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Supergravity, Brane Dynamics and String Duality ‡

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Abstract

In this review we show that a Clifford algebra possesses a unique irreducible representation; the spinor representation. We discuss what types of spinors can exist in Minkowski space-times and we explain how to construct all the supersymmetry algebras that contain a given space-time Lie algebra. After deriving the irreducible representations of the superymmetry algebras, we explain how to use them to systematically construct supergravity theories. We give the maximally supersymmetric supergravity theories in ten and eleven dimensions and discuss their properties. We find which superbranes can exist for a given supersymmetry algebra and we give the dynamics of the superbranes that occur in M theory. Finally, we discuss how the properties of supergravity theories and superbranes provide evidence for string duality.

In effect, we present a continuous chain of argument that begins with Clifford algebras and leads via supersymmetry algebras and their irreducible representations to supergravity theories, string duality, brane dynamics and M theory.

‡ This material is based on lectures presented at the EU conference on *Duality and Supersymmetric Theories*, the Issac Newton Institute, Cambridge, UK and at the TASI 1997 Summer School, Boulder, Colorado, USA.

Contents

0. Introduction

In this review we begin with an account of Clifford algebras and show that they each possess only one irreducible representation, the spinor representation. We discuss the types of spinors can exist in Minkowski space-times. Equipped with this knowledge, we show how to systematically construct the supersymmetry algebras that contain a given space-time Lie group, such as the Poincare group. In the context of the four-dimensional supersymmetry algebras, we illustrate how to find all irreducible representations of the supersymmetry algebras and so arrive at a listing of all possible supersymmetric theories.

We then turn to the construction of theories of local supersymmetry, that is supergravity theories and show how to derive these theories from a knowledge of the corresponding irreducible representation of the supersymmetry algebra, (i.e. on-shell states). Of the three methods given for the construction of supergravity theories two are rather systematic in that they always lead to the desired theory with very little additional information. In section 5 and we give the unique supergravity theory in eleven dimensions while in section 6 we describe that two maximally supersymmetric theories in ten dimensions, the IIA and IIB supergravity theories. In addition to explaining how one constructs these theories according to the methods given in section 4, we describe the properties of these theories. These include the $SL(2,R)$ invariance of the IIB theory and the derivation of the IIA supergravity theory from the eleven-dimensional supergravity theory by compactification on a circle. The ten-dimensional IIA and IIB supergravity theories are the low energy effective actions of IIA and IIB string theories and we discuss the consequences of this relationship for string theories. The eleven-dimensional supergravity theory is thought to be the low energy effective action of a yet to be clearly defined theory called M theory.

The supergravity theories admit solitonic solutions to their classical field equations that correspond to static p-branes. A p-brane is an object which sweeps out a $p+1$-dimensional surface as it moves through space-time. They generalise strings which are 1-branes. However, from the string theory perspective p-branes for $p > 1$ are non-perturbative objects. In section 7, we give the dynamics of p-branes and discuss which p-branes can occur in the IIA, IIB and M theories. In particular, we find that in M theory we can have only twobranes and fivebranes and we give the equations of motion of these branes.

The supergravity theories in ten and eleven dimensions form the basis for most discussions of string duality and in section 8 we outline some of these arguments. In particular we discuss the non-perturbative string duality symmetries and the way they rotate perturbative string states into the p-branes discussed in section 7.

In effect this review traces a continuous chain of argument that begins with Clifford algebras and leads via supersymmetry algebras and their representations to supergravity theories, superbranes and then to string dualities and M theory.

1.Clifford Algebras and Spinors

In this section we define a Clifford algebra in an arbitrary dimension and find its irreducible representations and their properties. This enables us to find which types of spinors are allowed in a given Minkowski space-time.

The starting point for the construction of supersymmetric theories is the supersymmetry algebra which underlies it. Supersymmetry algebras contain supercharges which transform as spinors under the appropriate Lorentz group. Hence, even to construct the supersymmetry algebras, as we do in the section two, we must first find out what types of spinors are possible in a given dimension and what are their properties. We will find in subsequent sections that supersymmetric algebras and the supersymmetric theories on which they are based rely for their existence in an essential way on the detailed properties of Clifford algebras, that we will derive in this section.

As far as I am aware the first discussion of spinors in arbitrary dimensions was given in [100] and many of the steps in this section are taken from this paper. Use has also been made of the reviews of references [101], [102] and [194].

1.1. Clifford Algebras

A Clifford algebra in D dimensions is defined as a set containing D elements γ_m which satisfy the relation

$$\{\gamma_m, \gamma_n\} =\equiv \gamma_m\gamma_n + \gamma_n\gamma_m = 2\eta_{mn} \tag{1.1.1}$$

where the labels $m, n, \ldots$ take D values and η_{mn} is the flat metric in $R^{s,t}$ $(s+t=D)$; that is the metric η_{mn} is a diagonal matrix whose first t entries down the diagonal are -1 and whose last s entries are $+1$. We can raise and lower the $m, n, \ldots$ indices using the metric $\eta_{mn} = \eta^{mn}$ in the usual way.

Under multiplication the D elements γ_n of the Clifford algebra generate a finite group denoted C_D which consists of the elements

$$C_D = \{\pm 1, \pm\gamma_m, \pm\gamma_{m_1,m_2}, \ldots, \pm\gamma_{m_1\ldots m_D}\} \tag{1.1.2}$$

The $\gamma_{m_1 m_2\ldots}$ is non-vanishing only if all indices $m_1, m_2, \ldots$ are different in which case it equals

$$\gamma_{m_1 m_2\ldots} = \gamma_{m_1}\gamma_{m_2}\cdots \tag{1.1.3}$$

The set of matrices $\gamma_{m_1 m_2 \ldots m_p}$ for all possible different values of the m's contains

$$\frac{D!}{(D-p)!p!} = \begin{pmatrix} D \\ p \end{pmatrix}$$

different elements. Hence the group C_D generated by the γ_m has order

$$2\sum_{p=0}^{D} \begin{pmatrix} D \\ p \end{pmatrix} = 2(1+1)^D = 2^{D+1} \tag{1.1.4}$$

1.2 Clifford Algebras in Even Dimensions

To find the representations of C_D is a standard exercise in representation theory of finite groups [116]. We will first consider the case of even D.

The number of irreducible representations of any finite dimensional group, G equals the number of its conjugacy classes. We recall that the conjugacy class $[a]$ of $a \in G$ is given by

$$[a] = \{gag^{-1} \quad \forall\, g \in G\} \tag{1.2.1}$$

For even D it is straightforward to show, using equation (1.1.1), that the conjugacy classes of C_D are given by

$$[+1], [-1], [\gamma_m], [\gamma_{m_1 m_2}], \ldots, [\gamma_{m_1 \ldots m_D}] \tag{1.2.2}$$

Hence for D even there are $2^D + 1$ inequivalent irreducible representations of C_D.

Next we use the fact that the number of inequivalent one-dimensional representations of any finite group G is equal to the order of G divided by the order of the commutator group of G. We denote the commutator group of G by $Com(G)$. It is defined to be the group $Com(G) = aba^{-1}b^{-1}, \ \forall \ a, b \ \in \ G$. For D even the commutant of C_D is just the elements ± 1 and so has order 2. As a result, the number of inequivalent irreducible one-dimensional representations of C_D is 2^D. Since the total number of irreducible representations is $2^D + 1$, we conclude that there is only one irreducible representation whose dimension is greater that one.

Finally, we make use of the theorem that if we denote the order of any finite group by $ordG$ and it has p irreducible inequivalent representations of dimension n_p then

$$ordG = \sum_p (n_p)^2 \tag{1.2.3}$$

Applying this theorem to C_D we find that

$$2^{D+1} = 1^2 2^D + n^2 \tag{1.2.4}$$

where n is the dimension of the only irreducible representation whose dimension is greater than one. We therefore conclude that $n = 2^{\frac{D}{2}}$. These results are summarised in the following theorem.

Theorem For D even the group C_D has $2^D + 1$ inequivalent irreducible representations. Of these irreducible representations 2^D are one-dimensional and the remaining representation has dimension $2^{\frac{D}{2}}$.

This means that we can represent the γ_m as $2^{\frac{D}{2}}$ by $2^{\frac{D}{2}}$ matrices for the irreducible representation with dimension greater than one. Our next task is to find the properties of this representation under complex conjugation and transpose.

That the above are irreducible representations of the group C_D means that they provide a representation of the group which consists of the elements given in equation (1.1.2) together with a group composition law which is derived from the Clifford algebra relations using only the operation of multiplication. In particular, the group operations do not include the operations of addition and subtraction which also occur in the Clifford algebra defining condition of equation (1.1.1). Hence, the irreducible representations of C_D are not necessarily irreducible representations of the Clifford algebra itself. In fact, all the one-dimensional irreducible representations of C_D do not extend to be also representations of the Clifford algebra as they do not obey the rules for addition and subtraction. As such, the only representation of C_D and the Clifford algebra is the unique irreducible representation of dimension greater than one described above. It follows that Clifford algebra itself has only one irreducible representation and this has dimension $2^{\frac{D}{2}}$. It is of course the well known spinor representation. In fact, the one-dimensional representations are not faithful representations of C_D and we shall not consider them in what follows.

Given an irreducible representation of the Clifford algebra, also denoted γ_m, with dimension greater than one we can take its complex conjugate. Denoting the complex conjugate of the representation by γ_m^*, it is obvious that γ_m^* also satisfies equation (1.1.1) and so form a representation of the same Clifford algebra. It follows that they also form a representation of C_D. However, there is only one irreducible representation of C_D of dimension greater than one and as a result, the complex conjugate representation and the original representation must be equivalent. Consequently, there exist a matrix B such that

$$\gamma_m^* = B\gamma_m B^{-1} \tag{1.2.5}$$

We can choose the scale of B such that $|\det B| = 1$. Taking the complex conjugate of equation (1.2.5) we find that

$$\gamma_m = (\gamma_m^*)^* = +B^* B\gamma^m B^{-1} B^{-1*} \tag{1.2.6}$$

Hence, we conclude that B^*B commutes with the irreducible representation and by Schur's Lemma must be a constant times the identity matrix, i.e.

$$B^*B = \epsilon I \tag{1.2.7}$$

Taking the complex conjugate of the above relation we find that $BB^* = \epsilon^* I$ and so $BB^*BB^{-1} = \epsilon^* I = \epsilon I$ thus $\epsilon = \epsilon^*$ i.e. ϵ is real. Since we have chosen $|\det B| = 1$ we conclude that that $|\epsilon| = 1$ and so $\epsilon = \pm 1$.

We can also consider the transpose of the irreducible representation γ_m that has dimension greater than one. Denoting the transpose of γ_m by γ_m^T, we find, using a very similar argument, that γ_m and γ_m^T are equivalent representations and so there exists a matrix C, called the charge conjugation matrix, such that

$$\gamma_m^T = -C\gamma_m C^{-1} \tag{1.2.8}$$

We denote the Hermitian conjugate of γ_m by $\gamma_m^\dagger = \gamma_m^{*T}$. We can relate C to B if we know the Hermiticity properties of the γ_m. For simplicity, and because this is the case of most interest to us, from now on we assume that we are in a Lorentzian space-time whose metric η_{mn} is given by $\eta = \text{diag}\,(-1, +1, +1, \ldots, +1)$. Any finite-dimensional representation of a finite group G can be chosen to be unitary. Making this choice for our group C_D we have $\gamma_m\gamma_m^\dagger = 1$. Taking into account the relationship $\gamma_n\gamma_n = \eta_{nn}$ we conclude that

$$\gamma_0{}^\dagger = -\gamma_0, \quad \gamma_m^\dagger = \gamma_m; \quad m = 1, \ldots, D-1 \tag{1.2.9}$$

We could, as some texts do, regard this equation as part of the definition of the Clifford algebra. We may rewrite equation (1.2.9) as

$$\gamma_m^\dagger = \gamma_0\gamma_m\gamma_0 \tag{1.2.10}$$

We may take C to be given by $C = -B^T\gamma_0$ as then

$$C\gamma_m C^{-1} = B^T\gamma_0\gamma_m(-\gamma_0((B)^T)^{-1}) = -B^T\gamma_m^\dagger (B)^{T^{-1}} = -(B^{-1}\gamma_m^* B)^T = -\gamma_m^T \tag{1.2.11}$$

as required.

Further restrictions on B can be found by computing γ_m^T in two ways: we see that $\gamma_m^T = (\gamma_m^*)^\dagger = (\gamma_m^\dagger)^*$ implies

$$(B^{-1})^\dagger \gamma_0 \gamma_m \gamma_0 B^\dagger = B \gamma_0 \gamma_m \gamma_0 B^{-1} \tag{1.2.12}$$

Using Schur's Lemma we deduce that $-\gamma_0 B^\dagger B \gamma_0$ is proportional to the unit matrix and as a result so is $B^\dagger B$, i.e. $B^\dagger B = \mu I$. Since $|\det B| = 1$ we find that $|\mu| = 1$, but taking the matrix element of $B^\dagger B = \mu I$ with any vector we conclude that μ is real and positive. Hence $\mu = 1$ and consequently B is unitary, i.e. $B^\dagger B = I$. This result and the previously derived equation $BB^* = \epsilon I$ imply that

$$B^T = \epsilon B, \quad C^T = -\epsilon C \tag{1.2.13}$$

We now wish to determine ϵ in terms of the space-time dimension D. Consider the set of matrices

$$I, \gamma_m, \gamma_{m_1 m_2 m_3}, \gamma_{m_1 m_2 m_3}, \gamma_{m_1 \ldots m_D}. \tag{1.2.14}$$

There are $2^D = \sum_p \binom{D}{p} = (1+1)^D$ such matrices and as they are linearly independent they form a basis for the space of all $2^{\frac{D}{2}}$ by $2^{\frac{D}{2}}$ matrices. Using equation (1.1.1) we can relate $\gamma_{m_1 \ldots m_p}$ to $\gamma_{m_p \ldots m_1}$, to find that the sign change required to reverse the order of the indices is given by

$$\gamma_{m_1 \ldots m_p} = (-1)^{p\frac{(p-1)}{2}} \gamma_{m_p \ldots m_1}. \tag{1.2.15}$$

This equation together with equation (1.2.11) imply that

$$C \gamma_{m_1 \ldots m_p} C^{-1} = (-1)^p (-1)^{p\frac{(p-1)}{2}} \gamma_{m_1 \ldots m_p}{}^T \tag{1.2.16}$$

or equivalently

$$(C \gamma_{m_1 \ldots m_p}) = \epsilon (-1)^{\frac{(p-1)(p-2)}{2}} (C \gamma_{m_1 \ldots m_p})^T \tag{1.2.17}$$

Using this result we can calculate the number of anti-symmetric matrices in the complete set of equation (1.2.14) when multiplied from the left by C; It is given by is given by

$$\sum_{p=0}^{D} \frac{1}{2} \left(1 - \epsilon (-1)^{\frac{(p-1)(p-2)}{2}}\right) \binom{D}{p} \tag{1.2.18}$$

Using the relationship

$$(-1)^{\frac{(p-1)(p-2)}{2}} = -\frac{1}{2}\left[(1+i) i^n + (1-i)(-i)^n\right], \tag{1.2.19}$$

we can carry out the sum in equation (1.2.18). We know, however, that the number of anti-symmetric matrices is $2^{\frac{D}{2}}(2^{\frac{D}{2}} - 1)\frac{1}{2}$. Equating these two methods of evaluating the number of antisymmetric matrices we find that

$$\epsilon = -\sqrt{2} \cos \frac{\pi}{4}(D+1). \tag{1.2.20}$$

Put another way $\epsilon = +1$ for $D = 2, 4$; mod 8 and $\epsilon = -1$ for $D = 6, 8$; mod 8. It follows from equation (1.2.13) that for $D = 2, 4$; mod 8, B is a symmetric unitary matrix. Writing B in terms of its real and imaginary parts $B = B_1 + iB_2$ where B_1 and B_2 are symmetric and real, the unitarity condition becomes $B_1^2 + B_2^2 = 1$ and $[B_1, B_2] = 0$. Under a change of basis of the γ_m matrices; $\gamma^{m\prime} = A\gamma^n A^{-1}$ we find that the matrix B changes as $B' = A^* B A^{-1}$. In fact, we can use A to diagonalize B_1 and B_2 which, still being unitary, must be of the form $B = diag(e^{i\alpha_1}, \ldots, e^{i\alpha_D})$. Carrying out another A transformation of the form $A = diag(e^{i\frac{\alpha_1}{2}}, \ldots, e^{i\frac{\alpha_D}{2}})$ we find the new B equals one. Hence if $D = 2, 4$; mod 8 the γ_m matrices can be chosen to be real and $C = \gamma^0$.

1.3 Spinors in Even Dimensions

By definition a spinor λ transforms under $Spin(1, D-1)$ as

$$\delta\lambda = \frac{1}{4} w^{mn} \gamma_{mn} \lambda \tag{1.3.1}$$

where $w^{mn} = -w^{nm}$ are the parameters of the Lorentz transformation. The group $Spin(1, D-1)$ is by definition the group generated by $\frac{1}{4} w^{mn}\gamma_{mn}$ and it is the covering group of $SO(1, D-1)$. The Dirac conjugate denoted $\bar{\lambda}^D$ must transform such that $\bar{\lambda}^D \lambda \equiv \bar{\lambda}^{D\alpha}\lambda_\alpha$ is invariant and so transforms under a Lorentz transformation as

$$\delta\bar{\lambda}^D = \bar{\lambda}^D \left(-\frac{1}{4} w^{mn}\gamma_{mn}\right). \tag{1.3.2}$$

Using the relation $\gamma_{mn}^\dagger = -\gamma_0\gamma_{nm}\gamma_0 = \gamma_0\gamma_{mn}\gamma_0$ we find that

$$\delta(\lambda^+\gamma^0) = (\lambda^+\gamma^0)\left(-\frac{1}{4} w^{mn}\gamma_{mn}\right) \tag{1.3.3}$$

Consequently, we can take the Dirac conjugate to be defined by

$$\bar{\lambda}^D \equiv \lambda^\dagger \gamma^0 \tag{1.3.4}$$

The Majorana conjugate, denoted $\bar{\lambda}^M$ is defined by

$$\bar{\lambda}^m = \lambda^T C. \tag{1.3.5}$$

Using the relationship $\gamma_{mn}^T = C\gamma_n\gamma_m C^{-1} = -C\gamma_{mn}C^{-1}$ we find that

$$\delta\bar{\lambda}^M = \lambda^T \frac{1}{4} w^{mn}\gamma_{mn}^T C = -\lambda^T C \left(\frac{1}{4} w^{mn}\gamma_{mn}\right) = \bar{\lambda}^M \left(-\frac{1}{4} w^{mn}\gamma_{mn}\right) \tag{1.3.6}$$

Hence the Majorana conjugate transforms like the Dirac conjugate under $Spin(1, D-1)$ transformations and as a result we can define a Majorana spinor to be one is whose Dirac and Majorana conjugates are equal:

$$\bar{\lambda}^D = \bar{\lambda}^M \tag{1.3.7}$$

The above condition can be rewritten as $\lambda^* = -(\gamma^0)^T C^T\lambda$ and, using the relation $C = B^T\gamma^0$, it becomes

$$\lambda^* = B\lambda \tag{1.3.8}$$

We could have directly verified that $B^{-1}\lambda^*$ transforms under $Spin(1, D-1)$ in the same way as λ by using the equation $B\gamma_{mn}B^{-1} = (\gamma_{mn})^*$ and as a result have imposed this Majorana condition without any mention of the Dirac conjugate.

We are finally in a position to discover the dimensions in which Majorana spinors exist. If we impose the relationship $\lambda^* = B\lambda$, then, taking the complex conjugate we find that it implies the relationship $\lambda = B^*\lambda^*$. Substituting this condition into the first relation we find that

$$\lambda = B^*B\lambda = \epsilon\lambda \tag{1.3.9}$$

since $B^*B = \epsilon I$. Consequently, Majorana spinors can only exist if $\epsilon = +1$, which is the case only in the dimensions $D = 2, 4;$ mod 8, i.e. $D = 2, 4, 10, 12\ldots$.

In an even-dimensional space-time we can construct the matrix

$$\gamma^{D+1} = \gamma_0\gamma_1\ldots\gamma_{D-1} = \gamma_{01\ldots D-1} \tag{1.3.10}$$

This matrix anticommutes with γ_m and so commutes with the generators $\left(-\frac{1}{4}\gamma_{mn}\right)$ of spin$(1, D-1)$, the covering group of $SO(1, D-1)$. Hence $\gamma^{D+1}\chi$ transforms like a spinor if χ does. A straightforward calculation shows that

$$(\gamma^{D+1})^2 = (-1)^{\frac{D(D-1)}{2}}(-1) = (-1)^{\frac{D}{2}-1} \tag{1.3.11}$$

Hence $(\gamma^{D+1})^2 = 1$ for $D = 2$ mod 4 while $(\gamma^{D+1})^2 = -1$ for $D = 4$ mod 4. In either case we can define Weyl spinors

$$\gamma^{D+1}\chi = \pm\chi \qquad \text{if } D = 2 \bmod 4 \tag{1.3.12}$$

and

$$i\gamma^{D+1}\chi = \pm\chi \qquad \text{if } D = 4 \bmod 4 \tag{1.3.13}$$

We can now consider when Majorana-Weyl spinors exist. We found that Majorana spinors (i.e. $\chi^* = B\chi$) exist if $\epsilon = 1$, i.e. when $D = 2, 4;$ mod 8. Taking the complex conjugate of the above Weyl conditions and using the relationship $(\gamma^{D+1})^* = B\gamma^{D+1}B^{-1}$ we find we get a non-vanishing solution only if $D = 2$ mod 4. Hence Majorana-Weyl spinors only exist if $D = 2$ mod 8 i.e. $D = 2, 10, 18, 26, \ldots$. The factor of i is necessary for $D = 4$ mod 4 as the chirality condition must have an operator that squares to one, however it is this same factor of i that gets a minus sign under complex conjugation and so rules out the possibility of having Majorana-Weyl spinors in these dimensions. We note that these are the dimensions in which self-dual Lorentzian lattices exist and, except for 18 dimensions, these are the dimensions in which critical strings exist.

Corresponding to the above chiral conditions we can defined projectors onto the spaces of positive and negative chiral spinors. These projectors are given by $P_\pm = \frac{1}{2}(1 \mp a\gamma^{D+1})$ where $a = 1$ if $D = 2$ mod 4 and $a = i$ if $D = 4$ mod 4. It is easy to verify that they are indeed projectors; i.e. $P_\pm P_\mp = 0$, $P_\pm^2 = P_\pm$, $P_\mp^2 = P_\mp$ and $P_\pm + P_\mp = 1$. Under complex conjugation the projectors transform as

$$P_\pm^* = \begin{cases} BP_\pm B^{-1}, & \text{if } D = 2 \bmod 4 \\ BP_\mp B^{-1}, & \text{if } D = 4 \bmod 4 \end{cases} \tag{1.3.14}$$

This equation places restrictions on the form that the matrices B and the chiral projectors can take. For example, let us write the $2^{\frac{D}{2}}$ by $2^{\frac{D}{2}}$ γ-matrices in terms of $2^{\frac{D}{2}-1}$ by $2^{\frac{D}{2}-1}$ blocks. We also choose our basis of spinor such that the projection operators are diagonal and such that P_+ has only its upper diagonal block non-vanishing and equal to the identity matrix and P_- with only its lower diagonal block non-zero and equal to the identity matrix space. Applying equation (1.3.14), we find that if $D = 2 \bmod 4$, the matrix B has only its two diagonal blocks non-zero and if $D = 0 \bmod 4$, only its off-diagonal blocks non-zero.

Under complex conjugation the chiral spinors transform as

$$B^{-1}(P_\pm\lambda)^* = \begin{cases} P_\pm B^{-1}\lambda^*, & \text{if } D = 2 \bmod 4 \\ P_\mp B^{-1}\lambda^*, & \text{if } D = 4 \bmod 4 \end{cases} \tag{1.3.15}$$

Hence, complex conjugation and multiplication by B^{-1} relates the same chirality spinors if $D = 2 \bmod 4$ and opposite chirality spinors if $D = 4 \bmod 4$. As such, in the dimensions $D = 2, 4;\ \bmod 8$ where we can define the Majorana spinors, the Majorana condition relates same chirality spinors if $D = 2 \bmod 4$ and opposite chirality spinors if $D = 4 \bmod 4$.

We now investigate how a matrix transformation on λ acts on its chiral components. Under the matrix transformation $\lambda \to A\lambda$ we find that $B^{-1}\lambda^* \to (B^{-1}A^*B)B^{-1}\lambda^*$ and so the equivalent transformation on $B^{-1}\lambda^*$ is $B^{-1}A^*B$. Clearly if A is a polynomial in the γ-matrices with real coefficients then this transformation is the same on λ and λ^*. As we have already discussed this is the case with Lorentz transformations which are generated by $J^{mn} = \frac{1}{2}\gamma^{mn}$. However, if $A = EP_\pm$ where E is a polynomial in the γ-matrices with real coefficients then the transformation becomes

$$B^{-1}(EP_\pm)^*B = \begin{cases} CP_\pm, & \text{if } D = 2 \bmod 4 \\ P_\mp, & \text{if } D = 4 \bmod 4 \end{cases} \tag{1.3.16}$$

As an example of the latter let us consider the chiral projections of Lorentz transformations which are given by $J_\pm^{mn} \equiv \frac{1}{2}\gamma^{mn}P_\pm$. In this case the above equation becomes $B^{-1}J_\pm^{mn*}B = J_\pm^{mn}$ for $D = 2 \bmod 4$ and $B^{-1}J_\pm^{mn*}B = J_\mp^{mn}$ for $D = 4 \bmod 4$. Hence for $D = 2 \bmod 4$ we find that the representation generated by $J_\pm^{mn}$ generate a representation which for a given chirality is conjugate to the complex conjugate representation of the same chirality. On the other hand, for $D = 4 \bmod 4$ we find that the complex conjugate of the chiral Lorentz transformations of a given chirality is conjugate to that for the opposite chirality. In dimensions $D = 2, 4;\ \bmod 8$ we can choose $B = 1$ and it is straightforward to interpret the corresponding constraints. For example, if $D = 2 \bmod 8$ then $(J_\pm^{mn})^* = J_\pm^{mn}$ and the representation they generate is contained in the group $SL(2^{\frac{D}{2}-1}, \mathbf{R})$.

However, for $D = 6, 8;\ \bmod 8$ the matrix B is anti-symmetric and so can not be chosen to be one. If in addition we take $D = 2 \bmod 4$ that is $D = 6 \bmod 8$ then then we conclude that the chiral Lorentz transformations are contained in the group $SU^*(2^{\frac{D}{2}-1})$. The group $SU^*(N)$ is the group of N by N complex matrices of determinant one that commute with the operation of complex conjugation and multiplication by an anti-symmetric matrix B which obeys $B^\dagger B = 1$ [103]. That is if $A \in SU^*(N)$ then $BAB^{-1} = A^*$. Taking such an infinitesimal transformation $A = I + K$ we find that $SU^*(N)$ has real dimension $N^2 - 1$. The first such case is $D = 6$, since $SU^*(4)$ and $Spin(1,5)$ both have dimension 15 we must conclude that the group of six-dimensional chiral Lorentz transformations generated by $J^{mn}_\pm$ is isomorphic to the group $SU^*(4)$.

If the spinors carry internal spinor indices that transform under a pseudo-real representation of an internal group then we can also define a kind of Majorana spinor when $D = 6, 8$; mod 8 by the condition

$$(\lambda_i)^* = \Omega^{ij} B \lambda_j \tag{1.3.17}$$

In this equation Ω^{ij} is a real anti-symmetric matrix which also obeys the relations $\Omega^{ij} = \Omega_{ij}$ and $\Omega^{ij}\Omega_{jk} = -\delta^i_k$. We call spinors that satisfy this type of Majorana condition symplectic Majorana spinors. Taking the complex conjugate of this symplectic Majorana condition, using the above relations and the fact that $B^* B = -1$ we find that it is indeed a consistent condition. The symplectic Majorana condition should also be such that the internal group, which acts on the internal indices $i, j, \ldots$, acts in the same way on the left and right hand sides of the symplectic Majorana condition. This requires λ_i to carry a pseudo-real representation of the internal group.

The vector representation of the group $USp(N)$ provides one of the most important examples of a pseudo-real representation. The group $USp(N)$ consists of unitary matrices that in addition preserve Ω^{ij}; that is matrices A which satisfy $A^\dagger A = 1$ and $A^T \Omega A = \Omega$. Taking such an infinitesimal transformation we find that this group has dimension $\frac{1}{2}N(N+1)$. Under $\lambda \to A\lambda$, we find that $\lambda^* \to A^*\lambda^*$, but $\Omega\lambda \to \Omega A \lambda = -\Omega A \Omega \Omega \lambda$. However, using the defining conditions of $USp(N)$ we can show that

$$-\Omega A \Omega = -\left(A^T\right)^{-1} \Omega\Omega = \left(A^T\right)^{-1} = \left(A^\dagger\right)^T = A^* \tag{1.3.18}$$

which means that the symplectic Majorana condition preserves $USp(N)$. We note that $USp(2) = SU(2)$.

In addition to the symplectic Majorana condition of equation (1.3.17) which requires $D = 6, 8$; mod 8 we can also impose a Weyl constraint if $D = 2$ mod 4. That is dimensions $D = 6$ mod 8 symplectic Majorana-Weyl spinors exist.

An important example of a symplectic Majorana-Weyl spinors are those in six dimensions that transform under $USp(4)$. These spinors naturally arise when we reduce that Majorana eleven-dimensional spinors to six dimensions. The eleven-dimensional spinors transform under $Spin(1,10)$ which under the reduction to six dimensions becomes $Spin(1,5) \times Spin(5)$. The $Spin(5)$ which is isomorphic to $USp(4)$ and becomes the internal group in six dimensions. In fact we get symplectic Majorana-Weyl spinors of both chirality each of which has $4 \times 4 = 16$ components.

1.4 Clifford Algebras in Odd Dimensions

We now take the dimension of space-time D to be odd. The group C_D of equation (1.1.2) is generated by the γ_n and has order 2^{D+1}. The irreducible representations can be found using the same arguments as we did for the case of a space-time of even dimensions. However, there are some differences which are a consequence of the fact that the conjugacy classes are not given by the obvious generalisation of those for even-dimensional case which were listed in equation (1.2.2). From all the γ-matrices, $\gamma_m,\ m=0,1,\ldots,D-1$ we can form the matrix $\gamma_D \equiv \gamma_0\gamma_1\ldots\gamma_{D-1}$. This matrix commutes with all the $\gamma_m\ m=0,1,\ldots,D-1$ and so all products of the γ_m. As such, $\pm\gamma_D$ form conjugacy classes by themselves and as a result the full list of conjugacy classes is given by

$$[1],[-1],[\gamma_m],[\gamma_{m_1m_2}],\ldots,[\gamma_{m_1\ldots m_D}],[-\gamma_{m_1\ldots m_D}] \tag{1.4.1}$$

There are 2^D+2 conjugacy classes and so 2^D+2 inequivalent irreducible representations of C_D.

The commutator group of C_D is given by $\{\pm 1\}$ and so has order 2. As such, the number of inequivalent irreducible one-dimensional representations of C_D is 2^D. Hence, in an odd-dimensional space-time we have two inequivalent irreducible representations of C_D of dimension greater than one. In either of these two irreducible representations, the matrix γ_D commutes with the entire representation and so by Schur's Lemma must be a multiple of the identity i.e. $\gamma_D = a^{-1}I$ where a is a constant. Multiplying both sides by γ_{D-1} we find the result

$$\gamma_{D-1} = a\gamma_0\gamma_1\ldots\gamma_{D-2} = a\gamma_{01\ldots D-2} \tag{1.4.2}$$

Using equation (1.3.11), we conclude that $(\gamma_{01\ldots D-2})^2 = -(-1)^{\frac{(D-1)}{2}}$ as the matrix $\gamma_{01\ldots D-2}$ is the same as that denoted by γ^{D+1} for the even-dimensional space-time with one dimension lower. However, as $\gamma_{D-1}^2 = +1$ we must conclude that $a=\pm 1$ for $D=3$ mod 4 and $a=\pm i$ for $D=5$ mod 4. The $\gamma_m;\ m=0,1,\ldots,D-2$ generate an even-dimensional Clifford algebra and we recall that the corresponding subgroup C_{D-1} has a unique irreducible representation of dimension greater than one, the dimension being $2^{\frac{(D-1)}{2}}$. It follows that the two irreducible representations for D odd which have dimension greater than one must coincide with this irreducible representation when restricted to C_{D-1}. Hence, the two inequivalent irreducible representations for D odd are generated by the unique irreducible representation for the $\gamma_m,\ m=0,1,\ldots,D-2$, with the remaining γ-matrix being given by $\gamma_{D-1} = a\gamma_0\gamma_1\ldots\gamma_{D-2}$. The two possible choices of a, given above, corresponding to the two inequivalent irreducible representations. Clearly, these two inequivalent irreducible representations both have dimension $2^{\frac{(D-1)}{2}}$ as this is the dimension of the unique irreducible representation with dimension greater than one in the space-time with one dimension less. We can check that this is consistent with the relationship between the order, 2^{D+1}, of the group and the sum of the dimension squared of all irreducible representations. The latter is given by $1^2.2^D + (2^{\frac{(D-1)}{2}})^2 + (2^{\frac{((D-1)}{2}})^2 = 2^{D+1}$ as required.

We now extend the complex conjugation and transpose properties discussed previously for even-dimensional space-time to the case of an odd-dimensional space-time. Clearly, for the matrices $\gamma_m,\ m=0,1,\ldots,D-2$, these properties are the same and are given in equations (2.1.5) and (2.1.8). It only remains to consider $\gamma_{D-1} = a\gamma_0\gamma_1\ldots\gamma_{D-2} \equiv a\gamma_{01\ldots D-2}$. It follows from the previous section that

$$\gamma^*_{01\ldots D-2} = B\gamma_{01\ldots D-2}B^{-1} \tag{1.4.3}$$

and

$$C\gamma_{01\ldots D-2}C^{-1} = (-1)^{\frac{(D-1)}{2}}\gamma^T_{01\ldots D-2} \tag{1.4.4}$$

We may also write this last equation as

$$C\gamma_{01\ldots D-2} = -\epsilon(-1)^{\frac{(D-1)}{2}}(C\gamma_{01\ldots D-2})^T \tag{1.4.5}$$

as $C^T = -\epsilon C$. Taking into account the different possible values of a discussed above we conclude that

$$\gamma^*_{D-1} = -(-1)^{\frac{(D-1)}{2}}B\gamma_{D-1}B^{-1} \tag{1.4.6}$$

and

$$\gamma^T_{D-1} = (-1)^{\frac{(D-1)}{2}}C\gamma_{D-1}C^{-1} \tag{1.4.7}$$

As we did for the even-dimensional case we can adopt the choice $C = B^T\gamma^0$, whereupon we find that $\gamma^\dagger_{D-1} = \gamma^0\gamma_{D-1}\gamma^0$. The representation is automatically unitary as a consequence of being unitary on the C_{D-1} subgroup.

For $D = 3 \bmod 4$ γ_{D-1} has the same relationships under complex conjugation and transpose as do the $\gamma_m,\ m = 0, 1, \ldots, D-2$ and as a result for $D = 3 \bmod 4$

$$(C\gamma_{m_1\ldots m_p})^T = \epsilon(-1)^{\frac{(p-1)(p-2)}{2}}(C\gamma_{m_1\ldots m_p}) \quad m_1, \ldots, m_p = 0, \ldots, D-1 \tag{1.4.8}$$

For $D = 5 \bmod 4$, we get an additional minus sign in this relationship if one of the $m_1, \ldots, m_p$ takes the value $D-1$.

Let us now consider which types of spinors can exist in odd dimensional space-times. Clearly in odd-dimensional space-times the Weyl condition is not a Lorentz invariant condition and so one cannot define such spinors. However, we can ask which odd-dimensional space-times have Majorana spinors, which we take to be defined by $\chi^* = B\chi$. Since either of the two inequivalent irreducible representations coincides with the unique irreducible representation of dimension greater than one when restricted to the subgroup C_{D-1}, the matrix B is the same as in the even-dimensional case. It follows that we require $\epsilon = +1$ which is the case for D=3,5 mod 8. We must, however, verify that the Majorana condition is preserved by all Lorentz transformations. Those that are generated by $\frac{1}{2}\gamma_{mn},\ m, n = 0, 1\ldots, D-2$ are guaranteed to work; however, carrying out the Lorentz transformation $\delta\chi = \frac{1}{4}\gamma_{mD-1}\chi, m = 0, 1\ldots, D-2$ we find it preserves the Majorana constraint only if $D = 3 \bmod 4$. Hence, Majorana spinors exist in odd D-dimensional space-time if $D = 3 \bmod 8$. We note that these odd dimensions are precisely one dimension higher than those where Majorana-Weyl spinors exist. This is not a coincidence as the reduction of a Majorana spinor in $D = 3 \bmod 8$ dimensions leads to two Majorana Weyl spinors of opposite chirality and, since the resulting matrix γ_{D+1}, is real we may Weyl project to find a Majorana-Weyl spinor.

1.5 Central Charges

One important application of the above theory is to find what central charges can appear in a supersymmetry algebra. That is what generators can appear in the anti-commutator $\{Q_\alpha,\ Q_\beta\}$ where Q_α is the generator of supersymmetry transformations. As we shall see, the result depends on the dimension of space-time and on whether the spinor Q_α is Weyl or Majorana-Weyl. To begin with we take the dimension of space-time to be even.

The right-hand side of the anti-commutator of the supercharges takes the form [117]

$$\{Q_\alpha,\ Q_\beta\} = (\gamma_m C^{-1})_{\alpha\beta} P^m + \sum (\gamma_{m_1\ldots m_p} C^{-1})_{\alpha\beta} Z^{m_1\ldots m_p} \tag{1.5.1}$$

where $Z^{m_1\ldots m_p}$ are the central charges and P^m is the generator of translations. The sum is over all possible central terms. Clearly, the matrix $(\gamma_{m_1\ldots m_p} C^{-1})_{\alpha\beta}$ must be symmetric in α, β. Examining equation (1.2.17) we find that this will be the case if

$$\epsilon(-1)^{\frac{(p-1)(p-2)}{2}} = 1 \tag{1.5.2}$$

For $D = 2, 4$; mod 8, $\epsilon = 1$ and so we find central charges for $p = 1, 2$; mod 4. In these dimensions we can define Majorana spinors and adopting this constraint still allows these central charge although the Majorana condition will place reality conditions on them. For $D = 6, 8$; mod 8 $\epsilon = -1$ and so we find central charges of rank p for $p = 3, 4$ mod 4.

Let us now consider the case when the spinors are Weyl spinors that is satisfy $(P_\pm Q)_\alpha = 0$. in this case we must modify the terms on the right hand-side of the anti-commutator to be given by

$$\{Q_\alpha,\ Q_\beta\} = (\gamma_{m_1\ldots m_p} P_\pm C^{-1})_{\alpha\beta} Z^{m_1\ldots m_p} \tag{1.5.3}$$

In this case $(\gamma_{m_1\ldots m_p} C^{-1})_{\alpha\beta}$ and $(\gamma_{m_1\ldots m_p} \gamma^{D+1} C^{-1})_{\alpha\beta}$ must be symmetric. However, the latter matrix is equal to ϵ times $(\gamma_{m_1\ldots m_{D-p}} C^{-1})_{\alpha\beta}$. Hence central charges of rank p are possible if $p = 1, 2$; mod 4 and $D - p = 1, 2$; mod 4.

Finally, we can consider Majorana-Weyl spinors which only exist in $D = 2$ mod 8. In this case we have the condition on p of the Weyl case above, which must be taken with $D = 2 + 8n$,for $n \in\ \mathbf{Z}$, allows only central charges of rank $p = 1$ mod 4. An example of this latter case is provided by $N = 1$ $D = 10$ which, if we take Majorana-Weyl spinors, is the algebra that underlies the $N = 1$ Yang-Mills theory and the type I supergravity theory that exists in ten dimensions. The algebra is then given by

$$\{Q_\alpha,\ Q_\beta\} = (\gamma_m P_\pm C^{-1})_{\alpha\beta} P^m + (\gamma_{m_1\ldots m_5} P_\pm C^{-1})_{\alpha\beta} Z^{m_1\ldots m_5} + (\gamma_{m_1\ldots m_9} P_\pm C^{-1})_{\alpha\beta} Z^{m_1\ldots m_9} \tag{1.5.4}$$

Clearly P^m is the usual generator of translations.

The above discussion can be generalised to the case of odd-dimensional space-times although in this case we do not have Weyl spinors. For the case of $D = 3$ mod 4, γ_{D-1} behaves exactly like the other γ-matrices under complex conjugation and transpose and as a result we find from equation (1.4.8) precisely the same condition for the existence of central charges i.e. equation (1.5.2) with the value of ϵ being the same as that in one dimension less. The case of $D=5$ mod 4 can be deduced in a similar way by taking into account the discussion below equation (1.4.8).

A very important example is the supersymmetry algebra in eleven dimensions for Majorana supercharges. This algebra underlies the supergravity theory in this dimension. For this algebra we can have central charges of rank p with $p=1,2$; mod 4 and so the supersymmetry algebra takes the form [117].

$$\{Q_\alpha, Q_\beta\} = (\gamma^m C^{-1})_{\alpha\beta} P_m + (\Gamma^{mn} C^{-1})_{\alpha\beta} Z_{mn} + (\Gamma^{mnpqr} C^{-1})_{\alpha\beta} Z_{mnpqr} \qquad (1.5.5)$$

We need only go up to rank five thanks to the identity

$$\epsilon^{m_1 \ldots m_p m_{p+1} \ldots m_{D-1}} \gamma_{n_{p+1} \ldots n_{D-1}} \propto \gamma^{m_1 \ldots m_p}$$

which is true in all odd-dimensional spaces.

To find the possible central charges is the obvious part of constructing the supersymmetry algebra we must also find their relations with the rest of the generators of the algebra. This involves enforcing the super Jacobi identities as explained in the next section.

It is straightforward to extend the discussion to supersymmetry algebras which contain supersymmetry generators Q^i_α that carry a representation of an internal algebra. The anti-commutator $\{Q^i_\alpha, Q^j_\beta\}$ is now symmetric under interchange in α, i and β, j.

For an account of Clifford algebras and spinors in non-Lorentzian space-times the reader is encouraged to consult reference [102].

2 The Supersymmetry Algebra in Four dimensions

The starting point for the construction of any supersymmetric theory which is invariant under either rigid or local supersymmetry is the supersymmetry algebra which underlies it. In section this section we demonstate how to systematically construct supersymmetry algebras from some very mild assumptions.

Supersymmetry algebras contain generators, called supercharges, that are Grassmann odd, transform under the Lorentz group as spinors and obey anti-commutation relations. It is a remarkable fact that supersymmetric algebras can be constructed from very little information. We must specify the spinorial character of the supercharges and what space-time Lie algebra is contained in the supersymmetry algebra. In section one, we deduced the possible spinors that can exist in a given dimension and we will see that the different choices lead to different superysymmetry algebras. The most important of the space-time Lie subalgebras is the Poincare algebra, but other possible choices include the de-Sitter algebra or the conformal algebra. Given a choice of the spinorial character of the supercharges and the space-time Lie subalgebra, the deduction of the corresponding supersymmetry algebra relies on a generalisation of the Jacobi identites that occur in Lie algebras.

In this section, we show how to systematically construct the supersymmetry algebras in four dimensions. In particular, we demonstrate how to construct all supersymmetry algebras that contain the Poincaré Lie algebra. We also state the supersymmetry algebras that contain the conformal and de Sitter Lie algebras. The generalisation to the sytematic construction of supersymmetry algebras in higher dimensions is straightforward. The only difference is that the no-go theorem must be used with more caution as its proof contains a number of assumptions that are not valid for higher-dimensional theories.

This material is essentially the same as that given in reference [0] and we have kept the same equation numbers as in that reference. We thank World Scientific publishing for their kind permission to reproduce this material.

In the 1960's, with the growing awareness of the significance of internal symmetries such as $SU(2)$ and larger groups, physicists attempted to find a symmetry which would combine in a non-trivial way the space-time Poincaré group with an internal symmetry group. After much effort it was shown that such an attempt was impossible within the context of a Lie group. Coleman and Mandula[4] showed on very general assumptions that any Lie group which contained the Poincaré group P, whose generators P_a and J_{ab} satisfy the relations

$$\begin{aligned}
[P_a, P_b] &= 0 \\
[P_a, J_{bc}] &= (\eta_{ab}P_c - \eta_{ac}P_b) \\
[J_{ab}, J_{cd}] &= -(\eta_{ac}J_{bd} + \eta_{bd}J_{ac} - \eta_{ad}J_{bc} - \eta_{bc}J_{ad})
\end{aligned} \tag{2.1}$$

and an internal symmetry group G with generators T_s such that

$$[T_r, T_s] = f_{rst}T_t \tag{2.2}$$

must be a direct product of P and G; or in other words

$$[P_a, T_s] = 0 = [J_{ab}, T_s] \tag{2.3}$$

They also showed that G must be of the form of a semisimple group with additional $U(1)$ groups.

It is worthwhile to make some remarks concerning the status of this no-go theorem. Clearly there are Lie groups that contain the Poincaré group and internal symmetry groups in a non-trivial manner; however, the theorem states that these groups lead to trivial physics. Consider, for example, two-body scattering; once we have imposed conservation of angular momentum and momentum the scattering angle is the only unknown quantity. If there were a Lie group that had a non-trivial mixing with the Poincaré group then there would be further generators associated with space-time. The resulting conservation laws will further constrain, for example, two-body scattering, and so the scattering angle can only take on discrete values. However, the scattering process is expected to be analytic in the scattering angle, θ, and hence we must conclude that the process does not depend on θ at all.

Essentially the theorem shows that if one used a Lie group that contained an internal group which mixed in a non-trivial manner with the Poincaré group then the S-matrix for all processes would be zero. The theorem assumes among other things, that the S-matrix exists and is non-trivial, the vacuum is non-degenerate and that there are no massless particles. It is important to realise that the theorem only applies to symmetries that act on S-matrix elements and not on all the other many symmetries that occur in quantum field theory. Indeed it is not uncommon to find examples of the latter symmetries. Of course, no-go theorems are only as strong as the assumptions required to prove them.

In a remarkable paper Gol'fand and Likhtman[1] showed that provided one generalised the concept of a Lie group one could indeed find a symmetry that included the Poincaré group and an internal symmetry group in a non-trivial way. In this section we will discuss this approach to the supersymmetry group; having adopted a more general notion of a group, we will show that one is led, with the aid of the Coleman-Mandula theorem, and a few assumptions, to the known supersymmetry group. Since the structure of a Lie group, at least in some local region of the identity, is determined entirely by its Lie algebra, it is necessary to adopt a more general notion than a Lie algebra. The vital step in discovering the supersymmetry algebra is to introduce generators Q^i_α, which satisfy anti-commutation relations, i.e.

$$\begin{aligned}\{Q^i_\alpha, Q^j_\beta\} &= Q^i_\alpha Q^j_\beta + Q^j_\beta Q^i_\alpha \\ &= \text{ some other generator}\end{aligned} \tag{2.4}$$

The significance of the i and α indices will become apparent shortly. Let us therefore assume that the supersymmetry group involves generators $P_a,\ J_{ab},\ T_s$ and possibly some other generators which satisfy commutation relations, as well as the generators Q^i_α ($i = 1, 2, \ldots, N$). We will call the former generators which satisfy Eqs. (2.1), (2.2) and (2.3) even and those satisfying Eq. (2.4) odd generators.

Having let the genie out of the bottle we promptly replace the stopper and demand that the supersymmetry algebras have a Z_2 graded structure. This simply means that the even and odd generators must satisfy the rules:

$$\begin{aligned}[\text{even}, \text{even}] &= \text{even} \\ \{\text{odd}, \text{odd}\} &= \text{even} \\ [\text{even}, \text{odd}] &= \text{odd}\end{aligned} \tag{2.5}$$

We must still have the relations

$$[P_a, T_s] = 0 = [J_{ab}, T_s] \tag{2.6}$$

since the even (bosonic) subgroup must obey the Coleman-Mandula theorem.

Let us now investigate the commutator between J_{ab} and Q^i_α. As a result of Eq.(2.5) it must be of the form

$$[Q^i_\alpha, J_{ab}] = (b_{ab})_\alpha{}^\beta Q^i_\beta \tag{2.7}$$

since by definition the Q^i_α are the only odd generators. We take the α indices to be those rotated by J_{ab}. As in a Lie algebra we have some generalised Jacobi identities. If we denote an even generator by B and an odd generator by F, we find that

$$\begin{aligned}\big[[B_1, B_2], B_3\big] + \big[[B_3, B_1], B_2\big] + \big[[B_2, B_3], B_1\big] &= 0 \\ \big[[B_1, B_2], F_3\big] + \big[[F_3, B_1], B_2\big] + \big[[B_2, F_3], B_1\big] &= 0 \\ \{[B_1, F_2], F_3\} + \{[B_1, F_3], F_2\} + [\{F_2, F_3\}, B_1] &= 0 \\ [\{F_1, F_2\}, F_3] + [\{F_1, F_3\}, F_2] + [\{F_2, F_3\}, F_1] &= 0\end{aligned} \tag{2.8}$$

The reader may verify, by expanding each bracket, that these relations are indeed identically true.

The identity

$$\big[[J_{ab}, J_{cd}], Q^i_\alpha\big] + \big[[Q^i_\alpha, J_{ab}], J_{cd}\big] + \big[[J_{cd}, Q^i_\alpha], J_{ab}\big] = 0 \tag{2.9}$$

upon use of Eq. (2.7) implies that

$$[b_{ab}, b_{cd}]_\alpha^\beta = -\eta_{ac}(b_{bd})_\alpha^\beta - \eta_{bd}(b_{ac})_\alpha^\beta + \eta_{ad}(b_{bc})_\alpha^\beta + \eta_{bc}(b_{ad})_\alpha^\beta \tag{2.10}$$

This means that the $(b_{cd})_\alpha^\beta$ form a representation of the Lorentz algebra or in other words that the Q^i_α carry a representation of the Lorentz group. We will select Q^i_α to be in the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation of the Lorentz group, i.e.

$$[Q^i_\alpha, J_{ab}] = \frac{1}{2}(\sigma_{ab})_\alpha^\beta Q^i_\beta \tag{2.11}$$

We can choose Q^i_α to be a Majorana spinor, i.e.

$$Q^i_\alpha = C_{\alpha\beta}\bar{Q}^{\beta i} \tag{2.12}$$

where $C_{\alpha\beta} = -C_{\beta\alpha}$ is the charge conjugation matrix (see Appendix A). This does not represent a loss of generality since, if the algebra admits complex conjugation as an involution we can always redefine the supercharges so as to satisfy (2.12) (see Note 1 at the end of this chapter).

The above calculation reflects the more general result that the Q^i_α must belong to a realization of the even (bosonic) subalgebras of the supersymmetry group. This is a simple consequence of demanding that the algebra be Z_2 graded. The commutator of any even generator B_1, with Q^i_α is of the form

$$[Q^i_\alpha, B_1] = (h_1)^{i\beta}_{\alpha j} Q^j_\beta \tag{2.13}$$

The generalised Jacobi identity

$$\big[[Q^i_\alpha, B_1], B_2\big] + \big[[B_1, B_2], Q^i_\alpha\big] + \big[[B_2, Q^i_\alpha], B_1\big] = 0 \tag{2.14}$$

implies that

$$[h_1, h_2]^{i\beta}_{\alpha j} Q^j_\beta = \big[Q^i_\alpha [B_1, B_2]\big] \tag{2.15}$$

or in other words the matrices h represent the Lie algebra of the even generators.

The above remarks imply that

$$[Q^i_\alpha, T_r] = (l_r)^i_j Q^j_\alpha + (t_r)^i_j (i\gamma_5)_\alpha^\beta Q^j_\beta \tag{2.16}$$

where $(l_r)^i_j + i\gamma_5 (t_r)^i_j$ represent the Lie algebra of the internal symmetry group. This results from the fact that δ^α_β and $(\gamma_5)_\alpha^\beta$ are the only invariant tensors which are scalar and pseudoscalar.

The remaining odd-even commutator is $[Q^i_\alpha, P_a]$. A possibility that is allowed by the generalised Jacobi identities that involve the internal symmetry group and the Lorentz group is

$$[Q^i_\alpha, P_a] = c(\gamma_a)_\alpha^\beta Q^i_\beta \tag{2.17}$$

However, the $\big[[Q^i_\alpha, P_a], P_b\big] + \ldots$ identity implies that the constant $c = 0$, i.e.

$$[Q^i_\alpha, P_a] = 0 \tag{2.18}$$

More generally we could have considered $(c\gamma_a + d\gamma_a\gamma_5)Q$, on the right-hand side of (2.17); however the above Jacobi identity and the Majorana condition imply that $c = d = 0$. (See Note 2 at the end of this chapter). Let us finally consider the $\{Q^i_\alpha, Q^j_\beta\}$ anticommutator. This object must be composed of even generators and must be symmetric under interchange of $\alpha \leftrightarrow \beta$ and $i \leftrightarrow j$. The even generators are those of the Poincaré group, the internal symmetry group and other even generators which, from the Coleman-Mandula theorem, commute with the Poincaré group, i.e. they are scalar and pseudoscalar. Hence the most general possibility is of the form

$$\{Q^i_\alpha, Q^j_\beta\} = r(\gamma^a C)_{\alpha\beta} P_a \delta^{ij} + s(\sigma^{ab} C)_{\alpha\beta} J_{ab} \delta^{ij} + C_{\alpha\beta} U^{ij} + (\gamma_5 C)_{\alpha\beta} V^{ij} \tag{2.19}$$

We have not included a $(\gamma^b\gamma_5 C)_{\alpha\beta} L^{ij}_b$ term as the (Q, Q, J_{ab}) Jacobi identity implies that L^{ij}_b mixes nontrivially with the Poincaré group and so is excluded by the no-go theorem.

The fact that we have only used numerically invariant tensors under the Poincaré group is a consequence of the generalised Jacobi identities between two odd and one even generators.

To illustrate the argument more clearly, let us temporarily specialise to the case $N = 1$ where there is only one supercharge Q_α. Equation (2.19) then reads

$$\{Q_\alpha, Q_\beta\} = r(\gamma^a C)_{\alpha\beta} P_a + s(\sigma^{ab} C)_{\alpha\beta} J_{ab}.$$

Using the Jacobi identity

$$\{[P_a, Q_\alpha], Q_\beta\} + \{[P_a, Q_\beta], Q_\alpha\} + [\{Q_\alpha, Q_\beta\}, P_a] = 0,$$

we find that

$$0 = s(\sigma^{cd} C)_{\alpha\beta}[J_{cd}, P_a] = s(\sigma^{cd} C)_{\alpha\beta}(-\eta_{ac} P_d + \eta_{ad} P_c),$$

and, consequently, $s = 0$. We are free to scale the generator P_a in order to bring $r = 2$.

Let us now consider the commutator of the generator of the internal group and the supercharge. For only one supercharge, Eq. (2.16) reduces to

$$[Q_\alpha, T_r] = l_r Q_\alpha + i(\gamma_5)_\alpha^\beta t_r Q_\beta.$$

Taking the adjoint of this equation, multiplying by $(i\gamma^0)$ and using the definition of the Dirac conjugate given in Appendix A, we find that

$$[\bar{Q}^\alpha, T_r] = l_r^* \bar{Q}^\alpha + \bar{Q}^\beta (i t_r^*)(\gamma_5)_\beta^\alpha.$$

Multiplying by $C_{\gamma\alpha}$ and using Eq. (2.12), we arrive at the equation

$$[Q_\alpha, T_r] = l_r^* Q_\alpha + i t_r^* (\gamma_5)_\alpha^\beta Q_\beta.$$

Comparing this equation with the one we started from, we therefore conclude that

$$l_r^* = l_r, \qquad t_r^* = t_r.$$

The Jacobi identity

$$[\{Q_\alpha, Q_\beta\}, T_r] + \big[[T_r, Q_\alpha], Q_\beta\big] + \big[[T_r, Q_\beta], Q_\alpha\big] = 0$$

results in the equation

$$\begin{aligned}[0 + \big(l_r \delta_\alpha^\gamma + i t_r (\gamma_5)\alpha^\gamma\big) 2(\gamma_a C)_{\gamma\beta} P_a] + (\alpha \;\leftrightarrow\; \beta) = \\ 2P_a\{l_r(\gamma_a C)_{\alpha\beta} + i t_r (\gamma_5 \gamma_a C)_{\alpha\beta}\} + (\alpha \;\leftrightarrow\; \beta) = 0.\end{aligned}$$

Since $(\gamma_a C)_{\alpha\beta}$ and $(\gamma_5\gamma_a C)_{\alpha\beta}$ are symmetric and antisymmetric in α, β respectively, we conclude that $l_r = 0$ but t_r has no constant placed on it. Consequently, we find that we have only one internal generator R and we may scale it such that

$$[Q_\alpha, R] = i(\gamma_5)_\alpha^\beta Q_\beta.$$

The $N = 1$ supersymmetry algebra is summarised in Eq. (2.27).

Let us now return to the extend supersymmetry algebra. The even generators $U^{ij} = -U^{ji}$ and $V^{ij} = -V^{ji}$ are called central charges[5] and are often also denoted by Z. It is a consequence of the generalised Jacobi identities $\big((Q, Q, Q)$ and $(Q, Q, Z)\big)$ that they commute with all other generators including themselves, i.e.

$$[U^{ij}, \text{anything}] = 0 = [V^{ij}, \text{anything}] \tag{2.20}$$

We note that the Coleman-Mandula theorem allowed a semi-simple group plus $U(1)$ factors. The details of the calculation are given in note 5 at the end of the chapter. Their role in supersymmetric theories will emerge in later chapters.

In general, we should write, on the right-hand side of (2.19),

$$(\gamma^a C)_{\alpha\beta} \omega^{ij} P_a + \ldots,$$

where ω^{ij} is an arbitrary real symmetric matrix. However, one can show that it is possible to redefine (rotate and rescale) the supercharges, whilst preserving the Majorana condition, in such a way as to bring ω^{ij} to the form $\omega^{ij} = r\delta^{ij}$ (see Note 3 at the end of this chapter). The $[P_a, \{Q_\alpha^i, Q_\beta^j\}] + \ldots = 0$ identity implies that $s = 0$ and we can normalise P_a by setting $r = 2$ yielding the final result

$$\{Q_\alpha^i, Q_\beta^j\} = 2(\gamma_a C)_{\alpha\beta} \delta^{ij} P^a + C_{\alpha\beta} U^{ij} + (\gamma_5 C)_{\alpha\beta} V^{ij} \tag{2.21}$$

In any case r and s have different dimensions and so it would require the introduction of a parameter with a non-zero dimension in order that they were both non-zero.

Had we chosen another irreducible Lorentz representation for Q^i_α other than $(j+\frac{1}{2}, j) \oplus (j, j+\frac{1}{2})$ we would not have been able to put P_a, i.e. a $(\frac{1}{2}, \frac{1}{2})$ representation, on the right-hand side of Eq. (2.21). The simplest choice is $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$. In fact this is the only possible choice (see Note 4).

Finally, we must discuss the constraints placed on the internal symmetry group by the generalised Jacobi identity. This discussion is complicated by the particular way the Majorana constraint of Eq. (2.12) is written. A two-component version of this constraint is

$$\bar{Q}_{\dot{A}i} = (Q^i_A)^*; \quad A, \dot{A} = 1, 2 \tag{2.22}$$

(see Appendix A for two-component notation). Equation (2.19) and (2.16) then become

$$\begin{aligned} \{Q^i_A, \bar{Q}_{\dot{B}j}\} &= -2i(\sigma^a)_{A\dot{B}}\delta^i_j P_a \\ \{Q^i_A, Q^j_B\} &= \varepsilon_{AB}(U^{ij} + iV^{ij}) \\ [Q^i_A, J_{ab}] &= +\frac{1}{2}(\sigma_{ab})_A{}^B Q^i_B \end{aligned} \tag{2.23}$$

and

$$[Q^i_A, T_r] = (l_r + it_r)^i_j Q^j_A \tag{2.24}$$

Taking the complex conjugate of the last equation and using the Majorana condition we find that

$$[Q_{\dot{A}i}, T_r] = Q_{\dot{A}k}(U^\dagger_r)^k_i \tag{2.25}$$

where $(U_r)^i_j = (l_r + it_r)^i_j$. The $(Q, \bar{Q}, T)$ Jacobi identity then implies that δ^i_j be an invariant tensor of G, i.e.

$$U_r + U^\dagger_r = 0 \tag{2.26}$$

Hence U_r is an anti-Hermitian matrix and so represents the generators of the unitary group $U(N)$. However, taking account of the central charge terms in the (Q, Q, T) Jacobi identity one finds that there is for every central charge an invariant antisymmetric tensor of the internal group and so the possible internal symmetry group is further reduced. If there is only one central charge, the internal group is $Sp(N)$ while if there are no central charges it is $U(N)$.

To summarise, once we have adopted the rule that the algebra be Z_2 graded and contain the Poincaré group and an internal symmetry group then the generalised Jacobi identities place very strong constraints on any possible algebra. In fact, once one makes the further assumption that Q^i_α are spinors under the Lorentz group then the algebra is determined to be of the form of equations (2.1), (2.6), (2.11), (2.16), (2.18) and (2.21).

The simplest algebra is for $N = 1$ and takes the form

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 2(\gamma_a C)_{\alpha\beta} P^a \\ [Q_\alpha, P_a] &= 0 \\ [Q_\alpha, J_{cd}] &= \frac{1}{2}(\sigma_{cd})_\alpha{}^\beta Q_\beta \\ [Q_\alpha, R] &= i(\gamma_5)_\alpha{}^\beta Q_\beta \end{aligned} \tag{2.27}$$

as well as the commutation relations of the Poincaré group. We note that there are no central charges (i.e. $U^{11} = V^{11} = 0$), and the internal symmetry group becomes just a chiral rotation with generator R.

We now wish to prove three of the statements above. This is done here rather than in the above text, in order that the main line of argument should not become obscured by technical points. These points are best clarified in two-component notation.

Note 1: Suppose we have an algebra that admits a complex conjugation as an involution; for the supercharges this means that

$$(Q^i_A)^* = b_i{}^j Q_{\dot{A}j}; \quad (Q_{\dot{A}j})^* = d^j{}_k Q^k_A$$

There is no mixing of the Lorentz indices since $(Q^i_A)^*$ transforms like $Q_{\dot{A}i}$, namely in the $(0, \frac{1}{2})$ representation of the Lorentz group, and not like Q^i_A which is in the $(\frac{1}{2}, 0)$ representation. The lowering of the i index under $*$ is at this point purely a notational device. Two successive $*$ operations yield the unit operation and this implies that

$$(b_i{}^j)^* d^j{}_k = \delta^i_k \tag{2.28}$$

and in particular that $b_i{}^j$ is an invertible matrix. We now make the redefinitions

$$Q'^i{}_A = Q^i_A, \; Q'{}_{\dot{A}i} = b_i{}^j Q_{\dot{A}j} \tag{2.29}$$

Taking the complex conjugate of $Q'{}_{Ai}$, we find

$$(Q'^i{}_A)^* = (Q^i_A)^* = b_i{}^j Q_{\dot{A}j} = Q'{}_{\dot{A}j}$$

while

$$(Q'{}_{\dot{A}i})^* = (b_i{}^j)^* (Q_{\dot{A}j})^* = (b_i{}^j)^* d^j{}_k Q^k_A = Q^i_A \tag{2.30}$$

using Eq. (2.28).

Thus the $Q'^i{}_A$ satisfy the Majorana condition, as required. If the Q's do not initially satisfy the Majorana condition, we may simply redefine them so that they do.

Note 2: Suppose the $[Q_A, P_a]$ commutator were of the form

$$[Q_A, P_a] = e(\sigma_a)_{A\dot{B}} Q^{\dot{B}} \tag{2.31}$$

where e is a complex number and for simplicity we have suppressed the i index. Taking the complex conjugate (see Appendix A), we find that

$$[Q_{\dot{A}}, P_a] = -e^* (\sigma_a)_{B\dot{A}} Q^B \tag{2.32}$$

Consideration of the $\big[[Q_A, P_a], P_b\big] + \ldots = 0$ Jacobi identity yields the result

$$-|e|^2 (\sigma_a)_{A\dot{B}} (\sigma^b)^{C\dot{B}} - (a \leftrightarrow b) = 0 \tag{2.33}$$

Consequently $e = 0$ and we recover the result

$$[Q_A, P_a] = 0 \tag{2.34}$$

Note 3: The most general form of the $Q^{Ai}, Q^{\dot{B}}_j$ anticommutator is

$$\{Q^{Ai}, Q^{\dot{B}}_j\} = -2iU^i_j(\sigma^m)^{A\dot{B}}P_m + \text{terms involving other Dirac matrices} \tag{2.35}$$

Taking the complex conjugate of this equation and comparing it with itself, we find that U is a Hermitian matrix

$$(U^i_j)^* = U^j_i \tag{2.36}$$

We now make a field redefinition of the supercharge

$$Q'^{Ai} = B^i_j Q^{Aj} \tag{2.37}$$

and its complex conjugate

$$Q'^{\dot{A}}_i = (B^i_j)^* Q^{\dot{A}}_j \tag{2.38}$$

Upon making this redefinition in Eq. (2.35), the U matrix becomes replaced by

$$U'^i_j = B^i_k U^k_l (B^j_l)^* \quad \text{or} \quad U' = BUB^\dagger \tag{2.39}$$

Since U is a Hermitian matrix, we may diagonalise it in the form $c_i\delta^i_j$ using a unitarity matrix B. We note that this preserves the Majorana condition on Q^{Ai}. Finally, we may scale $Q^i \to (1/\sqrt{c}^i)Q^i$ to bring U to the form $U = d_i\delta^i_j$, where $d_i = \pm 1$. In fact, taking $A = B = 1$ and $i = j = k$, we realise that the right-hand side of Eq. (2.35) is a positive-definite operator and since the energy $-iP_0$ is assumed positive definite, we can only find $d_i = +1$. The final result is

$$\{Q^{Ai}, Q^{\dot{B}}_j\} = -2i\delta^i_j(\sigma^m)^{A\dot{B}}P_m \tag{2.40}$$

Note 4: Let us suppose that the supercharge Q contains an irreducible representation of the Lorentz group other than $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$, say, the representation $Q_{A_1 \ldots A_n, \dot{B}_1 \ldots \dot{B}_m}$ where the A and B indices are understood to be separately symmetrised and $n + m$ is odd in order that Q be odd and $n + m > 1$. By projecting the $\{Q, Q^\dagger\}$ anti-commutator we may find the anti-commutator involving $Q_{A_1 \ldots A_n, \dot{B}_1 \ldots \dot{B}_m}$ and its Hermitian conjugate. Let us consider in particular the anti-commutator involving $Q = Q_{11\ldots 1, \dot{1}\dot{1}\ldots\dot{1}}$, this must result in an object of spin $n + m > 1$. However, by the Coleman-Mandula no-go theorem no such generator can occur in the algebra and so the anti-commutator must vanish, i.e. $QQ^\dagger + Q^\dagger Q = 0$.

Assuming the space on which Q acts has a positive definite norm, one such example being the space of on-shell states, we must conclude that Q vanishes. However if $Q_{11\ldots 1, \dot{1}\dot{1}\ldots\dot{1}}$ vanishes, so must $Q_{A_1 \ldots A_n, \dot{B}_1 \ldots \dot{B}_m}$ by its Lorentz properties, and we are left only with the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation.

Note 5: We now return to the proof of equation (2.20). Using the (Q,Q,Z) Jocobi identity it is straightforward to show that the supercharge Q commutes with the central charges Z. The (Q,Q,U) Jacobi identity then implies that the central charges commute with themselves. Finally, one considers the (Q, Q, T_r) Jacobi identity, which shows that the commutator of T_r and Z takes the generic form $[T_r,\ Z] = \ldots Z$. However, the generators T_r and Z form the internal symmetry group of the supersymmetry algebra and from the no-go theorem we know that this group must be a semisimple Lie group times $U(1)$ factors. We recall that a semisimple Lie group is one that has no normal Abelian subgoups other that the group itself and the identity element. As such, we must conclude that T_r and Z commute, and hence our final result that the central charges commute with all generators, that is they really are central.

Although the above discussion started with the Poincaré group, one could equally well have started with the conformal or (anti-) de Sitter groups and obtained the superconformal and super (anti-)de Sitter algebras. For completeness, we now list these algebras. The superconformal algebra which has the generators $P_n, J_{mn}, D, K_n, A, Q^{\alpha i}$, $S^{\alpha i}$ and the internal symmetry generators T_r and A is given by the Lorentz group plus:

$$\begin{aligned}
[J_{mn}, P_k] &= \eta_{nk} P_m - \eta_{mk} P_n \\
[J_{mn}, K_k] &= \eta_{nk} K_m - \eta_{mk} K_n \\
[D, P_K] &= -P_K, \quad [D, K_K] = +K_K \\
[P_m, K_n] &= -2J_{mn} + 2\eta_{mn} D \quad [K_n, K_m] = 0, \quad [P_n, P_m] = 0 \\
[Q^i_\alpha, J_{mn}] &= \frac{1}{2}(\gamma_{mn})_\alpha^\beta Q^i_\beta, \quad [S^i_\alpha, J_{mn}] = \frac{1}{2}(\gamma_{mn})_\alpha^\beta S^{\beta i} \\
\{Q^i_\alpha, Q^j_\beta\} &= -2(\gamma^n C^{-1})_{\alpha\beta} P_n \delta^{ij} \\
\{S^i_\alpha, S^j_\beta\} &= +2(\gamma^n C^{-1})_{\alpha\beta} K_n \delta^{ij} \\
[Q^i_\alpha, D] &= \frac{1}{2} Q^i_\alpha, \quad [S^i_\alpha, D] = -\frac{1}{2} S^i_\alpha \\
[Q^i_\alpha, K_n] &= -(\gamma_n)_\alpha^\beta S^i_\beta, \quad [S^i_\alpha, P_n] = (\gamma_n)_\alpha^\beta Q^i_\beta \\
[Q^i_\alpha, T_r] &= \big(\delta_\alpha^\beta (\tau_{r_1})^i_j + (\gamma_5)_\alpha^\beta (\tau_{r_2})^i_j\big) Q^j_\beta \\
[S^i_\alpha, T_r] &= \big(\delta_\alpha^\beta (\tau_{r_1})^i_j - (\gamma_5)_\alpha^\beta (\tau_{r_2})^i_j\big) Q^j_\beta \\
[Q^i_\alpha, A] &= -i(\gamma_5)_\alpha^\beta Q^i_\beta \left(\frac{4-N}{4N}\right) \\
[S^i_\alpha, A] &= \frac{4-N}{4N} i(\gamma_5)_\alpha^\beta S^i_\beta \\
\{Q^i_\alpha, S^j_\beta\} &= -2(C^{-1}_{\alpha\beta}) D\delta^{ij} + (\gamma^{mn} C^{-1})_{\alpha\beta} J_{mn} \delta^{ij} + 4i(\gamma_5 C^{-1}_{\alpha\beta}) A \delta^{ij} \\
&\quad - 2(\tau_{r_1})^{ij} (C^{-1})_{\alpha\beta} + \big((\tau_{r_2})^{ij} (\gamma_5 C^{-1})_{\alpha\beta}\big) T_r
\end{aligned} \tag{2.41}$$

The T_r and A generate $U(N)$, and $\tau_1 + \gamma_5 \tau_2$ are in the fundamental representation of $SU(N)$.

The case of $N = 4$ is singular and one can have either

$$[Q^i_\alpha, A] = 0 \quad \text{or} \quad [Q^i_\alpha, A] = -i(\gamma_5)_\alpha^\beta Q^i_\beta$$

and similarly for S^i_α and A. One may verify that both possibilities are allowed by the $N=4$ Jacobi identities and so form acceptable superalgebras.

The anti-de Sitter superalgebra has generators M_{mn}, $T_{ij} = -T_{ji}$ and $Q^{\alpha i}$, and is given by

$$
\begin{aligned}
[M_{mn}, M_{pq}] &= \eta_{np} M_{mq} + 3 \text{ terms} \\
[M_{mn}, T_{ij}] &= 0 \quad [Q^i_\alpha, M_{mn}] = \frac{1}{2}(\gamma_{mn})_\alpha{}^\beta Q^i_\beta \\
[Q^i_\alpha, T^{jk}] &= -2i(\delta^{ij} Q^k_\alpha - \delta^{ik} Q^j_\alpha) \\
\{Q^i_\alpha, Q^j_\beta\} &= \delta^{ij} (\gamma_{mn} C^{-1})_{\alpha\beta} i M_{mn} + (C^{-1})_{\alpha\beta} T^{ij} \\
[T^{ij}, T^{kl}] &= -2i(\delta^{jk} T^{il} + 3 \text{ terms})
\end{aligned}
\tag{2.42}
$$

3. Irreducible Representations of Four-Dimensional Supersymmetry

It is a relatively straightforward proceedure to find the irreducible representations of any supersymmetry algebra. These representations tell us which supersymmetric theories are possible for a given supersymmetry algebra. To be precise, they provide a list of all the particles that occur in each of the theories that have a given supersymmetric algebra as a symmetry. In this section, we carry out this proceedure for the four-dimensional supersymmetry algebras. Using a very similar procedure one can find the irreducible representations of the supersymmetry algebras in other dimensions. The reader is referred to reference [160] where this is procedure is sketched in an arbitary dimension and where lists of irreducible representations are given.

One of the most interesting features of the construction of these irreducible representations is when the central charges of the supersymmetry algebra are non-trivial. In this case, and when the central charges take particular values, the massive representations contain many fewer states that those contained in the generic massive representation. When this occurs the states in the representation are called BPS states and they play an important role in discussions of string duality.

The first part of this section is taken from reference [0] and we have kept the equation numbers the same as in that reference.

In this chapter we wish to find the irreducible representations of supersymmetry [11], or, put another way, we want to know what is the possible particle content of supersymmetric theories. As is well known the irreducible representations of the Poincaré group are found by the Wigner method of induced representations [12]. This method consists of finding a representation of a subgroup of the Poincaré group and boosting it up to a representation of the full group. In practice, one adopts the following recipe: we choose a given momentum q^μ which satisfies $q^\mu q_\mu = 0$ or $q^\mu q_\mu = -m^2$ depending which case we are considering. We find the subgroup H which leaves q^μ intact and find a representation of H on the $|q^\mu\rangle$ states. We then induce this representation to the whole of the Poincaré group P, in the usual way. In this construction there is a one-to-one correspondence between points of P/H and four-momentum which satisfies $P_\mu P^\mu = 0$ or $P_\mu P^\mu = -m^2$. One can show that the result is independent of the choice of momentum q^μ one starts with.

In what follows we will not discuss the irreducible representations in general, but only that part applicable to the rest frame, i.e. the representations of H in the states at rest. We can do this safely in the knowledge that once the representation of H on the rest-frame states is known then the representation of P is uniquely given and that every irreducible representation of the Poincaré group can be obtained by considering every irreducible representation of H.

In terms of physics, the procedure has a simple interpretation, namely, the properties of a particle are determined entirely by its behaviour in a given frame (i.e. for given q^μ). The general behaviour is obtained from the given q^μ by boosting either the observer or the frame with momentum q^μ to one with arbitrary momentum.

The procedure outlined above for the Poincaré group can be generalised to any group of the form $S \otimes_s T$ where the symbol $\otimes_s$ denotes the semi-direct product of the groups S and T where T is Abelian. It also applies to the supersymmetry group and we shall take it for granted that the above recipe is the correct procedure and does in fact yield all irreducible representations of the supersymmetry group.

Let us first consider the massless case $q_\mu q^\mu = 0$, for which we choose the standard momentum $q_s^\mu = (m, 0, 0, m)$ for our "rest frame". We must now find H whose group elements leave $q_s^\mu = (m, 0, 0, m)$ intact. Clearly this contains Q_α^i, P_μ and T_s, since these generators all commute with P_μ and so rotate the states with q_s^μ into themselves. As we will see in the last section one can not have non-vanishing central charges for the massless case.

Under the Lorentz group the action of the generator $\frac{1}{2}\Lambda^{\mu\nu}J_{\mu\nu}$ creates an infinitesimal transformation $q^\mu \to \Lambda^\mu_\nu q^\nu + q^\mu$. Hence q_s^μ is left invariant provided the parameters obey the relations

$$\Lambda_{30} = 0, \quad \Lambda_{10} + \Lambda_{13} = 0, \quad \Lambda_{20} + \Lambda_{23} = 0 \tag{8.1}$$

Thus the Lorentz generators in H are

$$T_1 = J_{10} + J_{13}, \quad T_2 = J_{20} + J_{23}, \quad J = J_{12} \tag{8.2}$$

These generators form the algebra

$$\begin{aligned} [T_1, J] &= -T_2 \\ [T_2, J] &= +T_1 \\ [T_1, T_2] &= 0 \end{aligned} \tag{8.3}$$

The reader will recognise this to be the Lie algebra of E_2, the group of translations and rotations in a two-dimensional plane.

Now the only unitary representations of E_2 which are finite dimensional have T_1 and T_2 trivially realised, i.e.

$$T_1|q_s^\mu\rangle = T_2|q_s^\mu\rangle = 0 \tag{8.4}$$

This results from the theorem that all non-trivial unitary representations of noncompact groups are infinite-dimensional. We will assume we require finite-dimensional representations of H.

Hence for the Poincaré group, in the case of massless particles, finding representations of H results in finding representations of E_2 and consequently for the generator J alone. We choose our states so that

$$J|\lambda\rangle = i\lambda|\lambda\rangle \tag{8.5}$$

Our generators are anti-Hermitian. In fact, J is the helicity operator and we select λ to be integer or half-integer (i.e. $J = \underline{q}\cdot\underline{J}/|\underline{q}|$ evaluated at $\underline{q} = (0, 0, m)$ where $J_i = \varepsilon_{ijk}J_{jk},\ i, j = 1, 2, 3$).

Let us now consider the action of the supercharges Q^i_α on the rest-frame states, $|q^\mu_s\rangle$. The calculation is easiest when performed using the two-component formulation of the supersymmetry algebra of Eq. (2.23). On rest-frame states we find that

$$\begin{aligned}\{Q^{Ai}, Q^{\dot{B}}_j\} &= -2\delta^i_j(\sigma_\mu)^{A\dot{B}}q^\mu_s \\ &= -2\delta^i_j(\sigma_0 + \sigma_3)^{A\dot{B}}m = +4m\delta^i_j\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}^{A\dot{B}}\end{aligned} \tag{8.6}$$

In particular these imply the relations

$$\{Q^{1i}, Q^{\dot{1}}_j\} = 0$$
$$\{Q^{2i}, Q^{\dot{2}}_j\} = 4m\delta^i_j$$
$$\{Q^i_{\dot{i}}, Q^{2j}\} = \{Q^1_{\dot{i}}, Q^{\dot{2}j}\} = 0(8.7)$$

The first relation implies that

$$\langle q^\mu_s|\big(Q^{1i}(Q^{1i})^* + (Q^{1i})^*Q^{1i}\big)|q^\mu_s\rangle = 0 \tag{8.8}$$

Demanding that the norm on physical states be positive definite and vanishes only if the state vanishes, yields

$$Q^i_2|q^\mu_s\rangle = Q_{\dot{2}i}|q^\mu_s\rangle = 0 \tag{8.9}$$

Hence, all generators in H have zero action on rest-frame states except $J,\ T_s,\ P_\mu,\ Q^i_1$ and $Q_{\dot{1}i}$. Using Eq. (2.23) we find that

$$\begin{aligned}[Q^i_1, J] &= \frac{1}{2}(\sigma_{12})^1_1 Q^i_1 \\ &= -\frac{i}{2}Q^i_1\end{aligned} \tag{8.10}$$

Similarly, we find that complex conjugation implies

$$[(Q^i_1)^*, J] = +\frac{i}{2}(Q^i_1)^* \tag{8.11}$$

The relations between the remaining generators summarised in Eqs. (8.7), (8.10), (8.11) and (2.24) can be summarised by the statement that Q^i_1 and $(Q^i_1)^*$ form a Clifford algebra, act as raising and lowering operators for the helicity operator J and transform under the N and $\bar{N}$ representation of $SU(N)$.

We find the representations of this algebra in the usual way; we choose a state of given helicity, say λ, and let it be the vacuum state for the operator $(Q_1^i)^*$, i.e.

$$\begin{aligned} Q_1^i|\lambda\rangle &= 0 \\ J|\lambda\rangle &= i\lambda|\lambda\rangle \end{aligned} \tag{8.12}$$

The states of this representation are then

$$\begin{aligned} |\lambda\rangle &= |\lambda\rangle \\ |\lambda - \frac{1}{2}, i\rangle &= (Q_1^i)^*|\lambda\rangle \\ |\lambda - 1, [ij]\rangle &= (Q_1^i)^*(Q_1^j)^*|\lambda\rangle \end{aligned} \tag{8.13}$$

etc. These states have the helicities indicated and belong to the $[ijk\ldots]$ anti-symmetric representation of $SU(N)$. The series will terminate after the helicity $\lambda - (N/2)$, as the next state will be an object antisymmetric in $N+1$ indices. Since there are only N labels, this object vanishes identically. The states have helicities from λ to $\lambda - (N/2)$, there being $N!/\big(m!(N-m)!\big)$ states with helicity $\lambda - (m/2)$.

To obtain a set of states which represent particles of both helicities we must add to the above set the representations with helicities from $-\lambda$ to $-\lambda + (N/2)$. The exception is the so-called CPT self-conjugate sets of states which automatically contain both helicity states.

The representations of the full supersymmetry group are obtained by boosting the above states in accordance with the Wigner method of induced representations.

Hence the massless irreducible representation of $N = 1$ supersymmetry comprises only the two states

$$\begin{aligned} &|\lambda\rangle \\ &|\lambda - \frac{1}{2}\rangle = (Q_1)|\lambda\rangle \end{aligned} \tag{8.14}$$

with helicities λ and $\lambda - \frac{1}{2}$ and since

$$Q_1 Q_1 |\lambda\rangle = 0 \tag{8.15}$$

there are no more states.

To obtain a CPT-invariant theory we must add states of the opposite helicities, i.e. $-\lambda$ and $-\lambda + \frac{1}{2}$. For example, if $\lambda = \frac{1}{2}$ we get on-shell helicity states 0 and $\frac{1}{2}$ and their CPT conjugates with helicities $-\frac{1}{2}, 0$, giving a theory with two spin 0 and one Majorana spin $\frac{1}{2}$. Alternatively, if $\lambda = 2$ then we get on-shell helicity states 3/2 and 2 and their CPT self conjugates with helicity $-3/2$ and -2; this results in a theory with one spin 2 and one spin 3/2 particles. These on-shell states are those of the Wess-Zumino model and $N = 1$

supergravity respectively. Later in this discussion we will give a complete account of these theories.

For $N = 4$ with $\lambda = 1$ we get the massless states

$$|1\rangle,\ |\frac{1}{2}, i\rangle,\ |0, [ij]\rangle,\ |-\frac{1}{2}, [ijk]\rangle,\ |-1, [ijkl]\rangle \tag{8.16}$$

This is a CPT self-conjugate theory with one spin 1, four spin $\frac{1}{2}$ and six spin 0 particles.

Table 8.1 below gives the multiplicity for massless irreducible representations which have maximal helicity 1 or less.

Table 8.1 Multiplicities for massless irreducible representations with maximal helicity 1 or less

N / Spin	1	1	2	2	4
Spin 1	–	1	1	–	1
Spin $\frac{1}{2}$	1	1	2	2	4
Spin 0	2	–	2	4	6

We see that as N increases, the multiplicities of each spin and the number of different types of spin increases. The simplest theories are those for $N = 1$. The one in the first column in the Wess-Zumino model and the one in the second column is the $N = 1$ supersymmetric Yang-Mills theory. The latter contains one spin 1 and one spin $\frac{1}{2}$, consistent with the formula for the lowest helicity $\lambda - (N/2)$, which in this case gives $1 - \frac{1}{2} = \frac{1}{2}$. The $N = 4$ multiplet is CPT self conjugate, since in this case we have $\lambda - (N/2) = 1 - 4/2 = -1$. The Table stops at N equal to 4 since when N is greater than 4 we must have particles of spin greater than 1. Clearly, $N > 4$ implies that $\lambda - (N/2) = 1 - (N/2) < -1$. This leads us to the well-known statement that the $N = 4$ supersymmetric theory is the maximally extended Yang-Mills theory.

The content for massless on-shell representations with a maximum helicity 2 is given in Table 8.2.

Table 8.2 Multiplicity for massless on-shell representations with maximal helicity 2.

N / Spin	1	2	3	4	5	6	7	8
Spin 2	1	1	1	1	1	1	1	1
Spin $\frac{3}{2}$	1	2	3	4	5	6	8	8
Spin 1		1	3	6	10	16	28	28
Spin $\frac{1}{2}$			1	4	11	26	56	56
Spin 0				2	10	30	70	70

The $N = 1$ supergravity theory contains only one spin 2 graviton and one spin 3/2 gravitino. It is often referred to as simple supergravity theory. For the $N = 8$ supergravity theory, $\lambda - (N/2) = 2 - \frac{8}{2} = -2$. Consequently it is CPT self-conjugate and contains all particles from spin 2 to spin 0. Clearly, for theories in which N is greater than 8, particles of spin higher than 2 will occur. Thus, the $N = 8$ theory is the maximally extended supergravity theory.

It has sometimes been claimed that this theory is in fact the largest possible consistent supersymmetric theory. This contention rests on the widely held belief that it is impossible to consistently couple massless particles of spin $\frac{5}{2}$ to other particles. In fact superstring theories do include spin $\frac{5}{2}$ particles, but these are massive.

We now consider the massive irreducible representations of supersymmetry. We take our rest-frame momentum to be

$$q_s^\mu = (m, 0, 0, 0) \tag{8.17}$$

The corresponding little group is then generated by

$$P_m, Q^{\alpha i}, T^r, Z_1^{ij}, Z_2^{ij}, J_m \equiv \frac{1}{2}\varepsilon_{mnr}J^{nr} \tag{8.18}$$

where $m,n,r = 1,2,3$ for the present discussion. The J_m generate the group $SU(2)$. Let us first consider the case where the central charges are trivially realised.

When acting on the rest-frame states the supercharges obey the algebra

$$\begin{aligned}\{Q^{Ai}, (Q^{Bj})^*\} &= 2\delta^A_B\delta^i_j m \\ \{Q^{Ai}, Q^{Bj}\} &= 0\end{aligned} \tag{8.19}$$

The action of the T^r is that of $U(N)$ with the $SU(2)$ rotation generators satisfy

$$\begin{aligned}[J_m, J_n] &= \varepsilon_{mnr}J_r \\ [Q^{Ai}, J_m] &= i(\sigma_m)^A{}_B Q^{Bi}\end{aligned} \tag{8.20}$$

where (σ_m) are the Pauli matrices. We note that as far as $SU(2)$ is concerned the dotted spinor $Q^{\dot{A}i}$ behaves like the undotted spinor Q_{Ai}.

We observe that unlike the massless case none of the supercharges are trivially realised and so the Clifford algebra they form has $4N$ elements, that is, twice as many as those for the massless case. The unique irreducible representation of the Clifford algebra is found in the usual way. We define a Clifford vacuum

$$Q^i_A|q^\mu_s\rangle = 0, \quad A = 1,2\ , i = 1,\dots,N \tag{8.21}$$

and the representation is carried by the states

$$|q^\mu_s\rangle,\ (Q^i_A)^*|q^\mu_s\rangle,\ (Q^i_A)^*(Q^j_B)^*|q^\mu_s\rangle, \dots \tag{8.22}$$

Thanks to the anti-commuting nature of the $(Q^j_A)^*$ this series terminates when Q^* is applied $2N+1$ times.

The structure of the above representation is not particularly apparent since it is not clear how many particles of a given spin it contains. The properties of the Clifford algebra are more easily displayed by defining the real generators

$$\begin{aligned}\Gamma^i_{2A-1} &= \frac{1}{2m}\big(Q^{Ai} + (Q^{Ai})^*\big) \\ \Gamma^i_{2A} &= \frac{i}{2m}\big(Q^{Ai} - (Q^{Ai})^*\big)\end{aligned} \tag{8.23}$$

where the

$$\Gamma^i_p = (\Gamma^i_1, \Gamma^i_2, \Gamma^i_3, \Gamma^i_4) \tag{8.24}$$

are Hermitian. The Clifford algebra of Eq. (8.19) now becomes

$$\{\Gamma^i_p, \Gamma^j_q\} = \delta^{ij}\delta_{pq} \tag{8.25}$$

The $4N$ elements of the Clifford algebra carry the group $SO(4N)$ in the standard manner' the $4N(4N-1)/2$ generators of $SO(4N)$ being

$$O^{ij}_{mn} = \frac{1}{2}[\Gamma^i_m, \Gamma^j_n] \tag{8.26}$$

As there are an even number of elements in the basis of the Clifford algebra, we may define a "parity" (γ_5) operator

$$\Gamma_{4N+1} = \prod_{p=1}^{4} \prod_{i=1}^{N} \Gamma^i_p \tag{8.27}$$

which obeys the relations

$$\begin{aligned} (\Gamma_{4N+1})^2 &= +1 \\ \{\Gamma_{4N+1}, \Gamma^i_p\} &= 0 \end{aligned} \tag{8.28}$$

Indeed, the irreducible representation of Eq. (8.22) is of dimension 2^{2N} and transforms according to an irreducible representation of $SO(4N)$ of dimension 2^{2N-1} with $\Gamma_{4N+1} = -1$ and another of dimension 2^{2N-1} with $\Gamma_{4N+1} = +1$. Now any linear transformation of the Q, Q^* (for example $\delta Q = rQ$) can be represented by a generator formed from the commutator of the Q and Q^* (for example, $r[Q, Q^*]$). In particular the $SU(2)$ rotation generators are given by

$$s_k = -\frac{i}{4m}(\sigma_k)^A{}_B[Q^{jB}, (Q^{jA})^*] \tag{8.29}$$

One may easily verify that

$$[Q^{jA}, s_k] = i(\sigma_k)^A_B Q^{Bj} \tag{8.30}$$

The states of a given spin will be classified by that subgroup of $SO(4N)$ which commutes with the appropriate $SU(2)$ rotation subgroup of $SO(4N)$. This will be the group generated by all generators bilinear in Q, Q^* that have their two-component index contracted, i.e.

$$\begin{aligned} \Lambda^i_j &= \frac{i}{2m}[Q^{Ai}, (Q^j_A)^*] \\ k^{ij} &= \frac{i}{2m}[Q^{Ai}, Q^j_A] \end{aligned} \tag{8.31}$$

with $(k^{ij})^\dagger = k_{ij}$. It is easy to verify that the Λ^i_j, k^{ij} and k_{ij} generate the group $USp(2N)$ and so the states of a given spin are labelled by representations of $USp(2N)$. That the group is $USp(2N)$ is most easily seen by defining

$$Q^a_A = \begin{cases} Q^i_A \delta^a_i & a = 1, \ldots, N \\ \varepsilon_{AB}(Q^{Bi})^* & a = N+1, \ldots, 2N \end{cases} \tag{8.32}$$

for then the generators Λ^i_j, k^{ij} and k_{ij} are given by

$$s^{ab} = \frac{i}{2m}[Q^{Aa}, Q^b_A]. \tag{8.33}$$

Using the fact that

$$\{Q_A^a, Q_B^b\} = \varepsilon_{AB}\Omega^{ab} \tag{8.34}$$

where

$$\Omega^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we can verify that

$$[s^{ab}, s^{cd}] = \Omega^{ac}s^{bd} + \Omega^{ad}s^{bc} + \Omega^{bc}s^{ad} + \Omega^{bd}s^{ac} \tag{8.35}$$

which is the algebra of $USp(2N)$.

The particle content of a massive irreducible representation is given by the following

Theorem [21]: If our Clifford vacuum is a scalar under the $SU(2)$ spin group and the internal symmetry group, then the irreducible massive representation of supersymmetry has the following content

$$2^{2N} = \left[\frac{N}{2}, (0)\right] + \left[\frac{N-1}{2}, (1)\right] + \ldots + \left[\frac{N-\kappa}{2}, (\kappa)\right] + \ldots + [0, (N)] \tag{8.36}$$

where the first entry in the bracket denotes the spin and the last entry, say (k), denotes which kth-fold antisymmetric traceless irreducible representation of $USp(2N)$ that this spin belongs to.

Table 8.3 Some massive representations (without central charges) labelled in terms of the $USp(2N)$ representations.

N / Spin	1				2			3		4
Spin 2				1			1		1	1
Spin $\frac{3}{2}$			1	2		1	4	1	6	8
Spin 1		1	2	1	1	4	$5+1$	6	$14+1$	27
Spin $\frac{1}{2}$	1	2	1		4	$5+1$	4	14	$14'+6$	48
Spin 0	2	1			5	4	1	$14'$	14	42

Consider an example with two supercharges. The classifying group is $USp(4)$ and the 2^4 states are one spin 1, four spin 1/2, and five spin 0 corresponding to the $\underline{1}-$, $\underline{4}-$ and $\underline{5}-$dimensional representations of $USp(4)$. For more examples see Table 8.3.

Should the Clifford vacuum carry spin and belong to a non-trivial representation of the internal group $U(N)$, then the irreducible representation is found by taking the tensor product of the vacuum and the representation given in the above theorem.

Massive Representations with a Central Charge

We now consider the case of particles that are massive, but which also possess a central charge. We take the particles to be in their rest-frame with momentum $q^\mu \equiv (M, 0, 0, 0)$.

The isotropy group, H contains $(P^a, Q^i_A, Q^{\dot{A}}_i, \underline{J}, T_r$ and $Z^{ij})$. In the rest frame of the particles, that is for the momentum q^μ, the algebra of the supercharges is given by

$$\{Q^{Ai},\ (Q^{Bj})^*\} = 2\delta^A_B \delta^i_j M \tag{1}$$

and

$$\{Q^i_A,\ Q^j_B\} = \epsilon_{AB} Z^{ij} \tag{2}$$

To discover what is the particle content in a supermultiplet we would like to rewrite the above algebra as a Clifford algebra. The first step in this proceedure is to carry out a unitary transformation on the internal symmetry index of the supercharges i.e. $Q^i_A \to U^i_j Q^j_A$ or $Q_A \to U Q^i_A$ with $U^\dagger U = 1$. Such a transformation preserves the form of the first relation of equation (1). However, the unitary transformation can be chosen [104] such that the central charge ,which transforms as $Z \to UZU^T$, can be brought to the form of a matrix which has all its entries zero except for the 2 by 2 matices down its diagonal. These 2 by 2 matrices are anti-symmetric as a consequence of the anti-symmetric nature of Z^{ij} which is preserved by the unitary transformation. This is the closest one can come to diagonalising an anti-symmetric matrix. Let us for simplicity restrict our attention to even N. We can best write down this matrix we replace the $i, j = 1, 2, \ldots N$ internal indices by $i = (a, m),\ j = (b, n),\ a, b = 1, 2,\ m, n = 1, \ldots, \frac{N}{2}$ whereupon

$$Z^{(a,m)(b,n)} = 2\epsilon^{ab}\delta^{mn} Z_n \tag{3}$$

In fact one also show that $Z_n \geq 0$. The supercharges in the rest-frame satisfy the relations

$$\{Q^{A(am)},\ (Q^{B(bn)})^*\} = 2\delta^A_B \delta^a_b \delta^m_n M \tag{4}$$

and

$$\{Q^{(am)}_A,\ Q^{(bn)}_B\} = 2\epsilon_{AB}\epsilon^{ab}\delta^{mn} Z_n \tag{5}$$

We now define the supercharges

$$S^{Am}_1 = \frac{1}{\sqrt{2}}(Q^{A1m} + (Q^{B2m}\epsilon_{BA})^*) \tag{6}$$

$$S^{Am}_2 = \frac{1}{\sqrt{2}}(Q^{A1m} - (Q^{B2m}\epsilon_{BA})^*) \tag{7}$$

in terms of which all the anti-commutators vanish except for

$$\{S^{Am}_1,\ (S^{Bn}_1)^*\} = 2\delta^{AB}\delta^{mn}(M - Z_n) \tag{8}$$

$$\{S^{Am}_2,\ (S^{Bn}_2)^*\} = 2\delta^{AB}\delta^{mn}(M + Z_n) \tag{9}$$

This algebra is a Clifford algebra formed from the $2N$ operators S^{Am}_1 and S^{Am}_2 and their complex conjugates. It follows from equation (9) that if we take the same indices on each supercharge that the right-hand side is positive definite and hence $Z_n \leq M$.

To find the irreducible representation of supersymmetry we follow a similar procedure to that which we followed for massive and massless particles. The result crucially depends on whether $Z_n < M,\ \forall\ n$ or if one or more values of n we saturate the bound $Z_n = M$.

Let us first consider $Z_n < M,\ \forall\ n$. In this case, the right-hand sides of both equations (8) and (9) are non-zero. Taking S_1^{Am} and S_2^{Am} to annihilate the vaccum the physical states are given by the creation opperators $(S_1^{Am})^*$ and $(S_2^{Am})^*$ acting on the vacuum. The resulting representation has 2^{2N} states and has the same structure as for the massive case in the absence of a central charge. The states are classified by $USP(2N)$ as for the massive case.

Let us now suppose that q of the Z_n's saturate the bound i.e $Z_n = M$. For these values of n the right-hand side of equation (8) vanishes; taking the expectation value of this relation for any physical state we find that

$$< phys|S_1^{An}(S_1^{An})^*|phys > + < phys|(S_1^{An})^*(S_1^{An})|phys >= 0 \tag{10}$$

The scalar product on the space of physical states satisfies all the axioms of a scalar product and hence we conclude that both of the above terms vanish and as a result

$$(S_1^{Bn})^*|phys >= 0 = (S_1^{An})|phys > \tag{11}$$

This argument is the same as that used to eliminate half of the supercharges and their complex conjugates in the massless case; however in the case under consideration here it only eliminates q of the supercharges and their complex conjugates. There remain the $\frac{N}{2}$ supercharges (S_2^{Bm}) and the $\frac{N}{2} - q$ supercharges (S_1^{Bm}) for the values of m for which Z_m do not saturate the bound as well as their complex conjugates. These supercharges form a Clifford algebra and we can take the $\frac{N}{2}$ supercharges (S_2^{Bm}) and the $\frac{N}{2} - q$ supercharges (S_1^{Bm}) to annihilate the vacuum and their complex conjugates to be creation operators. The resulting massive irreducible representation of supersymmetry has $2^{2(N-q)}$ states and it has the same form as a massive representation of $N - q$ extended supersymmetry. The states will be classified by $USp(2N - 2q)$.

Clearly, a representation in which some or all of the central charges are equal to their mass has fewer states than the massive representation formed when none of the central charges saturate the mass, or a massive representation for which all the central charges vanish. This is a consequence of the fact that the latter Clifford algebra has more of its supercharges active in the irreducible representation. In almost all cases, the representation with some of its central charges saturated contains a smaller range of spins than the massive representation with no central charges. This feature plays a very important role in discussions of duality in supersymmetric theories.

Let us consider the irreducible representations of $N = 4$ supersymmetry which has both of its two possible central charges saturated. These representations are like the corresponding $N = 2$ massive representations. An important example has a $\underline{1}$ of spin one, a $\underline{4}$ of spin one-half and $\underline{5}$ of spin zero. The underlined numbers are their $USp(4)$ representations. This representation arises when the $N = 4$ Yang-Mills theory is spontaneously broken by one of its scalars acquiring a vaccum expectation value. The theory before being spontaneously broken, has a massless representation with one spin one, 2 spin 1/2 , and six spin

0. Examining the massive representations for $N = 4$ in the absence of a central charge one finds that the representation with the smallest spins has all spins from spin 2 to spin 0. Hence the spontaneously broken theory can only be supersymmetric if the representation has a central charge. Another way to get the count in the above representation is to take the massless representation and recall that when the theory is spontaneously broken one of the scalars has been eaten by the vector as a result of the Higgs mechanism.

We close this section by answering a question which may have arisen in the mind of the reader. For the N extended supersymmetry algebra the supersymmetry algebra in the rest frame of equation (3) representation has $\frac{N}{2}$ possible central charges. This makes one central charge for the case of $N = 2$. However, this number conflicts with our understanding that a particle in $N = 2$ supersymmetric Yang-Mills theory can have two central charges corresponding to its electric and magnetic fields. The resolution of this conundrum is that although one can use a unitary transformation to bring the central charge of a given irreducible representation (i.e. particle) to have only one independent component one can not do this simultaneously for all irreducible multiplets or particles.

Some examples of massive representations with central charges are given in the table below.

Table 8.4 Some massive representations with one central charge ($|Z| = m$). All states are complex.

N / Spin	2		4		6		8
Spin 2						1	1
Spin $\frac{3}{2}$				1	1	6	8
Spin 1		1	1	4	6	$14 + 1$	27
Spin $\frac{1}{2}$	1	2	4	$5 + 1$	14	$14' + 6$	48
Spin 0	2	1	5	4	$14'$	14	42

We have seen that when the central charges take on special values the irreducible representation contain fewer states than the generic irreducible massive representation. When this occurs the states in the representation are called BPS states and the multiplet of states is called a BPS or short multiplet. Quantum corrections do not, except in very unusual circumstances, alter the number of states in a theory, and as such one expects the number and type of BPS multiplets to be the same in the classical and quantum theories. This follows from the observation that for a BPS multiplet to become a generic massive multiplet, the presence of more states would be required in the theory and one does not expect to suddenly find new states in the theory as one smoothly alters the parameters of the theory. As such, we can expect this result to hold regardless of the value of the coupling constant of the quantum theory.

This property is very useful since one can verify the existence of certain BPS states when the coupling constant of the theory is small by conventional techniques and it then follows that these BPS states will be present in the same theory when the coupling constant is large. In fact, the presence of BPS states is one of the few things that one can reliably establish in the strong coupling regime of a supersymmetric theory. BPS multiplets play an important role in discussions on duality; if one suspects that two theories are dual one of the things one can reliably check is the correspondence between the BPS states in each theory.

The account of the massive irreducible representations of supersymmetry given here is along similar lines to the review by Ferrara and Savoy given in [21].

4. Three Ways to Construct a Supergravity Theory

In this section it is explained how to construct a supergravity theory from the knowledge of its on-shell states.

There are three main ways. The method which has been used most often is the Noether method. It was used to construct the four-dimensional $N = 1$ supergravity in its on-shell [14][15] and off-shell formulations [16][17]. This method was also used in the construction of the d=11 supergravity theory [106] and a variant of it was used to find many of the properties of the IIB theory [111]. Although this method does not make use of any sophistocated mathematics and can be rather lengthy, it is very powerful. Starting from the linearised theory for the relavent supermultiplet, it gives a systematic way of finding the final non-linear theory. To illustrate the method clearly and without undue technical complexity we explain, in section 4.1.1, how to construct the Yang-Mills theory from the linearised theory. In section 4.1.2 we find the linearised supergravity theory in four dimensions and apply the Noether proceedure to find the full non-linear theory. In fact, we will only carry out the first steps in this Noether procedure, but these clearly illustrate how to find the final result, most of whose features are already apparent at an early stage of the process. We then give the $N = 1$ supergravity theory and show it is invariant.

The second method uses the superspace description of supergravity theories and in section 4.2 we begin by summarising this approach. Supergravity theories in superspace share a number of similarities with the usual theory of general relativity. They are built from a supervierbein and a spin-connection and are invariant under superdiffeomorphisms. Using the supervierbein and spin-connection we can construct covariant derivatives and then define torsions and curvatures in the usual way. The superspace formulation differs from the usual formulation of general relativity in the nature of the tangent space group. The tangent space of the superspace formulation of supergravity contains fermionic and bosonic subspaces; however, the tangent space group is only the Lorentz group which rotates the odd (bosonic) and even (fermionic) subspaces seperately. In general relativity the tangent space group is also the Lorentz group but in this case it rotates all vectors of a given length into each other.

The use of a restricted tangent space group in the superspace formulation allows us to take some of the torsions and curvatures to be zero since these constraints are respected by the Lorentz tangent space group. In fact the torsions and curvatures form a highly reducible representation of the Lorentz group. Indeed, the imposition of such constraints is precisely what is required to find the correct theory of supergravity. Hence the problem of finding the superspace formulation of supergravity is to find which of the torsions and curvatures are zero. We require different sets of constriants for the on-shell and off-shell supergravity theory. Clearly one gets from the latter to the former by imposing more constraints. To find the constraints for the off-shell theory, when this exists, can be rather difficult; however it turns out that to find the constraints for the on-shell theory is very straightforward and remarkably requires only dimensional analysis. Given a knowledge of the on-shell states in x-space, which are determined in a straightforward way from the supersymmetry algebra, as explained in section three, we can deduce the dimensions of all the gauge-invariant quantities. By introducing an notion of geometric dimension, which in effect absorbs all factors of Newton's constant κ into the fields, one finds that for sufficently low dimensions and certain Lorentz character there are no gauge-invariant tensors in x-space. The superspace torsions and curvatures are gauge-invariant and so the superspace tensors with these dimensions and Lorentz character must then vanish as there is no available x-space tensor that their lowest component could equal. We can substitute these superspace constraints into the the Bianchi idenitites satisfied by the torsions and curvatures to find constraints on the torsions and curvatures of higher dimension. From this set of constraints one can find an x-space theory and it turns out that in all known cases this theory is none other than the corresponding on-shell supergravity theory. In other words, the constraints deduced from dimensional analysis and the use of the Bianchi identities are sufficient to find the on-shell supergravity in superspace and hence also in x-space. In section 4.2.2 we explicity carry out this programme for the four-dimensional $N = 1$ supergravity theory and recover the theory we found by the Noether method. This procedure was used to find the full IIB supergravity in superspace and in x-space [111]. We refer the reader to reference [111] for the details of this construction.

Finally, we briefly mention in section 4.3 the third method of finding supergravity theories by the gauging certain space-time groups.

Several parts of this section are taken from reference [0] and we keep the same equation numbering as in this reference.

4.1 The Noether Method

4.1.1 Yang-Mills Theory and the Noether Technique

Any theory whose nonlinear form is determined by a gauge principle can be constructed by a Noether procedure [9]. Because of the importance of the Noether technique in constructing theories of supergravity we will take this opportunity to illustrate the technique within the framework of the simpler supersymmetric Yang-Mills theory [10].

Let us begin by considering the construction of the Yang-Mills theory itself from the linearised (free) theory. At the free level the theory is invariant under two distinct transformations: rigid and local Abelian transformations. Rigid transformations belong to a group S with generators R_i which satisfy

$$[R_i, R_j] = s_{ij}{}^k R_k \tag{7.1}$$

The structure constants $s_{ij}{}^k$ may be chosen to be totally antisymmetric and the indices $i, j, k, \ldots$ can be raised and lowered with the Kronecker delta δ_i^j. Under these rigid transformations the vector fields A_a^i transform as

$$\delta A_a^i = s_{jk}{}^i T^j A_a^k \tag{7.2}$$

where T_j are the infinitesimal group parameters. The other type of transformations are local Abelian transformations

$$\delta A_a^i = \partial_a \Lambda^i \tag{7.3}$$

Clearly both these transformations form a closed algebra† The linearised theory which is invariant under the transformations of Eqs. (7.2) and (7.3) is given by

$$A^{(0)} = \int d^4x \left\{ -\frac{1}{4} f_{ab}^i f_{ab}^i \right\} \tag{7.4}$$

where $f_{ab}^i = \partial_a A_b^i - \partial_b A_a^i$ The nonlinear theory is found in a series of steps, the first of which is to make the rigid transformations local, i.e. $T^j = T^j(x)$. Now, $A^{(0)}$ is no longer invariant under

$$\delta A_a^i = s_{jk}{}^i T^j(x) A_a^k \tag{7.5}$$

but its variation may be written in the form

$$\delta A^{(0)} = \int d^4x \{ (\partial_a T^k(x)) j^{ak} \} \tag{7.6}$$

where

$$j^{ak} = -s_{ij}{}^k A_b^i f^{abj} \tag{7.7}$$

Now consider the action A_1

$$A_1 = A^{(0)} - \frac{1}{2} g \int d^4x (A_a^i j^{ai}) \tag{7.8}$$

where g is the gauge coupling constant; it is invariant *to order* g^0 *provided* we combine the local transformation $T^i(x)$ with the local transformation $\Lambda^i(x)$ with the identification $\Lambda^i(x) = (1/g) T^i(x)$. That is, the initially separate local and rigid transformations of the linearised theory become knitted together into a single local transformation given by

$$\delta A_a^i = \frac{1}{g} \partial_a T^i(x) + s^{ijk} T_j(x) A_a^k(x) \tag{7.9}$$

The first term in the transformation of δA_a^i yields in the variation of the last term in A_1, a term which cancels the unwanted variation of $A^{(0)}$.

We now continue with this process of amending the Lagrangian and transformations order-by-order in g until we obtain an invariant Lagrangian. The variation of A_1 under the second term of the transformation of Eq. (7.9) is of order g and is given by

$$\delta A_1 = \int d^4x \{ -g (A_a^i A_b^j s_{ij}{}^k)(A_b^l \partial_a T^m s_{lm}{}^k) \} \tag{7.10}$$

† In particular one finds $[\delta_\Lambda, \delta_T] A_a^i = \partial_a (s_{jk}{}^i T^j \Lambda^k)$.

An action invariant to order g is

$$A_2 = A_1 + \int d^4x \frac{g^2}{4} (A_a^i A_b^j s_{ij}{}^k)(A^{bl} A^{am} s_{lm}{}^k)$$
$$= \frac{1}{4} \int d^4x (F_{ab}^i)^2 \tag{7.11}$$

where

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i - g s_{jk}{}^i A_a^j A_b^k \tag{7.12}$$

In fact the action A_2 is invariant under the transformations of Eq. (7.9) to all orders in g and so represents the final answer, and is, of course, the well-known action of Yang-Mills theory. The commutator of two transformations on A_a^i is

$$[\delta_{T_1}, \delta_{T_2}] A_a^i = s_{ij}{}^k T_2^j \left(\frac{1}{g} \partial_a T_1^k + s_{lm}{}^k T_1^l A_a^m \right) - (1 \leftrightarrow 2)$$
$$= \frac{1}{g} \partial_a T_{12}^i + s_{ij}{}^k T_{12}^j A_a^k \tag{7.13}$$

where

$$T_{12}^i = s_{jk}{}^i T_2^j T_1^k$$

and so the transformations form a closed algebra. In the last step we used the Jacobi identity in terms of the structure constants.

For supergravity and other local theories the procedure is similar, although somewhat more complicated. The essential steps are to first make the rigid transformations local and find invariant Lagrangians order by order in the appropriate gauge coupling constant. This is achieved in general not only by adding terms to the action, but also adding terms to the transformation laws of the field. If the latter process occurs one must also check the closure order by order in the gauge coupling constant.

Although one can use a Noether procedure which relies on the existence of an action, one can also use a Noether method which uses the transformation laws alone. This works, in the Yang-Mills case, as follows: upon making the rigid transformation local as in Eq. (7.5) one finds that the algebra no longer closes, i.e.

$$[\delta_\Lambda, \delta_T] A_a^i = \partial_a \left(s_{ij}{}^i (T^j \Lambda^k) \right) - s_{jk}{}^i (\partial_a T^j) \Lambda^k \tag{7.14}$$

The cure for this is to regard the two transformations as simultaneous and knit them together as explained above. Using the new transformation for A_a^i of Eq. (7.9) we then test the closure to order g^0. In fact, in this case the closure works to all orders in g and the process stops here; in general however, one must close the algebra order by order in the coupling constant modifying the transformation laws and the closure relations for the algebra. Having the full transformations it is then easy to find the full action when that exists.

Let us apply the above Noether proceedure to find the $N = 1$ supersymmetric Yang-Mills theory from its linearised theory. In the linearised theory the fields A_a^i, λ^i, D^i have the rigid transformations T_i and local transformations $\Lambda^i(x)$ given by

$$\begin{aligned} &\delta A_a^i = s_{jk}{}^i T^j A_a^k, \ \ \delta\lambda^i = s_{jk}{}^i T^j \lambda^k \\ &\delta D^i = s_{jk}{}^i T^j D^k \end{aligned} \tag{7.15}$$

and

$$\delta A_a^i = \partial_a \Lambda^i \quad \delta D^i = 0, \quad \delta\lambda^i = 0 \tag{7.16}$$

The supersymmetry transformations of the linearised theory are given by

$$\begin{aligned} \delta A_a^i &= \bar{\varepsilon}\gamma_a \lambda^i \\ \delta\lambda^i &= -\frac{1}{2}\sigma^{cd} f_{cd}^i \varepsilon + i\gamma_5 D^i \varepsilon \\ \delta D^i &= i\bar{\varepsilon}\gamma_5 \partial\!\!\!/ \lambda^i \end{aligned} \tag{7.17}$$

These transformations form a closed algebra, and leave invariant the following linearised Lagrangian

$$L = -\frac{1}{4}(f_{cd}^i)^2 - \frac{1}{2}\bar{\lambda}^i \partial\!\!\!/ \lambda^i + \frac{1}{2}(D^i)^2 \tag{7.18}$$

Let us use the Noether method on the algebra to find the nonlinear theory. Making the rigid transformation on A_a^i local we must, as in the Yang-Mills case, knit the rigid and local transformations together (i.e. $\Lambda^i(x) = (1/g)T^i(x)$) to gain closure of gauge transformations on A_μ^i. Closure of supersymmetry and gauge transformations implies that the rigid transformations on λ^i and D^i also become local. This particular closure also requires that all the supersymmetry transformations are modified to involve covariant quantities. For example, we find that on D^i

$$[\delta_T, \delta_\varepsilon] D^i = i\bar{\varepsilon}\gamma_5\gamma^a s_{jk}{}^i (\partial_a T^j)\lambda^k \tag{7.19}$$

and as a result we must replace $\partial_a \lambda^i$ by $\mathcal{D}_a \lambda^i = \partial_a \lambda^i - g s_{jk}{}^i A_a^j \lambda^k$ in the δD^i of Eq. (7.17) and then the commutator $[\delta_\Lambda, \delta_\varepsilon]$ is zero to all orders in g. The algebra then takes the form

$$\begin{aligned} \delta A_a^i &= \bar{\varepsilon}\gamma_a \lambda^i \\ \delta\lambda^i &= \left(-\frac{1}{2}\sigma^{cd} F_{cd}^i + i\gamma_5 D^i\right)\varepsilon \\ \delta D^i &= i\bar{\varepsilon}\gamma_5 \mathcal{D}\!\!\!/ \lambda^i \end{aligned} \tag{7.20}$$

where

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i - g s_{jk}{}^i A_a^j A_b^k$$

We must now verify that the above supersymmetry transformations close. For other supersymmetric gauge theories we must add further terms to the supersymmetry transformations in order to regain closure. However, in this case gauge covariance and dimensional analysis ensure that there are no possible terms that one can add to these supersymmetry transformations and so the transformations of Eqs. (7.20) must be the complete laws for the full theory. The reader may verify that there are no inconsistencies by showing that the algebra does indeed close.

The action invariant under these transformations is

$$A = \int d^4x \left\{ -\frac{1}{4}(F^i_{ab})^2 - \frac{1}{2}\bar{\lambda}^i \not{D}\lambda^i + \frac{1}{2}(D^i)^2 \right\} \tag{7.21}$$

One could also have used the Noether procedure on the action. Gauge invariance implies that the action is that given in Eq. (7.21). Demanding that this gauge invariant action be supersymmetric requires us to modify the supersymmetry transformations to those of Eq. (7.20).

4.1.2 $N = 1$ $D = 4$ Supergravity

We will now construct $N = 1$ $D = 4$ supergravity using the Noether method. The starting point is the linearised theory which we now construct along similar lines to the method used to find the Wess-Zumino model and $N = 1$ super QED in chapters 5 and 6 in reference [0].

The Linearised Theory

We will start with the on-shell states and construct the linearized theory, without and then with auxiliary fields. The $N = 1$ irreducible representations of supersymmetry which include a spin 2 graviton contain either a spin 3/2 or a spin 5/2 fermion. The spin 5/2 particle would seem to have considerable problems in coupling to other fields and so we will choose the spin 3/2 particle.

As in the Yang-Mills case, the linearized theory possesses rigid supersymmetry and local Abelian gauge invariances. The latter invariances are required, in order that the fields do describe the massless on-shell states alone without involving ghosts. We recall that a rigid symmetry is one whose parameters are space-time independent while a local symmetry has space-time dependent parameters.

These on-shell states are represented by a symmetric second rank tensor field, $h_{\mu\nu}$ ($h_{\mu\nu} = h_{\nu\mu}$) and a Majorana vector spinor, $\psi_{\mu\alpha}$. For these fields to represent a spin 2 particle and a spin 3/2 particle they must possess the infinitesimal gauge transformations

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) \\ \partial \psi_{\mu\alpha} &= \partial_\mu \eta_\alpha(x) \end{aligned} \tag{9.1}$$

The unique ghost-free gauge-invariant, free field equations are

$$E_{\mu\nu} = 0, \quad R^\mu = 0 \tag{9.2}$$

where $E_{\mu\nu} = R^L{}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^L$, $R^{Lab}{}_{\mu\nu}$ is the linearized Riemann tensor given by

$$R^{Lab}{}_{\mu\nu} = -\partial_a\partial_\mu h_{b\nu} + \partial_b\partial_\mu h_{a\nu} + \partial_a\partial_\nu h_{b\mu} - \partial_b\partial_\nu h_{a\mu}$$

and

$$
\begin{aligned}
R^{\mu} &= \varepsilon^{\mu\nu\rho\kappa} i\gamma_5\gamma_\nu\partial_\rho\psi_\kappa \\
R^{Lb}{}_{\mu} &= R^{Lab}{}_{\mu\nu}\delta^{\mu}_{a} \\
R^{L} &= R^{La}{}_{\mu}\delta^{\mu}_{a}
\end{aligned}
\tag{9.3}
$$

For an explanation of this point see van Nieuwenhuizen. [13].

We must now search for the supersymmetry transformations that form an invariance of these field equations and represent the supersymmetry algebra on-shell. On dimensional grounds the most general transformation is

$$
\begin{aligned}
\delta h_{\mu\nu} &= \frac{1}{2}(\bar{\varepsilon}\gamma_\mu\psi_\nu + \bar{\varepsilon}\gamma_\nu\psi_\mu) + \delta_1\eta_{\mu\nu}\bar{\varepsilon}\gamma^\kappa\psi_\kappa \\
\delta\psi_\mu &= +\delta_2\sigma^{ab}\partial_a h_{b\mu}\varepsilon + \delta_3\partial_\nu h^{\nu}{}_{\mu}\varepsilon
\end{aligned}
\tag{9.4}
$$

The parameters δ_1, δ_2 and δ_3 will be determined by the demanding that the set of transformations which comprise the supersymmetry transformations of equation (9.4) and the gauge transformations of equation (9.1) should form a closing algebra when the field equations of equation (9.2) hold. At the linearized level the supersymmetry transformations are linear rigid transformations, that is, they are *first order* in the fields $h_{\mu\nu}$ and $\psi_{\mu\alpha}$ and parametrized by *constant* parameters ε^α.

Carrying out the commutator of a Rarita-Schwinger gauge transformation, $\eta_\alpha(x)$ of Eq. (9.1) and a supersymmetry transformation, ε of Eq. (9.4) on $h_{\mu\nu}$, we get:

$$
[\delta_\eta, \delta_\varepsilon]h_{\mu\nu} = \frac{1}{2}(\bar{\varepsilon}\gamma_\mu\partial_\nu\eta + \bar{\varepsilon}\gamma_\nu\partial_\mu\eta) + \delta_1\eta_{\mu\nu}\bar{\varepsilon}\partial\!\!\!/\eta
\tag{9.5}
$$

This is a gauge transformation with parameter $\frac{1}{2}\bar{\varepsilon}\gamma_\mu\eta$ on $h_{\mu\nu}$ provided $\delta_1 = 0$. Similarly, calculating the commutator of a gauge transformation of $h_{\mu\nu}$ and a supersymmetry transformation on $h_{\mu\nu}$ automatically yields the correct result, i.e., zero. However, carrying out the commutator of a supersymmetry transformation and an Einstein gauge transformation on $\psi_{\mu\alpha}$ yields

$$
[\delta_{\xi_\mu}, \delta_\varepsilon]\psi_\mu = +\delta_2\sigma^{ab}\partial_a(\partial_\mu\xi_b)\varepsilon + \delta_3\partial_\nu\partial^\nu\xi_\mu\varepsilon + \delta_3\partial_\nu\partial_\mu\xi^\mu\varepsilon
\tag{9.6}
$$

which is a Rarita-Schwinger gauge transformation on ψ_μ provided $\delta_3 = 0$. Hence we take $\delta_1 = \delta_3 = 0$.

We must test the commutator of two supersymmetries. On $h_{\mu\nu}$ we find the commutator of two supersymmetries to give

$$
\begin{aligned}
[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]h_{\mu\nu} &= +\frac{1}{2}\{\bar{\varepsilon}_2\gamma_\mu\delta_2\sigma^{ab}\partial_a h_{b\nu}\varepsilon_2 + (\mu \leftrightarrow \nu)\} - (1 \leftrightarrow 2) \\
&= \delta_2\{\bar{\varepsilon}_2\gamma^b\varepsilon_1\partial_\mu h_{b\nu} - \bar{\varepsilon}_2\gamma^a\varepsilon_1\partial_a h_{\mu\nu} - (\mu \leftrightarrow \nu)\}
\end{aligned}
\tag{9.7}
$$

This is a gauge transformation on $h_{\mu\nu}$ with parameter $\delta_2\bar{\varepsilon}_2\gamma^b\varepsilon_1 h_{b\nu}$ as well as a spacetime translation. The magnitude of this translation coincides with that dictated by the supersymmetry group provided $\delta_2 = -1$ which is the value we now adopt.

It is important to stress that linearized supergravity differs from the Wess-Zumino model in that one must take into account the gauge transformations of Eqs. (9.1) as well as the rigid supersymmetry transformations of Eq. (9.4) in order to obtain a closed algebra. The resulting algebra is the $N = 1$ supersymmetry algebra when supplemented by gauge transformations. This algebra reduces to the $N = 1$ supersymmetry algebra only on gauge-invariant states.

For the commutator of two supersymmetries on ψ_μ we find

$$
\begin{aligned}
[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\psi_\mu &= -\sigma^{ab}\partial_a \varepsilon_2 \frac{1}{2}(\varepsilon_1 \gamma_b \psi_\mu + \bar{\varepsilon}_1 \gamma_\mu \psi_b) - (1 \leftrightarrow 2) \\
&= +\frac{1}{2.4}\sum_R c_R \bar{\varepsilon}_1 \gamma_R \varepsilon_2 \sigma^{ab} \partial_a \gamma^R (\gamma_b \psi_\mu + \gamma_\mu \psi_b) - (1 \leftrightarrow 2) \\
&= \frac{1}{8}\bar{\varepsilon}_1 \gamma_R \varepsilon_2 \sigma^{ab} \gamma^R \left(\gamma_b \psi_{a\mu} + \frac{1}{2}\gamma_\mu \psi_{ab} \right) \\
&\quad + \partial_\mu \left(\frac{1}{8}\bar{\varepsilon}_1 \gamma_R \varepsilon_2 \sigma^{ab} \gamma^R \gamma_b \psi_a \right) - (1 \leftrightarrow 2)
\end{aligned} \tag{9.8}
$$

where $\psi_{\mu\nu} = \partial_\mu \psi_\nu - \partial_\nu \psi_\mu$. Using the different forms of the Rarita-Schwinger equation of motion, given by

$$
\begin{aligned}
&R^\mu = 0 \\
&\Leftrightarrow \gamma^\mu \psi_{\mu\nu} = 0 \\
&\Leftrightarrow \psi_{\mu\nu} + \frac{1}{2} i \gamma_5 \varepsilon_{\mu\nu\rho\kappa} \psi^{\rho\kappa} = 0
\end{aligned} \tag{9.9}
$$

we find the final result to be

$$
\begin{aligned}
[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\psi_\mu = 2\bar{\varepsilon}_2 \gamma^c \varepsilon_1 \partial_c \psi_\mu + \partial_\mu \big(&- \bar{\varepsilon}_2 \gamma^c \varepsilon_1 \psi_c \\
&+ \sum_R c_R \frac{1}{8} \bar{\varepsilon}_1 \gamma_R \varepsilon_2 \sigma^{ab} \gamma^R \gamma_b \psi_a - (1 \leftrightarrow 2) \big)
\end{aligned} \tag{9.10}
$$

This is the required result: a translation and a gauge transformation on ψ_μ.

The reader can verify that the transformations of Eq. (9.4) with the values of $\delta_1 = \delta_3 = 0,\ \delta_2 = -1$ do indeed leave the equations of motion of $h_{\mu\nu}$ and $\psi_{\mu\alpha}$ invariant.

Having obtained an irreducible representation of supersymmetry carried by the fields $h_{\mu\nu}$ and $\psi_{\mu\alpha}$ when subject to their field equations we can now find the algebraically on-shell Lagrangian. The action (Freedman, van Nieuwenhuizen and Ferrara [14], Deser and Zumino [15]) from which the field equations of Eq. (9.2) follow, is

$$
A = \int d^4x \left\{ -\frac{1}{2} h^{\mu\nu} E_{\mu\nu} - \frac{1}{2} \bar{\psi}_\mu R^\mu \right\} \tag{9.11}
$$

It is invariant under the transformations of Eq. (9.4) provided we adopt the values for the parameters $\delta_1,\ \delta_2$ and δ_3 found above. This invariance holds without use of the field equations, as it did in the Wess-Zumino case.

We now wish to find a linearized formulation which is built from fields which carry a representation of supersymmetry without imposing any restrictions (i.e., equations of motion), namely, we find the auxiliary fields. As a guide to their number we can apply our Fermi-Bose counting rule which, since the algebra contains gauge transformations, applies only to the gauge-invariant states. On-shell, $h_{\mu\nu}$ has two helicities and so does $\psi_{\mu\alpha}$; however off-shell, $h_{\mu\nu}$ contributes $(5 \times 4)/2 = 10$ degrees of freedom minus 4 gauge degrees of freedom giving 6 bosonic degrees of freedom. On the other hand, off-shell $\psi_{\mu\alpha}$ contributes $4 \times 4 = 16$ degrees of freedom minus 4 gauge degrees of freedom, giving 12 fermionic degrees of freedom. Hence the auxiliary fields must contribute 6 bosonic degrees of freedom. If there are n auxiliary fermions there must be $4n + 6$ bosonic auxiliary fields.

Let us assume that a minimal formulation exists, that is, there are no auxiliary spinors. Let us also assume that the bosonic auxiliary fields occur in the Lagrangian as squares without derivatives (like F and G) and so are of dimension two. Hence we have 6 bosonic auxiliary fields; it only remains to find their Lorentz character and transformations. We will assume that they consist of a scalar M, a pseudoscalar N and a pseudovector b_μ, rather than an antisymmetric tensor or 6 spin-0 fields. We will give the motivating arguments for this later.

Another possibility is the two fields A_μ and $a_{\kappa\lambda}$ which possess the gauge transformations $\delta A_\mu = \partial_\mu \Lambda$; $\delta a_{\kappa\lambda} = \partial_\kappa \Lambda_\lambda - \partial_\lambda \Lambda_\kappa$. A contribution $\varepsilon_{\mu\nu\rho\kappa} A^\mu \partial^\nu a^{\rho\kappa}$ to the action would not lead to propagating degrees of freedom.

The transformation of the fields $h_{\mu\nu}$, $\psi_{\mu\alpha}$, M, N and b_μ must reduce on-shell to the on-shell transformations found above. This restriction, dimensional arguments and the fact that if the auxiliary fields are to vanish on-shell they must vary into field equations gives the transformations to be [16,17]

$$
\begin{aligned}
\delta h_{\mu\nu} &= \frac{1}{2}(\bar{\varepsilon}\gamma_\mu \phi_\nu + \bar{\varepsilon}\gamma_\nu \psi_\mu) \\
\delta \psi_{\mu\alpha} &= -\sigma^{ab}\partial_a h_{b\mu}\varepsilon - \frac{1}{3}\,\gamma_\mu (M + i\gamma_5 N)\varepsilon + b_\mu i\gamma_5 \varepsilon + \delta_6 \gamma_\mu \not{b} i\gamma_5 \varepsilon \\
\delta M &= \delta_4 \bar{\varepsilon}\gamma \cdot R \\
\delta N &= \delta_5 i\bar{\varepsilon}\gamma_5 \gamma \cdot R \\
\delta b_\mu &= +\delta_7 i\bar{\varepsilon}\gamma_5 R_\mu + \delta_8 i\bar{\varepsilon}\gamma_5\gamma_\mu \gamma \cdot R
\end{aligned}
\tag{9.12}
$$

The parameters δ_4, δ_5, δ_6, δ_7 and δ_8 are determined by the restriction that the above transformations of Eq. (9.12) and the gauge transformations of Eq. (9.1) should form a closed algebra. For example, the commutator of two supersymmetries on M gives

$$
[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]M = \delta_4\{-\bar{\varepsilon}_2\gamma^\mu\varepsilon_1\partial_\mu M + 16\bar{\varepsilon}_2\sigma^{\mu\nu} i\gamma_5\varepsilon_1(1 + 3\delta_6)\partial_\mu b_\nu\}
\tag{9.13}
$$

which is the required result provided $\delta_4 = -\frac{1}{2}$ and $\delta_6 = -\frac{1}{3}$. Carrying out the commutator of two supersymmetries on all fields we find a closing algebra provided

$$
\delta_4 = -\frac{1}{2}, \quad \delta_5 = -\frac{1}{2}, \quad \delta_6 = -\frac{1}{3}, \quad \delta_7 = \frac{3}{2} \quad \text{and} \quad \delta_8 = -\frac{1}{2}
\tag{9.14}
$$

We henceforth adopt these values for the parameters. An action which is constructed from the fields $h_{\mu\nu}$, $\psi_{\mu\alpha}$, M, N and b_μ and is invariant under the transformations of Eq. (9.12) with the above values of the parameters is [16],[17]

$$A = \int d^4x \left\{ -\frac{1}{2} h_{\mu\nu} E^{\mu\nu} - \frac{1}{2} \bar{\psi}_\mu R^\mu - \frac{1}{3}(M^2 + N^2 - b^\mu b_\mu) \right\} \tag{9.15}$$

This is the action of linearized $N = 1$ supergravity and upon elimination of the auxiliary field M, N and b_μ it reduces to the algebraically on-shell Lagrangian of Eq. (9.11).

The Nonlinear Theory

The full nonlinear theory of supergravity can be found from the linearized theory discussed above by applying the Noether technique discussed at the beginning of this section. Just as in the case of Yang-Mills the reader will observe that the linearized theory possesses the local Abelian invariances of Eq. (9.1) as well as the rigid (i.e., constant parameter) supersymmetry transformations of Eq. (9.4).

We proceed just as in the case of the Yang-Mills theory and make the parameter of rigid transformations space-time dependent, i.e., set $\varepsilon = \varepsilon(x)$ in Eq. (9.4). The linearized action of Eq. (9.15) is then no longer invariant, but its variation must be of the form

$$\delta A_0 = \int d^4x \partial_\mu \bar{\varepsilon}^\alpha j^\mu{}_\alpha \tag{9.16}$$

since it is invariant when $\bar{\varepsilon}^\alpha$ is a constant. The object $j^\mu{}_\alpha$ is proportional to $\psi_{\mu\alpha}$ and linear in the bosonic fields $h_{\mu\nu}$, M or N and b_μ. As such, on dimensional grounds, it must be of the form

$$j_{\mu\alpha} \propto \partial_\tau h_{\rho\mu} \psi_{\nu\beta} + \ldots$$

Consider now the action, A, given by

$$A_1 = A_0 - \frac{\kappa}{4} \int d^4x \bar{\psi}^\mu j_\mu \tag{9.17}$$

where κ is the gravitational constant. The action A is invariant to order κ^0 *provided* we combine the now local supersymmetry transformation of Eq. (9.12) with a local Abelian Rarita-Schwinger gauge transformation of Eq. (9.1) with parameter $\eta(x) = (2/\kappa)\varepsilon(x)$. That is, we make a transformation

$$\begin{aligned} \delta\psi_\mu = & \frac{2}{\kappa}\partial_\mu \varepsilon(x) - \partial_a h_{b\mu} \sigma^{ab} \varepsilon(x) - \frac{1}{3}\gamma_\mu (M + i\gamma_5 N)\varepsilon(x) \\ & + i\gamma_5 \left(b_\mu - \frac{1}{3}\gamma_\mu \gamma^\nu b_\nu \right) \varepsilon(x) \end{aligned} \tag{9.18}$$

the remaining fields transforming as before except that ε is now space-time dependent.

As in the Yang-Mills case the two invariances of the linearized action become knitted together to form one transformation, the role of gauge coupling being played by the gravitational constant, κ. The addition of the term $(-\kappa/4)\bar{\psi}^{\mu}j_{\mu}$ to A_0 does the required job; its variation is

$$-\frac{\kappa}{4}\cdot 2\cdot\left(\frac{2}{\kappa}\right)(\partial_{\mu}\bar{\varepsilon})j^{\mu} + \text{terms of order } \kappa^{1} \tag{9.19}$$

The order κ^1 terms do not concern us at the moment. We note that $j_{\mu\alpha}$ is linear in $\psi_{\mu\alpha}$ and so we get a factor of 2 from $\delta\psi_{\mu\alpha}$.

In fact, one can carry out the Noether procedure in the context of pure gravity where one finds at the linearized level the rigid translation

$$\delta h_{\mu\nu} = \zeta^{\lambda}\partial_{\lambda}h_{\mu\nu} \tag{9.20}$$

and the local gauge transformation

$$\delta h_{\mu\nu} = \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} \tag{9.21}$$

These become knitted together at the first stage of the Noether procedure to give

$$\delta h_{\mu\nu} = \frac{1}{\kappa}\partial_{\mu}\zeta_{\nu} + \frac{1}{\kappa}\partial_{\nu}\zeta_{\mu} + \zeta^{\lambda}\partial_{\lambda}h_{\mu\nu} \tag{9.22}$$

since $\xi_{\nu} = (1/\kappa)\zeta_{\nu}$. This variation of $h_{\mu\nu}$ contains the first few terms of an Einstein general coordinate transformation of the vierbein which is given in terms of $h_{\mu\nu}$ by

$$e_{\mu}{}^{a} = \eta_{\mu}{}^{a} + \kappa h_{\mu}{}^{a} \tag{9.23}$$

We proceed in a similar way to the Yang-Mills case. We obtain order by order in κ an invariant Lagrangian by adding terms to the Lagrangian and in this case also adding terms to the transformations of the fields. For example, if we added a term to $\delta\psi_{\mu}$ say, $\delta\bar{\psi}_{\mu} = \ldots + \bar{\varepsilon}X_{\mu}\kappa$, then from the linearized action we receive a contribution $-\kappa\bar{\varepsilon}X_{\mu}R^{\mu}$ upon variation of ψ_{μ}. It is necessary at each step (order of κ) to check that the transformations of the fields form a closed algebra. In fact, any ambiguities that arise in the procedure are resolved by demanding that the algebra closes.

The final set of transformations [16,17] is

$$\begin{aligned}
\delta e_{\mu}{}^{a} &= \kappa\bar{\varepsilon}\gamma^{a}\psi_{\mu}\\
\delta\psi_{\mu} &= 2\kappa^{-1}D_{\mu}\big(w(e,\psi)\big)\varepsilon + i\gamma_5\left(b_{\mu} - \frac{1}{3}\gamma_{\mu}\not{b}\right)\varepsilon - \frac{1}{3}\gamma_{\mu}(M + i\gamma_5 N)\varepsilon\\
\delta M &= -\frac{1}{2}e^{-1}\bar{\varepsilon}\gamma_{\mu}R^{\mu} - \frac{\kappa}{2}i\bar{\varepsilon}\gamma_5\psi_{\nu}b^{\nu} - \kappa\bar{\varepsilon}\gamma^{\nu}\psi_{\nu}M + \frac{\kappa}{2}\bar{\varepsilon}(M + i\gamma_5 N)\gamma^{\mu}\psi_{\mu}\\
\delta N &= -\frac{e^{-1}}{2}i\bar{\varepsilon}\gamma_5\gamma_{\mu}R^{\mu} + \frac{\kappa}{2}\bar{\varepsilon}\psi_{\nu}b^{\nu} - \kappa\bar{\varepsilon}\gamma^{\nu}\psi_{\nu}N - \frac{\kappa}{2}i\bar{\varepsilon}\gamma_5(M + i\gamma_5 N)\gamma^{\mu}\psi_{\mu}\\
\delta b_{\mu} &= \frac{3i}{2}e^{-1}\bar{\varepsilon}\gamma_5\left(g_{\mu\nu} - \frac{1}{3}\gamma_{\mu}\gamma_{\nu}\right)R^{\nu} + \kappa\bar{\varepsilon}\gamma^{\nu}b_{\nu}\psi_{\mu} - \frac{\kappa}{2}\bar{\varepsilon}\gamma^{\nu}\psi_{\nu}b_{\mu}\\
&\quad - \frac{\kappa}{2}i\bar{\psi}_{\mu}\gamma_5(M + i\gamma_5 N)\varepsilon - \frac{i\kappa}{4}\varepsilon_{\mu}{}^{bcd}b_b\bar{\varepsilon}\gamma_5\gamma_c\psi_d
\end{aligned} \tag{9.24}$$

where

$$R^\mu = \varepsilon^{\mu\nu\rho\kappa} i\gamma_5\gamma_\nu D_\rho\big(w(e,\psi)\big)\psi_\kappa$$

$$D_\mu\big(w(e,\psi)\big) = \partial_\mu + w_{\mu ab}\frac{\sigma^{ab}}{4}$$

and

$$\begin{aligned}
w_{\mu ab} = {}& \frac{1}{2}e^\nu{}_a(\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}) - \frac{1}{2}e_b{}^\nu(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}) \\
& - \frac{1}{2}e_a{}^\rho e_b{}^\sigma(\partial_\rho e_{\sigma c} - \partial_\sigma a_{\rho c})e_\mu{}^c \\
& + \frac{\kappa^2}{4}(\bar\psi_\mu\gamma_a\psi_b + \bar\psi_a\gamma_\mu\psi_b - \bar\psi_\mu\gamma_b\psi_a)
\end{aligned} \tag{9.25}$$

They form a closed algebra, the commutator of two supersymmetries on any field being

$$\begin{aligned}
[\delta_{\varepsilon_1},\delta_{\varepsilon_2}] = {}& \delta_{\text{supersymmetry}}(-\kappa\xi^\nu\psi_\nu) + \delta_{\text{general coordinate}}(2\xi_\mu) \\
& + \delta_{\text{Local Lorentz}}\bigg(-\frac{2\kappa}{3}\varepsilon_{ab\lambda\rho}b^\lambda\xi^\rho \\
& -\frac{2\kappa}{3}\bar\varepsilon_2\sigma_{ab}(M + i\gamma_5 N)\varepsilon_1 + 2\xi^d w_d{}^{ab}\bigg)
\end{aligned} \tag{9.26}$$

where

$$\xi_\mu = \bar\varepsilon_2\gamma_\mu\varepsilon_1$$

The transformations of Eq. (9.24) leave invariant the action [16], [17]

$$A = \int d^4x \left\{\frac{e}{2\kappa^2}R - \frac{1}{2}\bar\psi_\mu R^\mu - \frac{1}{3}e(M^2 + N^2 - b_\mu b^\mu)\right\} \tag{9.27}$$

where

$$R = R_{\mu\nu}{}^{ab}e_a{}^\mu e_b{}^\nu$$

and

$$R_{\mu\nu}{}^{ab}\frac{\sigma_{ab}}{4} = [D_\mu, D_\nu]$$

The auxiliary fields $M,\ N$ and b_μ may be eliminated to obtain the nonlinear algebraically on-shell Lagrangian which was the form in which supergravity was originally found in Refs. 14 and 15.

As discussed at the beginning of this section one could also build up the non-local theory by working with the algebra of field transformations alone.

Invariance of $N = 1$ Supergravity

We refer the reader to chapter 10 of reference [0] for a demonstration of the invariance of the supergravity under local supersymmetry transformations. This proof [18,19] uses the 1.5 order formalism.

4.2 On-Shell $N = 1$ $D = 4$ Superspace

In this section we will construct the on-shell superspace formulation of $N = 1$ $D =$ supergravity, from which we recover the equations of motion in x-space.

4.2.1 Geometry of Local Superspace

The geometrical framework [69] of superspace supergravity has many of the constructions of general relativity, but also requires additional input. A useful guide in the construction of local superspace is that it should admit rigid superspace as a limit.

We begin with an eight-dimensional manifold $z^\Pi = (x^u, \theta^{\underline{\alpha}})$ where x^u is a commuting coordinate and $\theta^{\underline{\alpha}}$ is a Grassmann odd coordinate. On this manifold a super-general coordinate reparametrization has the form

$$z^\Pi \to z'^\Pi = z^\Pi + \xi^\Pi \tag{16.1}$$

where $\xi^\Pi = (\xi^u, \xi^{\underline{\alpha}})$ are arbitrary functions of z^Π.

Just as in general relativity we can consider scalar superfields, that is, fields for which

$$\phi'(z') = \phi(z) \tag{16.2}$$

and superfields with superspace world indices φ_Λ; for example

$$\varphi_\Lambda = \frac{\partial \phi}{\partial z^\Lambda} \tag{16.3}$$

The latter transform as

$$\varphi'_\Lambda(z') = \frac{\partial z^\Pi}{\partial z'^\Lambda} \varphi_\Pi(z) \tag{16.4}$$

The transformation properties of higher order tensors is obvious.

We must now specify the geometrical structure of the manifold. For reasons that will become apparent, the superspace formulation is essentially a vierbien formulation. We introduce supervierbiens $E_\Pi{}^N$ which transform under the supergeneral coordinate transformations as

$$\delta E_\Pi{}^N = \xi^\Lambda \partial_\Lambda E_\Pi{}^N + \partial_\Pi \xi^\Lambda E_\Lambda{}^N \tag{16.5}$$

The N-index transforms under the tangent space group which is taken to be just the Lorentz group; and so $\delta E_\Pi{}^N = E_\Pi{}^M \Lambda_M^{\ N}$ where

$$\Lambda_M^{\ N} = \begin{pmatrix} \lambda_m{}^n & 0 & 0 \\ 0 & -\frac{1}{4}(\sigma_{mn})_A{}^B \Lambda^{mn} & 0 \\ 0 & 0 & +\frac{1}{4}(\sigma_{mn})_{\dot{A}}{}^{\dot{B}} \Lambda^{mn} \end{pmatrix} \tag{16.6}$$

The matrix $\Lambda_m^{\ n}$ is an arbitrary function on superspace and it governs not only the rotation of the vector index, but also the rotation of the spinorial indices. Since we are dealing with an eight-dimensional manifold one could choose a much larger tangent space group. For example, Λ_M^N could be an arbitrary matrix that preserves the metric

$$g_{NM} = \mathrm{diag}(a_1\eta_{mn}, a_2\varepsilon_{AB}, a_3\varepsilon_{\dot{A}\dot{B}}) \tag{16.7}$$

where a_1, a_2 and a_3 are non-zero arbitrary constants. Demanding reality of the metric implies $a_2^* = a_3$ and we may scale away one factor. This corresponds to taking the tangent space group to be $OSp(4,1)$. In such a formulation one could introduce a metric $g_{\Pi\Lambda} = E_{\Pi}{}^{N} g_{NM} E_{\Lambda}{}^{M}$ and one would have a formulation which mimicked Einstein's general relativity at every step [70].

Such a formulation, however, would not lead to the x-space component $N=1$ supergravity given earlier. One way to see this is to observe that the above tangent space group does not coincide with that of rigid superspace (super Poincaré/Lorentz), which has the Lorentz group, as given in Eq. (16.6) with Λ_m^n a constant matrix, as its tangent space group. As linearized superspace supergravity must admit a rigid superspace formulation, any formulation based on an $OSp(4,1)$ tangent group will not coincide with linearized supergravity. In fact the $OSp(4,1)$ formulation has a higher derivative action.

An important consequence of this restricted tangent space group is that tangent supervectors $V^N = V^{\Pi} E_{\Pi}{}^{N}$ belong to a reducible representation of the Lorentz group. This allows one to write down many more invariants. The objects $V^m V_m$, $V^A V^B \varepsilon_{AB}$, $V^{\dot{A}} V^{\dot{B}} \varepsilon_{\dot{B}\dot{A}}$ are all separately invariant.

In other words, in the choice of metric in Eq. (16.7) the constants a_1, a_2, a_3 can have any value including zero.

We define a Lorentz valued spin connection

$$\Omega_{\Lambda M}{}^{N} = \begin{pmatrix} \Omega_{\Lambda m}{}^{n} & 0 & 0 \\ 0 & -\frac{1}{4}\Omega_{\Lambda}{}^{mn}(\sigma_{mn})_A{}^B & 0 \\ 0 & 0 & \frac{1}{4}\Omega_{\Lambda}{}^{mn}(\bar{\sigma}_{mn})_{\dot{A}}{}^{\dot{B}} \end{pmatrix} \tag{16.8}$$

This object transforms under super general coordinate transformations as

$$\delta\Omega_{\Lambda M}{}^{N} = \xi^{\Pi}\partial_{\Pi}\Omega_{\Lambda M}{}^{N} + \partial_{\Lambda}\xi^{\Pi}\Omega_{\Pi M}{}^{N} \tag{16.9}$$

and under tangent space rotation as

$$\delta\Omega_{\Lambda M}{}^{N} = -\partial_{\Lambda}\Omega_{M}{}^{N} + \Omega_{\Lambda M}{}^{S}\Lambda_{S}{}^{N} + \Omega_{\Lambda R}{}^{N}\Lambda_{M}{}^{R}(-1)^{(M+R)(N+R)} \tag{16.10}$$

The covariant derivatives are then defined by

$$D_{\Lambda} = \partial_{\Lambda} + \frac{1}{2}\Omega_{\Lambda}{}^{mn} J_{mn} \tag{16.11}$$

where J_{mn} are the appropriate Lorentz generators (see Appendix A of reference [0]). The covariant derivative with tangent indices is

$$D_N = E_N{}^{\Lambda} D_{\Lambda} \tag{16.12}$$

where E_N^{Λ} is the inverse vierbien defined by

$$E_N^{\Lambda} E_{\Lambda}{}^M = \delta_N{}^M \tag{16.13}$$

or

$$E_{\Lambda}{}^M E_M^{\Pi} = \delta_{\Lambda}^{\Pi} \tag{16.14}$$

Equipped with super-vierbien and spin-connection we define the torsion and curvature tensors as usual

$$[D_N, D_M\} = T_{NM}^{\ R} D_R + \frac{1}{2} R_{NM}{}^{mn} J_{mn} \tag{16.15}$$

Using Eqs. (16.11) and (16.12) we find that

$$T_{NM}{}^R = E_M^{\Lambda} \partial_{\Lambda} E_N^{\Pi} E_{\Pi}{}^R + \Omega_{MN}^{\ \ R} - (-1)^{MN}(M \leftrightarrow N) \tag{16.16}$$

$$R_{MN}{}^{rs} = E_M^{\Lambda} E_N^{\Pi} (-1)^{\Lambda(N+\Pi)} \{\partial_{\Lambda} \Omega_{\Pi}^{\ rs} + \Omega_{\Lambda}{}^{rk} \Omega_{\Pi k}^{\ \ s} - (-1)^{\Lambda\Pi}(\Lambda \leftrightarrow \Pi)\} \tag{16.17}$$

The super-general coordinate transformations can be rewritten using these tensors

$$\delta E_{\Lambda}{}^M = -E_{\Lambda}{}^R \xi^N T_{NR}^{\ \ M} + D_{\Lambda} \xi^M \tag{16.18}$$

$$\delta \Omega_{\Lambda M}{}^N = E_{\Lambda}{}^R \xi^P R_{PRM}{}^N \tag{16.19}$$

where $\xi^N = \xi^{\Lambda} E_{\Lambda}{}^N$ and we have discarded a Lorentz transformation.

The torsion and curvature tensors satisfy Bianchi identities which follow from the identity

$$\big[D_M, [D_N, D_R\}\big\} - \big[[D_M, D_N\}, D_R\big\} + (-1)^{RN} \big[[D_M, D_R\}, D_N\big\} = 0 \tag{16.20}$$

They read

$$\begin{aligned} 0 = I^{(1)}_{RMN}{}^{F} &= [-(-1)^{(M+N)R} D_R T_{MN}{}^F + T_{MN}{}^S T_{SR}{}^F + R_{MNR}{}^F] \\ &\quad + [+(-1)^{MN} D_N T_{MR}{}^F - (-1)^{NR} T_{MR}{}^S T_{SN}{}^F - (-1)^{NR} R_{MRN}{}^F] \\ &\quad + [-D_M T_{NR}{}^F \\ &\quad + (-1)^{(N+R)M} T_{NR}{}^S T_{SM}{}^F + (-1)^{(N+R)M} R_{NRM}{}^F] \end{aligned} \tag{16.21}$$

and

$$\begin{aligned} I^{(2)}_{RMN}{}^{mn} &= [(-1)^{(M+N)R} D_R R_{MN}{}^{mn} + T_{MN}^{\ \ S} R_{SR}{}^{mn}] \\ &\quad - (-1)^{NR} (R \to N,\ N \to R,\ M \to M \text{ in the first bracket}) \\ &\quad + (-1)^{(N+R)M} (M \to N,\ N \to R,\ R \to M \text{ in the first bracket}) = 0 \end{aligned} \tag{16.22}$$

It can be shown that if $I^{(1)}_{MNR}{}^F$ holds then $I^{(2)}_{RMN}{}^{mn}$ is automatically satisfied. This result holds in the presence of constraints on $T_{MN}^{\ \ R}$ and $R_{MN}{}^{mn}$ and is a consequence of the restricted tangent space choice. We refer to this as Dragon's theorem [71]. For all fermionic indices we find that

$$\begin{aligned} I_{ABC}^{\ \ N} &= -D_A T_{BC}{}^N + T_{AB}{}^S T_{SC}{}^N + R_{ABC}{}^N - D_C T_{AB}{}^N + T_{CA}{}^S T_{SB}{}^N \\ &\quad + R_{CAB}{}^N - D_B T_{CA}{}^N + T_{BC}^{\ \ S} T_{SA}{}^N + R_{BCA}{}^N = 0 \end{aligned} \tag{16.23}$$

and for fermionic indices with one bosonic index

$$I_{ABr}{}^{N} = -D_A T_{Br}{}^{N} + T_{AB}{}^{S} T_{Sr}{}^{N} + R_{ABr}{}^{N} - D_r T_{AB}{}^{N} + T_{rA}{}^{S} T_{SB}{}^{N} \\ + R_{rAB}{}^{N} + D_B T_{rA}{}^{N} - T_{Br}{}^{S} T_{SA}{}^{N} - R_{BrA}{}^{N} = 0 \tag{16.24}$$

while

$$I_{Anr}{}^{N} = -D_A T_{nr}{}^{N} + T_{An}{}^{s} T_{sr}{}^{N} + R_{Anr}{}^{N} - D_r T_{An}{}^{N} + T_{rA}{}^{s} T_{sn}{}^{N} \\ + R_{rAn}{}^{N} - D_n T_{rA}{}^{N} + T_{nr}{}^{s} T_{sA}{}^{N} + R_{nrA}{}^{N} = 0 \tag{16.25}$$

Clearly one can replace any undotted index by a dotted index and the signs remain the same. We recall that for rigid superspace all the torsions and curvatures vanish except for $T_{A\dot{B}}{}^{n} = -2i(\sigma^n)_{A\dot{B}}$. Clearly this is inconsistent with an $OSp(4,1)$ tangent space group.

The dimensions of the torsions and curvature can be deduced from the dimensions of D_N. If F and B denote fermionic and bosonic indices respectively then

$$[D_F] = \frac{1}{2}; \quad [D_B] = 1 \tag{16.26}$$

and

$$[T_{FF}{}^{B}] = 0; \quad [T_{FF}{}^{F}] = [T_{FB}{}^{B}] = \frac{1}{2} \\ [T_{FB}{}^{F}] = [T_{BB}{}^{B}] = 1; \quad [T_{BB}{}^{F}] = \frac{3}{2} \tag{16.27}$$

while

$$[R_{FF}{}^{mn}] = 1; \quad [R_{FB}{}^{mn}] = \frac{3}{2}; \quad [R_{BB}{}^{mn}] = 2 \tag{16.28}$$

It is useful to consider the notion of the geometric dimension of fields. This is the dimension of the field as it appears in the torsions and curvature. Such expressions never involve κ and as they are nonlinear in certain bosonic fields, such as the vierbien $e_\mu{}^n$, these fields must have zero geometric dimensions. The dimensions of the other fields are determined in relation to $e_\mu{}^n$ to be given by

$$[e_\mu{}^n] = 0 \quad [\psi_\mu{}^\alpha] = \frac{1}{2} \quad [M] = [N] = [b_\mu] = 1 \tag{16.29}$$

These dimensions can,for example, be read of from the supersymmetry transformations. These dimensions differ from the canonical assignment of dimension by one unit. The difference comes about as we have absorbed factors of κ into the fields.

4.2.2 On-shell Derivation of $N = 1$ $D = 4$ Superspace Supergravity

Having set up the appropriate geometry of superspace we are now in a position to derive on-shell $N = 1$ $D = 4$ supergravity using its superspace setting and solely from a knowledge of the on-shell states of a given spin in the irreducible representation. The result is derived by using dimensional analysis and the Bianchi identities in superspace discussed above. We now illustrate this procedure for $N = 1$ supergravity. It was this method that was used [111] to find the full equations of motion in superspace and in x-space of IIB supergravity.

The on-shell states are represented by $h_{\mu\nu}$ ($h_{\mu\nu} = h_{\nu\mu}$) and ${\psi_\mu}^\alpha$ which have the gauge transformations

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \\ \delta \psi_{\mu\alpha} &= \partial_\mu \eta_\alpha \end{aligned} \tag{16.115}$$

We have omitted the nonlinear terms, as only the general form is important. The geometric dimension of $h_{\mu\nu}$ is zero while that of ${\psi_\mu}^\alpha$ is one half. The lowest dimension gauge covariant objects are of the form

$$\partial\psi \quad \text{and} \quad \partial\partial h \tag{16.116}$$

which have dimensions 3/2 and 2 respectively.

Consider now the super torsion and curvature; these objects at $\theta = 0$ must correspond to covariant x-space objects. If there is no such object then the corresponding tensor must vanish at $\theta = 0$ and hence to all orders in θ. The only dimension-0 tensors are $T_{AB}^{\ \ n}$ and $T_{A\dot{B}}^{\ \ n}$. There are no dimension-0 covariant objects except the numerically invariant tensor $(\sigma^n)_{A\dot{B}}$. Hence, we must conclude that

$$T_{AB}{}^n = 0 \quad T_{A\dot{B}}{}^n = c(\sigma^n)_{A\dot{B}} \tag{16.117}$$

where c is a constant. We choose $c \neq 0$ in order to agree with rigid superspace. The reality properties of $T_{A\dot{B}}{}^n$ imply that c is imaginary and we can normalize it to take the value $c = -2i$.

There are no dimension-$\frac{1}{2}$ covariant tensors in x-space and so

$$T_{A\dot{B}}{}^{\dot{C}} = T_{AB}{}^C = T_{Am}{}^n = 0 \tag{16.118}$$

There are no dimension-1 covariant objects in x-space. This would not be the case if one had an independent spin connection, ${w_\mu}^{rs}$, for $\partial e + w + \ldots$ would be a covariant quantity. When ${w_\mu}^{rs}$ is not an independent quantity it must be given in terms of ${e_\mu}^n$ and ${\psi_\mu}^\alpha$ in such a way as to render the above dimension-1 covariant quantity zero. Hence, for a dependent spin connection, i.e., in second-order formalism, we have

$$T_{nA}{}^{\dot{B}} = T_{nA}{}^B = R_{AB}{}^{mn} = 0 = R_{A\dot{B}}{}^{mn} \tag{16.119}$$

In other words, every dimension 0-, $\frac{1}{2}$–, 1- torsion and curvature vanishes with the exception of $T_{A\dot{B}}{}^n = -2i(\sigma^n)_{A\dot{B}}$.

The reader who is familiar with the off-shell constraints for $N = 1$ supergravity can compare them with the on-shell constraints found here. The set of constraints of off-shell supergravity is given in Section 16.2 of reference [0]. We find that that the extra constraints are $T_{nA}{}^B = T_{nA}{}^{\dot{B}} = 0 = R_{AB}{}^{AB}$. In terms of the superfields R, $W_{(ABG)}$ and $G_{A\dot{B}}$ this is equivalent to

$$R = G_{A\dot{B}} = 0$$

Returning to the on-shell theory. The dimension 3/2 tensors can involve at $\theta = 0$ the spin 3/2 object $\partial\psi$ and so these will not all be zero. The only remaining non-zero tensors are $T_{mn}{}^{A}$, $R_{Ar}{}^{mn}$ and $R_{st}{}^{mn}$ and of course $T_{A\dot{B}}{}^{n} = -2i(\sigma^n)_{A\dot{B}}$. However, the previous constraints of Eqs. (16.117)-(16.119) are sufficient to specify the entire theory, as we will now demonstrate. The first nontrivial Bianchi identity has dimension 3/2 and is

$$\begin{aligned} I_{nB\dot{D}}{}^{\dot{C}} = -D_n T_{B\dot{D}}{}^{\dot{C}} + T_{nB}{}^{F} T_{F\dot{D}}{}^{\dot{C}} + R_{nB\dot{D}}{}^{\dot{C}} + D_{\dot{D}} T_{nB}{}^{\dot{C}} - T_{\dot{D}n}{}^{F} T_{FB}{}^{\dot{C}} \\ - R_{\dot{D}nB}{}^{\dot{C}} - D_B T_{\dot{D}n}{}^{\dot{C}} + T_{B\dot{D}}{}^{\dot{F}} T_{Fn}{}^{\dot{C}} + R_{B\dot{D}n}{}^{\dot{C}} = 0 \end{aligned} \tag{16.120}$$

Using the above constraints this reduces

$$-2i(\sigma^m)_{B\dot{D}} T_{mn}{}^{\dot{C}} - R_{nB\dot{D}}{}^{\dot{C}} = 0 \tag{16.121}$$

Tracing on $\dot{D}$ and $\dot{C}$ then yields

$$(\sigma^m)_{B\dot{D}} T_{mn}{}^{\dot{D}} = 0 \tag{16.122}$$

This is the Rarita-Schwinger equation as we will demonstrate shortly.

The spin 2 equation must have dimension two and is contained in the I_{Bmn}^{A} Bianchi identity.

$$\begin{aligned} I_{Bmn}^{A} = -D_B T_{mn}{}^{A} + T_{Bm}{}^{F} T_{Fn}{}^{A} + R_{Bmn}{}^{A} - D_n T_{Bm}{}^{A} + T_{nB}{}^{F} T_{Fm}{}^{A} \\ + R_{nBm}{}^{A} - D_m T_{nB}{}^{A} + T_{mn}{}^{F} T_{FB}{}^{A} + R_{mnB}{}^{A} = 0 \end{aligned} \tag{16.123}$$

Application of the constraints gives

$$-D_B T_{mn}{}^{A} + T_{mnB}^{A} = 0 \tag{16.124}$$

On contracting with $(\sigma^m)_{\dot{B}A}$ we find

$$(\sigma^m)_{\dot{B}A} D_B T_{mn}{}^{A} = 0 = R_{mnB}{}^{A} (\sigma^m)_{\dot{B}A} \tag{16.125}$$

Using the fact that $R_{mnB}{}^{A} = -\frac{1}{4} R_{mn}{}^{pq} (\sigma_{pq})_B^{A}$ yields the result

$$R_{mn} - \frac{1}{2}\eta_{mn} R = 0$$

or

$$R_{mn} = 0 \quad \text{where} \quad R_{mn} = R_{msn}{}^{s} \tag{16.126}$$

We now wish to demonstrate that these are the spin 3/2 and spin 2 equations. The $\theta = 0$ components of $E_\mu{}^n$ and $E_\mu{}^A$ are denoted as follows:

$$E_\mu{}^n(\theta = 0) = e_\mu{}^n, \quad E_\mu{}^A(\theta = 0) = \frac{1}{2}\psi_\mu{}^A \tag{16.127}$$

At this stage the above equation is simply a definition of the fields $e_\mu{}^n$ and $\psi_\mu{}^A$. The $\theta=0$ components of $E_{\underline{A}}^n$ may be gauged away by an appropriate super general coordinate transformation. As

$$\delta E_{\underline{A}}{}^n(\theta=0) = \xi^\Pi \partial_\Pi E_{\underline{A}}{}^n|_{\theta=0} + \partial_{\underline{A}} \xi^\Pi E_\Pi{}^n|_{\theta=0} = \ldots + \partial_{\underline{A}}\xi^\mu e_\mu{}^n + \ldots \tag{16.128}$$

we may clearly choose $\partial_{\underline{A}}\xi^\mu$ so that $E_{\underline{A}}{}^n = 0$. Similarly we may choose

$$E_{\underline{\dot{A}}}{}^{\dot{B}} = \delta_{\dot{A}}{}^{\dot{B}}, \quad E_{\underline{A}}{}^{B} = \delta_{A}{}^{B}, \quad E_{\underline{A}}{}^{\dot{B}} = 0 \tag{16.129}$$

To summarize

$$E_\Pi{}^m(\theta=0) = \begin{pmatrix} e_\mu{}^n & \frac{1}{2}\psi_\mu{}^A & \frac{1}{2}\psi_\mu{}^{\dot{A}} \\ 0 & \delta_B{}^A & 0 \\ 0 & 0 & \delta_{\dot{B}}{}^{\dot{A}} \end{pmatrix} \tag{16.130}$$

For the spin connection $\Omega_\Pi{}^m$ we define

$$\Omega_\mu{}^{mn}(\theta=0) = w_\mu{}^{mn} \tag{16.131}$$

and we use a Lorentz transformation to gauge

$$\Omega_{\underline{\alpha}}{}^{mn}(\theta=0) = 0 \tag{16.132}$$

At $\theta=0$ we then find

$$T_{\mu\nu}{}^{\dot{A}} = -\frac{1}{2}\partial_\mu \psi_\nu{}^{\dot{A}} + \Omega_{\mu\nu}{}^{\dot{A}} - (\mu \leftrightarrow \nu) = -\frac{1}{2}\psi_{\mu\nu}{}^{\dot{A}} \tag{16.133}$$

where

$$\psi_{\mu\nu}{}^{\dot{A}} \equiv D_\mu \psi_\nu{}^{\dot{A}} - (\mu \leftrightarrow \nu)$$

and

$$D_\mu \psi_\nu{}^{\dot{A}} = \partial_\mu \psi_\nu{}^{\dot{A}} - \psi_\nu{}^{\dot{B}} w_{\mu\dot{B}}{}^{\dot{A}} \tag{16.134}$$

Here we have used the results

$$\Omega_{\mu\nu}{}^{\dot{A}} = E_\nu{}^N w_{\mu N}{}^{\dot{A}} = \frac{1}{2}\psi_\nu{}^{\dot{B}} w_{\mu\dot{B}}{}^{\dot{A}} \tag{16.135}$$

The torsion with all tangent indices is given in terms of $T_{\mu\nu}{}^{\dot{A}}$ by the relation

$$\begin{aligned} T_{\mu\nu}{}^{\dot{A}}(\theta=0) &= E_\mu{}^N(\theta=0) E_\nu{}^M(\theta=0) T_{NM}{}^{\dot{A}}(\theta=0)(-1)^{NM} \\ &= e_\mu{}^n e_\nu{}^m T_{nm}{}^{\dot{A}}(\theta=0) \end{aligned} \tag{16.136}$$

where we have used the constraints $T_{Bn}{}^{\dot{A}} = T_{\dot{B}C}{}^{\dot{A}} = 0$. Consequently

$$0 = (\sigma^m)_{A\dot{B}} T_{mn}{}^{\dot{B}}(\theta=0) = -\frac{1}{2}(\sigma^m)_{A\dot{B}} e_m{}^\mu e_n{}^\nu \psi_{\mu\nu}{}^{\dot{B}} \tag{16.137}$$

and we recognize the Rarita-Schwinger equation on the right-hand side.

Actually to be strictly rigorous we must also show that $w_\mu{}^{mn}$ is the spin connection given in terms of $e_\mu{}^n$ and $\psi_\mu{}^A$. In fact, this follows from the constraint $T_{nm}{}^r = 0$. We note that

$$T_{\mu\nu}{}^r(\theta=0) = -\partial_\mu e_\nu{}^r + w_{\mu\nu}{}^r - (\mu \leftrightarrow \nu) \tag{16.138}$$

However,

$$\begin{aligned} T_{\mu\nu}{}^r(\theta=0) &= E_\mu{}^N(\theta=0)E_\nu{}^M(\theta=0)T_{NM}{}^r(\theta=0)(-1)^{MN} \\ &\quad - \frac{1}{4}\psi_\mu{}^A\psi_\nu{}^{\dot B}T_{A\dot B}{}^r(\theta=0) - \psi_\mu{}^{\dot B}\psi_\nu{}^A T_{\dot B A}{}^r(\theta=0) \\ &= +\frac{1}{2}i\psi_\nu{}^{\dot B}(\sigma^r)_{A\dot B}\psi_\mu{}^A - (\mu \leftrightarrow \nu) \end{aligned} \tag{16.139}$$

Consequently we find that

$$w_{\mu n}{}^m e_\nu{}^n - \partial_\nu e_\mu{}^m - (\mu \leftrightarrow \nu) = +\frac{i}{2}\psi_\nu{}^{\dot B}(\sigma^m)_{A\dot B}\psi_\mu{}^A - (\mu \leftrightarrow \nu) \tag{16.140}$$

which can be solved in the usual way to yield the correct expression for $w_{\mu n}{}^m$.

The spin 2 equation is handled in the same way:

$$R_{\mu\nu}{}^{mn}(\theta=0) = \partial_\mu w_\nu{}^{mn} + w_\mu{}^{mr}w_{\nu r}{}^n - (\mu \leftrightarrow \nu) \tag{16.141}$$

However

$$\begin{aligned} R_{\mu\nu}{}^{nm}(\theta=0) &= E_\mu{}^N(\theta=0)E_\nu{}^M(\theta=0)R_{NM}{}^{mn}(\theta=0)(-1)^{mN} \\ &= e_\mu{}^p e_\nu{}^q R_{pq}{}^{nm}(\theta=0) + \frac{1}{2}\big(\psi_\mu{}^{\dot A}e_\nu{}^p R_{\dot A p}{}^{nm} \\ &\quad + \psi_\mu{}^A e_\nu{}^p R_{Ap}{}^{nm}(\theta=0) - (\mu \leftrightarrow \nu)\big) \end{aligned} \tag{16.142}$$

The object $R_{Ap}{}^{nm}$ can be found from the Bianchi identity $I_{Anr}{}^s$

$$\begin{aligned} 0 = I_{Anr}{}^s &= -D_A T_{nr}{}^s + T_{An}{}^F T_{Fr}{}^s + R_{Anr}{}^s - D_r T_{An}{}^s + T_{rA}{}^F T_{Fn}{}^s \\ &\quad + R_{rAn}{}^s - D_n T_{rA}{}^s + T_{nr}{}^F T_{FA}{}^s + R_{nrA}{}^s \end{aligned} \tag{16.143}$$

Using the constraints we find that

$$R_{Anr}{}^s + R_{rAn}{}^s = +2iT_{nr}{}^{\dot B}(\sigma^s)_{A\dot B} \tag{16.144}$$

From Eq. (16.137) we find that

$$R_{Anr}{}^s + R_{rAn}{}^s = -i(\sigma^s)_{A\dot B}e_n{}^\mu e_r{}^\nu \psi_{\mu\nu}{}^{\dot B} \tag{16.145}$$

Contracting Eq. (16.142) with $e^\nu{}_m$ we find

$$e_m{}^\nu R_{\mu\nu}{}^{nm}(\theta=0) \equiv R_\mu{}^n = e_\mu{}^p R_{pm}{}^{nm} + \frac{1}{2}(\psi_\mu{}^A R_{Am}{}^{nm} + \psi_\mu{}^{\dot A}R_{\dot A m}{}^{nm}) \tag{16.146}$$

Equation (16.145) then gives

$$e_\mu{}^p R_{pm}{}^{nm} = R_\mu{}^n - \left(\frac{i}{2}\psi_\mu{}^A(\sigma^m)_{A\dot{B}} e_m{}^\lambda e^{n\tau}\psi_{\lambda\tau}{}^{\dot{B}} + \text{h.c.}\right) \tag{16.147}$$

Equation (16.126) ($R_{mn}=0$) then yields the spin 2 equation of $N=1$ supergravity, which is the left-hand side of the above equation.

At first sight it appears that the task is not finished; one should also analyze all the remaining Bianchi identities and show that they do not lead to any inconsistencies. However, it can be shown that the other Bianchi identities are now automatically satisfied [71,82].

4.3 Gauging of Space-time Groups

It has been know for many years that Einstein's theory of general relativity contains a local Lorentz symmetry. When the action is expressed in firstorder formalism the spin-connection is the gauge field and the Riemann tensor the field strength for Lorentz group [114]. It was only in reference [18] that a space-time group was gauged and general relativity theory was deduced from a gauge theory viewpoint. In fact, reference [18] gauged the super Poincaé group and deduced the supergravity theory from this view point. It is straightforward to restrict the calculation to that for the Poincaé group and deduce just general relativity. Of course at that time supergravity had been constructed [14],[15], but the gauging proceedure provided the first analytical proof [18] of its invariance under local supersymmetry. The theory of supergravity with a cosmological constant by gauging the the super de Sitter group was independently found in reference [113].

Let us consider gauging the $N=1$ super Poincaé group; corresponding to the generators J_{ab}, P_a and Q^α we introduce the gauge fields w_μ^{ab}, e_μ^a and $Q_{\mu\alpha}$ which will become the spin-connection, the vierbein and gravitino. It is straightforward to calculate the field strengths $R_{\mu\nu}^{ab}, C_{\mu\nu}^a$ and $\phi_{\mu\nu\alpha}$ and the gauge transformations of the gauge fields. Supergravity is not the gauge theory of the super Poincaé group in an obvious way and we must proceed by setting the field strength $C_{\mu\nu}^a$ associated to translations to zero. We now construct an action to the appropriate order in derivatives that is invariant under the gauge transformations of the super Poincaé group subject to the constraint $C_{\mu\nu}^a$. In particular, we start from the most general action which is first-order in the field strengths i.e.

$$\int d^4x\ \epsilon^{\mu\nu\rho\lambda}(\epsilon_{abcd}e_\mu^a e_\nu^b R_{\rho\lambda}^{cd} + if\bar{\psi}_\mu\gamma_5\gamma_\nu\phi_{\rho\lambda}).$$

The constant f is readily fixed by demanding the invariance of this action. Hence, we rapidly arrive at the supergravity action. In fact the constraint $C_{\mu\nu}^a = 0$ is just that required to correctly express the spin-connection in terms of the vierbein and gravitino, that is to go from first- to second-order formalism. The constraint is also just that required to convert gauge transformations associated with translations into general coordinate trasformations.

When carrying out the variation of the action subject to the constraint the variation of the spin-connection is irrelevant, since its variation multiplies its equation of motion which vanishes due to the constraint $C_{\mu\nu}^a = 0$. However, this constriant is none other than the condition for expressing the spin-connection in terms of the vierbein, that is the transition to second-order formalism. Thus the invarinace of the above action subject to the constraint provides an analytical proof of the invariance of supergravity under local supersymmetry [18],[19]. This way of proceeding became known as the 1.5-order formalism and it is reviewed in chapter 10 of reference [0].

The construction of the theories of conformal supergravity were carried out using this gauging method [118]. The key to getting the gauge method to work is to guess the appropriate constraints. However, since these constraints break the original gauge transformations they are not always easy to find. Much effort has been devoted to developing the method discussed into a systematic procedure. One such work was that of reference [115] where the full gauge symmetry was realised, but was spontaneously broken.

The gauge techninque has not been used to construct the theories of supergravity in ten and eleven dimensions and it may be instructive to derive them using this method. It cannot be a coincidence that gravity and supergravity admit such simple formulations as a gauge theories and this connection suggests that there is something to be understood at a deeper level.

5. Eleven-dimensional Supergravity

In this section, we give the eleven-dimensional supergravity theory and describe its properties. Eleven dimensional supergravity is thought to be the low energy effective action of a new kind of theory called M theory which is believed to underlie string theory. Little is known about M theory apart from its relation to eleven dimensional supergravity.

The non-trivial representation of the Clifford algebra in eleven dimensions is inherited from that in ten dimensions and so has dimension $2^{\frac{10}{2}} = 32$. We also inherit the properties $\epsilon = 1$ and so $B^T = B$ and $C = -C^T$ which were discussed in section one. The resulting properties of the γ matrices are given in equations (1.4.6) and (1.4.8).

Eleven-dimensional supergravity is based on the $D = 11$ supersymmetry algebra with Majorana spinor Q_α which has 32 real components. As we discussed in chapter two the algebra takes the form [117]

$$\{Q_\alpha, Q_\beta\} = (\gamma^m C^{-1})_{\alpha\beta} P_m + (\Gamma^{mn} C^{-1})_{\alpha\beta} Z_{mn} + (\Gamma^{mnpqr} C^{-1})_{\alpha\beta} Z_{mnpqr} \tag{5.1}$$

where P_m is the translation operator and Z_{mn} and Z_{mnpqr} are central charges. Although these play little apparent role in the construction of the supergravity theory they are very important in M theory.

Eleven dimensions is the maximal dimension in which one can have a supergravity theory [105]. By a supergravity theory we mean a theory with spins two and less. This observation follows from the study of the massless irreducible representations of supersymmetry, which can be deduced in a straightforward way from the relevant supersymmetry algebras. The irreducible representations of the four-dimensional supersymmetry algebras were given in section three. We found that the maximal supergravity theory in four dimensions corresponded to $N = 8$ supersymmetry. This is the algebra with eight Majorana supercharges, each of which has four real components. In fact, this theory can be obtained the eleven dimensional supergravity theory by dimensional reduction.

We now explain this result and show that it implies that a supergravity theory can live in at most eleven dimensions. The four dimensional result follows in an obvious way from the fact that in the massless case from each four component supercharge only two of the components act non-trivially on the physical states. Further, these two components form a Clifford algebra, one of which raises the helicity by 1/2 and one of which lowers the helicity by 1/2. Choosing the supercharges that raises the helicity to annihilate the vacuum, the physical states are given by the action of the remaining supercharges. If the supersymmetry algebra has N Majorana supercharges, the physical states are given by the action of N creation operators each of which lowers the helicity by 1/2. Consequently, if we take the vacuum to have helicity two the lowest helicity state in the representation will be $2 - \frac{N}{2}$. To have a supergravity theory we cannot have less than helicity -2 and hence the limit $N \leq 8$. Given a supergravity theory in a dimension greater than four we can reduce it in a trivial way by taking all the fields to be independent of the extra dimensions, to find a supergravity theory in four dimensions. However, the number of supercharges is unchanged in the reduction and so the maximal number is $4 \times 8 = 32$. Hence the supergravity in the higher dimension must arise in a dimension whose spinor representation has dimension 32 or less. Thus we find the desired result; eleven dimensions is the highest dimension in which a supergravity theory can exist. It also follows that any supergravity theory must have 32, or fewer supercharges and that the maximal, or largest, supergravity theory in a given dimension has 32 supercharges.

The irreducible representation, or particle content, of eleven dimension supergravity was found in reference [105] by analysing the irreducible representations of the supersymmetry algebra of equation (5.1). One could also deduce it by requiring that it reduce to the irreducible representation of the four-dimensional $N = 8$ supergravity theory given in section three. We now give a more intuitive argument for the particle content.

Eleven-dimensional supergravity must be invariant under general coordinate transformations (i.e. local translations) and local supersymmetry transformations. To achieve these symmetries it must possess the equivalent "local gauge" fields, the vielbein e_μ^a and the gravitino $\psi_{\mu\alpha}$. The latter transforms as $\delta\psi_{\mu\alpha} = \partial_\mu \eta_\alpha + \dots$ and so must be the same type of spinor as the supersymmetry parameter which in this case is a Majorana spinor.

For future use we now give the on-shell count of degrees of freedom of the graviton and gravitino in D dimensions. The relevant bosonic part of the little group which classifies the irreducible representation is $SO(D-2)$. The graviton encoded in e_μ^a is a second-rank symmetric traceless tensor of $SO(D-2)$ and as such has $\frac{1}{2}(D-2)(D-1) - 1$ degrees of freedom on-shell. The gravitino has $(D-3)cr$ real components. Here c is the dimension of the Clifford algebra in dimension $D-2$ and so is given by $c = 2^{\frac{D}{2}-1}$ if D is even and $c = 2^{\frac{(D-1)}{2}-1}$ if D is odd. The quantity r is 2,1 or 1/2 if $\psi_{\mu\alpha}$ is a Dirac, Majorana or Majorana-Weyl spinor respectively. In terms of little group representations, the gravitino is a vector spinor $\phi_i, i = 1, \dots, D-2$ which is γ-traceless $\gamma^i \phi_i = 0$,. A general vector spinor in $D-2$ dimensions has $(D-2)cr$ components, but the γ-trace subtracts another spinor's worth of components (i.e. cr).

For eleven dimensions we find that the graviton and gravitino have 44 and 128 degrees of freedom on-shell. However, in any supermultiplet of on-shell physical states the fermionic and bosonic degrees of freedom must be equal. Assuming that there are no further fermionic degrees of freedom we require another 84 bosonic on-shell degrees of freedom. If we take these to belong to an irreducible representation of $SO(9)$ then the unique solution would be a third rank anti-symmetric tensor. This can only arise from a third rank gauge field $A_{\mu_1\mu_2\mu_3}$ whose fourth rank gauge field $F_{\mu_1\mu_2\mu_3\mu_4} \equiv 4\partial_{[\mu_1}A_{\mu_2\mu_3\mu_4]}$, the anti-symmetry being with weight one. We note that in D dimensions a rank p anti-symmetric gauge field belongs to the anti-symmetric rank p tensor representation of $SO(D-2)$ and so has $\frac{(D-2)\ldots(D-p-1)}{p!}$ degrees of freedom on-shell.

The eleven-dimensional supergravity Lagrangian was constructed in reference [106] and is given by

$$L = -\frac{e}{4\kappa^2}R\big(\Omega(e,\psi)\big) - \frac{e}{48}F_{\mu_1\ldots\mu_4}F^{\mu_1\ldots\mu_4} - \frac{ie}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\varrho}D_\nu\left(\frac{1}{2}(\Omega+\hat{\Omega})\right)\psi_\varrho$$
$$+\frac{1}{192}e\kappa(\bar{\psi}_{\mu_1}\Gamma^{\mu_1\ldots\mu_6}\psi_{\mu_2} + 12\bar{\psi}^{\mu_3}\Gamma^{\mu_4\mu_5}\psi^{\mu_6})(F_{\mu_3\ldots\mu_6} + \hat{F}_{\mu_3\ldots\mu_6})$$
$$+\frac{2\kappa}{(12)^4}\epsilon^{\mu_1\ldots\mu_{11}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}A_{\mu_9\mu_{10}\mu_{11}} \tag{5.2}$$

where

$$F_{\mu_1\ldots\mu_4} = 4\partial_{[\mu_1}A_{\mu_2\mu_3\mu_4]}$$
$$\hat{F}_{\mu_1\ldots\mu_4} = F_{\mu_1\ldots\mu_4} - 3\bar{\psi}_{[\mu_1}\Gamma_{\mu_2\mu_3}\psi_{\mu_4]}, \tag{5.3}$$

and

$$\Omega_{\mu mn} = \hat{\Omega}_{\mu mn} - \frac{i}{4}\bar{\psi}_\nu\Gamma_{\mu mn}{}^{\nu\lambda}\psi_\lambda,$$
$$\hat{\Omega}_{\mu mn} = \Omega^0_{\mu mn}(e) + \frac{i}{2}(\bar{\psi}_\nu\Gamma_n\psi_m - \bar{\psi}_\nu\Gamma_m\psi_n + \bar{\psi}_n\Gamma_\mu\psi_m) \tag{5.4}$$

The symbol $\Omega^0_{\mu mn}(e)$ is the usual expression for the spin-connection in terms of the vielbein e_μ^n which can be found in section four.

It is invariant under the local supersymmetry transformations

$$\delta e_\mu^m = -i\kappa\bar{\epsilon}\Gamma^m\psi_\mu$$
$$\delta\psi_\mu = \frac{1}{\kappa}D_\mu(\hat{\Omega})\epsilon + \frac{i}{12^2}(\Gamma_\mu{}^{\nu_1\ldots\nu_4} - 8\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4})F_{\nu_1\ldots\nu_4}\epsilon \tag{5.5}$$
$$\delta A_{\mu_1\mu_2\mu_3} = \frac{3}{2}\bar{\epsilon}\Gamma_{[\mu_1\mu_2}\psi_{\mu_3]}$$

Although the result may at first sight look complicated most of the terms can be understood if one were to consider constructing the action using the Noether method. In this method, which we discussed in chapter four, we start from the linearized theory for the graviton, the gravitino and the gauge field $A_{\mu_1\mu_2\mu_3}$. The linearized action is bilinear in the fields and is invariant under a set of rigid supersymmetry transformations which are linear in the fields as well as the local Abelian transformations $\delta\psi_{\mu\alpha} = \partial_\mu\eta_\alpha$, $A_{\mu_1\mu_2\mu_3} = \partial_{[\mu_1}\Lambda_{\mu_2\mu_3]}$ as well as an appropriate analogous transformation for the linearised vielbein.

The linearised supersymmetry transformations are found by using dimensional analysis and closure of the linearised supersymmetry algebra. We now let the rigid supersymmetry parameter ϵ depend on space-time and identify the two spinor parameters by $\eta_\alpha = \frac{1}{\kappa}\epsilon_\alpha$. We know that the final result will be invariant under general coordinate transformations and so we may at each step in the Noether procedure insert the vielbein in all terms so as to ensure this invariance. Even at this stage in the Noether procedure we recover all the terms in the transformations in the fields given above in equation (5.5) except for some of the terms in the spin-connection $\hat{\Omega}$. For the action, we find all the terms except the last two terms and again some terms in the spin-connection of the gravitino. The action at this stage is not invariant under the now local supersymmetry transformations and as explained in chapter four we can gain invariance at order κ^0 by adding a term of the form $\bar{\psi}^{\mu\alpha} j_{\mu\alpha}$ where $j_{\mu\alpha}$ is the Noether current for the supersymmetry of the linearised theory. This term is none other than the second to last term in the above action. To gain invariance to order κ^1 we must cancel the variations of this second to last term under the supersymmetry transformation. This is achieved if we add the last term to the action. Hence even at this stage in the procedure we have accounted for essentially all the terms in the action and transformation laws. While one can pursue the Noether procedure to the end, to find the final form of the action and transformations laws, it is perhaps best to guess the final form of the connection and verify that the action is invariant and the local supersymmetry transformations close.

The eleven-dimensional action contains only one coupling constant, Newton's constant κ, which has the dimensions of $mass^{-\frac{9}{2}}$ and so defines a Planck mass m_p by $\kappa = m_p^{-\frac{9}{2}}$. We note that there are no scalars in the action whose expectation value could be used to define another coupling constant. If we scale the fields by $\psi_\mu \to \kappa^{-1}\psi_\mu$ and $A_{\mu_1\mu_2\mu_3} \to \kappa^{-1}A_{\mu_1\mu_2\mu_3}$ we find that all factors of κ drop out of the action except for a prefactor of κ^{-2} and all factors of κ drop out of the supersymmetry transformation laws. As such, when expressed in terms of these variables, the classical field equations do not contain κ. In fact, the value of the coupling constant κ has no physical meaning. One way to see this fact is to observe that if, after carrying out the above redefinitions, we Weyl-scale the fields by

$$e_\mu^m \to e^{-\alpha} e_\mu^m,\ \psi_\mu \to e^{-\frac{\alpha}{2}}\psi_\mu,\ \text{and,}\ A_{\mu_1\mu_2\mu_3} \to e^{-3\alpha}A_{\mu_1\mu_2\mu_3} \tag{5.6}$$

as well as scale the coupling constant by $\kappa \to e^{-\frac{9\alpha}{2}}\kappa$, we then find that the action is invariant. In deriving this result we used the equation

$$R(e^\tau e_\mu^m) = e^{-2\tau}(R(e_\mu^m) + 2(D-1)D^\mu D_\mu \tau + (D-1)(D-2)g^{\mu\nu}\partial_\mu\tau\partial_\nu\tau) \tag{5.7}$$

where R is the Ricci tensor in dimension D. Of course this is not a symmetry of the action in the usual sense as we have rescaled the coupling constant. However, as the coupling constant only occurs as a prefactor multiplying the entire action, it is a symmetry of the classical equations of motion. Hence, we can only specify the value of the constant κ with respect to a particular metric. These transformations are not a symmetry of the quantum theory where, in the path integral, the prefactor of κ^{-2} which multiplies all terms in the action becomes $\kappa^{-2}\hbar$ where $\hbar$ is Planck's constant. In this case we can absorb the rescaling either in κ or $\hbar$. The above Weyl scaling of the vielbein implies that the proper distance d^2s

scales as $d^2s \to e^{-\alpha}d^2s$. Taking the scaling to be absorbed by $\hbar$ we find that $\hbar \to e^{-9\alpha}\hbar$ and so small proper distance corresponds to small $\hbar$. Put another small κ, or equivalently $\hbar$, is the same as working at small proper distance.

Although we have constructed the supergravity theories in ten and eleven dimensions in this section we have omitted many of the significant formulae, such as the supersymmetry transformations. Since these are contained in the original papers [105],[106],[107][108][109] on the subject, we have used the same metric as in these papers, that is the tangent space metric $\eta_{mn} = diag(+1, -1, \dots, -1)$. Since many practitioners nowadays prefer the other signature we now give the rules to change to the tangent space metric which is mainly plus i.e. $\eta_{nm} = diag(-1, 1, \dots, 1)$. To go to the latter metric we must take $\eta_{nm} \to -\eta_{nm}$, $\gamma^a \to i\gamma^a$, $e_\mu^n \to e_\mu^n$. Using this rule it is easy to carry out the change. One finds, for example, that $g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma \to -g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma$ and $R \to -R$.

6. IIA and IIB Supergravity

In this section we give the supergravity theories in ten dimensions which have the maximal supersymmetry. There are two such theories, called IIA and IIB, and they are the effective low energy actions for the IIA and IIB string theories respectively. We describe the properties of these supergravity theories that are relavent for the string theories and play an important role in string duality.

6.1 Supergravity Theories in Ten Dimensions

In ten dimensions the non-trivial representation of the Clifford algebra has dimension $2^{\frac{10}{2}} = 32$. As we found in section one the matrices B and C associated with the complex conjugation and transpose of the γ-matrices obey the properties $B^T = B$ and $C = -C^T$ (i.e. $\epsilon = 1$). The properties of the γ matrices are given in equations (1.2.16) and (1.2.17). In ten dimensions a Majorana spinor has 32 real components, however, we can also have Majorana-Weyl spinors and these only have 16 real components.

The supersymmetry algebra for a single Majorana-Weyl supercharge which has 16 real components is given in equation (1.5.4). There are two supersymmetric theories that are based on this algebra, the so called $N = 1$ Yang-Mills theory [119] and the $N = 1$ supergravity theory [120] which is more often called type I supergravity. The coupling between the two theories was given in references [121].

In the discussion at the beginning of section five we found that a supergravity theory can be based on a supersymmetry algebra with 32 or fewer supercharges. If we consider the supersymmetry algebra with a 32 component Majorana spinor we find the IIA supergravity theory which was constructed in references [107],[108] and [109]. Clearly, when we decomposed the Majorana spinor into Majorana-Weyl spinors we get two such spinors: one of each chirality. The other ten-dimensional supersymmetry algebra with 32 supercharges has two Majorana-Weyl spinors of the same chirality and the corresponding supergravity is IIB supergravity. This theory was constructed in references [110],[112] and [111]. Unlike the other supergravity theories in ten dimensions IIB supergravity has an internal symmetry which is the group $SL(2, \mathbf{R})$ [110].

Upon reduction of eleven-dimensional supergravity to ten dimensions by taking the eleventh dimension to be a circle we will obtain a ten-dimensional theory that possesses a supersymmetry algebra based on a 32 component Majorana supercharge which decomposes into two Majorana-Weyl spinors of opposite Weyl chiralities. Thus the resulting theory can only be IIA supergravity. Indeed, this was how IIA supergravity was found [107],[108] [109].

The importance of the IIA and IIB supergravity theories, which was the main motivation for their construction, is that they are the low energy effective theories of the corresponding IIA and IIB closed string theories in ten dimensions. Type I supergravity coupled to $N=1$ Yang-Mills theory is the effective action for the low energy limit of the $E_8 \otimes E_8$ or $SO(32)$ heterotic string. All these supergravity theories were constructed at a time when string theory was deeply unpopular and when it did become fashionable little interest was taken in supergravity theories. However, they now form the basis for many of the discussions of duality in string theories.

6.2 IIA Supergravity

As we have mentioned, this theory is based on a supersymmetry algebra which contains one Majorana spinor Q_α $\alpha = 1, \ldots, 32$. Following the discussion of section 2.5, we conclude that the anti-commutator of two supersymmetry generators can have central charges of rank p where $p = 1, 2$; mod 4 and so the corresponding anti-commutator is given by [117]

$$\{Q_\alpha,\ Q_\beta\} = (\gamma^m C^{-1})_{\alpha\beta} P_m + (\gamma^{mn} C^{-1})_{\alpha\beta} Z_{mn} + (\gamma^{m_1 \ldots m_5} C^{-1})_{\alpha\beta} Z_{m_1 \ldots m_5}$$

$$+ (\gamma^{m_1 \ldots m_4} \gamma_{11} C^{-1})_{\alpha\beta} Z_{m_1 \ldots m_4} + (\gamma^m \gamma_{11} C^{-1})_{\alpha\beta} Z_m + (\gamma_{11} C^{-1})_{\alpha\beta} Z \tag{6.2.1}$$

We need only take $p \leq 5$ since we may use the equation

$$\gamma^{m_1 \ldots m_s} \gamma_{11} = \frac{1}{(10-s-1)!} \eta \epsilon^{m_1 \ldots m_s m_{s+1} \ldots m_{10}} \gamma_{m_{s+1} \ldots m_{10}}, \tag{6.2.2}$$

where $\eta = \pm 1$, to eliminate terms with $p \geq 6$.

The IIA theory was obtained in references [107],[108] and [109] from the $D=11$ theory by compactification and this is the method of construction we now follow. We consider the eleven-dimensional supergravity of equation (5.2) and take the eleventh dimension to be a circle S^1 of radius R. To be precise, we take the eleventh coordinate x^{10} to be such that $x^{10\prime} \sim x^{10}$ if $x^{10\prime} = x^{10} + 2\pi n R,\ n \in \mathbf{Z}$ where $2\pi R$ parameterizes the range of x^{10}. We will also write x^{10} as $x^{10} = \theta R$ for $0 \leq \theta < 2\pi$. We adopt the convention that hatted indices run over all eleven dimensional indices, but unhatted indices only run over the ten dimensional indices, for example $\hat{\mu}, \hat{\nu} = 0, 1, \ldots, 10$ while $\mu, \nu = 0, 1, \ldots, 9$. Given any of the fields in the eleven-dimensional supergravity we can take its Fourier transform on x^{10}. In particular, if ϕ represents any of these fields whose Lorentz and possible spinor indices are suppressed, we find that

$$\phi(x^\mu, x^{10}) = \phi(x^\mu) + \sum_{n, n \neq 0} e^{in\theta} \phi_n(x^\mu) \tag{6.2.3}$$

Thus from each particle in ten dimensions we find an infinite number of particles in ten dimensions. The non-zero modes (i.e. $n \neq 0$), however, will lead to massive particles whose masses are given by their momentum in the eleventh direction. Such massive particles are called Kaluza-Klein particles. In the limit when the radius of the circle is large these particles become infinitely massive and can be neglected, whereupon one is left with a finite set of massless particles which form a supergravity theory. Discarding the massive particles can be achieved by taking all the eleven-dimensional fields to be independent of x^{10} and this we now do.

This reduction proceeds in the following generic manner

$$\begin{array}{cccc} D=11 & e_{\hat{\mu}}{}^{\hat{m}} & A_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3} & \psi_{\hat{\mu}\alpha} \\ & \downarrow & \downarrow & \downarrow \\ D=10 & e_{\mu}{}^{m}, B_\mu, \phi & A_{\mu_1\mu_2\mu_3}, A_{\mu_1\mu_2} & \psi_{\mu\alpha}, \lambda_\alpha \end{array} \tag{6.2.4}$$

While one can reduce the fields in many ways only some definitions of the ten-dimensional fields will lead to a final result which is in the generic form in which a supergravity theory is usually written. The three-form gauge field reduces in an obvious way $A_{\mu_1\mu_2\mu_3} = A_{\mu_1\mu_2\mu_3}$, $A_{\mu_1\mu_2} = A_{\mu_1\mu_2 10}$ where $\mu_1, \mu_2\mu_3 = 0, 1 \ldots 9$. However, the useful reductions for the other fields are more subtle; the vielbein takes the form

$$e_{\hat{\mu}}{}^{\hat{m}} = \begin{pmatrix} e^{-\frac{1}{12}\sigma} e_\mu{}^m & 2e^{\frac{2}{3}\sigma} B_\mu \\ 0 & e^{\frac{2}{3}\sigma} \end{pmatrix}, \ (e^{-1})_{\hat{m}}^{\ \hat{\mu}} = \begin{pmatrix} e^{\frac{1}{12}\sigma} e_{\ m}^{\ \mu} & -2e^{\frac{1}{12}\sigma} B_\nu (e^{-1})_m^{\ \nu} \\ 0 & e^{-\frac{2}{3}\sigma} \end{pmatrix} \tag{6.2.5}$$

while the eleven-dimensional gravitino becomes

$$\psi_{\hat{m}} = (e^{-\frac{1}{24}\sigma} e^\mu_m \psi'_\mu, \ \frac{2}{3}\sqrt{2} e^{\frac{17}{24}\sigma}\lambda) \tag{6.2.6}$$

where $\psi'_\mu = e^{-\frac{1}{24}\sigma}(\psi_\mu - \sqrt{\frac{1}{72}}\Gamma_\mu \Gamma^{11}\lambda) - \sqrt{\frac{32}{9}} e^{\frac{3}{4}\sigma} B_\mu \lambda$ and $\Gamma^{11} = i\Gamma^1 \ldots \Gamma^{10}$. The above formulae and those below are related to those of reference [107] by carrying out the transformation $\sigma \to \frac{2}{3}\sigma$ on the latter.

The field B_μ is a gauge field whose gauge transformation has a parameter that came from the general coordinate transformations with parameter ξ^{10} in the eleven-dimensional theory. The component $e^m_{\hat{\mu}=10}$ of the vielbein can be chosen to be zero as a result of a local Lorentz transformation $w^m_{\ 10}$. The strange factors involving e^σ and other redefinitions are required in order to get the usual Einstein and spinor kinetic energy terms. The ten-dimensional Newtonian coupling constant κ is given in terms of the eleven-dimensional Newtonian constant κ_{11} by $\kappa^2 = \frac{(\kappa_{11})^2}{2\pi R}$ and has the dimensions of $(mass)^{-8}$. This equation follows from examining the κ that results from the-dimensional reduction. We have set $\kappa = 1$ in this section.

The resulting ten-dimensional IIA supergravity theory is given by

$$L = L^B + L^F \tag{6.2.7}$$

where the first term contains all the terms which are independent of the fermions and the second term is the remainder. The bosonic part is given by [107],[108],[109]

$$\begin{aligned} L^B = & -eR\big(w(e)\big) - \frac{1}{12} e e^{\frac{\sigma}{2}} F'_{\mu_1\ldots\mu_4} F'^{\mu_1\ldots\mu_4} + \frac{1}{3} e e^{-\sigma} F_{\mu_1\ldots\mu_3} F^{\mu_1\ldots\mu_3} \\ & - e e^{\frac{3}{2}\sigma} F_{\mu_1\mu_2} F^{\mu_1\mu_2} + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma \\ & + \frac{1}{2\cdot(12)^2} \epsilon^{\mu_1\ldots\mu_{10}} F_{\mu_1\ldots\mu_4} F_{\mu_5\ldots\mu_8} A_{\mu_9\mu_{10}} \end{aligned} \tag{6.2.8}$$

where

$$F_{\mu_1\mu_2} = 2\partial_{[\mu_1}B_{\mu_2]} \tag{6.2.9}$$

$$F_{\mu_1\mu_2\mu_3} = 3\partial_{[\mu_1}A_{\mu_2\mu_3]} \tag{6.2.10}$$

$$F'_{\mu_1\ldots\mu_4} = 4\partial_{[\mu_1}A'_{\ldots\mu_4]} + 12A_{[\mu_1\mu_2}G_{\mu_3\mu_4]} \tag{6.2.11}$$

In the last definition we have used the field $A'_{\mu_1\mu-2\mu_3} = A_{\mu_1\mu-2\mu_3} - 6B_{[\mu_1}A_{\mu_2\mu_3]}$ which is invariant under the gauge transformation with parameter ξ^{10}. The fermionic part of the Lagrangian is much more complicated and the first two terms are

$$L^F = -\frac{i}{2}e\psi_{\mu_1}\Gamma^{\mu_1\mu_2\mu_3}D_{\mu_2}\psi_{\mu_3} + \frac{i}{2}e\bar{\lambda}\Gamma^{\mu}D_{\mu}\lambda + \ldots \tag{6.2.12}$$

The transformations of the fields can be deduced in a similar fashion from the transformation of eleven-dimensional fields of equation (5.1). For example, the veilbein and dilaton transform as

$$\delta e_{\mu}{}^{m} = -i\bar{\epsilon}\Gamma^{m}\psi_{\mu},\ \delta\sigma = \sqrt{2}i\bar{\lambda}\Gamma^{11}\epsilon \tag{6.2.13}$$

where ϵ is the suitably defined parameter of local supersymmetry transformations. We refer the reader to references [107],[108] and [109] for the transformations of the other fields and the terms in the fermionic part of the action.

The IIA action has an $SO(1,1)$ invariance with parameter c that transforms the fields as

$$\sigma \to \sigma + c,\ B_{\mu} \to e^{-\frac{3}{4}c}B_{\mu},\ A_{\mu\nu} \to e^{\frac{1}{2}c}A_{\mu\nu},\ A_{\mu\nu\rho} \to e^{-\frac{1}{4}c}A_{\mu\nu\rho} \tag{6.2.14}$$

while the vielbein in ten dimensions is inert. This symmetry has its origin in the eleven-dimensional theory and in particular the Weyl scalings of equation (5.6) given in the previous section. Although these are not a symmetry of the action in eleven dimensions we can convert them into a symmetry of the action in ten dimensions provided we combine them with a diffeomorphism on x^{10}, in particular the diffeomorphism $x^{10} \to e^{-9\alpha}x^{10}$. To keep the range of x^{10} the same we also scale R by $R \to e^{-9\alpha}R$. This diffeomorphism is a symmetry of the eleven dimensional theory and from the active viewpoint transforms $\int dx^{10} \to e^{-9\alpha}\int dx^{10}$ and the Lagrangian L in eleven-dimensions Lagrangian to $L \to e^{9\alpha}L$. The theory in ten dimensions is obtained by substituting the field expansion of equation (6.2.3) into the action in eleven-dimensions. However, the effect of the $\int dx^{10}$ is just to extract the part of the Lagrangian L in eleven dimensions that is independent of x^{10}, which then becomes the Lagrangain in ten dimensions and gives a factor of $2\pi R$. The latter is then combined with the factor κ_{11}^{-2} to define the ten-dimensional Newtonian coupling constant $\kappa_{10}^{-2} = 2\pi R\kappa_{11}^{-2}$. Clearly, the coupling constant κ_{10} in ten dimensions is inert under the combined transformation as $R \to e^{-9\alpha}R$ under the diffeomorphism and $\kappa_{11}^{-2} \to e^{9\alpha}\kappa_{11}^{-2}$ under the Weyl scaling. Similarly the action in ten dimensions is inert as it scales by $e^{9\alpha}$ under the diffeomorphism and $e^{-9\alpha}$ under the Weyl scaling. Thus we have found that dimensional reduction has transformed a symmetry of the equations of motion into a symmetry of the action.

An alternative way of writing the IIA Lagrangian is to use the so-called string metric $g^s_{\mu\nu}$. This metric is the one that occurs in the sigma model approach to string theory which starts with the sigma model action in two dimensions:

$$-\frac{1}{4\pi\alpha'}\int d^2\xi(\sqrt{-g}g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu g^s_{\mu\nu}+\epsilon^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu A_{\mu\nu})+\frac{1}{4\pi}\int d^2\xi\sigma R^{(2)} \qquad (6.2.15)$$

where $R^{(2)}$ is the curvature scalar in two dimensions . The fields $A_{\mu\nu}$ and σ are the anti-symmetric tensor gauge field and dilaton which appear in the massless string spectrum and occur in the IIA action given above. The constant α' has the dimensions of $(mass)^{-2}$ and defines the mass scale m_s of the string by $m_s^2=\frac{1}{4\pi\alpha'}$. The combination in front of the first two terms of the string action is often called the string tension T (i.e. $T=\frac{1}{4\pi\alpha'}$). One recovers the tree-level string equations and thus at lowest order in α' we find the supergravity equations of motion, by demanding conformal invariance. The corresponding string vielbein $e^{s\ m}_{\mu}$ is related to the above vielbein by $e^{s\ m}_{\mu}=e^{\frac{1}{4}\sigma}e_\mu^{\ m}$.

The last term in the above sigma model action, for constant σ, takes the form $<\sigma>\chi$ where χ is the Euler number and is given by $\chi=\frac{1}{4\pi}\int d^2\xi R^{(2)}$. For a closed Riemann surface of genus g it is given by $\chi=2-2g$.

Making this change in the bosonic part of the IIA Lagrangian of equation (6.2.8) and dropping the "s" superscript on the string vielbein we find that the Lagrangian becomes

$$L_B=ee^{-2\sigma}\{-R+4\partial_\mu\sigma\partial^\mu\sigma-\frac{1}{3}F_{\mu_1\mu_2\mu_3}F^{\mu_1\mu_2\mu_3}\}$$

$$+\left\{-\frac{1}{12}eF'_{\mu_1\ldots\mu_4}F'^{\mu_1\ldots\mu_4}-eF_{\mu_1\mu_2}F^{\mu_1\mu_2}\right\}+\frac{2}{12^2}\epsilon^{\mu_1\ldots\mu_{10}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}A_{\mu_9\mu_{10}} \qquad (6.2.16)$$

As we have mentioned, the IIA action is the lower energy limit of the IIA string theory, which is obtained as the string tension goes to zero (i.e. $\alpha'\to\infty$). In this limit one is left with only the massless particles of the IIA supergravity theory. It will be very useful to know how these particles arise in the IIA string. This closed string theory in its formulation with manifest world surface supersymmetry, that is the Neveu-Schwarz-Rammond formulation [7], has four sectors; the $NS\otimes NS$, the $R\otimes R$, $R\otimes NS$ and the $NS\otimes R$ corresponding to the different boundary conditions that can be adopted for the two dimensional spinor in the theory. Clearly, the $NS\otimes R$ and $R\otimes NS$ sectors contain the fermions while the $NS\otimes NS$ and the $R\otimes R$ sectors contain the bosons. It is straightforward to solve the physical state conditions in these sectors and one finds that the bosonic fields of the IIA supergravity arise as

$$\underbrace{e_\mu^a, A_{\mu\nu},\sigma;}_{NS\otimes NS}\quad \underbrace{A_{\mu\nu\varrho}, B_\mu;}_{R\otimes R}\quad \underbrace{\psi_{\mu\alpha},\lambda_\alpha}_{NS\otimes R\ and\ R\otimes NS} \qquad (6.2.17)$$

Looking at the IIA Lagrangian in the string frame of equation (6.2.16) we find that all the fields that arise in the $NS\otimes NS$ sector occur in a different way from those in the $R\otimes R$ sector; While the former have a factor of $e^{-2\sigma}$ the latter do not have such a factor. We will find the same phenomenon for the IIB string.

The IIA supergravity has two parameters. It has the Newtonian coupling constant κ, which we have suppressed, and the parameter $< e^\sigma >$. The IIA string also has two parameters the string tension T, or equivalently the string mass scale m_s, and the string coupling constant g_s. Since the low energy effective action of the string is IIA supergravity these two sets of parameters must be related.

We first consider how the parameters arise in the string theory. In a second quantized formulation of string theory, one finds that the action can be written in a way where the string coupling only occurs as a prefactor of g_s^{-2}. Examples of such formulations are the light-cone gauge action or the gauge covariant action. The parameter α' only occurs in these formulations through the masses of the the particles or equivalently the L_n operators that occur in these formulations. In the path integral formulation, the action becomes multiplied by $\hbar^{-1}$ and so we find that $\hbar$ and g_s only occur in the combination $\hbar g_s^2$. This situation is identical to the way in which the gauge coupling occurs in Yang-Mills theory. The power of Planck's constant measures the number of loops. Indeed if we have any Feynman graph with n loops, I propagators and V vertices, the power of $\hbar$ associated with an n-loop graph is given by $\hbar^{(I-V)}$, since each vertex carries a power of $\hbar$ and each propagator an inverse power of $\hbar$. Using the topological relation $n = I - V + 1$, the power of $\hbar$ for a n-loop diagram becomes $\hbar^{(n-1)}$. Our previous discussion then implies that each n-loop diagram has a power $g_s^{(2n-2)}$ associated with it.

Now let us examine how the parameters arise in the effective action. The first quantized string action in two dimensions of equation (6.2.15) contains the dilaton σ multiplied by the Euler number χ of the Riemann surface. A surface of genus n corresponds to a n-loop string amplitude and so in the path integral of this action one finds a factor of $e^{(2n-2)<\sigma>}$. As a result, we conclude that the IIA string coupling and the expectation value of the dilaton are related by $g_s =< e^\sigma >$ Since κ has the dimension of $(mass)^{-4}$ it must be proportional to $(\alpha')^2$ and is given by the relation $\kappa = (\alpha')^2 e^{<\sigma>}$.

We now finish our discussion of the IIA supergravity by reiterating some of the above features that will be useful for discussions of string duality. The IIA supergravity theory has the gauge fields $\sigma, B_\mu,\ A_{\mu\nu}$, and $A_{\mu\nu\varrho}$. This means the IIA theory has gauge fields of rank q where $q = 1, 2, 3$ and these have corresponding field strengths of rank $q + 1 = 2, 3, 4$. Given a field strength $F_{\mu_1 \ldots \mu_q}$ of rank $q + 1$ we can define a dual field strength by $F_{\mu_1 \ldots \mu_(D-1-q)} = \frac{1}{(q+1)!}\epsilon_{\mu_1 \ldots \mu_{10}} F^{\mu_{D-q} \cdots \mu_{10}}$. Hence the duals of the above field strengths are forms of rank $q = 6, 7, 8$. When the original field strengths are on-shell we can, at least at the linearized level write the dual field strengths in terms of dual gauge fields of ranks 5,6 and 7. Hence the IIA theory has gauge fields of ranks p=1,2,3,5,6,7 if we include the dual gauge fields as well as the original ones. It is instructive to list the above gauge fields according to the string sector in which they arise. In the $NS \otimes NS$ we find gauge fields of ranks 2 and 6 while in the $R \otimes R$ sector we find gauge fields of ranks 1,3,5 and 7. We observe that classifying the gauge fields according to the different sectors splits them into fields of odd and even rank.

From equation (6.2.5) we read off the component of the vielbein associated with the circle to be $e_{10}^{\ 10} = e^{\frac{2}{3}<\sigma>}$ with corresponding metric $g_{10\ 10} = e^{\frac{4}{3}<\sigma>}$. The parameter R introduced into the defining range of the variable x^{10} has no physical meaning as it only parametrizes the range of x^{10}. However, from the metric we can compute the radius R_{11}

of the circle in the eleventh dimension. We find that $R_{11} = Re^{\frac{2}{3}<\sigma>}$. We recall that the string coupling is given by $g_s = e^{<\sigma>}$ and as a result we find that

$$g_s = \left(\frac{R_{11}}{R}\right)^{\frac{3}{2}} \tag{6.2.18}$$

The above relationship between the radius R_{11} of compactification of the eleven-dimensional theory and the IIA string coupling constant implies in particular that as $R_{11} \to \infty$ we find that $g_s \to \infty$. Thus in the strong coupling limit of the IIA string the radius of the circle of compactification becomes infinite suggesting that the theory decompactifies [124].

We now consider some properties of the Kaluza-Klein modes which we have so far ignored in the reduction from eleven dimensions. Their massess are given by the action of $(e^{-1})_{11}{}^{\hat{\mu}}\partial_{\hat{\mu}} = e^{-\frac{2}{3}\sigma}\partial_{11}$ where we have used equation (6.2.5). Examining the expansion of equation (6.2.3) we find that the masses of Kaluza-Klein particles are given by $n\frac{e^{-\frac{2}{3}<\sigma>}}{R}$ for integer n. However, using the relationship between R and R_{11} given just above we find that the masses of the Kaluza-Klein particles are given by $\frac{n}{R_{11}}$. The gauge field B_μ which originated from the eleven-dimensional metric couples to the Kaluza-Klein particles in a way which is governed by the derivative

$$(e^{-1})_m{}^{\hat{\mu}}\partial_{\hat{\mu}} = e^{\frac{1}{12}\sigma}(e^{-1})_m{}^{\mu}(\partial_\mu - 2B_\mu\partial_{11})$$

where we have again used equation (6.2.5). As such, we find that the Kaluza-Klein particles have charges given by $\frac{2n}{R}$ for integer n. From the IIA string perspective, the B_μ field is in the $R \otimes R$ sector and so the Kaluza-Klein particles couple with these charges to the $R \otimes R$ sector. It turns out that the IIA supergravity possess solitonic particle solutions that have precisely the masses and charges of the Kaluza-Klein particles [123]. Thus the IIA supergravity and so in effect the IIA string knows about all the particle content of the theory in eleven dimensions and not only the massless modes that arise after the compactification.

It is these observations that underlie the conjecture [123],[124] that the strong coupling limit of the IIA string theory is an eleven-dimensional theory, called M-theory, whose low energy limit is eleven-dimensional supergravity.

6.3 IIB Supergravity

The IIB supergravity was found in references [110],[111] and [112] using two different methods. In reference [110], a variant, [126], of the Noether method was used: rather than working with an action and transformation laws, one can just use the transformation laws. One begins with the rigid supersymmetry transformations and local Abelian transformations of the linearised theory. Letting the supersymmetry parameter become space-time dependent, the transformations laws no longer close; however we may still close the supersymmetry algebra order-by-order in κ by adding terms to the transformation laws provided we also identify the now local spinor parameter of the supersymmetry transformation with the spinor parameter that occurs in the local Abelian transformation of the gravitino. In this way the supersymmetry transformations laws of the IIB theory and the fact that the scalars belong to the coset $SU(1,1)/U(1)$ were found [110]. Using the fact that the transformations laws only close on-shell this work was extended in reference [112] to find the equations of motion in the absence of fermions. In the independent work of reference [111] the full equations of motion in superspace and x-space were found using the on-shell superspace techniques of section 4.2. The third-order terms of the IIB theory were constructed for the light-cone gauge Hamiltonian in reference [125].

The strategy behind these calculations is explained in chapter 4 and although the ideas are straightforward the calculations themselves are technically complicated to the extend that they will not be reproduced here. Nonetheless we will describe the essential features of the IIB theory so that the reader will grasp some of the ideas involved and gain a feel for the IIB theory itself.

6.3.1 The Algebra

The IIB supergravity is based on a supersymmetry algebra whose two supercharges Q^i_α, $i = 1, 2$; $\alpha = 1, \ldots, 32$ are Majorana-Weyl spinors of the same chirality. They therefore obey the conditions

$$(Q^i_\alpha)^* = Q^i_\alpha, \quad \Gamma_{11} Q^i = Q^i \tag{6.3.1}$$

The supersymmetry algebra is given by

$$\{Q^i_\alpha, Q^j_\beta\} = (\gamma^\mu C^{-1})_{\alpha\beta}\delta^{ij} P_\mu + \ldots \tag{6.3.2}$$

where $+\ldots$ denote terms with central charges whose form the reader may readily find by following the discussion at the end of section 1.5.

It is more useful to work instead with the complex Weyl supercharges $Q_\alpha = Q^1_\alpha + iQ^2_\alpha$, $\bar{Q}_\alpha = Q^1_\alpha - iQ^2_\alpha = (Q_\alpha)^*$ The supersymmetry algebra also contains a $U(1)$ generator denoted R ($R^\dagger = -R$) which acts on the supercharges as

$$[Q_\alpha,\ R] = iQ_\alpha,\ [\bar{Q}_\alpha,\ R] = -i\bar{Q}_\alpha \tag{6.3.3}$$

6.3.2 The Particle Content

The field content of the IIB theory is

$$e_\mu{}^m,\ A_{\mu\nu},\ a,\ B_{\mu\nu\rho\kappa},\ \psi_{\mu\alpha},\ \lambda_\alpha \tag{6.3.4}$$

The fields $A_{\mu\nu}$ and a are complex while the gauge field $B_{\mu\nu\rho\kappa}$ is real. The spinors are complex Weyl spinors; the graviton $\psi_{\mu\alpha}$ is of the opposite chirality to λ_α, but has the same chirality as the supersymmetry parameter ϵ_α. Recalling our discussion above equation (5.2) we find that these fields lead to 35, 56, 2, 35 ,112 and 16 on-shell degrees of freedom respectively. The gauge field $B_{\mu_1\mu_2\mu_3\mu_4}$ defines the linearised five-rank field strength $g_{\mu_1\mu_2\mu_3\mu_4\mu_5} \equiv 5\partial_{[\mu_1} B_{\mu_2\mu_3\mu_4\mu_5]}$ which satisfies a self-duality condition. At the linearized level this self-duality condition is given by

$$g_{\mu_1\mu_2\mu_3\mu_4\mu_5} = \frac{1}{5!}\epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5\nu_1\nu_2\nu_3\nu_4\nu_5} g^{\nu_1\nu_2\nu_3\nu_4\nu_5} \equiv {}^*g_{\mu_1\mu_2\mu_3\mu_4\mu_5} \tag{6.3.5}$$

Without the self-duality condition this gauge field corresponds to a particle that belongs to the fourth-rank totally anti-symmetric representation of the little group $SO(8)$. The self-duality condition above corresponds to the constraint that this representation is self-dual and hence the 35 degrees of freedom given above. Thus the supermultiplet of IIB supergravity has 128 bosonic degrees of freedom and 128 fermionic degrees of freedom on-shell.

Most of the fields of equation (6.3.4) transform under the $U(1)$ transformations; their R weights are 0,2,4,0,1 and 3 respectively. Clearly, real fields must have R weight zero and the gravitino must have the opposite R weight as the supercharge Q_α since $\delta\psi_\mu = \partial_\mu \epsilon$.

Following the pattern of the standard action for gauge fields one may be tempted to use the linearised action

$$\int d^{10}x \quad g_{\nu_1\nu_2\nu_3\nu_4\nu_5} g^{\nu_1\nu_2\nu_3\nu_4\nu_5} \tag{6.3.6}$$

for the fourth-rank gauge field. However, if g is the five-form associated with the gauge field, the above action is given by $\int g \wedge {}^*g = \int g \wedge g = 0$. This discussion can be rephrased without using forms as follows: using the self-duality condition we can rewrite one of the field strengths in terms of ${}^*g$; swopping the indices on the ϵ symbol such that the last five indices are at the beginning, we incur a minus sign; using the self-duality condition once more we again find that the above action vanishes as its negative.

Clearly this result holds for any rank $2n+1$ self-dual gauge field strength in a space-time of dimension $4n+2$, $n \in$ **Z**. Indeed there is no simple action for the fourth-rank gauge field and so for the IIB theory itself. Although some actions have been suggested there are reasons to believe that they do not correctly capture all the physics of the theory. As a result we will content ourselves with deriving the equations of motion.

The IIB supergravity is the theory that describes the effective action of the low energy limit of the IIB string. The massless fields in the IIB string being those that occur in the IIB supergravity. To find the massless fields we must examine the physical state conditions for the IIB string which, being a closed superstring, has in the Neveu-Schwarz- Rammond formulation, the usual $NS\otimes NS$, $R\otimes R$, $R\otimes NS$ and $NS\otimes R$ corresponding to the possible boundary conditions for the two dimensional spinor in the theory. Clearly, the last two sectors contains the fermions while the $NS\otimes NS$ and the $R\otimes R$ sectors contain the bosons. The bosons arise as

$$\underbrace{e^a_\mu,\ A^1_{\mu\nu},\ \sigma}_{NS\otimes NS} \quad \underbrace{A^2_{\mu\nu},\ l,\ B_{\mu\nu\rho\tau}}_{R\otimes R} \tag{6.3.7}$$

where $A^1_{\mu\nu}$ and $A^2_{\mu\nu}$, are real fields which make up the complex fields $A_{\mu\nu}$, and we replace the complex scalar a by two real fields l and σ. The precise way in which these decompositions are defined will be specified later. The fact that the two rank-two gauge fields are split between the $NS\otimes NS$ and $R\otimes R$ sectors has important consequences for discussions of string duality. Since the physical state condition in the $NS\otimes NS$ sector are exactly the same as in the IIA string we should not be surprised to find that this sector contains exactly the same bosonic field content.

6.3.3 The Scalars

The gauge fields, the graviton and the gravitini possess gauge transformations as a result of which they can only occur in gauge invariant quantities, which in the sense of section 4, have geometric dimensions greater than zero. At first sight, this is not the case for the scalars of the theory since they have geometric dimension zero. As we explained in section, dimensional analysis plays an important role in the construction of supergravity theories and as such it might seem that the role of the scalars in the theory is difficult to determine. Fortunately, however, the scalars belong [110] to the coset space $\frac{SU(1,1)}{U(1)}$ and as a consequence the way they can can occur in the theory is strongly constrained.

The use of coset spaces to describe scalar fields was described in reference [127]. Since it is just as simple to describe the general theory [127] we will give the construction for a general coset space. Let G be any group, H one of its subgroups and denote the coset space by G/H. Let us consider any space-time dependent $\mathcal{V} \in G$ which is taken to transform as

$$\mathcal{V} \ \rightarrow \ g\mathcal{V}h \tag{6.3.8}$$

where $h \in H$ is a local (i.e. space-time dependent) transformation and $g \in G$ is a rigid transformation. We may use the local H transformations to gauge away $\dim H$ scalar fields leaving $\dim G - \dim H$ scalar fields in $\mathcal{V}$. The object $\Omega_\mu \equiv \mathcal{V}^{-1}\partial_\mu \mathcal{V}$ belongs to the Lie algebra of G and so can be written in the form

$$\Omega_\mu = \mathcal{V}^{-1}\partial_\mu \mathcal{V} = f_\mu^a T_a + w_\mu^i H_i \tag{6.3.9}$$

where H_i are generators of H and T_a are the remaining generators of G. The object ω_μ is invariant under the rigid transformations $g \in \ G$, but transforms under local H transformations as

$$\omega_\mu \ \rightarrow \ h^{-1}\partial_\mu h + h^{-1}\omega_\mu h \tag{6.3.10}$$

The theory simplifies if we restrict ourselves to reductive cosets which are those for which the commutator $[T_a, \ H_i]$ can be written in terms of only the coset generators T_a. In this case the above transformation rule implies that $f_\mu \equiv f_\mu^a T_a \rightarrow h^{-1} f_\mu h$ and $w_\mu \equiv w_\mu^i H_i \rightarrow h^{-1} w_\mu h + h^{-1}\partial_\mu h$. We can think of f_μ^a as a vielbein on G/H defining a set of preferred frames and w_μ^i as the connection associated with local H transformations.

An invariant Lagrangian is given by

$$\eta^{\mu\nu} Tr(f_\mu f_\nu) \tag{6.3.11}$$

The corresponding equation of motion is given by

$$D_\mu f_a^\mu = 0 \tag{6.3.12}$$

where we have introduced the covariant derivative $D_\mu f_\nu \equiv \partial_\mu f_\nu + [w_\mu, \ f_\nu]$. It is straightforward to verify that the above Lagrangian and equation of motion are invariant under both local H transformations and rigid G transformations.

Let us work out the above expressions for the case of interest, namely for $G = SU(1,1)$ and $H = U(1)$. The group $G = SU(1,1)$ is the set of two by two matrices g of determinant one which acts on the column vector $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ by $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to g \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ in such a way as to preserve $|z_1|^2 - |z_2|^2$. The most general element of $SU(1,1)$ can be written in the form

$$U = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix} \tag{6.3.13}$$

subject to $uu^* - vv^* = 1$. An infinitesimal element of $G = SU(1,1)$ can therefore be written in the form $g = I + A$ where A is given by

$$A = -2i\hat{a}\sigma_3 + b_1\sigma_1 - b_2\sigma_2 = \begin{pmatrix} -2i\hat{a} & b \\ b^* & +2i\hat{a} \end{pmatrix} \tag{6.3.14}$$

where $\hat{a}$, b_1 and b_2 are real, $b = b_1 + ib_2$, and σ_i, $i = 1,2,3$, are the Pauli matrices. An alternative parameterisation of elements of $G = SU(1,1)$ is given by exponentiating the above infinitesimal element;

$$U = e^A = \begin{pmatrix} cosh\rho - 2i\hat{a}\frac{sinh\rho}{\rho} & b\frac{sinh\rho}{\rho} \\ b^*\frac{sinh\rho}{\rho} & cosh\rho + 2i\hat{a}\frac{sinh\rho}{\rho} \end{pmatrix} \tag{6.3.15}$$

where $\rho^2 = b^*b - 4\hat{a}^2$. The $U(1)$ subgroup is generated by $i\sigma_3$ and its elements take the form

$$h = \begin{pmatrix} e^{-2i\hat{a}} & 0 \\ 0 & e^{+2i\hat{a}} \end{pmatrix} \tag{6.3.16}$$

Taking $\mathcal{V}$ to be a general elements of $SU(1,1)$ we find that ω_μ takes the form

$$\omega_\mu = \begin{pmatrix} 2iQ_\mu & P_\mu \\ \bar{P}_\mu & -2iQ_\mu \end{pmatrix} \tag{6.3.17}$$

where $Q_\mu = (Q_\mu)^*$. Under an infinitesimal local $U(1)$ transformation of equation (6.3.16) we find from equation (6.3.10) that the vielbein and $U(1)$-connection transform as

$$\delta Q_\mu = -\partial_\mu \hat{a}, \quad \delta P_\mu = 4i\hat{a}P_\mu \tag{6.3.18}$$

The last equation corresponds to the $U(1)$ weight 4 assignment given earlier to the scalars. If we write $\mathcal{V} = (C_+, C_-)$, where C_+ and C_- are two column vectors, then $C_\pm$ transforms as $C_\pm \to gC_\pm e^{\mp 2i\hat{a}}$ under local and rigid transformations. If we further take the ratio of the top and bottom component of the column vectors $C_\pm$ and denoted it by $c_\pm$, then it follows that $c_\pm$ is inert under local H transformations but transforms as

$$c_\pm \to \frac{uc_\pm + v}{v^*c_\pm + u^*} \tag{6.3.19}$$

under a rigid transformation of the form of equation (6.3.13).

We can use a local $U(1)$ transformation to bring $\mathcal{V}$ to be of the form

$$\mathcal{V} = \frac{1}{\sqrt{1-\phi\phi^*}} \begin{pmatrix} 1 & \phi \\ \phi^* & 1 \end{pmatrix} \tag{6.3.20}$$

This choice is most easily achieved using the form of $SU(1,1)$ elements given in equation (6.3.15). Examining the second column vector in $\mathcal{V}$ we find that $c_- = \phi$ and so ϕ transforms as under G as

$$\phi \to \frac{u\phi + v}{v^*\phi + u^*} \tag{6.3.21}$$

For the choice of $\mathcal{V}$ of equation (6.3.20) we find that

$$Q_\mu = -\frac{i}{4}\frac{(-\phi\partial_\mu\phi^* + \phi^*\partial_\mu\phi)}{(1-\phi\phi^*)},\ P_\mu = \frac{\partial_\mu\phi}{(1-\phi\phi^*)} \tag{6.3.22}$$

The corresponding equation of motion is found by substituting equation (6.3.17) into equation (6.3.12) to find

$$D_\mu P^\mu \equiv \partial_\mu P^\mu + 4iQ_\mu P^\mu = 0 \tag{6.3.23}$$

The actual IIB equations of motion of the scalars must be of this form, but will also include other terms containing the superpartners of the scalars. We observe that the scalars only occur through P_μ or, for derivatives of other fields with non-zero $U(1)$ weight, through Q_μ. Both these fields are given by equation (6.3.9) which contains the space-time derivatives acting on the group element and so they have geometric dimension one. This fact plays a crucial role in the way the scalars are encoded into the IIB theory as it allows the use of dimensional analysis to restrict the way the scalars can occur in the theory.

It has been found that the scalar fields that occur in supergravity theories always belong to a coset space [169],[170],[171]. Some of the other interesting cases are the $N=4$ and $N=8$ supergravity theories in four dimensions where the coset spaces are $SU(1,1)/U(1)$[169] and $E_7/SU(8)$ [170]. In table 6.1 we give the coset spaces associated with the maximal supergravities. this table was taken from references [167] and [181]. All these theories except for the IIB theory arise from the dimensional reduction of the eleven-dimensional supergravity theory.

6.1 Coset spaces of the Maximal Supergravities

D	G	H
11	1	1
10, IIB	$SL(2)$	SO(2)
10, IIA	$SO(1,1)/Z_2$	1
9	$GL(2)$	$SO(2)$
8	$E_3 \sim SL(3) \times SL(2)$	$U(2)$
7	$E_4 \sim SL(5)$	$USp(4)$
6	$E_5 \sim SO(5,5)$	$USp(4) \times USp(4)$
5	E_6	$USp(8)$
4	E_7	$SU(8)$
3	E_8	$S0(16)$

6.3.4 The Gauge Fields

Let us now examine how the gauge fields associated with the rank-two tensor gauge field $A_{\mu\nu}$ can occur in the theory. We define the field strength of $A_{\mu\nu}$ to be $\Im_{\mu_1\mu_2\mu_3} = 3\partial_{[\mu_1} A_{\mu_2\mu_3]}$. This gauge field must be inert under the local $U(1)$ transformations associated with the coset $\frac{SU(1,1)}{U(1)}$. If it were to have a non-trivial $U(1)$ transformation then it could only transform covariantly, but in this case the field strength would not transform covariantly. One could attempt to avoid this latter conclusion by including the $U(1)$ connection Q_μ in the definition of the field strength, however then the corresponding field strength would not be invariant under the standard $U(1)$ gauge transformations of the gauge field. Under rigid $g \in SU(1,1)$ it transforms as

$$(\bar{\Im}, \Im) \;\rightarrow\; (\bar{\Im}, \Im) g^{-1} \tag{6.3.24}$$

We can define a $SU(1,1)$ invariant field strength by

$$(\bar{F}, F) = (\bar{\Im}, \Im)\mathcal{V} \tag{6.3.25}$$

Using equations (6.3.8) and (6.3.16) we find that under a local $U(1)$ transformation the new fields $(\bar{F}, F)$ have weights $(-2, 2)$ and so transform as

$$(\bar{F}, F) \;\rightarrow\; (\bar{F}, F)h \tag{6.3.26}$$

Using the form of $\mathcal{V}$ given in equation (6.3.20) we can readily find $(\bar{\Im}, \Im)$ in terms of $(\bar{F}, F)$.

We note that the coset space description of the scalar fields plays an important part in the formulation of the gauge field which will occur in the equations of motion. It will follow that the equations of motion will admit duality transformations. Since almost all maximal supergravity have gauge fields and coset space scalars this is a general feature of supergravity theories. We refer the reader to the lectures of M-K. Gaillard and B. Zumino in this volume for a complete discussion of this topic.

6.3.5 The Equations of Motion

As we have discussed, the derivation of the equations of motion was carried out using a variant of the Noether method (see section 4.1) and the on-shell superspace technique discussed in section 4.2. These calculations are too involved to reproduce here; however, many features of the equations can be deduced using the features of the IIB theory discussed above. For example, the equation of motion must reduce to the correct linearised equations, obey the requirements of dimensional analysis and contain terms of the same $U(1)$ weight. In addition the gauge fields can only occur in terms of their field strengths $F^{\mu_1\mu_2\mu_3}$ and $\bar{F}^{\mu_1\mu_2\mu_3}$ as well as a five rank field strength $G_{\mu_1\ldots\mu_5}$. These field strengths all have geometric dimension one. The vielbein must occur through the usual Ricci tensor $R_{\mu\nu}$ which has geometric dimension two and as we discussed above the scalars belong to the coset $SU(1,1)/U(1)$ and hence are only contained in the geometric dimension one objects P_μ and Q_μ. The latter can only occur as part of a covariant derivative.

Let us begin with the scalars; their equation of motion must generalise equation (6.3.23) and hence the terms in the equation must have $U()$ weight 4 and geometric dimension two. The only possible candidate is $F_{\mu_1\mu_2\mu_3}F^{\mu_1\mu_2\mu_3}$. The equation for $A_{\mu\nu}$ must contain $D^{\mu_3}F_{\mu_1\mu_2\mu_3}$ and so it has geometric dimension two and $U(1)$ weight 2. The only such terms we can add are $F_{\mu_1\mu_2\mu_3}P^{\mu_3}$ and $G_{\mu_1\ldots\mu_5}F^{\mu_3\cdots\mu_5}$. The equation of motion for the fourth rank gauge field is just the self-duality condition on the five-rank field strength and so this equation has geometric dimension one and $U(1)$ weight zero. There are no terms one can add to this equation other than the duality condition on he field strength itself. Analysing the veilbein equation in the same way we essentially determine the field equations up to constants.

The equations of motion of IIB supergravity in the absence of fermions is given by [111],[112]

$$D^\mu P_\mu = \frac{1}{6}F_{\mu_1\mu_2\mu_3}F^{\mu_1\mu_2\mu_3} \tag{6.3.27}$$

$$D^{\mu_3}F_{\mu_1\mu_2\mu_3} = \bar{F}_{\mu_1\mu_2\mu_3}P^{\mu_3} - \frac{i}{6}G_{\mu_1\ldots\mu_5}F^{\mu_3\cdots\mu_5} \tag{6.3.28}$$

$$R_{\mu\nu} = -2\bar{P}_{(\mu}P_{\nu)} - F_{(\mu}{}^{\nu_1\nu_2}F_{\nu)\nu_1\nu_2} + \frac{1}{12}g_{\mu\nu}\bar{F}_{\mu_1\mu_2\mu_3}F^{\mu_1\mu_2\mu_3} - \frac{1}{96}G_\mu{}^{\mu_1\ldots\mu_4}G_{\nu\mu_1\ldots\mu_4} \tag{6.3.29}$$

$$G_{\mu_1\ldots\mu_5} = {}^*G_{\mu_1\ldots\mu_5} \tag{6.3.30}$$

where

$$G_{\mu_1\ldots\mu_5} = 5\partial_{[\mu_1}A_{\mu_2\ldots\mu_5]} + 20i(A_{[\mu_1\mu_2}\Im^*_{\mu_3\ldots\mu_5]} - A^*_{[\mu_1\mu_2}\Im_{\mu_3\ldots\mu_5]}) \tag{6.3.31}$$

The reader is referred to reference [111] for the fermionic contribution. They are invariant [111,112] under the local supersymmetry and $U(1)$ transformations of reference [110].

6.3.6 The $SL(2,\mathbf{R})$ Version

The group $SU(1,1)$ is isomorphic to the group $SL(2,\mathbf{R})$. For some purposes it is better to formulate the theory in a manner where the $SL(2,\mathbf{R})$ form of the invariance is manifest rather than as above where the $SU(1,1)$ symmetry is apparent. As we explained above, $g \in SU(1,1)$ acts on the column vector $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ by $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to g \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. If we denote the ratios of the column vector by $z = \frac{z_1}{z_2}$ then the action of $SU(1,1)$ becomes

$$z \to \frac{uz + v}{v^* z + u^*} \tag{6.3.32}$$

This action is such that it takes the unit disc $|z| \le 1$ to itself. We can map the unit desk to the upper half plane $H = \{w : Imw \ge 0\}$ by the transformation

$$z \ \to \ w = i \left(\frac{1 - z}{1 + z} \right) \tag{6.3.33}$$

The action induced by the transformation of equation (6.3.32) on H is given by

$$w \ \to \ \frac{aw + b}{cw + d} \tag{6.3.34}$$

where $ad - bc = 1$ and a, b, c, d are real. In this last transformation we recognise the action of the group $SL(2, \mathbf{R})$, corresponding to the element

$$\hat{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}). \tag{6.3.35}$$

It is well known that $SL(2, \mathbf{R})$ is the largest group which maps the upper half plane to itself and so we should not be surprised that in mapping from the unit disc onto the upper half plane the action of $SU(1,1)$ becomes that of $SL(2, \mathbf{R})$. The precise relationship between the parameters of the two groups is given by

$$a = \frac{1}{2}(u + u^* - v - v^*), \ \ b = \frac{i}{2}(-u + u^* - v + v^*),$$

$$c = \ -\frac{i}{2}(-u + u^* + v - v^*), \ \ d = \ \frac{1}{2}(u + u^* + v + v^*) \tag{6.3.36}$$

For the $SU(1,1)$ formulation of the IIB theory given above we found that the scalar field ϕ of equation (6.3.20) transformed under $SU(1,1)$ in the same way as the variable z of equation (6.3.21). Hence if we make the transformation from ϕ to the variable φ by

$$\phi \ \to \varphi = i \left(\frac{1 - \phi}{1 + \phi} \right) \tag{6.3.37}$$

then φ transforms under $SL(2, \mathbf{R})$ just like w, that is

$$\varphi \to \frac{a\varphi + b}{c\varphi + d} \tag{6.3.38}$$

It remains to find new variables for the rank-two gauge field that transform in a recognisable way under $SL(2, \mathbf{R})$. Let us write the field strength $\Im_{\mu\nu\rho}$ as $\Im_{\mu\nu\rho} = \Im^2_{\mu\nu\rho} + i\Im^1_{\mu\nu\rho}$ then an explicit calculation shows that the transformation law of equation (6.3.24) for $\Im_{\mu\nu\rho}$ becomes

$$\begin{pmatrix} \Im^1_{\mu\nu\rho} \\ \Im^2_{\mu\nu\rho} \end{pmatrix} \rightarrow \hat{g} \begin{pmatrix} \Im^1_{\mu\nu\rho} \\ \Im^2_{\mu\nu\rho} \end{pmatrix} \tag{6.3.39}$$

It is straightforward to substitute for the new variables into the equations of motion (6.3.27) to (6.3.31) to find a formulation that is manifestly $SL(2, \mathbf{R})$ invariant. Carrying out this transformation and also making the substitution $\varphi = l + ie^{\sigma}$, one finds that the $NS \otimes NS$ fields, (i.e. $e_{\mu}{}^{a}$, $A^1_{\mu\nu}$ and σ) have identical equations of motion as the $NS \otimes NS$ sector of the IIA supergravity. In fact, since these fields do not include the rank four gauge field we can formulate the dynamics of the $NS \otimes NS$ sector of the theory in terms of an action and this action will have a Lagrangian which is none other than the first term of equation (6.2.16), if we choose to work with the string metric.

We close with some comments on some of the features of the IIB theory discussed that are most relevant to our discussion on string duality. Like the IIA theory, the IIB theory has two coupling constants; the Newtonian constant κ, whose dependence we have suppressed, and the expectation value of $< e^{\sigma} >$. As we have explained in the previous section the latter plays the role of the IIB string coupling constant, i.e. $g_s =< e^{\sigma} >$. In general, an $SL(2, \mathbf{R})$ transformation changes from weak to strong string coupling. For example, the transformation $\varphi = l + ie^{\sigma} \;\rightarrow\; \varphi' = l' + ie^{\sigma'} = -\frac{1}{\varphi}$ implies that

$$g_s' = \langle e^{\sigma'} \rangle = \frac{1}{< e^{\sigma} >} = \frac{1}{g_s} \tag{6.3.40}$$

maps from the weak to the strong regime.

The $SL(2, \mathbf{R})$ transformation also mixes the two real rank-three field strengths $\Im^i_{\mu\nu\rho}$, $i = 1, 2$ one of which arises from the $NS \otimes NS$ sector and one from the $R \otimes R$ sector of the string theory. Hence a generic $SL(2, \mathbf{R})$ transformation transforms fields in the $NS \otimes NS$ sector into those in the $R \otimes R$ sector and vice versa.

The IIB theory cannot be obtained from the eleven-dimensional supergravity by a reduction; however if we were to reduce the IIB theory on a circle to nine dimensions then the resulting supergravity theory would have an underlying supersymmetry algebra with 32 supercharges which, being an odd dimension, would be of no fixed chirality. This is precisely the same supersymmetry algebra that would result if we were to reduce the IIA theory on a circle to nine dimensions. In particular, they form two 16 component Majorana spinors. Since this algebra uniquely determines the maximal nine-dimensional supergravity theory we must conclude that reducing the IIA and IIB supergravity theories to nine dimensions leads to the same supergravity theory. [107].

6.4 Type I Supergravity

The type I Supergravity was in fact the first supergravity theory in ten dimensions to be constructed [120]. Its underlying algebra contains a Majorana-Weyl spinor supercharge whose anti-commutator was given in equation (1.5.4). The field content is given by

$$\underbrace{e_\mu^a, \phi}_{NS\otimes NS} \; ; \quad \underbrace{A_{\mu\nu};}_{R\otimes R} \quad \text{plus} \quad \underbrace{\psi_{\mu\alpha}, \lambda_\alpha}_{NS\otimes R} \tag{6.4.1}$$

where we have also indicated the sectors in the type I string from which they come. The gravitino $\psi_{\mu\alpha}$ and the spinor λ_α are Majorana-Weyl spinors of opposite chirality and all the bosonic fields are real.

This theory can be obtained form the either the IIA or the IIB theory by truncation. From the IIA theory we impose the obvious Weyl conditions on the gravitino and spinor in the IIA theory, and to be consistent with supersymmetry, we must also set $A_{\mu\nu\rho} = 0 = B_\mu$. It is straightforward to find the Lagrangian for the bosonic fields by truncating the corresponding action for the IIA theory of equation (6.2.8).

We can also obtain the type I theory from the IIB theory by a truncation. In the IIB theory we consider the operator Ω which changes the sign of $\Im^2_{\mu\nu\rho}$, l and $B_{\mu\nu\rho\sigma}$, which are in the $NS \otimes NS$, $R \otimes R$ and $R \otimes R$ sectors respectively, but leaves inert e_μ^a, $\Im^1_{\mu\nu\rho}$ and σ which are in $NS \otimes NS$, $R \otimes R$ and $NS \otimes NS$ sectors respectively. To recover the type I supergravity we keep only fields which are left inert by Ω and so we find only the latter fields. On the spinors we impose a Majorana condition which leads to a gravitini and another fermion which are both Majorana-Weyl although of opposite chirality. In fact Ω corresponds in string theory to world sheet parity, that is it exchanges left and right moving modes.

The $N = 1$ Yang-Mills theory is also based on the supersymmetry algebra with one Majorana-Weyl supercharge. It consists of a gauge field A_μ and one Majorana-Weyl spinor λ_α. This theory [119] is easily derived. We first write down the linearised transformation laws that are determined up to two constants by dimensional analysis. The constants are then fixed by demanding that the supersymmetry transformations and the linearised gauge transformations form a closed algebra. The full theory is uniquely found by demanding that the algebra closes and that the action be invariant under the usual non-Abelian gauge transformation of the gauge field. The result is given by

$$\int d^{10}x(-\frac{1}{4}F^i_{\mu\nu}F^{\mu\nu i} - \frac{i}{2}\bar{\lambda}^i\gamma^\mu\mu(D_\mu\lambda)^i) \tag{6.4.2}$$

which is invariant under

$$\delta A^i_\mu = i\bar{\epsilon}\gamma_\mu\lambda^i, \;\; \delta\lambda^i = -\frac{1}{2}F^i_{\mu\nu}\gamma^{\mu\nu}\epsilon \tag{6.4.3}$$

where

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - gf_{jk}{}^i A^j_\mu A^k_\nu \tag{6.4.4}$$

is the Yang-Mills field strength and

$$D_\mu\lambda^i = \partial_\mu\lambda^i - gf_{jk}{}^i A^j_\mu\lambda^k \tag{6.4.5}$$

We can also consider the coupling between the $N=1$ Yang-Mills theory and the type I supergravity. This was found in reference [121]. If we take the gauge group to be $SO(32)$ or $E_8 \otimes E_8$ we find the theory that results from the low energy limit of the corresponding heterotic string theory or the type I string theory.

7. Brane Dynamics

7.1 Bosonic Branes

Super p-branes are extended objects that sweep out a $p+1$-dimensional space-time manifold in a background superspace-time. A 0-brane is just a particle and a 1-brane is a string. However p-branes for $p \geq 2$ also occur in string theory as solitons and are thus non-perturbative objects. In this section we find what possible superbranes can exist and give their dynamics. Although p-branes with $p \geq 2$ are intrinsically non-perturbative objects, they are related by duality symmetries to perturbative particle and string states. As such, they play an important role in discussions of string duality.

We first consider a brane that has no supersymmetry. A p-brane sweeps out a $p+1$-dimensional world surface M, with coordinates ξ^m, $m=0,1,\ldots,p$ in a D-dimensional target space $\underline{M}$ with coordinates $X^{\underline{n}}$, $\underline{n}=0,1,\ldots,D-1$. As the symbols imply we use $m,n,p,\ldots$ for the embedded surface world indices and $\underline{m},\underline{n},\underline{p},\ldots$ for target space world indices. The corresponding tangent space indices are $a,b,c,\ldots$ for world surface indices and $\underline{a},\underline{b},\underline{c},\ldots$ for target space world indices. This notation is used extensively in the literature in this subject. The reader should have no difficulty making the transition from the $\mu,\nu\ldots$ and $m,n,p\ldots$ used in the previous section for the world and tangent target space indices respectively. The surface M swept out by the p-brane in the target space $\underline{M}$ is specified by the functions $X^{\underline{n}}(\xi^n)$ which extremise the action

$$-T\int d^{p+1}\xi\sqrt{-det g_{mn}} \tag{7.1.1}$$

where

$$g_{mn}=\partial_n X^{\underline{n}}\partial_m X^{\underline{m}} g_{\underline{nm}} \tag{7.1.2}$$

and $g_{\underline{mm}}$ is the metric of the target space-time often referred to as the background metric. The constant T is the brane tension and has the dimensions of $(mass)^{p+1}$. The action of equation (7.1.1) is invariant under reparameterisations of both the target space $\underline{M}$ and the world surface M. The metric g_{mn} is the metric induced on the world surface M by the background metric of the target space. As such, we recognize the action in equation (7.1.1) as the area swept out by the p-brane. Hence, like the string and point particle, a p-brane moves so as to extremise the volume of the surface it sweeps out. If the target space is flat the background metric is just the Minkowski metric $g_{\underline{mm}}=\eta_{\underline{mm}}$. A 0-brane is just a point particle and if it has mass m then $T=m$. A 1-brane is just the usual bosonic string and the action of equation (7.1.1) is the Nambu action for the string if we take the background metric to be flat. In this case we often write $T=\frac{1}{2\pi\alpha'}$ where α' is the string Regge slope parameter.

The bosonic brane does not have enough symmetry to determine its couplings to the fields in the target space. However, a p-brane naturally couples to a $p+1$ gauge field $A_{\underline{m}_1\cdots\underline{m}_{p+1}}$ of the target space by a term of the form

$$\int d^{p+1}\xi \epsilon^{n_1\dots n_{p+1}}\partial_{n_1}X^{\underline{m}_1}\dots\partial_{n_{p+1}}X^{\underline{m}_{p+1}}A_{\underline{m}_1\cdots\underline{m}_{p+1}} \tag{7.1.3}$$

So for example, the motion of a 0-brane, that is a point particle, is described by the functions $X^{\underline{n}}(\tau)$, where $\xi^0=\tau$, and it naturally couples to a vector field $A_{\underline{n}}$ in the form

$$\int d\tau \frac{dX^{\underline{n}}}{d\tau}A_{\underline{n}} \tag{7.1.4}$$

If we couple this expression to that in equation (7.1.1) for a flat target space then the equations of motion for $X^{\underline{n}}$ are nothing but the Lorentz force law for a charged particle in an electromagnetic field. A 1-brane, i.e. string, couples to a two-form $A_{\underline{nm}}$ in the manner

$$\int d^2\xi \epsilon^{mn}\partial_m X^{\underline{m}_1}\partial_n X^{\underline{m}_2}A_{\underline{m}_1\underline{m}_2} \tag{7.1.5}$$

We can split the target space indices $\underline{n},\underline{m},$ into those associated with the directions longitudinal to the brane and those with directions which are transverse to the brane. We denote the former by $n,m\ldots=0,\ldots,p$, and the latter by $n',m'\ldots=p+1,\ldots,D-1$. A useful gauge is the static gauge in which we use the reparameterisation transformations of the world surface to identify the $p+1$ longitudinal coordinates $X^n(\xi),\ n=0,1,\ldots,p$, with the coordinates $\xi^n,\ n=0,1,\ldots,p$ of the p brane; in other words

$$X^n(\xi)=\xi^n,\ n=0,1,\ldots,p \tag{7.1.6}$$

This leaves the transverse coordinates $X^{n'}(\xi),\ n'=p+1,\ldots,D-1,$ to describe the dynamics of the brane. We can think of the $D-p-1$ transverse coordinates as the Goldstone bosons or zero modes of the broken translations due to the presence of the p-brane.

As we have explained in section 6, the low energy effective action of a string theory is a supergravity theory. It has been found that p-brane solutions arise from this supergravity theory. For such a static solution the supergravity fields do not depend on p of the spatial coordinates which are the spatial coordinates of the p-brane world surface, but do depend on the $D-p-1$ coordinates trnasverse to the brane. It turns out that the supergravity fields usually depend on functions which are harmonic functions of the transverse coordinates.

It will be instructive to consider a simpler and better understood example of solitons, namely the monopoles (i.e. a 0-brane) that occur in four dimensions. Monopoles arise as static solutions in the $N=2$ supersymmetric Yang-Mills field theory. The space of parameters required to specify the solution is called the moduli space. If there are K monopoles then the moduli space has dimension $4K$. For one monopole this four-dimensional space is $\mathbf{R}^3\otimes S^1$. It is made up of the three spatial coordinates $\mathbf{R}^3$ which specify the position of the monopole and a further parameter associated with gauge transformations which have a non-trivial behaviour at infinity. For $K \geq 2$ the moduli space is more complicated, but an explicit metric is known for the case of $K = 2$ [128]. Since there exist static monopoles solutions, it follows that monopoles do not experience any forces when at rest. However, when they are set in motion they do experience velocity dependent forces. In general the behaviour of the monopoles is described by the quantum field theory in which they arise. However, if the monopoles have only a small amount of energy above their rest masses then we can approximate their motion in terms of the coordinates of their moduli space [129]. The motion is then described by an action whose fields are the coordinates of the moduli space which are given a time dependence corresponding to the monopole motion. If we had only one monopole then three of the moduli $X^i,\ i = 1, 2, 3$, could be interpreted as the Goldstone bosons resulting from the breaking of the spatial translations due to the presence of the monopole and the fourth moduli η would be related to the existence of non-trivial gauge transformations at infinity. As such, the motion is described by $X^i(\tau),\ i = 1, 2, 3,\ \eta(\tau)$ where $\xi^0 = \tau$. A much more detailed account of these ideas can be found in the lectures of N. Manton in this volume.

In a similar spirit we can consider the moduli space and action which describes the low energy behaviour of p-brane solitons. The moduli space of the p-brane solitons contains the positions of the p-branes and to describe their low energy motion we let the moduli depend on the world-volume coordinates of the p-brane. Indeed, we can think of the action of equation (7.1.1) when we take only its terms to lowest order in derivatives as the effective action which describes the low energy behaviour of a single p-brane soliton.

There is an alternative interpretation of the action of equation (7.1.1). Were we believe that a certain p-brane was a fundamental object then we might take the action of equation (7.1.1) to describe its dynamics completely. This was precisely the viewpoint of for the string for many years.

7.2 Types of Superbranes

A super p-brane can be viewed as a $p+1$-dimensional **bosonic** sub-manifold M, with coordinates $\xi^n,\ n = 0, 1, \ldots, p$, that moves through a target superspace $\underline{M}$ with coordinates

$$Z^{\underline{N}} = (X^{\underline{n}}, \Theta^{\underline{\alpha}}) \tag{7.2.1}$$

We use the superspace index convention that $\underline{N}, \underline{M}, \ldots$ and $\underline{A}, \underline{B}, \ldots$ represent the world and tangent space indices of the target space. Later, in subsection 7.5, we use the same symbols without the underlining to represent the corresponding indices of the world surface superspace.

In this section we wish to find which types of branes can exist in which space-time dimensions with particular emphasis on ten and eleven dimensions. We will also discuss the features of brane dynamics which are generic to all branes, leaving to the next three subsections a more detailed discussion of the dynamics of the specific types of branes.

Superbranes come in various types. The simplest are those whose dynamics can be described entirely by specifying the superworld surface $Z^{\underline{N}}(\xi^n)$ that the p-brane sweeps out in the target space. We refer to these branes as **simple superbranes**; they were also previously called type I branes' not to be confused with type I strings. As we shall see there are other types of branes that have higher spin fields living on their world surfaces.

In particular, we will discuss branes that have vectors and second rank anti-symmetric tensor gauge fields on their world surface.

A super brane has an action of the form

$$A = A_1 + A_2 \tag{7.2.2}$$

The first term is given by

$$A_1 = -T \int d^{p+1}\xi \sqrt{-det g_{mn}} + \ldots \tag{7.2.3}$$

where

$$g_{mn} = \partial_n Z^{\underline{N}} \partial_m Z^{\underline{M}} g_{\underline{NM}}, \tag{7.2.4}$$

$$g_{\underline{NM}} = E_{\underline{N}}{}^{\underline{a}} E_{\underline{M}}{}^{\underline{b}} \eta_{\underline{ab}} \tag{7.2.5}$$

and $E_{\underline{N}}{}^{\underline{a}}$ is the supervielbein on the target superspace. By abuse of notation we use the same symbol for the world surface metric as for the bosonic case; the reader will be able to distinguish between the two as a result of the context. The $+\ldots$ denotes terms involving possible world surface fields as well as those that depend on the other background fields. The constant T is the p-brane tension and has the dimensions of $(mass)^{p+1}$ since the action is dimensionless and $X^{\underline{n}}$ has the dimension of $(mass)^{-1}$. If we consider a brane that is static we can think of the tension T as the mass per unit spatial volume of the brane.

The symbol $g_{\underline{NM}}$ is not really a background metric in the usual sense since the sum on the tangent space indices is restricted to be only over the bosonic part. Such a restricted summation is possible as a consequence of the fact that the superspace tangent space group is just the Lorentz group. In fact, as we discussed in section 4, the theory of local superspace is formulated in terms of the supervielbein since the metric is not uniquely defined.

The second part of the action A_2 of equation (7.2.2) contains, in addition to others, the term

$$\int d^{p+1}\xi \epsilon^{n_1 \ldots n_{p+1}} \partial_{n_1} X^{\underline{m}_1} \ldots \partial_{n_{p+1}} X^{\underline{m}_{p+1}} A_{\underline{m}_1 \ldots \underline{m}_{p+1}} \tag{7.2.6}$$

where $A_{\underline{m}_1 \ldots \underline{m}_{p+1}}$ is a background gauge field.

The background space-time fields can only belong to a supermultiplet that exists in the target superspace. In this review, we will take these supermultiplets to be the supergravity theory that has the background supersymmetry algebra possessed by the p-brane. The field content of the possible supergravity theories can be deduced from the supersymmetry algebra using the methods given in section three. However, as we discussed in section five, the supergravity theory is essentially unique if the supersymmetry algebra has 32 supercharges and the type of spinors contained in the supersymmetry algebra are specified. Thus, unlike bosonic branes, the background fields are specified by the background supersymmetry of the brane, if that supersymmetry has 32 supercharges. If the brane has 16 supercharges then although one cannot generally specify the background fields uniquely, the possible supergravity theories are very limited. The general coupling of a p-brane to the background fields is complicated, but it always has a coupling to a $(p+1)$-gauge field in the form of equation (7.1.3). However, since this $(p+1)$-gauge field must be one of the background fields of the supergravity theory, this places an important restriction on which superbranes can arise for a given dimension and target space supersymmetry algebra.

If we consider the case of a super 1-brane, the action of equation (7.2.2) is just the Green-Schwarz action [130] for the superstring. The case $p = 2$ is often referred to as the membrane and the action for the supermembrane in eleven dimensions was found in [151,157].

The action of equation (7.2.2) is invariant under super reparameterisations of the target superspace $\underline{M}$, but only bosonic reparameterisations of the world surface M since the embedded manifold M is a bosonic manifold. It is also invariant under a Fermi-Bose symmetry called κ-supersymmetry. From the viewpoint adopted here this is a complicated symmetry that ensures that the fermions have the correct number of degrees of freedom on-shell and we will discuss it in more detail in the next section. This symmetry relates the terms in A_1 to those in A_2 and vice versa, and in fact fixes uniquely the form of A_2 given the form of A_1.

Even if the target space is flat superspace the supervielbein has a non-trivial dependence on the coordinates given by

$$\partial_m Z^{\underline{N}} E_{\underline{N}}{}^{\underline{a}} = \partial_m X^{\underline{a}} - \frac{i}{2}\bar{\Theta}\gamma^{\underline{a}}\partial_m \Theta, \;\; \partial_m Z^{\underline{N}} E_{\underline{N}}{}^{\underline{\alpha}} = \partial_m \Theta^{\underline{\alpha}} \tag{7.2.7}$$

In this case, the target space super reparameterisation invariance reduces to just rigid supersymmetry'

$$\delta X^{\underline{a}} = \frac{i}{2}\bar{\epsilon}\gamma^{\underline{a}}\Theta, \;\; \delta\theta^{\underline{\alpha}} = \epsilon^{\underline{\alpha}} \tag{7.2.8}$$

Note that the action of equation (7.2.2) does not appear to possess world-surface supersymmetry. We recall for the case of a 1-brane (that is a string) there are two formulations, the Green-Schwarz formulation given here and the original Neveu-Schwarz-Ramond formulation. The latter is formulated in terms of $X^{\underline{n}}$ and a spinor which, in contrast to above, possess a target space vector index and is a spinor with respect to the two-dimensional world sheet. This formulation is manifestly invariant under world surface reparameterisations, but not under target space super reparameterisations. If one goes to light-cone gauge (i.e. in effect static gauge) then the two formulations become the same and so the Green-Schwarz formulation has a hidden world sheet supersymmetry and the Neveu-Schwarz-Ramond [172] a hidden target space supersymmetry provided we carry out the GSO projection.

It is thought [173] that all branes which in their Green-Schwarz formulation admit κ-supersymmetry actually have a hidden world surface supersymmetry. Although the analogue of a Neveu-Schwarz-Ramond formulation is not known for p-branes when $p > 1$, there does exist a superembedding formalism [131,132,133,134,137,138,139]. In this formulation the p-brane sweeps out a **supermanifold** which is embedded in the target superspace. Although this approach leads to equations of motions and not an action, it has the advantage that it possesses super reparmeterisation invariance in both the world surface and the target superspaces. The κ-symmetry is then just part of the super reparameterisations of the world surface. Its particular form is a result of the gauge fixing required to get from the super embedding formalism to the so-called Green-Schwarz formulation. This origin of κ-symmetry was first found in reference [178] within the context of the point particle. We will comment further on the superembedding formalism in section 7.5.

The superbranes also possesses a static gauge for which the bosonic coordinates take the form of equation (7.1.6) while κ-supersymmetry can be used to set half of the fermions $\Theta^{\underline{\alpha}} = (\Theta^{\alpha},\ \Theta^{\alpha'})$ to vanish i.e. $\Theta^{\alpha} = 0$. While the remaining $D-p-1$ bosonic coordinates $X^{n'}$ correspond to the Goldstone bosons associated with the breaking of translations by the p-brane, the remaining $\Theta^{\alpha'}$ Goldstone fermions correspond to the breaking of half of the supersymmetries by the brane.

The fields of the p-brane belong to a supermultiplet of the world surface and so must have equal numbers of fermionic and bosonic degrees of freedom on-shell. Let us first consider a p-brane that arises in a theory that has maximal supersymmetry. This would be the case if the brane describes the low-energy motion of a soliton of a maximal supergravity theory which breaks half the supersymmetry. In this case, if it breaks half of the 32 supersymmetries of the target space, it will, in static gauge, have only 16 $\Theta^{\alpha'}$ which will lead to 8 fermionic degrees of freedom on-shell. If we are dealing with a simple superbrane these must be matched by the coordinates $X^{n'}$ in static gauge which must therefore be eight in number. Thus if we are in eleven dimensions the only simple superbrane is a 2-brane while if we are in ten dimensions the only simple superbrane is a 1-brane.

There also exist D-branes whose dynamics requires a vector field $A_n,\ n = 0, 1, \ldots, p$, living on the brane in addition to the coordinates $X^{\underline{n}}$, $\Theta^{\underline{\alpha}}$ which describe the embedding of the brane in the target superspace. In this case, if we have a brane that arises in the background of a maximal supergravity theory and which breaks half of the this supersymmetry then we again have 8 fermionic degrees of freedom on-shell. This must be balanced by the vector which has $p-1$ degrees of freedom on-shell and the remaining $D-p-1$ transverse coordinates from which we deduce that $D = 10$. Hence such D-branes can only exist in ten dimensions. For D-branes this simple counting argument allows branes for all p, however, as we shall see, not all values of p occur for a given target space supersymmetry. In addition, we will discuss other branes with higher rank gauge fields living on their world surface.

To find further restrictions on which branes actually exist we can use the argument given above. A super p-brane couples to a $p+1$-gauge field which must belong to the background supergravity theory. Hence to see which branes can exist we need only see which gauge fields are present in the corresponding supergravity theory. When doing this we must bear in mind that if we have a rank $p+1$ gauge field which has a rank $p+2$-field strength $F_{(p+2)}$ we can take its dual to produce a rank $D-p-2$ field strength ${}^*F_{(D-p-2)}$ which, if the original field strength is on-shell, has a corresponding rank-$D-p-3$ dual gauge field ${}^*A_{D-p-3}$. The dual field strength is defined by

$$ {}^*F_{\underline{n}_1\cdots\underline{n}_{(D-p-2)}} = \frac{1}{(p+2)!}\epsilon_{\underline{n}_1\cdots\underline{n}_{(D-p-2)}\underline{n}_{(D-p-1)}\cdots\underline{n}_D} F^{\underline{n}_{(D-p-1)}\cdots\underline{n}_D} \tag{7.2.9} $$

Hence, if the original rank $p+1$ gauge field couples to a p-brane, the dual gauge field is rank $D-p-3$ and couples to a $D-p-4$ brane. In fact D-branes can only couple to the $R\otimes R$ sector of a string theory [135].

Let us begin with branes in eleven dimensions. This theory was described in section 5 and possesses a third-rank gauge field $A_{(3)}$ which should couple to a 2-brane. The dual potential has rank six i.e. ${}^*A_6$ and this couples to a 5-brane. Thus in eleven dimensions we expect only a 2-brane and a 5-brane. The 2-brane is just the simple superbrane we discussed above. The 5-brane has 16 Goldstone fermions and these lead to the same 8 degrees of freedom on-shell; however it has only five transverse coordinates leading to only five degrees of freedom on-shell. Clearly, we require another three degrees of freedom on-shell. These must form a representation of the little group $SO(4)$. If they are belong to an irreducible representation of $SO(4)$, this can only be a second rank self-dual antisymmetric tensor. On-shell this corresponds to a second rank antisymmetric gauge field whose field strength obeys a self duality condition. The dynamics of the fivebrane is significantly more complicated than that of simple branes and will be given in section 5.4.

Let us now consider the branes that can couple to the IIA string. In section six we found that the IIA theory has gauge fields and their duals of ranks $1,2,3,5,6,7$. Of these ranks 2 and 6 arise in the $NS\otimes NS$ sector of the string while those of ranks $1,3,5$ and 7 arise in the $R\otimes R$ sector of the string. This suggests the existence of p-branes for $p=1$ and 5 that can couple to the $NS\otimes NS$ sector of the IIA string and p-branes for $p=0,2,4$ and 6 that can couple to the $R\otimes R$ sector of the IIA string. The latter are the D-branes. Of these only the 1-brane can be a simple brane and it is this brane that couples to the $A_{\mu\nu}$ gauge field of the IIA string. It is in fact the IIA string itself and it is sometimes therefore called the fundamental string. The fivebrane is the straightforward dimensional reduction of the five eleven-dimensional fivebrane that will be discussed in the next section. In the literature an 8-brane is also discussed. This brane is associated with the massive supergravity theory constructed in reference [174]. This theory necessarily contains a cosmological constant c and so has a term $\int d^{10}x\ c\sqrt{-detg_{\underline{nm}}}$ in the action. However, we can write this term as $\int d^{10}x\ c\sqrt{-detg_{\underline{nm}}}F^{\underline{n}_1\cdots\underline{n}_{(10)}}F_{\underline{n}_1\cdots\underline{n}_{(10)}}$ where $F_{\underline{n}_1\cdots\underline{n}_{(10)}}$ is the curl of a rank 9 gauge field which suggests the existence of an 8-brane [185].

Let us now turn to the IIB theory and its branes. The original gauge fields in the IIB theory are of rank 2 in the $NS\otimes NS$ sector and ranks $0,2,4$ in the $R\otimes R$ sector. If we include their dual gauge fields we have gauge fields of ranks $2,6$ in the $NS\otimes NS$ sector and ranks $0,2,4,6,8$ in the $R\otimes R$ sector. We do not include two rank four gauge fields as their field strength is the five rank self-dual field strength of the IIB theory. This suggests that there exist 1 and 5-branes which couple to the $NS\otimes NS$ of the IIB theory and $-1,1,3,5,7$-branes which couple to the $R\otimes R$ sector. The latter are the D-branes. The 1-brane which couples to the rank two gauge field in the $NS\otimes NS$ sector is the IIB string itself. The 5-brane which couples to the $NS\otimes NS$ sector is a more complicated object. Note that it is the threebrane that couples to the rank four gauge field whose field strength satisfies a self-duality property. As one might expect this D-brane possesses a self-duality symmetry [136]. The $p=-1$ brane which couples to the $R\otimes R$ sector occupies just a point in space-time and so is an instanton.

In fact, all the branes discussed above exist and the possible branes in eleven and ten dimensions are listed in table 7.1. In this table a "D" or "S" subsrcipt denotes the brane to be a Dirichlet and simple brane respectively.

7.1 Super Brane Scan

\ p Theory \									
M-theory			2_S			5			
IIA	0_D	1_S	2_D		4_D	5	6_D		8_D
IIB		$1_S + 1_D$		3_D		$5 + 5_D$		7_D	

As we have mentioned above, one can also search for the p-brane solitons of the corresponding supergravity theories. The p-brane actions then correspond to the low-energy motions of these solitons and it has been shown that there exist p-brane solitons for all the above superbranes. We refer the reader to the lectures of G. Gibbons in this volume and the reviews of reference [176].

We now discuss one further guide to determining which superbranes occur in which theory. Given a p-brane one can construct the following current

$$j^{n\underline{m}_1\cdots\underline{m}_p} = \epsilon^{nn_1\ldots n_p}\partial_{n_1}X^{\underline{m}_1}\ldots\partial_{n_p}X^{\underline{m}_P} \tag{7.2.9}$$

which is obviously conserved. The charge associated with this current is given by

$$Z^{\underline{m}_1\cdots\underline{m}_p} = \int d^p\xi j^{0\underline{m}_1\cdots\underline{m}_p} \tag{7.2.10}$$

where the integral is over the p spatial coordinates of the brane. Our previous discussion on the coupling of the p-brane to a $p+1$-form gauge background gauge field can be restated as the $p+1$-form background gauge field couples to the current $j^n\partial_n X^{\underline{m}}$. When computing the space-time supersymmetry algebra in the presence of the p-brane it turns out [180] that the above charge occurs as a central charge. Hence, if a p-brane arises in a particular theory we expect to find its p-form central charge in the corresponding supersymmetry algebra. Clearly, if the supersymmetry algebra does not admit a p-form central charge the p-brane cannot occur unless further supersymmetry is broken. However, we can turn this argument around and find out which central charges can occur in the appropriate supersymmetry algebra and then postulate the existence of a p-brane for each corresponding p-form central charge. Which central charges can occur for a specified type of supercharge was discussed in section 1. For example, in equation (1.5.5), we found that the eleven-dimensional supersymmetry algebra with one Majorana spinor, which is the supersymmetry algebra appropriate to eleven-dimensional supergravity, had a two-form and a three-form central charge. Hence in M theory we expect from the above argument to find a twobrane and a fivebrane. This agrees with our previous considerations.

We close this section by giving some simple dimensional arguments concerning the tensions of various branes. In eleven dimensions, that is M theory we have only one scale the Planck scale m_p. As a result, up to constants, the tensions T_2 and T_5 of the 2-brane and 5-brane can only be given by $T_2 = (m_p)^3$ and $T_5 = (m_p)^6$ respectively. As we discussed in section 6, we can obtain the IIA supergravity by dimensional reduction of eleven-dimensional supergravity on a circle and it is conjectured that IIA string theory can be obtained by reduction of M theory in the same way. Hence for the IIA theory in ten dimensions we have two scales, the Planck mass m_p and the radius of compactification of the circle R_{11}. Since the string coupling is dimensionless it must be a function of $m_p R_{11}$ and we found in section 6 that it is given by $g_s = e^{<\sigma>} = (m_p R_{11})^{\frac{3}{2}}$.

From the 2-brane in eleven dimensions we can obtain a 2-brane in ten dimensions by simply ignoring the dependence of the fields of the eleven-dimensional 2-brane on x^{10}. We can also find a 1-brane, that is a string if we simultaneously reduce and wrap the 2-brane on the circle [175]. The tensions of these two objects in ten dimensions are therefore given by $T_2 = (m_p)^3$ and $T_1 = R_{11}(m_p)^3$ respectively. The factor of R_{11} occurs because the energy per unit length of the 1-brane arises from the energy of the 2-brane on the circle. Similarly, from the fivebrane in eleven dimensions we can obtain a fivebrane and a fourbrane in ten dimensions with tensions $T_5 = (m_p)^5$ and $T_4 = R_{11}(m_p)^5$. The 1-brane in ten dimensions is just the fundamental string (i.e. the IIA string) and its tension T_1 can be identified as the square of the string mass; that is $T_1 = R_{11}(m_p)^3 = \frac{1}{4\pi\alpha'} \equiv (m_s)^2$. It is instructive to express the tensions in terms of the variables g_s and m_s appropriate for the IIA string. Using our relationship $R_{11} = (m_p)^{-1}(g_s)^{\frac{2}{3}}$ we find that $R_{11} = \frac{g_s}{m_s}$ and $m_p^3 = \frac{m_s^3}{g_s}$. Substituting into the above tensions we find that $T_2 = \frac{(m_s)^3}{g_s}$, $T_4 = \frac{(m_s)^5}{g_s}$ and $T_2 = \frac{(m_s)^6}{g_s^2}$. We observe the characteristic inverse coupling constant dependence that is typical of non-perturbative solitons. The 2 and 4-branes are Dirichlet branes and have a $\frac{1}{g_s}$ dependence while the fivebrane has a $\frac{1}{g_s^2}$ dependence. This coupling constant dependence of the tension of Dirichlet branes is universal and is related to the occurance of open strings in the theory.

In fact, one can account from M theory for two of the remaining branes that are associated with the IIA string. The 0-brane and the 6-brane arise from the pp-wave and Kaluza-Klein monopole solutions in eleven-dimensional supergravity respectively. The eight brane is associated with massive supergravity in ten dimensions and its connection to eleven dimensions is unclear.

7.3 Simple Superbranes

In this section we give the complete dynamics of simple superbranes [151,157,173] which, we recall, depend only on the embedding coordinates $Z^{\underline{N}} = (X^{\underline{n}}, \Theta^{\underline{\alpha}})$. The action for a simple super p-brane is given by

$$A = A_1 + A_2 \tag{7.3.1}$$

The first term is given by

$$A_1 = -T \int d^{p+1}\xi \sqrt{-det g_{mn}} \tag{7.3.2}$$

where

$$g_{mn} = \partial_n Z^{\underline{N}} \partial_m Z^{\underline{M}} g_{\underline{NM}}, \tag{7.2.4}$$

$$g_{\underline{NM}} = E_{\underline{N}}{}^{\underline{a}} E_{\underline{M}}{}^{\underline{b}} \eta_{\underline{ab}} \tag{7.3.5}$$

where $E_{\underline{N}}{}^{\underline{A}}$ is the superviebein of the background superspace. The second term in the action of equation (7.3.1) is given by

$$A_2 = -\frac{T}{(p+1)!} \int d^{p+1}\xi \epsilon^{n_1 \ldots n_{p+1}} \partial_{n_1} Z^{\underline{M}_1} E_{\underline{M}_1}^{\underline{A}_1} \ldots \partial_{n_{p+1}} Z^{\underline{M}_{p+1}} E_{\underline{M}_{p+1}}^{\underline{A}_{p+1}} B_{\underline{A}_1 \cdots \underline{A}_{p+1}} \tag{7.3.6}$$

In this expression, $B_{\underline{A}_1 \cdots \underline{A}_{p+1}}$ is a background superspace $p+1$ gauge field referred to superspace tangent indices. Its corresponding superspace $p+1$ form is

$$B = \frac{1}{(p+1)!} E^{\underline{A}_1} \ldots E^{\underline{A}_{p+1}} B_{\underline{A}_1 \cdots \underline{A}_{p+1}} = dZ^{\underline{N}_{p+1}} \ldots dZ^{\underline{N}_1} B_{\underline{N}_1 \cdots \underline{N}_{p+1}} \tag{7.3.7}$$

where $E^{\underline{A}} = dZ^{\underline{N}} E_{\underline{N}}{}^{\underline{A}}$.

The geometry of the target superspace is described as in section 4 using supervielbeins, but now also with the addition of the gauge field B. The covariant objects are the torsions and curvatures as before, but now we have in addition the gauge field strength of the gauge field B. The corresponding superfield strength H is the exterior derivative, in superspace of course, acting on the superspace gauge field; i.e. $H = dB$. The superspace torsions and curvatures satisfy Bianchi identities as in section four, but in addition we have the identity

$$D_{\underline{A}_1} H_{\underline{A}_2 \cdots \underline{A}_{p+2}} + T_{\underline{A}_1 \underline{A}_2}{}^{\underline{B}} H_{\underline{B} \underline{A}_3 \cdots \underline{A}_{p+1}} + \text{super cyclic permutations} = 0 \tag{7.3.8}$$

In the case that the background superspace is flat then the supervielbein are given as in equation (7.2.7). However, the gauge field B also has a non-trivial form. The resulting torsions all vanish except for

$$T_{\underline{\alpha\beta}}{}^{\underline{a}} = i(\gamma^{\underline{a}} C^{-1})_{\underline{\alpha\beta}}, \tag{7.3.9}$$

and

$$H_{\underline{\alpha\beta}\underline{a}_1 \cdots \underline{a}_p} = -i(-1)^{\frac{1}{4}p(p-1)} (\gamma_{\underline{a}_1 \cdots \underline{a}_p} c^{-1})_{\underline{\alpha\beta}} \tag{7.3.10}$$

In terms of forms we may express these results as

$$T^{\underline{a}} = \frac{1}{2} i d\bar{\Theta} \gamma^{\underline{a}} d\Theta \tag{7.3.11}$$

and

$$H = \frac{i}{2p!} E^{a_p} \ldots E^{\underline{a}_1} d\bar{\Theta} \gamma_{\underline{a}_1 \cdots \underline{a}_p} d\Theta \tag{7.3.12}$$

Since H is an exact form $dH = 0$. Using this on equation (7.3.12) and from the form of $E^{\underline{a}}$ of equation (7.2.7) we find that $dH = 0$ is equivalent to the condition [173]

$$d\bar{\Theta} \gamma_{\underline{a}} d\Theta d\bar{\Theta} \gamma^{\underline{a} \underline{b}_1 \cdots \underline{b}_{(p-1)}} d\Theta = 0 \tag{7.3.13}$$

Since Θ is Grassmann odd, $d\Theta$ is Grassmann even and hence the above condition must hold for a Grassmann even spinor of the appropriate type that is complex, Weyl, Majorana or Majorana-Weyl, but is otherwise arbitrary. This means in effect that we can discard the Θ from the above equation provided we include a projector if the spinor is Weyl, and symmetrise on all the free spinor indices if it is Majorana and on the first and third and second and fourth indices separately if it is complex. For example, if Θ is Majorana then we find the identity

$$(\gamma_{\underline{a}}C^{-1})_{(\underline{\alpha}\underline{\beta}}(\gamma^{\underline{a}\underline{b}_1\ldots\underline{b}_{p-1}}C^{-1})_{\underline{\delta}\underline{\epsilon})} = 0 \tag{7.3.14}$$

These identities can be reformulated using the appropriate Fierz identity and it can then be shown [173] that they hold if the number of fermion and boson degrees of freedom of the brane are equal. We have already used this condition to find the simple branes in ten and eleven dimensions.

Finally it remains to discuss the κ-symmetry of the action of equation (7.3.1). The variation of the coordinates $Z^{\underline{N}}$ under this symmetry when referred to the tangent basis are given by

$$\delta Z^{\underline{N}}E_{\underline{N}}{}^{\underline{a}} = 0,\ \delta Z^{\underline{N}}E_{\underline{N}}{}^{\underline{\alpha}} = (1+\Gamma)^{\underline{\alpha}}{}_{\underline{\beta}}\kappa^{\underline{\beta}} \tag{7.3.15}$$

where the local parameter $\kappa^{\underline{\beta}}$ is a world surface scalar, but a tangent space spinor. The matrix Γ is given by

$$\Gamma = \frac{(-1)^{\frac{1}{4}(p-2)(p-1)}}{(p+1)!\sqrt{-detg_{nm}}}\epsilon^{n_1\ldots n_{p+1}}\partial_{n_1}Z^{\underline{N}_1}E^{\underline{a}_1}_{\underline{N}_1}\ldots\partial_{n_{p+1}}Z^{\underline{N}_{p+1}}E^{\underline{a}_{p+1}}_{\underline{N}_{p+1}}\gamma_{a_1\ldots a_{p+1}} \tag{7.3.16}$$

The matrix Γ obeys the remarkably simple property $\Gamma^2 = 1$ and as a result $\frac{1}{2}(1\pm\Gamma)$ are projectors. Clearly a κ of the form $\kappa = \frac{1}{2}(1-\Gamma)\kappa^1$ leads to no contribution to the symmetry to which therefore only half of the components of κ actually contribute. The remaining half allows us to gauge away the corresponding half of Θ. This property of κ symmetry is essential for getting the correct number of on-shell fermion degrees of freedom namely 8. In fact the action is only invariant under κ symmetry if the background fields obey their equations of motion.

For the case when the background superspace is flat the above κ-symmetry reduces to

$$\delta X^{\underline{n}} = \frac{i}{2}\bar{\Theta}\gamma^{\underline{n}}\delta\Theta,\ \delta\Theta = (1+\Gamma)\kappa \tag{7.3.17}$$

The form of the redundancy discussed above in the κ-invariance leads to very difficult problems when gauge fixing the κ-symmetry and so quantizing the theory. Imposing a gauge condition on the spinor Θ to fix the κ-symmetry can only fix part of the symmetry and leaves unfixed that part of κ which is given by κ^1. A little thought shows that as a result the ghost action itself will have a local symmetry which must be itself fixed. In fact, this process continues and results in an infinite number of ghosts for ghosts corresponding to the infinite set of invariances. This would not in itself be a problem, but it has proved impossible to find a Lorentz-covariant gauge-fixed formulation. The problem of quantizing can also be seen from the viewpoint of the Hamiltonian approach where one finds that the system possess first and second class constraints that cannot be separated in a Lorentz-covariant manner. The Green-Schwarz action, for the string has so far defied attempts to quantize it in a truly covariant manner.

7.4 D-branes Dynamics

As we discussed, D-branes have a vector field A_n living on their world surface in addition to the embedding coordinates $X^{\underline{n}}$, $\Theta^{\underline{\alpha}}$. The action for the D-branes is a generalization of that of the simple brane of equation (7.3.1) to include this vector field A_n. In the absence of background fields other than the background metric and when $\Theta^{\underline{\alpha}} = 0$ the action is just the Born-Infeld action

$$-T \int d^{p+1}\xi \sqrt{-det(g_{nm} + F_{nm})} \tag{7.4.1}$$

where $F_{mn} = 2\partial_{[m} A_{n]}$. The full action was found in references [147-150] by using κ-symmetry. Its precise form and the relationship to Dirichlet open strings can be found in the lectures of Costas Bachas in this volume.

7.5 Branes in M Theory

As we have discussed above, there exists a 2-brane and a 5-brane in eleven dimensions. The 2-brane is a simple brane and so its dymanics are those given in section 7.3. The dynamics of the fivebrane is the subject of this section. As we discussed in section 7.2 the fivebrane contains five scalar fields and a 16-component spinor corresponding to the breaking of translations and supersymmetry by the fivebrane. It also contains, living on its world surface, an antisymmetric second-rank tensor gauge field whose field strength obeys a self-duality condition. As this condition is required to obtain the correct number of degrees of freedom, we only need it to hold at the linearized level and it is an important feature of the fivebrane dynamics that the self-duality condition in the full theory is a very non-linear condition on the third-rank field strength.

As for other branes, the bosonic indices of the fields on the fivebrane can be decomposed into longitudinal and transverse indices (i.e. for the world target space indices $\underline{n} = (n, n'))$ according to the decomposition of the eleven-dimensional Lorentz group $SO(1,10)$ into $SO(1,5) \times SO(5)$. The corresponding decomposition of the eleven dimensional spin group $Spin(1,10)$ is into $Spin(1,5) \times Spin(5)$. These spin groups divided by Z_2 are isomorphic to their corresponding Lorentz groups in the usual way. In fact $Spin(5)$ is isomorphic to the group $USp(4)$ which is defined below equation (1.3.18). Thus the fivebrane possesses a $Spin(1,5) \times Spin(5)$ or $Spin(1,5) \times USp(4)$ symmetry.

We now assign the fields of the fivebrane to multiplets of this symmetry. The five real scalars are of course $Spin(1,5)$ singlets, but belong to the vector representation of $SO(5)$, $X^{n'}, n' = 6, \ldots 10$. This representation corresponds to the second-rank anti-symmetric tensor representation $\phi_{ij},\ i,j = 1, \ldots 4$, of the isomorphic $USp(4)$ group which is traceless with respect to the ant-symmetric metric Ω^{ij} of this group. The gauge field B_{mn} is a singlet under $USp(4)$. We recall from section one that in six dimensions one cannot have Majorana spinors, but can have symplectic Majorana spinors and even symplectic Majorana-Weyl spinors. The fermions of the fivebrane belong to the four-dimensional vector representation of $USp(4)$ and are $USp(4)$ symplectic Majorana-Weyl spinors and so obey equation (1.3.17). Since they are Weyl we can work with their Weyl projected components which take only four values as opposed to the usual $8 = 2^3$ components for a six-dimensional spinor. The spinor indices of the groups $Spin(1,5)$ and $USp(4)$ are denoted by $\alpha, \beta, ... = 1, ..., 4$ and $i, j, ... = 1, ..., 4$ respectively and thus the spinors carry the indices Θ^i_α, $\alpha = 1, \ldots 4$, $i = 1, \ldots 4$. The spinors therefore have the required sixteen real components.

It is instructive to examine how this spinor arises from the original eleven-dimensional spinor $\Theta^{\underline{\alpha}}$ in terms of which the fivebrane dynamics was first formulated. Although we began with spinor indices $\underline{\alpha}$ that take thirty-two values we can, as above for all branes, split these indices into two pairs of indices each taking sixteen values $\underline{\alpha} = (\alpha, \alpha')$. In the final six-dimensional expressions the spinor indices are further written according to the above decomposition of the spin groups and we take $\alpha \to \alpha i$ and $\alpha' \to {}^{i}_{\alpha}$ when appearing as superscripts and $\alpha \to \alpha i$ and $\alpha' \to {}^{\alpha}_{i}$ when appearing as subscripts. It should be clear whether we mean α to be sixteen- or four-dimensional depending on the absence or presence of $i, j, ...$ indices respectively. For example, we will write $\Theta^{\alpha'} \to \Theta^i_\alpha$.

The fields of the fivebrane belong to the so-called $(2,0)$ tensor multiplet which transforms under $(2,0)$ six-dimensional supersymmetry. The $(2,0)$ notation means that the supersymmetry parameter is a $USp(4)$ symplectic Majorana-Weyl spinor. By contrast, we note that $(1,0)$ supersymmetry means that the supersymmetry parameter is a $USp(2)$ symplectic Majorana-Weyl spinor while, if the parameter is just a $USp(4)$ symplectic Majorana spinor, the supersymmetry would be denoted by $(2,2)$.

The classical equations of motion of the fivebrane in the absence of fermions and background fields are [137]

$$G^{mn}\nabla_m\nabla_n X^{a'} = 0 \ , \tag{7.5.1}$$

and

$$G^{mn}\nabla_m H_{npq} = 0. \tag{7.5.2}$$

where the world surface indices are $m, n, p = 0, 1, ..., 5$ and $a, b, c = 0, 1, ..., 5$ for world and tangent indices respectively. The transverse indices are $a', b' = 6, 7, 8, 9, 10$. We now define the symbols that occur in the equation of motion. The usual induced metric for a p-brane is given, in static gauge and flat background superspace, by

$$g_{mn} = \eta_{mn} + \partial_m X^{a'}\partial_n X^{b'}\delta_{a'b'} \ . \tag{7.5.3}$$

The covariant derivative in the equations of motion is defined with the Levi-Civita connection with respect to the metric g_{mn}. Its action on a vector field T_n is given by

$$\nabla_m T_n = \partial_m T_n - \Gamma_{mn}{}^{p} T_p \tag{7.5.4}$$

where

$$\Gamma_{mn}{}^{p} = \partial_m\partial_n X^{a'}\partial_r X^{b'} g^{rs}\delta_{a'b'} \ . \tag{7.5.5}$$

We define the world surface vielbein associated with the above metric in the usual way $g_{mn} = e_m{}^a\eta_{ab}e_n{}^b$. There is another inverse metric G^{mn} which occurs in the equations of motion and it is related to the usual induced metric given above by the equation

$$G^{mn} = (e^{-1})_c{}^m \eta^{ca} m_a{}^d m_d{}^b (e^{-1})_b{}^m \ . \tag{7.5.6}$$

where the matrix m is given by

$$m_a{}^b = \delta_a{}^b - 2h_{acd}h^{bcd} \ . \tag{7.5.7}$$

The field h_{abc} is an anti-symmetric three-form which is self-dual:

$$h_{abc} = \frac{1}{3!}\varepsilon_{abcdef}h^{def} \ , \tag{7.5.8}$$

but it is not the curl of a three-form gauge field. It is related to the field $H_{mnp} = 3\partial_{[m}B_{np]}$ which appears in the equations of motion and is the curl of a gauge field, but H_{mnp} is not self-dual. The relationship between the two fields is given by

$$H_{mnp} = e_m{}^a e_n{}^b e_p{}^c (m^{-1})_c{}^d h_{abd} \ . \tag{7.5.9}$$

Clearly, the self-duality condition on h_{abd} transforms into a condition on H_{mnp} and vice versa for the Bianchi identity $dH = 0$.

The fivebrane equations of motion were found in reference [137] for arbitrary background fields and to first-order in the fermions; we refer the reader to this reference for the construction.

The above fivebrane equations of motion were found using the embedding formalism. This formalism including the superspace embedding condition of equation (7.5.12) were first given within the context of the superparticle in reference [178]. Further work on the superparticle was carried out in references [195] and [196]. The superembedding formalism was first applied to p-branes in references [131,132, 197, 133] and to the M theory fivebrane in [134]. The above fivebrane equations of motion were derived in [137]. The form of the fivebrane equations of motion and the relationship to the D4-brane of the IIA theory was discussed in reference [137]. Reference[138] contains a review of the embedding formalism. The super embedding approach was also used to find p-brane actions and equations of motion in references [141] and [198].

We now briefly explain the simple idea that underlies this formalism. We consider a target or background superspace $\underline{M}$ in which a brane sweeps out a superspace M. On each of these superspaces we have a set of preferred frames or supervielbeins. The frame vector fields on the target manifold $\underline{M}$ and the fivebrane world surface M are denoted by $E_{\underline{A}} = E_{\underline{A}}^{\underline{M}}\partial_{\underline{M}}$ and $E_A = E_A{}^M\partial_M$ respectively. We recall that we use the superspace index convention that $\underline{N}, \underline{M}, \ldots$ and $\underline{A}, \underline{B}, \ldots$ represent the world and tangent indices of the target superspace $\underline{M}$ while $N, M, \ldots$ and $A, B, \ldots$ represent the world and tangent space indices of the embedded superspace M.

Since the supermanifold M is embedded in the supermanifold $\underline{M}$, the frame vector fields of M must point somewhere in $\underline{M}$. Exactly where they point is encoded in the coefficients $E_A^{\underline{A}}$ which relate the vector fields E_A and $E_{\underline{A}}$, i.e.,

$$E_A = E_A^{\underline{A}}E_{\underline{A}} \tag{7.5.10}$$

Applying this relationship to the coordinate $Z^{\underline{M}}$ we find the equation

$$E_A^{\underline{A}} = E_A{}^N\partial_N Z^{\underline{M}}E_{\underline{M}}^{\underline{A}} \ . \tag{7.5.11}$$

It is now straightforward to express the torsion and curvature tensors of M in terms of those of $\underline{M}$ plus terms involving a suitable covariant derivative of $E_A{}^{\underline{A}}$; one finds that

$$\nabla_A E_B{}^{\underline{C}} - (-1)^{AB} \nabla_B E_A{}^{\underline{C}} + T_{AB}{}^C E_C{}^{\underline{C}} = (-1)^{A(B+\underline{B})} E_B{}^{\underline{B}} E_A{}^{\underline{A}} T_{\underline{AB}}{}^{\underline{C}} \tag{7.5.12}$$

where the derivative ∇_A is covariant with respect to both embedded and target superspaces, that is, it has connections which act on both underlined and non-underlined indices.

The tangent space of the superspaces $\underline{M}$ and M can be divided into their Grassmann odd and even sectors, that is the odd and even pieces of M are spanned by the vector fields E_α and E_a respectively and similarly for $\underline{M}$. The superembedding formalism has only one assumption; the odd tangent space of M should lie in the odd tangent space of $\underline{M}$. This means that

$$E_\alpha{}^{\underline{a}} = 0 \tag{7.5.13}$$

To proceed one substitutes this condition into the relationships of equation (7.5.12) between the torsions and curvatures of M in terms of $\underline{M}$ and analyses the resulting equations in order of increasing dimension. For example at dimension zero one finds the equation

$$E_{\underline{a}}{}^{\underline{a}} E_{\underline{b}}{}^{\underline{b}} T_{\underline{ab}}{}^{\underline{c}} = T_{\underline{ab}}{}^{c} E_c{}^{\underline{c}} \tag{7.5.14}$$

This procedure is much the same as for the usual superspace Bianchi identities for super Yang-Mills and supergravity theories. For the twobrane and fivebrane of M-theory this procedure yields:

(a) the equations of motion for the fields of the brane,
(b) the equations of motion for the background fields,
(c) the geometry of the embedded manifold, that is its torsions and curvatures.

Finding the equations of motion of the supergravity background fields means finding the superspace constraints on the target superspace torsions and curvatures.

Although the embedding condition of equation (7.5.13) is very natural in that, as we saw in section 4, all the geometry of superspace is contained in the odd sectors of the tangent space of supermanifolds, its deeper geometrical significance is unclear. However, the power of this approach became evident once it was shown to lead to the correct dynamics for the most sophisticated brane, the M-theory fivebrane. Although the above results hold for many branes, they do not hold for all branes unless the embedding condition is in general suplemented by a further condition.

The appearance of a metric in the equations of motion which is different to the usual induced metric has its origins in the fact that the natural metric that appears on the world surface of the fivebrane has an associated inverse vielbein denoted by $(E^{-1})_a{}^m$ which is related in the usual way through $G^{mn} = (E^{-1})_a{}^m (E^{-1})_b{}^n \eta^{ab}$. The relationship between the two inverse vielbeins is $(e^{-1})_a{}^m = (m^{-1})_a{}^b (E^{-1})_b{}^m$.

There is another formulation of the dynamics of the fivebrane given in references [141] and [142]. Although this formulation involves an action this is not necessarily an advantage as has been pointed out in reference [143]. Reference [141] used the superembedding formalism, but in a different way to that considered in this section.

The above equation of motion for the fivebrane admits two interesting solutions corresponding to onebranes (self-dual strings) [144] and threebranes [145]. The threebrane solution has been used [146,177] to derive the complete low energy effective action of the $N = 2$ Yang-Mills theory. Indeed, the fivebrane may self-intersect on a four manifold which can be considered as this 3-brane. Just like the monopoles discussed at the beginning of this section we may consider the moduli space of threebranes. The low-energy motion of several threebranes can then be described in terms of their moduli which now depend on the world surface of the threebrane. The corresponding action for the low energy motion of these threebranes is then a four dimensional quantum field theory with $N = 2$ supersymmetry which is none other that the complete chiral effective action of the spontaneously broken $N = 2$ supersymmetry gauge theory. Thus from the **classical** dynamics of the fivebrane we can deduce the chiral effective action of the spontaneously broken $N = 2$ Yang-Mills theory which includes an infinite number of instanton corrections, only the first two of which can be calculated using known instanton techniques.

8. String Duality

In this section we use our previous discussions on supergravity theories and brane dynamics to give some of the evidence for string dualities. We restrict our attention to the relationships between the IIA and IIB strings and M theory and only aim to give some introductory remarks. A more complete discussion can be found in the lectures of Ashoke Sen in this volume.

Dualities in string theory can be of several different types. There are dualities which relate one string theory to another and dualities which relate a given string theory to itself. We refer to the latter as self-dualities. Dualities can also be classified by whether they map the perturbative regime of the theory to the same perturbative regime or whether they map the perturbative regime to the non-perturbative regime. Roughly speaking, the perturbative and non-perturbative regimes correspond to the small and large coupling constant regimes of the theory respectively. Currently, our ability to calculate systematically in quantum field theories is limited to the perturbative regime where one can extract meaningful answers by using perturbation theory using the coupling constant as the parameter. In fact, in the absence of a special non-renormalization theorem, such expansions, which are derived from the path integral, are not convergent. However, for certain quantum field theories the first few terms give increasingly more accurate results and one can by resumming the series find the result to any desired accuracy as a matter of principle. To determine the range of values of the coupling constant for which the theory is in its pertubative regime is not always very straightforward and it is found by studying the behaviour of the amplitudes as a complex function of the coupling constant. In some theories there are no values of the coupling conatant for which the perturbation series can be resummed and so these theories have in effect no perturbative regime.

In fact, many interesting phenomena are outside the range of perturbation theory. Such phenomena are often related to the presence of instantons and solitons in the theory. It is of course one of the outstanding problems of theoretical physics to understand properly non-perturbative effects, such as quark confinement.

Dualities which map from the perturbative to the same perturbative regime of a theory are mapping from a regime of the theory which is relatively well understood to the same region. As a result, such dualities are straightforward to check. The best example of this type of duality in string theory is so called T-duality [183, 191, 192] which occurs in string theories that are compactified. The simplest example of T-duality occurs for the closed bosonic string theory compactified on a circle of radius R. One finds that the theory is invariant under the transformation $R \to \frac{\alpha'}{R}$. To be more precise, the mass spectrum of the physical states and their scattering amplitudes are invariant under this transformation [184].

More generally, we can consider the T-duality which occurs in a string theory which has been compactified on d of its dimensions. In string theory we can make separate compactifications for the left and right modes of the string and as a result the relavent torus is $T^{(d,d)} = \frac{\mathbf{R}^{(d,d)}}{\Lambda^{(d,d)}}$ where $\Lambda^{(d,d)}$ is the Lorentzian lattice that generates the torus. The lattice is formed from the momenta in the left and right directions (p_L, p_R). This lattice is just the momenta on the torus, while the Lorentzian signature arises from the scalar product encoded in the constraint $(L_0 - \bar{L}_0)\psi = 0$. This constraint also implies the lattice is even. Modular invariance of this string theory requires that $\Lambda^{(d,d)}$ be an even self-dual Lorentzian lattice. Clearly, the resulting string theory depends on the torus, or equivalently the self-dual lattice used in the compactification. It turns out that all even self-dual Lorentzian lattices that can occur in such string compactifications are uniquely classified by the coset [190]

$$O(d;\mathbf{R}) \otimes O(d;\mathbf{R}) \backslash O(d,d;\mathbf{R})/O(d,d;\mathbf{Z}) \tag{8.1}$$

Consequently, the resulting string theories are also classified by this coset. Clearly, we can act with the group $O(d,d;\mathbf{Z})$ on the basis vectors of a given lattice and it will take that lattice to itself. Hence we must also divide out by this group. The d^2 parameters of the coset can be accounted for by the d^2 expectation values of the metric and anti-symmetric tensor of the string in the compactified directions [161]. The corresponding string theory is invariant under the duality symmetry $O(d,d;\mathbf{Z})$. The T-dualities map a given theory to itself and are therefore an example of a self-duality. The T-duality symmetries are well understood and have been shown to hold to all orders in string perturbation theory [184].

Dualities which map from perturbative to non-perturbative regimes are not very well understood. The problem with verifying such dualities is that they relate quantities in the perturbative regime, which in general can be reliably computed, to quantities in the non-perturbative regime which in general can not be calculated. Therefore such dualities are in general difficult to test. However, for certain supersymmetric theories, some properties of the theory in the non-perturbative regime are reliably known. One example is the low-energy effective action of a string theory that possesses a supersymmetry with 32 supercharges. Such a low-energy effective action is none other than a supergravity theory. We recall that supergravity theories can have an underlying supersymmetry algebra that has at most 32 supercharges. Furthermore, for one of the maximal supergravity theories, the action is essentially uniquely determined by the underlying supersymmetry algebra. For example, if we are in ten dimensions then a supersymmetry algebra with 32 supercharges has either two Majorana-Weyl spinors of the same chirality or two Majorana-Weyl spinors of the opposite chirality and the corresponding unique supergravity theories are the IIB and IIA supergravities respectively. Therefore, given any string theory which is invariant under a space-time supersymmetry that has 32 supercharges, then the low energy effective action of this string theory must be the unique maximal supergravity in that dimension for the particular maximal supersymmetry algebra involved. Hence if we suspect that two string theories are related by a duality symmetry one test we can apply is to find if the suspected duality relates their low energy effective actions which are the unique corresponding maximal supergravities. We can turn this approach around and consider which string theories have low-energy effective theories (i.e supergravity theories) that can be related by a duality symmetry and then consider if these symmetries can be promoted to string dualities.

A further consequence of these considerations concerns string theories that are invariant under a space-time supersymmetry with 32 supercharges and that are self-dual. Since their low energy effective actions in both the strong and weak coupling limits are the same low-energy supergravity, it follows that this self-duality symmetry must be a symmetry of the maximal supergravity theory. Thus to look for self-duality symmetries we can examine the symmetries of the maximal supergravities and wonder which to promote to a string duality.

One other property that we can reliably establish in all coupling constant regimes of a supersymmetric theory is the existence of the BPS states discussed in section 3. Recall from there that these supermultiplets have fewer states than a supermultiplet with a generic mass and their existence relies on their mass being equal to one of the central charges that appear in the supersymmetry algebra. For these states to disappear as one changes the coupling constant would require the abrupt existence of additional degrees of freedom and this does not usually occur.

Equipped with this strategy we now examine the relationships between the maximal supergravity theories. For simplicity let us consider the relationships between the maximal supergravities in eleven, ten and nine dimensions. These are the eleven-dimensional supergravity and the IIA and IIB theories in ten dimensions and the single nine-dimensional maximal supergravity. Their relationships are set out in figure 8.1. We recall from section 6 that IIA supergravity can be obtained from eleven-dimensional supergravity by reduction on a circle and that the radius R of this circle and the coupling constant g_s of the IIA string theory are related by equation (6.2.11). The IIB theory has a supersymmetry algebra with two Majorana-weyl spinors of the same chirality and so can not be obtained from eleven-dimensional supergravity by dimensional reduction. It does, however, possess an $SL(2,\mathbf{R})$ symmetry. Finally, we found that if we reduce either the IIA or IIB theory to nine dimensions then we obtain the same unique supergravity.

Let us now consider what these relationships suggest for the corresponding string theories. Let us begin by considering which of the theories in ten dimensions could be self-dual. The obvious candidate is the IIB theory whose low-energy effective action has an $SL(2,\mathbf{R})$ symmetry [110]. It is natural to consider this group or one of its subgroups to be a symmetry of the IIB string theory. The IIB supergravity has two second-rank tensor gauge fields $B^1_{\mu\nu}$ and $B^2_{\mu\nu}$ which transform into each other under $SL(2,\mathbf{R})$ as in equation (6.3.39). From the viewpoint of the IIB string these fields belongs to the $NS\otimes NS$ and

$R \otimes R$ sectors respectively. As we explained in section 7.2 these target space gauge fields couple to the 1-branes which are the IIB string itself and the Dirichlet 1-brane.

Let us consider an object which is charged with respect to both gauge fields $B^1_{\mu\nu}$ and $B^2_{\mu\nu}$ and denote the charges by q^1 and q^2 respectively. Since the charges they carry are proportional to their corresponding field strengths they are rotated into each other under $SL(2,\mathbf{R})$. Using equation (6.3.39), we conclude that under an $SL(2,\mathbf{R})$ transformation the charges are changed according to [110]

$$\begin{pmatrix} q^{1\prime} \\ q^{2\prime} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} \tag{8.2}$$

However, the analogue of the Dirac quantization condition for branes [191] implies that the charges are always quantized in integers once we adopt suitable units. It is easy to see that the maximal subgroup of $SL(2,\mathbf{R})$ which preserves such a charge quantization is $SL(2,\mathbf{Z})$. The charge quantization condition can be expressed as the statement that one can only ever find whole fundamental strings and D1-branes and their bound states.

Thus we can at most choose the group $SL(2,\mathbf{Z})$ to be a symmetry of IIB string theory. This symmetry includes the transformation of equation (6.3.40) and so takes the weak coupling regime theory to the strong coupling regime of the IIB string.

Clearly, in the IIB string theory there exist elementary string states of unit charge with respect to the gauge field $B^1_{\mu\nu}$, but no charge with respect to the other gauge field $B^2_{\mu\nu}$. One such state has charge $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Acting with an $SL(2,\mathbf{Z})$ transformation we find, using equation (8.2), states with the charges $\begin{pmatrix} a \\ c \end{pmatrix}$ where a and c are integers and are the charges with respect $B^1_{\mu\nu}$ and $B^2_{\mu\nu}$ respectively. While the states charged with respect to just $B^1_{\mu\nu}$ are the elementary string states which occur in the perturbative domain, the $SL(2,\mathbf{Z})$ rotated states carry charge with respect to $B^1_{\mu\nu}$ and $B^2_{\mu\nu}$ and are non-perturbative in nature. Indeed, the elementary states of the IIB string cannot be charged with respect to $B^2_{\mu\nu}$. Hence, the existence of an $SL(2,\mathbf{Z})$ symmetry in the theory implies that there must exist states with the above charges for all integers a and c which are relatively prime, that is have no common factor. The latter condition follows from the $SL(2,\mathbf{Z})$ condition $ad-bc=1$, since in this case $ad-bc$ would contain a common factor and so could not equal one. It was shown [164], by considering parallel Dirichlet 1-branes, that these states actually exist and this provides us with an important consistency check on the conjectured $SL(2,\mathbf{Z})$ symmetry in the string theory.

It was in reference [152] that it was first suggested that string theory could possess a duality symmetry that transformed weak- to strong-coupling regimes. In particular, it was known that that the low-energy effective action for the heterotic string theory compactified on T^6 possessed an $SL(2,\mathbf{R})$ symmetry and the authors of reference [152,153] suggested that the $SL(2,\mathbf{Z})$ subgroup be a symmetry of the the heterotic string theory compactified on T^6. By considering the action of this symmetry on BPS states, in references [152-154] and [155] evidence for this conjecture was given. In reference [156], it was suggested that the $SL(2,\mathbf{Z})$ subgroup of the known $SL(2,\mathbf{R})$ [110] symmetry of the IIB low energy effective action be a symmetry of the full IIB theory. Indeed, reference [156] conjectured that if we changed the field from $\mathbf{R}$ to $\mathbf{Z}$ in the known coset symmetry of the low energy effective action (i.e. supergravity theory) then we will find a symmetry of the corresponding string theory. To be precise it was suggested that since the supergravity theories possessed scalars which were contained in coset spaces G/H that the corresponding string theory have a symmetry $G(\mathbf{Z})$ where $G(\mathbf{Z})$ is the group G with its field changed form $\mathbf{R}$ to $\mathbf{Z}$. The coset space symmetries of the maximal supergravity theories are listed in table 6.1. For the IIA or IIB string compactified on a six-dimensional torus, the low-energy effective action is the $N=8$ supergravity in four dimensions and so we expect to find a $E_7(\mathbf{Z})$ symmetry. These references followed the pattern that occurs in the $N=4$ Yang-Mills theory which was earlier given in reference [193].

Now let us turn our attention to the relationship between the eleven-dimensional supergravity [106] and the IIA supergravity [107-109]. As explained in section 6.2 the former theory results from the latter if we compactify on a circle [107-109]. We found in equation (6.2.18) that the IIA string coupling g_s and the radius of compactification R_{11} are related by $g_s \propto R_{11}^{\frac{3}{2}}$. Clearly, in the strong-coupling limit of the IIA string i.e. as $g_s \to \infty$ the radius $R_{11} \to \infty$. However, in this limit the circle of compactification becomes flat and one expects to recover an eleven-dimensional theory whose low-energy limit is eleven-dimensional supergravity. This realization has lead to the conjecture [123],[124] that the strong coupling limit of IIA theory defines a consistent theory called M-theory which possesses eleven-dimensional Poincaé invariance and has eleven-dimensional supergravity as its low-energy effective action.

We now summarise some of the evidence for the existence of M theory. When compactifying the eleven-dimensional supergravity in section 6.2 in addition to the massless modes of the IIA supergravity theory we found Kaluza-Klein modes. These Kaluza-Klein modes have a mass $\frac{n}{R_{11}}$ which have charge $\frac{n}{R} = \frac{n}{R_{11}} g_s^{\frac{2}{3}}$ with respect to the $U(1)$ gauge field B_μ in the IIA theory that arises from the graviton in eleven dimensions. Since the low-energy effective action of M theory is eleven-dimensional supergravity, these Kaluza-Klein modes are also present in M theory. However, if we are to believe that the strong-coupling limit of IIA string theory is M theory then these Kaluza-Klein modes should be also be present in the IIA string. In fact, the Kaluza-Klein states belong to massive supermultiplets that have fewer states than a supermultiplet with a generic mass and so they are the so called BPS states considered in section 3 whose mass is equal to their central charge. As we discussed at the end of that section, the existence of BPS states must be present in the strong coupling regime of a theory if they are present in the weak-coupling regime and vice versa. As such, we should find the analogue of the Kaluza-Klein states at all coupling constant regimes of the IIA string theory.

Hence an essential test of the conjecture is to find the analogues of the Kaluza-Klein states in the IIA theory. The first evidence for the existence of these states in the IIA string was the realization that the IIA supergravity admitted solitonic states of the correct mass and charge [123]. The elementary states, that is perturbative states of the IIA string, are not charged with respect to the gauge field B_μ since it belongs to the $R \otimes R$ sector of the IIA string. However, as we discussed in section 6, Dirichlet 0-branes do couple to the B_μ gauge field in the $R \otimes R$ sector of the IIA string. The Dirichlet 0-branes are non-perturbatitve in nature, but this identification is in accord with fact that the masses of the

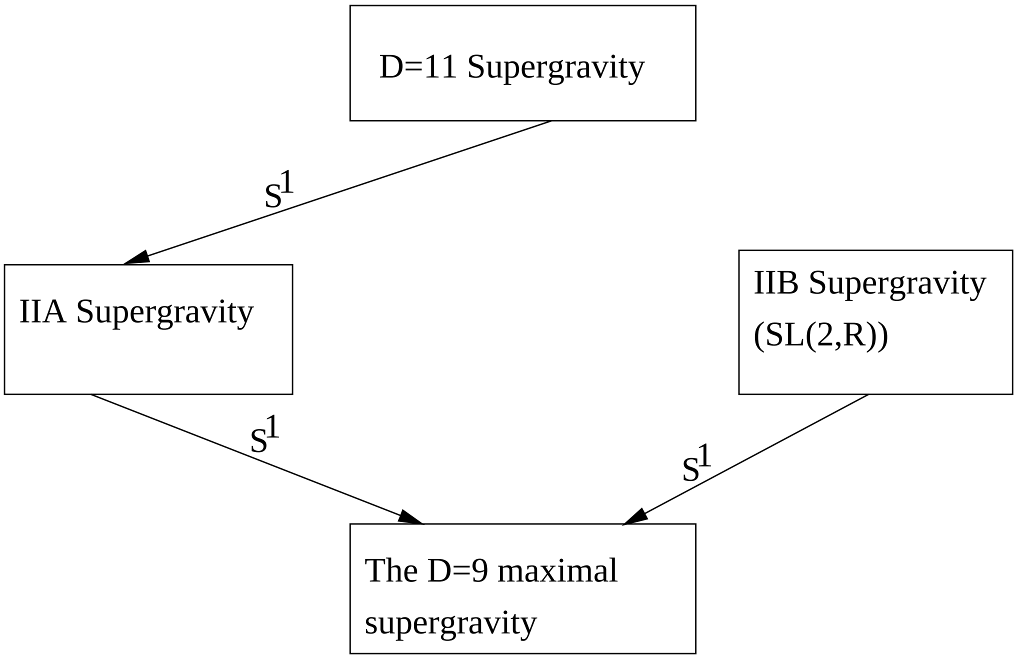
Kaluza-Klein states are given by $\frac{n}{R_{11}} = n\frac{m_s}{g_s}$ when expressed in terms of the string coupling constant. These states becomes very large as the coupling constant becomes small, in a manner typical of non-perturbative solitons. In the limit of large coupling they become massless. The appearance of an infinite number of massless modes indicates the transition to a theory in which a dimension has become decompactified.

The dynamics of a single such Dirichlet 0-brane was outline in section 7. In a flat background, it consists of a supersymmetric generalization of Born-Infeld action in one dimension. At low-energy this theory in just the dimensional reduction of $D = 10$ $U(1)$ gauge theory to one dimension. Such a single Dirichlet 0-brane has the correct mass and charge to be identified with the lowest mass Kaluza-Klein state. It is thought that the higher mass Kaluza-Klein states can be identified with bound states of Dirichlet 0-branes at threshold, that is, bound states that have the same energy as the lowest energy state of two particles. The dynamics of several parallel Dirichlet 0-branes is described by a supersymmetric generalization of Born-Infeld theory which now carries a $U(N)$ gauge group if N is the number of parallel Dirichlet 0-branes. At low energy, this theory is just a dimensional reduction of $D = 10$ Yang-Mills theory to one dimension. Just as one quantizes the point particle to discover that it corresponds to the Klein-Gordon equation in quantum field theory and quantizes a 1-brane (i.e. string) to discover its particle spectrum, one must quantize this supersymmetric generalization of non-abelian Born-Infeld theory to discover the states in the IIA string arising from the presence of the Dirichlet 0-branes. It is thought that this quantum mechanical system does indeed have bound states that have the correct mass and charge to be identified with the Kaluza-Klein states [182].

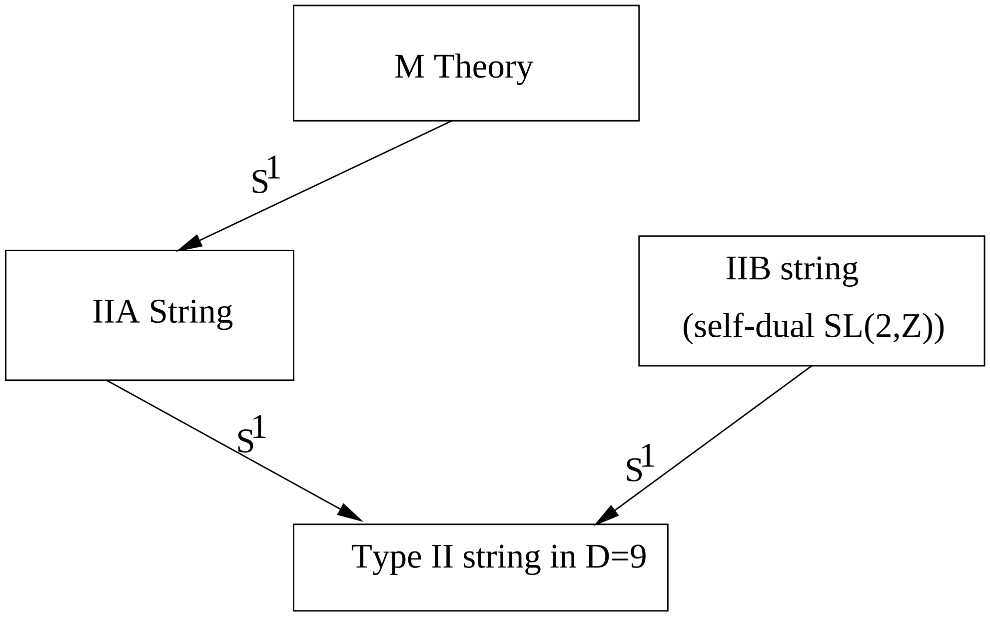
The last similarity between the maximal supergravity theories is the equivalence of the IIA and IIB supergravity theories when compactified on a circle [107]. Hence, if one compactifies the IIA and IIB string theories on circles of radius R_A and R_B respectively one obtains string theories in nine dimensions which have the same low effective action, since the maximal supergravity theory in this dimension is unique. This suggests that these two string theories in nine dimensions are really the same theory. However, the variables in which the two theories are found after compactification may be related in a non-trivial way. In fact, the two IIA and IIB string theories compactified on circles of radius R_A and R_B respectively are the same theory, but they are related by a T-duality transformation such that $R_A \to \frac{\alpha'}{R_B}$. [186-188]. In the limit $R_A \to \infty$ the theory decompactifies and one recovers the IIA string in ten dimensions. Just as for the reduction of the IIA theory, the radius of the circle is related to a scalar field in the theory in nine dimensions that appears from the metric in ten dimensions and as a result each value of the radius corresponds to a point in the moduli space of the theory in nine dimensions . The limit in which $R_A \to \infty$ is just a limit to a point in moduli space. Similarly, in the limit $R_B \to \infty$, which is also the limit $R_A \to 0$, we recover the IIB string theory in ten dimensions. Hence, different limits in the moduli space of the nine dimensional theory lead to different theories in ten dimensions. Starting with say the IIA theory in ten dimensions, one can compactify on a circle, carry out a T-duality transformation and then take the appropriate limit to recover the IIB theory in ten dimensions. Because of the need to take limits one cannot, in general, simply carry out the T-dualtiy directly in the ten-dimensional theories.

Of course the IIA supergravity theory in nine dimensions can be obtained from eleven-dimensional supergravity by compactification on a torus and so one may expect that the IIA string in nine dimensions can be obtained from compactification of M theory on a torus. The resulting nine-dimensional theories will inherit the isometries of the torus as symmetries. In fact it can be shown that the $SL(2, \mathbf{Z})$ which acts on the basis vectors of lattice which underlies the torus, can be identified with the $SL(2, \mathbf{Z})$ of the IIB theory after it has been compactified on a circle.

The relations between M theory and the IIA and IIB string theories in ten dimensions and the one string theory in nine dimensions discussed above are summarized in figure 8.2. The similarity, for the reasons explained above, between this table and talbe 8.1 for the corresponding supergravity theories is obvious. It is relatively straightforward, using similar arguments, to incorporate the heterotic and type I string theories into these pictures. We refer to the lectures of Ashoke Sen contained in this volume for a more complete treatment of string duality.



Relations between Maximal Supergravities



Relations between the IIA and IIB string theories and M Theory

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