar{D}^2 \bar{A} \tag{5.5.9}$$

Hence $K$ is not an independent superfield.

To summarise; the $N = 2$ superfield $\mathcal{A}$ decomposes into two $N = 1$ superfields $A$ and $W_B$ which are subject to the constraints

$$\bar{D}_{\dot{B}} A = 0,\ \bar{D}_{\dot{B}} W_c = 0,\ D^B W_B = \bar{D}^{\dot{B}} \bar{W}_{\dot{B}} \tag{5.5.10}$$

These $N = 1$ superfields contain the Wess-Zumino and $N = 1$ Yang-Mills multiplets.

We can further define the $x$-space component superfields as follows

$$A|_{\theta=0} = a,\ D_B A|_{\theta=0} = \chi_B,\ -\frac{1}{2}D^2 A|_{\theta=0} = f \tag{5.5.11}$$

and

$$W_B|_{\theta=0} = \lambda_B,\ D_C W_B|_{\theta=0} = \mathcal{F}^{\mu\nu}(\sigma_{\mu\nu})_{BC} + iD\epsilon_{CB} \tag{5.5.12}$$

where $\mathcal{F}^{\mu\nu} = F^{\mu\nu} - i{}^*F^{\mu\nu}$ and ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}F^{\rho\lambda}$

Let us now consider an $N = 2$ effective theory whose only massless particles are $N = 2$ $U(1)$ multiplets. In such a theory we can integrate over all the massive particles in the functional integral to find an effective action for the remaining $U(1)$ supermultiplets. The most simple example is the $SU(2)$ $N = 2$ Yang-Mills theory that is spontaneously broken to $U(1)$ by shifting one of the scalar fields of the theory. Although the usual action for such $U(1)$ mulitplets is a free theory, the effective action resulting from such a process is very non-trivial and must be of the form

$$Im \int d^4x d^4\theta F(\mathcal{A}) + \int d^4x d^4\theta d^4\bar{\theta} K(\mathcal{A}, \bar{\mathcal{A}}, D_B\mathcal{A}, \ldots) \tag{5.5.13}$$

For simplicity we have suppressed the index which would label the different $U(1)$ factors.

It can be argued as follows that the low energy part of such an action is just given by the first term. The integrand of the second term has mass dimension zero. As such, one must introduce a mass scale $\Lambda$ and it will contain terms such as $\frac{\mathcal{A}^{2n}}{\Lambda^{2n}}, n \in \mathbf{Z}$. Evaluating this in terms of the $x$-space component fields we find terms of the form

$$\int d^4x \frac{a^n(\partial_\nu\partial^\nu)^4\bar{a}^n}{\Lambda^{2n}} + \ldots \tag{5.5.14}$$

where $+\ldots$ means its supersymmetric completion. Although one can consider more complicated contributions to the effective action, the final result must seemingly contain the mass scale $\Lambda$ to the appropriate power and so be higher order in derivatives than the kinetic energy term for the fields. This is to be contrasted with the first term of equation (5.5.13), the integrand of which has mass dimension two and so the resulting $x$-space expressions are of the same order in derivatives as the standard kinetic energy terms. Thus if we are only interested in the low energy effective theory one can neglect the second term and only consider the chiral effective action is of the form

$$Im \int d^4x d^4\theta F(\mathcal{A}) \tag{5.5.15}$$

The above argument as given applies not just to $U(1)$ factors, but to any $N = 2$ theory. In fact, there is a flaw in this argument; the integrand of the second term of equation (5.5.11) could contain terms of the form $ln(\frac{\mathcal{A}_1\mathcal{A}_2}{\mathcal{A}_3\mathcal{A}_4})$ where $\mathcal{A}_i,\ i = 1,2,3,4$ are different superfields for the $U(1)$ factors or other superfields in the theory. Clearly, this term does not involve a mass scale $\Lambda$ and its $x$-space expression will not be higher order in derivatives. Indeed one finds [321],[322] that just such terms arise even in the one loop calculation of $N = 2$ $SU(2)$ Yang-Mills theory before it is spontaneously broken. Fortunately they can not occur in the effective action for the $N = 2$ $U(1)$ theory resulting from the spontaneous breaking of the $N = 2$ $SU(2)$ Yang-Mills theory discussed above.

Let us now assume that we are only interested in a low energy effective theory whose action is of the form given in equation (5.5.15). The solution to the above constraints of equation (5.5.1) and (5.5.2) for the Abelian theory are of the form [64]

$$\mathcal{A} = \bar{D}^4 D^{ij} V_{ij} \tag{5.5.16}$$

where $V_{ij}$ is an unconstrained superfield of mass dimension -2. Varying the above action with respect to $V_{ij}$ yields the equation of motion. We find that

$$Im \int d^4x d^4\theta \bar{D}^4 D^{ij} \delta V_{ij} \frac{dF(\mathcal{A})}{d\mathcal{A}} \propto Im \int d^4x d^8\theta \delta V_{ij} D^{ij} \frac{dF}{d\mathcal{A}} \tag{5.5.17}$$

and hence we find the equation of motion to be

$$D^{ij}\frac{dF}{d\mathcal{A}} = \bar{D}^{ij}\frac{d\bar{F}}{d\bar{\mathcal{A}}} \tag{5.5.18}$$

We observe that if we write $\mathcal{A}_D = \frac{dF}{d\mathcal{A}}$ and $\mathcal{A}$ as a doublet i.e. $\begin{pmatrix}\mathcal{A}_D\\ \mathcal{A}\end{pmatrix}$. Then the equation (5.5.2) which encodes the Bianchi identity and the equation of motion (5.5.18) can be written as

$$D^{ij}\begin{pmatrix}\mathcal{A}_D\\ \mathcal{A}\end{pmatrix} = \bar{D}^{ij}\begin{pmatrix}\mathcal{A}_D\\ \mathcal{A}\end{pmatrix} \tag{5.5.19}$$

Clearly, this system has a set of equations of motion invariant under

$$\begin{pmatrix}\mathcal{A}_D\\ \mathcal{A}\end{pmatrix} \to \Omega\begin{pmatrix}\mathcal{A}_D\\ \mathcal{A}\end{pmatrix} \tag{5.5.20}$$

where $\Omega \in SL(2,R)$. This symmetry is restricted to $SL(2,\mathbf{Z})$ when the theory is quantized. It is clear form the free theory that this symmetry interchanges the equation of motion with the Bianchi identity and thus is a duality trasnformation. This symmetry was first discussed in this theory in reference [324]. The $N=2$ superspace formulation of the theory makes the appearance of this symmetry particularly apparent [325].

For the free theory $F = i\mathcal{A}^2$ and we find the equation of motion and constraint imply that $D^{ij}\mathcal{A} = 0 = \bar{D}^{ij}\mathcal{A}$. As a result at $\theta^{\mathcal{A}i} = 0$ the triplet auxiliary fields of the $N=2$ Yang-Mills theory vanish ensuring the correct equations of motion.

We can now evaluate the chiral effective action [318]:

$$\begin{aligned} Im\int d^4xd^4\theta F &= Im\int d^4xd^2\theta d^2\eta F = \int d^4xd^2\theta\left(-\frac{\nabla}{4}\right)^2 F \\ &= Im\int d^4xd^2\theta\left\{-\frac{(\nabla^2\mathcal{A})}{4}\frac{dF}{d\mathcal{A}} - \frac{1}{4}\nabla^B\mathcal{A}\nabla_B\mathcal{A}\frac{d^2F}{d\mathcal{A}^2}\right\}_{\eta=0} \\ &= Im\int d^4xd^2\theta\left\{(-\frac{1}{4}\bar{D}^2\bar{A})\frac{dF}{dA} - \frac{1}{4}\frac{d^2F}{dA^2}W^BW_B\right\} \\ &= Im\int d^4xd^4\theta\bar{A}\frac{dF}{dA} - \frac{1}{4}\int d^4xd^2\theta\frac{d^2F}{dA^2}W^BW_B \end{aligned} \tag{5.5.21}$$

One can further evaluate the above action in terms of the $x$-space component fields by writing the $d^2\theta = -\frac{1}{4}D^2$ and $d^2\bar{\theta} = -\frac{1}{4}\bar{D}^2$ and using the definitions of the $x$-space component fields of equations (5.5.11) and (5.5.12). The result is

$$A_1 + A_2 \tag{5.5.22}$$

where the fermion independent part is given by

$$A_1 = Im\int d^4x\frac{d^2F}{da^2}\{-\partial_\mu\bar{a}\partial^\mu a - \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{4}(f\bar{f} + D^2)\} \tag{5.5.23}$$

and

$$A_2 = Im\int d^4x\{-\frac{i}{4}\frac{d^2F}{da^2}(\bar{\chi}_{\dot{A}}(\sigma^\mu)^{B\dot{A}}\partial_\mu\chi_B + \lambda_{\dot{A}}(\sigma^\mu)^{B\dot{A}}\partial_\mu\lambda_B)$$

$$-\frac{1}{8}\frac{d^3F}{da^3}(\bar{f}\chi^A\chi_a + f\lambda^A\lambda_A - iD\chi^A\lambda_A) - \frac{1}{8}\frac{d^3F}{da^3}\chi^A\lambda^B(\sigma_{\mu\nu})_{AB}\mathcal{F}_{\mu\nu} + \frac{1}{16}\frac{d^4F}{da^4}\chi^A\chi_A\lambda^A\lambda_A) \tag{5.5.24}$$

We now wish to find the complete form of the perturbative expression for the chiral effective action of equation (5.5.15). This result can be derived as a consequence of the way the theory breaks superconformal invariance. For simplicity let us begin with the theory of an $N = 2$ $U(1)$ mulitplet which is a free theory. In this case, the effective action and the original action coincide and are given by $F = i\mathcal{A}^2$. This action is invariant under the $N = 2$ superconformal group discussed in section 6. In particular, it is invariant under the internal $U(2) = SU(2) \otimes U(1)$ transformations. The $U(1)$ transformations are called $R$ transformations and they can be taken to act on the Grassmann odd coordinates $\theta$ as $\theta \to e^{i\alpha}\theta$ where $\alpha$ is the parameter of the transformation (see section 6). The $R$ transformations act on $\mathcal{A}$ as $\mathcal{A} \to e^{2i\alpha}\mathcal{A}$. It is easy to see that with the choice of $F = i\mathcal{A}^2$ then $F \to e^{4i\alpha}F$ which cancels the variation of $\int d^4\theta$ leaving the action invariant. It is straightforward to deduce the action of $R$ transformations on the component fields, it acts as a chiral rotations on the spinors $\lambda$ and acts on the scalars $a$ as $a \to e^{2i\alpha}a$. The dilations can be handled in a similar manner. Under a dilation with parameter $w$ one finds that $x^\mu \to e^{-w}x^\mu, \theta^A \to e^{\frac{w}{2}}\theta^A$ and $\mathcal{A} \to e^{2w}\mathcal{A}$ which leaves invariant the free action.

The $N = 2$ superconformal currents are contained in the superfield $J$ which is subject to the constraint

$$D^{ij}J = 0 \tag{5.5.25}$$

For the free $U(1)$ theory discussed just above $J = \mathcal{A}\bar{\mathcal{A}}$ and we find that it does indeed satisfy the above equation as $D^{ij}(\mathcal{A}\bar{\mathcal{A}}) = D^{ij}(\mathcal{A})\bar{\mathcal{A}} = 0$ by virtue of the equation of motion of $\mathcal{A}$. The superfield $J$ has mass dimension two, $R$ weight 0 and the $U(2)$ Currents are the $\theta^A = 0$ component of $D^i_A\bar{D}^j_{\dot{A}}J$.

For the theories we are considering the effective action does not in general possess a superconformal symmetry as the underlying theory is anomalous. The anomaly must modify equation (5.5.25) by a term on the right hand side that is constructed from $\mathcal{A}$ and has mass dimension three, $R$ weight $-2$ and the correct $SU(2)$ transformation property. The only possible choice is [121]

$$D^{ij}J = -\frac{1}{3}\frac{\beta(g)}{g}\bar{D}^{ij}(\bar{\mathcal{A}})^2 \tag{5.5.26}$$

where $\beta(g)$ is a function of the gauge coupling constant $g$ that will turn out to be the $\beta$-function, but at this point is an unknown quantity. At first sight, one might think one could add a term of the form $\bar{D}^{ij}(\bar{\mathcal{A}})\mathcal{A}$ however using the Bianchi identity of equation (5.5.2) we can manipulate this expression to be of the form of $J$ and so absorb it on the left hand side as a redefinition of the current. A review of supercurrent multiplets and their anomalies is given in chapter 20 of reference [0].

By acting with spinorial covariant derivatives on $J$ in equation (5.5.26) we find that it implies that the divergence of the $R$ current $j^5_\mu$ is of the form

$$\partial^\mu j^5_\mu = \frac{2}{3}\frac{\beta(g)}{g}(-\frac{1}{4}F^{\mu\nu *}F_{\mu\nu} + \ldots) \tag{5.5.27}$$

and corresponding to the breaking of dilations the trace of the energy-momentum tensor

$$\theta^\mu_\mu = -2\frac{\beta(g)}{g}(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \ldots) \tag{5.5.28}$$

In the these equation $+\ldots$ means terms involving fermions and scalars. It is known [320] that the latter equation for $\theta^\mu_\mu$ is proportional to the beta-function times $F^{\mu\nu}F_{\mu\nu}$. Taking into account the constant of proportionality we conclude that $\beta$ is indeed that the beta-function of the theory.

It was shown in references [120] with related work in references [121],[114],[115] that the perturbative contribution to beta function in any renormalizable $N = 2$ Yang-Mills theory coupled to $N = 2$ matter has only a one loop contribution. The derivation of this result is too complicated to give here, but we refer the reader to chapter 18 of reference [0] for a review. The full perturbative beta function in such a theory is then [120]

$$\beta = \frac{g^3}{(4\pi)^2}(T(R_\sigma) - 3C_2(G)) \equiv \frac{3g^3}{4\pi^2}\beta_1 \tag{5.5.30}$$

Here $(T^s)^a{}_b(T^s)^b{}_a = \delta^{st}T(R)$ where $(T^s)^a{}_b$ are the generators of the gauge group $G$ in the representation $R$ to which the $N = 2$ matter belongs and $f_{rst}f_{qst} = \delta_{rq}C_2(G)$ where $f_{rst}$ are the structure constants of the group $G$. For example for the case of $SU(N)$ we find that $C_2 = N$ and that if the matter is in the fundamental representation $T(R) = \frac{1}{2}$. The reader may readily construct some of the finite quantum field theories discovered in reference [120].

Finally we can consider the constraints placed on the perturbative contribution to the effective action by the anomaly equation (5.5.26). In particular, let us consider the underlying theory to be an $SU(2)$ $N = 2$ Yang-Mills theory coupled to $N = 2$ matter which is spontaneously broken to $U(1)$. The beta function that occurs in equation (5.5.26) is that for the underlying theory and so in these equations we can substitute the beta function of equation (5.5.30). The integrated Ward identity tells us that the variation of the effective action is equal to the anomaly. Carrying this out for the dilations and $R$ transformations one finds that

$$\delta(Im\int d^4xd^4\theta F) = Im\int d^4xd^4\theta(2i\alpha + w)(\mathcal{A}\frac{dF}{d\mathcal{A}} - 2F) = Im\int d^4xd^4\theta(2i\alpha + w)8\pi i\beta_1 u \tag{5.5.31}$$

where $u = -\frac{A^2}{4\pi^2}$. Although this equation is very natural in that both sides have chiral integrands and are of the correct dimension and $R$ weight it may be unclear to the reader that the right-hand side of this equation should really be of this precise form. To derive the equation one must formulate the Ward-identity for the $N = 2$ theory. The Ward identity contains three terms the variation of the effective action, the divergence condition for the currents and the anomaly term. To find the above integrated form we must integrate the Ward identity in such a way as to eliminate the term which depends on the supercurrent. This process is rather complicated, however the reader may verify that at the component level the above equation does contain the right hand side of equations (5.5.27) and (5.5.28) in the way dictated from the usual Ward identity in $x$-space. Only by carrying out this proceedure can one determine the somewhat elisve constant. From equation (5.5.31) we can extract the equation

$$\mathcal{A}\frac{dF}{d\mathcal{A}} - 2F = 8\pi i\beta_1 u \tag{5.5.32}$$

Upon making the substitution $F = \mathcal{A}^2 G$ and solving the resulting equation for $G$ we find that [319]

$$F = -\frac{i}{\pi}\beta_1\mathcal{A}^2 ln\frac{\mathcal{A}^2}{\Lambda^2} \tag{5.5.33}$$

Thus using the anomaly equation and the knowledge of the perturbative beta function in the $N = 2$ theories [120] we can determine the perturbative part of the chiral effective action completely [319].

In the above we have used the fact that in a supersymmetric theory the currents of symmetries of the theory belong in a supermultiplet. As such, knowledge of the properties about one of the currents can be used to deduce the related behaviour of the other currents and so deduce consequences for the theory. This type of argument was first used [112] to show that the $N = 4$ Yang-Mills theory is finite [112-117].

In fact, equation (5.5.32) holds for the full non-perturbative quantum theory except in this case the anomaly (i.e. $u$) is not the simple function of $\mathcal{A}$ given above. The demonstration of this fact and a more careful derivation of the last equations given in this section can be found in reference [323].

## Lecture 6. Superconformal Theories

### 6.1 The Geometry of Superconformal Transformations

The superspace that we used in lecture 4 was defined as the coset space of the super-Poincare group divided by the Lorentz group and internal symmetry group. This superspace is called Minkowski superspace. For the case of the $N = 1$ super-Poincare group, the superspace is parameterised by the coordinates $Z^M = (x^\mu, \theta^A, \theta^{\dot{A}})$ corresponding to the generators $P^\mu$ and $Q^A, Q^{\dot{A}}$ which generate transformations that are not contained in the isotropy subgroup. We can construct on superspace a set of preferred frames with supervierbeins ${E_M}^\pi$ . The covariant derivatives are given by $D_M = E_M^\pi \frac{\partial}{\partial Z^\pi}$. Their precise form being

$$D_A = \frac{\partial}{\partial\theta^A} - i(\sigma^m)_{A\dot{B}}\theta^{\dot{B}}\partial_m,\ D_{\dot{A}} = \frac{\partial}{\partial\theta^{\dot{A}}} - i(\sigma^m)_{B\dot{A}}\theta^B\partial_m, \tag{6.1}$$

and

$$D_m = \frac{\partial}{\partial x^m} \equiv \partial_m \tag{6.2}$$

We can read off the components of the inverse supervierbien from these equations.

For superconformal theories it is more natural to consider a superspace which is constructed from the coset space found by dividing the superconformal group by the subgroup which is generated by Lorentz transformations $J_{\mu\nu}$, dilations $D$, special translations $K_\mu$ and special supersymmetry transformations $S_{Ai}, S_{\dot{A}i}, i = 1, \ldots, N$ and the internal symmetry generators. The internal group for the superconformal algebra contains the group $U(N) = SU(N) \times U(1)$ although in the case of $N = 4$ the $U(1)$ factor does not act on the supercharges.

This coset construction leads to the same Minkowski superspace with the same transformations for the super-Poincare group, but it has the advantage that it automatically encodes the action of the superconformal transformations on the superspace. These transformations were first calculated by Martin Sohnius in reference [306].

The purpose of this section is to give an alternative method of calculating the superconformal transformations in four dimensions which will enable us to give a compact superspace form for the superconformal transformations. In particular, all the parameters of the transformations will be encode in one superfield which we can think of as the superspace equivalent of a conformal Killing vector. This formulation was first given by B. Conlong and P. West and can be found in reference [307]. One reason for reviewing this work here is that there is still not a readable account readily available in the literature. This section was written in collaboration with B. Conlong. Some reviews on this subject can be found in [308].

Conformal transformations in Minkowski space are defined to be those transformations which preserve the Minkowski metric up to scale ( see chapter 25 of reference [0] for a review). However, superspace does not have a natural metric since the tangent space group contains the Lorentz group which does not relate the bosonic to the fermionic sectors of the tangent space. The treatment we now give follows that given in chapter 25 of reference [0] for the case of two dimensional superconformal transformations.

There are two methods to define a superconformal transformation.

[a] We can demand that it is a superdiffeomorphism which preserves part of the bosonic part of the supersymmetric line element

$$dL^m dL^n \eta_{mn} \tag{6.3}$$

where

$$dL^m \equiv dZ^M E_M^{\ m} = dx^m + i\bar{\theta}\gamma^m d\theta \tag{6.4}$$

up to an arbitrary local scale factor.

[b] We can alternatively demand that it is a superdiffeomorphism which preserves the spinor components of the superspace covariant derivatives up to an arbitrary local scale factor. More precisely, a superconformal transformation is one such that

$$D_A \mapsto D'_A = f_A{}^B(z) D_B, \quad \bar{D}_{\dot{A}} \mapsto \bar{D}'_{\dot{A}} = \bar{f}_{\dot{A}}{}^{\dot{B}}(z) \bar{D}_{\dot{B}} \tag{6.5}$$

We note that the transformation must preserve each chirality spinor derivative separately. In fact, these two definitions are equivalent and we will work with only the second definition.

Carrying out the super-reparameterisation $z^M = (x^\mu, \theta^A, \bar\theta^{\dot A}) \mapsto z^{M\prime} = (x^{\mu\prime}, \theta^{A\prime}, \bar\theta^{\dot A\prime})$ upon the spinorial covariant derivatives we find that a finite superconformal transformation obeys the constraints

$$D_A \bar\theta'^{\dot B} = 0, \quad D_A x'^\mu + i\sigma^\mu_{B\dot B}\bar\theta'^{\dot B}(D_A\theta'^B) = 0 \tag{6.6}$$

and

$$\bar D_{\dot A}\theta'^B = 0, \quad \bar D_{\dot A} x'^\mu + i\sigma^\mu_{B\dot B}\theta'^B(\bar D_{\dot A}\bar\theta'^{\dot B}) = 0 \tag{6.7}$$

The corresponding transformation of the covariant derivatives being

$$D_A = (D_A\theta'^B)D'_B, \quad \bar D_{\dot A} = (\bar D_{\dot A}\bar\theta'^{\dot B})\bar D'_{\dot B} \tag{6.8}$$

We now consider an infinitesimal transformation

$$z^\pi \mapsto z'^\pi = z^\pi + G^\pi \tag{6.9}$$

where $G^\pi = (G^\mu, G^A, G^{\dot A})$ is a set of infinitesimal superfields. Equations (6.6) and (6.7) then become

$$D_A G^{\dot A} = 0, \quad D_A G^\mu + i\sigma^\mu_{A\dot B}G^{\dot B} + i\sigma^\mu_{B\dot B}\bar\theta^{\dot B}D_A G^B = 0 \tag{6.10}$$

and

$$\bar D_{\dot A}\bar G^B = 0, \quad \bar D_{\dot A}G^\mu + i\sigma^\mu_{A\dot A}G^A + i\sigma^\mu_{B\dot B}\theta^B\bar D_{\dot A}\bar G^{\dot B} = 0 \tag{6.11}$$

The vector field corresponding to such an infinitesimal transformation is given by $V = G^\pi\partial_\pi$. However, this can also be written as $V = F^M D_M$ where the change of basis corresponds to the relation $F^M = G^\pi E_\pi^M$. In terms of components this change is given by

$$F^\mu = G^\mu - i\sigma^\mu_{A\dot A}\bar\theta^{\dot A}G^A - i\sigma^\mu_{A\dot A}\theta^A G^{\dot A} \tag{6.12}$$

as well as

$$F^A = G^A, \; F^{\dot A} = G^{\dot A} \tag{6.13}$$

We shall denote the vector component by $F^\mu, \ldots$ or $F^n, \ldots$ even though it should strictly speaking carry the latter $m, n, \ldots$ indices. It is straightforward to verify that equations (6.10) and (6.11) now take the neater form

$$D_A F^\mu = -2i\sigma^\mu_{A\dot A}\bar F^{\dot A}, \; \bar D_{\dot A}F^\mu = -2i\sigma^\mu_{A\dot A}F^A \tag{6.14}$$

and

$$D_{\dot A}F^B = 0 = D_A F^{\dot B}. \tag{6.15}$$

A somewhat quicker derivation of this result can be given by first writing the infinitesimal change in the covariant derivatives under an infinitesimal superdiffeomorphism in the form $D_M \to D_M + [V, D_M]$. Using the form for $V$ given above which contains the covariant derivatives and then using the fact that the only non-zero commutator or anti-commutator, where appropriate, of the covariant derivatives is $\{D_A, D_{\dot B}\} = -2i(\sigma^n)_{A\dot B}\partial_n$ we recover equations (6.14) and (6.15).

Equation (6.14) can be rewritten as

$$F^{\dot{A}} = -\frac{1}{8i}\sigma_\mu^{A\dot{A}}D_A F^\mu,\ F^A = -\frac{1}{8i}\sigma_\mu^{A\dot{A}}\bar{D}_{\dot{A}}F^\mu \tag{6.16}$$

from which it is apparent that all transformations may be expressed in terms of $F^\mu$ alone. Using equations (6.12) and (6.13) we find that the explicit transformations of the coordinates are given by

$$\begin{aligned}
x'^\mu &= x^\mu + (F^\mu - \frac{1}{8}\theta^A D_A F^\mu + \frac{1}{8}\theta^A(\sigma^\mu{}_\nu)_{AC}D^C F^\nu - \frac{1}{8}\bar{\theta}^{\dot{A}}\bar{D}_{\dot{A}}F^\mu + \frac{1}{8}\bar{\theta}^{\dot{A}}(\bar{\sigma}^\mu{}_\nu)_{\dot{A}\dot{C}}\bar{D}^{\dot{C}}F^\nu) \\
\theta'^A &= \theta^A - \frac{1}{8i}\sigma_\mu^{A\dot{A}}\bar{D}_{\dot{A}}F^\mu \\
\bar{\theta}'^{\dot{A}} &= \bar{\theta}^{\dot{A}} - \frac{1}{8i}\sigma_\mu^{A\dot{A}}D_A F^\mu
\end{aligned} \tag{6.18}$$

Let us define $F_{B\dot{B}} \equiv \sigma^\mu_{B\dot{B}}F_\mu$, whereupon equation (6.14) becomes

$$D_A F_{B\dot{B}} = -4i\varepsilon_{AB}F_{\dot{B}},\ \ \bar{D}_{\dot{A}}F_{B\dot{B}} = -4i\varepsilon_{\dot{A}\dot{B}}F_B \tag{6.19}$$

from which we may deduce the constraint

$$D_{(A}F_{B)\dot{B}} = 0,\ \ \bar{D}_{(\dot{A}}F_{|B|\dot{B})} = 0 \tag{6.20}$$

Acting with $D_C$ on $D_A F_{B\dot{B}}$ and using equation (6.19) we conclude that

$$D_C D_A F_{B\dot{B}} = -4i\varepsilon_{AB}D_C F_{\dot{B}} = -D_A D_C F_{B\dot{B}} = -4i\varepsilon_{CB}D_A F_{\dot{B}} = 0. \tag{6.21}$$

The last step follows by tracing with $\varepsilon^{AB}$. Consequently, we find that equation (6.15) follows from equation (6.14) or equivalently equation (6.20) and so the superconformal transformations are encoded in $F_{B\dot{B}}$ subject to equation (6.20). We shall refer to equation (6.20) as the superconformal Killing equation, and the field $F_{B\dot{B}}$ as the superconformal Killing vector, these being the natural analogues of the conformal Killing equation and the usual Killing vector in Minkowski space.

To find the consequences for the $x$-space component fields within $F_{B\dot{B}}$ we expand the superfield $F_{B\dot{B}}$ as a Taylor series in $\theta$ and solve the superconformal Killing equation order by order in $\theta$. Writing $F_{B\dot{B}}$ as

$$\begin{aligned}
F_{B\dot{B}}(x,\theta,\bar{\theta}) =& \zeta_{B\dot{B}}(x) + \theta^A\chi_{AB\dot{B}}(x) + \bar{\theta}^{\dot{A}}\bar{\chi}_{\dot{A}B\dot{B}}(x) + \theta^A\bar{\theta}^{\dot{A}}A_{AB\dot{B}\dot{A}}(x) \\
&+ \frac{1}{2}\theta^2 f_{B\dot{B}}(x) + \frac{1}{2}\bar{\theta}^2 g_{B\dot{B}}(x) \\
&+ \frac{1}{2}\theta^2\bar{\theta}^{\dot{A}}\lambda_{B\dot{B}\dot{A}}(x) + \frac{1}{2}\bar{\theta}^2\theta^A\mu_{BA\dot{B}}(x) + \frac{1}{4}\theta^2\bar{\theta}^2\kappa_{B\dot{B}}(x)
\end{aligned} \tag{6.22}$$

and substituting this expression into the superconformal Killing equation we find that the resulting constraints are solved by the solution

$$\begin{aligned}
F_{B\dot{B}} =& \zeta_{B\dot{B}} + \theta_B\bar{\chi}_{\dot{B}} - \bar{\theta}_{\dot{B}}\chi_B + i\theta^A\bar{\theta}^{\dot{A}}\sigma^\mu_{B\dot{A}}\partial_\mu\zeta_{A\dot{B}} \\
&+ i\theta_B\bar{\theta}_{\dot{B}}\alpha - \frac{i}{2}\theta^2\bar{\theta}^{\dot{A}}\sigma^\mu_{B\dot{B}}\partial_\mu\bar{\chi}_{\dot{B}} \\
&+ \frac{i}{2}\bar{\theta}^2\theta^A\sigma^\mu_{A\dot{B}}\partial_\mu\chi_B
\end{aligned} \tag{6.23}$$

In this equation $\alpha$ is constant, $\zeta_{B\dot{B}}$ is a conformal Killing vector which satisfies

$$\partial_{(A}{}^{(\dot{A}}\zeta_{B)}{}^{\dot{B})} = 0 \tag{6.24}$$

and $\chi_B$ and $\bar{\chi}_{\dot{B}}$ are conformal spinors which obey the relation

$$\partial_{(A}{}^{\dot{A}}\chi_{B)} = 0, \quad \partial_{(\dot{A}}{}^{A}\bar{\chi}_{\dot{B})} = 0 \tag{6.25}$$

The solutions to equations (6.24) and (6.25) are given by

$$\begin{aligned} \zeta_\mu &= a_\mu + \lambda x_\mu + \Lambda_\mu{}^\nu x_\nu + k_\mu x^2 - 2(k\cdot x)x_\mu \\ \chi_B &= 4(i\varepsilon_B + \bar{\eta}^{\dot{B}}\sigma^\mu_{B\dot{B}}x_\mu) \\ \bar{\chi}_{\dot{B}} &= -4(i\bar{\varepsilon}_{\dot{B}} + \eta^B\sigma^\mu_{B\dot{B}}x_\mu) \end{aligned} \tag{6.26}$$

where $a_\mu$, $\lambda$, $\Lambda_{\mu\nu}$, $\varepsilon_B$ and $\eta_B$ are constant parameters.

Combining equations (6.26) and (6.23) it is clear that the parameters $a_\mu$, $\lambda$, $\Lambda_{\mu\nu}$, $k_\mu$, $\alpha$, $\varepsilon_B$ and $\eta_B$ are translations, dilations, Lorentz rotations, special conformal transformations, chiral transformations, chiral rotations, supersymmetry transformations and special supersymmetry transformations respectively.

Having found the superconformal transformations on superspace we now turn our attention to the transformations of superfields under a superconformal transformation. If $\varphi$ is a general superfield, which may carry Lorentz indices, then, its transformation is of the form

$$\delta\varphi(z) = \delta z^\pi\partial_\pi\varphi(z) + J\varphi(z) \tag{6.27}$$

where $J$ is a superfield which arises from the non-trivial action of generators from the isotropy group acting on $\varphi$ at the origin of the superspace. This factor is most pedagogically worked out by considering the superfields as induced representations. However, here we content ourselves with the final result which for a general superfield $\varphi$ is given by

$$\begin{aligned} \delta\varphi =&(F\cdot\partial)\varphi - \frac{1}{8i}\sigma_\mu^{A\dot{A}}(\bar{D}_{\dot{A}}F^\mu D_A\varphi + D_AF^\mu\bar{D}_{\dot{A}}\varphi) \\ &+\left\{\frac{1}{4}(\partial\cdot F)\Delta + \frac{1}{96i}\sigma^{\mu A\dot{A}}([D_A,\bar{D}_{\dot{A}}]F_\mu)\mathcal{A} - 2(\partial^{[\mu}F^{\nu]})\sigma_{\mu\nu}\right. \\ &\left.+\frac{1}{8}\partial^\mu(D^AF_\mu)\mathcal{S}_A + \frac{1}{8}\partial^\mu(\bar{D}^{\dot{A}}F_\mu)\bar{\mathcal{S}}_{\dot{A}} - \frac{1}{8}\partial^\mu(\partial\cdot F)\kappa_\mu\right\}\varphi \end{aligned} \tag{6.28}$$

In this equation the symbols $\{\Delta, \Sigma_{\mu\nu}, \mathcal{A}, \mathcal{S}_A, \bar{\mathcal{S}}_{\dot{A}}, \kappa_\mu\}$ are constants that are the values of the corresponding generators of the isotropy group acting on the superfield when it is taken to be at the origin of superspace. For almost all known situations, only the parameters $\Delta$, $\Sigma_{\mu\nu}$ and $\mathcal{A}$, which correspond to the dilation, Lorentz and $U(1)$ transformations respectively, are non-zero. The first part of the result is just the shift in the coordinates which is given by

$$\delta z^\pi\partial_\pi\varphi = (F\cdot\partial)\varphi - \frac{1}{8i}\sigma_\mu^{A\dot{A}}(\bar{D}_{\dot{A}}F^\mu D_A\varphi + D_AF^\mu\bar{D}_{\dot{A}}\varphi) \tag{6.29}$$

while $J$ is given by

$$J = \frac{1}{4}(\partial \cdot F)\Delta + \frac{1}{96i}\sigma^{\mu A\dot{A}}[D_A, \bar{D}_{\dot{A}}]F_\mu \mathcal{A} - 2\partial^{[\mu}F^{\nu]}\sigma_{\mu\nu}$$
$$+ \frac{1}{8}\partial^\mu(D^A F_\mu)\mathcal{S}_A + \frac{1}{8}\partial^\mu(\bar{D}^{\dot{A}} F_\mu)\bar{\mathcal{S}}_{\dot{A}} - \frac{1}{8}\partial^\mu(\partial \cdot F)\kappa_\mu \tag{6.30}$$

We can verify that equation (6.28) reproduces some of the known results. Let us consider dilations which are generated by taking $F_\mu = \lambda x_\mu$. For this case, equation (6.28) becomes

$$\delta\varphi = \lambda(x^\mu\partial_\mu + \frac{1}{2}\theta^A\partial_A + \frac{1}{2}\bar{\theta}^{\dot{A}}\bar{\partial}_{\dot{A}})\varphi + (\lambda)\Delta\varphi \tag{6.31}$$

which we recognise as the well known result. In fact, by writing $J$ as the most general form possible which is linear in $F_{B\dot{B}}$, contains covariant derivatives, is consistent with dimensional analysis and then evaluating the result for particular transformations we can also arrive a the correct $J$.

We can apply equation (6.28) to the case of a chiral and anti-chiral superfield. For simplicity, let us consider a lorentz scalar chiral superfield whose $\mathcal{S}_A, \bar{\mathcal{S}}_{\dot{A}}, \kappa_\mu$ values also vanish. The result is

$$\delta\varphi = (F\cdot\partial)\varphi - \frac{1}{8i}\sigma_\mu^{A\dot{A}}\bar{D}_{\dot{A}}F^\mu D_A\varphi + \Delta(\frac{1}{4}(\partial\cdot F)\varphi + \frac{1}{48i}\sigma^{\mu A\dot{A}}([D_A, \bar{D}_{\dot{A}}]F_\mu)\varphi) \tag{6.32}$$

and

$$\delta\bar{\varphi} = (F\cdot\partial)\bar{\varphi} - \frac{1}{8i}\sigma_\mu^{A\dot{A}}D_A F^\mu \bar{D}_{\dot{A}}\bar{\varphi} + \Delta(\frac{1}{4}(\partial\cdot F)\bar{\varphi} - \frac{1}{48i}\sigma^{\mu A\dot{A}}([D_A, \bar{D}_{\dot{A}}]F_\mu)\bar{\varphi}) \tag{6.33}$$

The reader will observe that the dilation and $A$ weights of the chiral superfield are tied together, a fact that can be established by taking the straightforward reduction of equation (6.28) and making sure the transformed superfield is still chiral or anti-chiral as appropriate. We will discuss this result from a more general perspective in the next section.

### 6.2 Anomalous Dimensions of Chiral Operators at a Fixed Point

Let us consider a supersymmetric theory at a fixed point of the renormalisation group, i.e. $\beta = 0$. Such a theory should be invariant under superconformal transformations. As in all supersymmetric theories some of the observables are given by chiral operators which by definition obey the equation

$$D_{\dot{A}}\varphi = 0 \tag{6.34}$$

where $\varphi$ denotes the chiral operator involved. It follows that this equation must itself be invariant under any superconformal transformation i.e. $D_{\dot{A}}\delta\varphi = 0$. Choosing a special supersymmetry transformation we conclude that

$$\{D_{\dot{A}},\ S_{\dot{B}}\}\varphi = 0 \tag{6.35}$$

In this equation we can swop the covariant derivative for the generator of supersymmetry transformations using the equation

$$D_{\dot{A}} = \frac{\partial}{\partial\theta^{\dot{A}}} - i(\sigma^m)_{A\dot{A}}\theta^A\partial_m = Q_{\dot{A}} - 2i(\sigma^m)_{A\dot{A}}\theta^A\partial_m \tag{6.36}$$

We then conclude that

$$\{Q_{\dot{A}},\ S_{\dot{B}}\}\varphi = 0 \tag{6.37}$$

plus terms that contain space-time derivatives. However, in this equation the condition must hold separately on the parts of the equation containing space-time derivatives and those that do not. The advantage of writing the equation in this form is that the anti-commutator is one of the defining relations of the superconformal algebra, namely

$$\{Q_{\dot{A}},\ S_{\dot{B}}\} = \epsilon_{\dot{A}\dot{B}}(2D - 4iA) + (\bar{\sigma}^{\mu\nu})_{\dot{A}\dot{B}}J_{\mu\nu} \tag{6.38}$$

where $D$ and $A$ are the generators of the dilations and U(1) transformations in the superconformal algebra which we gave this algebra in lecture one. If we restrict the superconformal algebra to just its super-Poincare subgroup then the $A$ generator is identified with the generator of $R$ transformations. The latter satisfies the relation $[Q_A,\ R] = iQ_A$ comparing this with the equivalent commutator in the superconformal group (i.e. $[Q_A,\ A] = -i\frac{3}{4}Q_A$) we thus find that the generators are related by $A = -\frac{3}{4}R$.

Consequently for a Lorentz invariant chiral operator we conclude that $D = -i\frac{3}{2}R$. One can also find this result by substituting the explicit expressions for $D_{\dot{A}}$ and $S_{\dot{A}}$ in equation (6.35) and setting $\theta = 0$. We summarise the result in the theorem

**Theorem** [309]

Any Lorentz invariant operator in a four dimensional supersymmetric theory at a fixed point has its anomalous dimension $\Delta$ and chiral $R$ weight, related by the equation

$$\Delta = -i\frac{3}{2}R. \tag{6.39}$$

In any conformal theory we can determine the two and three point Green's functions using conformal invariance alone. However, one can not normally use this symmetry alone to fix the anomalous weights of any operators. Since non-trivial fixed points are outside the range of usual perturbation theory, these must be calculated using techniques such as the $\epsilon$-expansion. The result so obtained are approximations and in some case one can not reliably calculate the anomalous dimensions at all. However, in supersymmetric theories at a fixed point one can determine the anomalous dimensions of chiral operators in superconformal theories exactly in terms of their $R$ weight. However, in many situations one does know the $R$ weight of the chiral operators of interest and we so can indeed exploit the above theorem to find their anomalous dimensions exactly [309]. We shall shortly demonstrate this procedure with some examples.

We must first fix the normalisation of the dilation and $R$ weights that is implied by the superconformal algebra. The relation $[P_\mu,\ D] = P_\mu$ implies that $P_\mu$ has dilation weight one. On the other hand, the relationship $[Q_A,\ R] = iQ_A$ implies that $Q_A$ has $R$ weight 1.

Consequently, $\theta^A$ has $R$ weight $-1$ meaning that it transforms as $\theta^A \to e^{-i\alpha}\theta^A$ where $\alpha$ is the parameter of $R$ transformations.

As our first example, let us consider the Wess-Zumino model in four dimensions and suppose that it had a non-trivial fixed point at which the interaction was of the usual form;

$$\int d^4x d^2\theta \varphi^3 \tag{6.40}$$

Using the above scaling of $\theta$ we find that $\varphi$ transforms as $\varphi \to e^{i\frac{2}{3}\alpha}\varphi$ and as a result $\varphi$ has $R$ weight $\frac{2}{3}$. Using our theorem we find that $\varphi$ had dilation weight one. This is the canonical dilation weight of $\varphi$, that is, the weight it would have in the free theory. It can be argued that if $\varphi$ has its canonical weight then the theory must be free and so such a non-trivial fixed point can not exist [310]. It can also be argued that this result implies the the Wess-Zumino model is a trivial field theory meaning that the only consistent value of the coupling constant as we remove the cutoff is zero [311].

Now let us consider the Wess-Zumino model in three dimensions and suppose it has a non-trivial fixed point at which the interaction is given by

$$\int d^3x d^2\theta \varphi^3. \tag{6.41}$$

This is the supersymmetric generalisation of the Ising model. Running through the same argument as above, but taking into account the modified form of the three dimensional superconformal algebra, we find that $\varphi$ has anomalous dimension $\frac{1}{6}$. The scaling weight of a quantum operator is made up of the sum of its canonical weight and its anomalous weight. The operator $\varphi$ has canonical weight 1/2 and hence its scaling weight is $\frac{2}{3}$. Such a non-trivial fixed point is known to exist by using the epsilon expansion which also gives an anomalous dimension in agreement with this result [312]. Since the anomalous dimension is non-zero the theory can not be a trivial theory, as is the case for the Wess-Zumino model in four dimensions.

We now briefly summarise the alternative argument given in reference [312] for calculating the anomalous dimensions of chiral operators using the epsilon expansion. Let us consider a $d$ dimensional theory whose action is given by

$$\int d^dx d^4\theta \varphi_0 \bar{\varphi}_0 + (\int d^dx d^2\theta g_0 \varphi_0^n + c.c) \tag{6.42}$$

where the action is given in terms of bare quantities which are denoted with a subscript 0. The dimension of $\varphi_0$ and $g_0$ are readily found to be $\frac{(d-2)}{2}$ and $d-1-\frac{n}{2}(d-2)$ respectively. The critical dimension, $d_c$ of the theory is the one where the coupling $g_0$ is dimensionless and so is given by $d_c = 2\frac{(n-1)}{(n-2)}$. To renormalize the theory we introduce the wavefunction and coupling renoramalization constants $Z$ and $Z_g$ respectively which relate the bare quantities to the renormalized quantities. The latter are denoted by the same symbols, but without a subscript 0. The relationships between the bare and renormalized quantities are given by the equations

$$\varphi_0 = Z^{\frac{1}{2}}\varphi, \quad \text{and} \quad g_0 = \mu^{\epsilon\frac{(n-2)}{2}} Z_g g \tag{6.43}$$

where the constant $\mu$ is the renormalization scale. It is raised to the above power in order that $g$ be dimensionless.

In the epsilon expansion method one carries out a double perturbation expansion in $\epsilon \equiv d_c - d$ and $g$. The beta-function, $\beta$ and anomalous dimension of $\varphi$, $\gamma$ are defined, as usual, by

$$\beta = \mu \frac{\partial}{\partial \mu} g, \quad \gamma = \mu \frac{\partial}{\partial \mu} lnZ \tag{6.44}$$

where the differentiation is carried out for $g_0$ and the regulator, $\Lambda$ held fixed. The non-renormalisation theorem implies that $Z_g Z^{\frac{n}{2}} = 1$ and as a result we find that

$$\beta = -\frac{1}{2}(n-2)\epsilon g + ng\gamma \tag{6.45}$$

Since we are interested in the anomalous dimension at a fixed point (i.e. $\beta = 0$) we must conclude that

$$\gamma = \epsilon \frac{(n-2)}{2n} = \frac{(n-1)}{n} - d\frac{(n-2)}{2n} \tag{6.46}$$

Using the renormalization group to calculate the two point function of $\varphi$ we find that the actual scaling dimension of $\varphi$ is the sum of its anomalous dimension and its canonical dimension and so its scaling dimension is the given by

$$\frac{(n-1)}{n} + \frac{d}{n} - 1 \tag{6.47}$$

Substituting $d = n = 3$ we recover the above result for the three dimensional Ising model, while for the four dimensional Ising model $\varphi$ has zero anomalous dimension. The calculation of the anomalous dimensions of chiral operators in supersymmetric theories is the only known case for which the epsilon expansion gives exact results. However, even in this case one must go further and establish the existence of the non-trivial fixed point [312].

The theorem in this section can also be used to fix the anomalous dimensions for the chiral operators in the two dimensional $N = 2$ supersymmetric Landau-Ginsburg models whose superpotential at the fixed point take the form

$$\int d^2x d^2\theta \varphi^n + c.c \tag{6.48}$$

The anomalous dimensions agree with the correspondence between these models at their fixed points and the $N = 2$ minimal series of superconformal models. This result was first conjectured in [317] and shown by using the epsilon expansion in reference [312]. Using equation (6.45) we find that the anomalous dimension of $\varphi$ is $\frac{1}{n}$.

The theorem can also be applied to four dimensional gauge invariant operators composed form the $N = 1$ Yang-Mills field strength $W_A$. Such a connection was used to argue that $N = 1$ super QED is trivial [309] and has been used extensively by Seiberg in recent work on dualities between certain $N = 1$ supersymmetric theories.

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