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Covariant Path-Integral Approach to String Theory

(Lectures at the 3rd Jerusalem Winter School of Theoretical Physics)

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Preface

These lectures deal with the covariant path integral formalism, specifically as applied to the theory of closed oriented bosonic strings. There are other approaches to string theory, such as the covariant operator method, and both the path-integral and operator methods in light cone gauge, as well as the newer string field theories now under development. Most of these other approaches are described by the other lecturers at this school.

Each approach has its advantages and disadvantages. The operator methods allow a very straightforward description of the particle spectrum, but are terribly difficult to extend beyond one-loop order. On the other hand, the covariant path integral formalism in principle makes a clear statement about higher-order effects, and with a little effort one can use it to figure out the particle spectrum. One notable disadvantage of covariant methods is that the fermion vertex operators are complicated enough to have eluded description until this year. In any case, my own reading and research this year has concentrated on the covariant path integral method, and I will leave the other methods to the other lecturers.

This is a revised version of the lectures I gave in Jerusalem. The treatment of the measure for integrals over Teichmüller variables has been greatly expanded, incorporating work of Moore, Nelson, and Polchinski published in 1986. Little or nothing has been added here to their work, but I have tried to make some aspects of the presentation somewhat more explicit and concrete. I have also expanded the discussions of vertex functionals for massless particles. To make room for these expansions, I have cut back on the discussion of some of my own work on vertex functionals and radiative corrections, which was presented in Jerusalem but has been published since then (in Refs. 18 and 21).

I am grateful to J. Hughes for taking accurate notes of my lectures in Jerusalem, and to J. Polchinski for his invaluable help and advice in preparing this revised version.

1. History

I'll begin with a capsule history of string theory, as it led to the development of the covariant path integral formalism. First, before this formalism was initiated, Veneziano[1] in 1968 proposed a model 2-body scattering amplitude with various desirable properties:

$$\Gamma \propto A(s,t) + A(t,u) + A(u,s)$$
$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} = \int_0^1 dz z^{-1-\alpha(s)}(1-z)^{-1-\alpha(t)} \tag{1.1}$$

Here s,t,u are the Mandelstam kinematic variables (the center-of-mass frame squared energy in the channels $12 \to 34$, $13 \to 24$, and $14 \to 23$, respectively), and $\alpha(s)$ is the "Regge trajectory", taken here to be a linear function of s. The integral form of A was rapidly generalized to multiparticle amplitudes by Koba and Nielsen and others[2], with a final version given in 1970 by Donino and Sciuto[3]:

$$A(p_1 \ldots p_N) = |z_I - z_{II}||z_{II} - z_{III}||z_{III} - z_I| \times \int dz_1 dz_2 \ldots dz_N \prod_{i<j} |z_i - z_j|^{2\alpha' p_i \cdot p_j} \tag{1.2}$$
$$z_i \text{ cyclically ordered}$$
$$z_I, z_{II}, z_{III} \text{ fixed}$$

(Here z_I, z_{II}, and z_{III} are any three of the $z_1 \ldots z_N$, which are given arbitrary fixed values, and α' is the slope of Regge trajectories in this model. The full scattering amplitude Γ is given by summing (1.2) over all cyclically inequivalent permutations.) It was almost immediately realized that this amplitude is the solution to a dynamical problem in an electrostatic analogue model, and can be expressed as a path integral[4]

$$A(p_1 \ldots p_N) \propto \int \prod_{\mu,\sigma} dx^\mu(\sigma) \int_{cyclic\ order} dz_1 \ldots dz_N \times \exp\left(-\frac{T}{2}\int_{\sigma_2>0} d^2\sigma \frac{\partial x^\mu}{\partial \sigma^\alpha}\frac{\partial x^\nu}{\partial \sigma^\alpha}\eta_{\mu\nu}\right) \prod_{I=1}^{N} e^{ip_I \cdot x(z_I)} \tag{1.3}$$

(Here (σ^1, σ^2) are the coordinates of a point in the upper half-plane; the $x^\mu(\sigma)$ are integrated over all functions of σ, subject to the Neumann boundary condition that $\partial x^\mu / \partial \sigma^2 = 0$ for $\sigma^2 = 0$; T is a constant later called the string tension, related to the Regge slope by $\alpha' = 1/2\pi T$, and $x^\mu(z)$ is the value of $x^\mu(\sigma^1, \sigma^2)$ on the line $\sigma^1 = z, \sigma^2 = 0$.)

A little earlier, Virasoro[5] in 1969 had offered an alternative to (1.1) with one term instead of three. Path-integral techniques were then used in 1970 by Shapiro[6] to give the generalization of Virasoro's model to multiparticle amplitudes:

$$\Gamma(p_1 \ldots p_N) \propto \int \prod_{\mu,\sigma} dx^\mu(\sigma) \int d^2\sigma_1 \ldots d^2\sigma_N \times \exp\left(-\frac{T}{2}\int d^2\sigma \frac{\partial x^\mu}{\partial \sigma^\alpha}\frac{\partial x^\nu}{\partial \sigma^\alpha}\eta_{\mu\nu}\right)\prod_{I=1}^{N} e^{ip_I \cdot x(\sigma_I)} \tag{1.4}$$

The difference is that here σ^α runs over the whole plane, instead of the upper half plane, and the $e^{ip\cdot x}$ are integrated over the whole plane instead of the boundary $\sigma^2 = 0$.

Today we call (1.3) and (1.4) the tree approximation to the theories of the open and closed strings, respectively. It is the tree approximation, because the only singularities of (1.3) and (1.4) as functions of the momentum variables are poles, not cuts. It became clear early[7] that the way to include higher order corrections was to add topological complications to the plane or the upper half-plane: cut out holes with circular boundaries, or holes with opposite sides identified, or pairs of holes with corresponding points on their boundaries identified. This was observed to give a cut structure consistent with unitarity. However, it was now necessary to introduce certain "Teichmüller parameters", that characterize these topological complications. The question then arises, how to choose the appropriate weight function in integrating over these parameters? In this early period of string theory, the early 1970s, this could be done only by using the operator method or by invoking unitarity. A pure path-integral calculation of the weight functions for the Teichmüller parameters had to wait until the development of a two-dimensionally generally covariant formalism.

As it happened, a two-dimensionally generally covariant formalism was developed not in order to calculate the weight functions for integrals over Teichmüller parameters, but for several other purposes. First, in sorting out the particle states that contribute to poles in physical scattering amplitudes, it was found that the underlying two-dimensional field theory possessed an infinite Lie algebra of broken symmetries, the Virasoro algebra[8]. Formally, this could be recognized as the algebra of the group of infinitesimal coordinate transformations in one dimension (for open strings) or conformal transformations of the complex plane (for closed strings). In order to understand these symmetries as something more than a mathematical accident, it was necessary to replace the action, the argument of the exponentials in (1.3) and (1.4), with a two-dimensionally generally covariant expression. This was first done by Nambu in 1970[9]. His proposed action (for the closed string) was

$$\begin{aligned} I_{NAMBU} &= T \int d^2\sigma (\mathrm{Det} G)^{1/2} \\ G_{\alpha\beta}(\sigma) &\equiv \frac{\partial x^\mu(\sigma)}{\partial \sigma^\alpha} \frac{\partial x_\mu(\sigma)}{\partial \sigma^\beta} \end{aligned} \tag{1.5}$$

However, it is difficult to apply quantum mechanics to an action that cannot be expressed as a power series in the fields and their derivatives.

The modern form of the string action was introduced[10] rather later, in 1976, as an incidental by-product of the effort to introduce supersymmetry into the underlying two-dimensional field theory. A 2×2 metric $g_{\alpha\beta}(\sigma)$ was introduced along with the coordinate $x^\mu(\sigma)$, and used to write the argument of the exponential in Eq. (1.4) as a generally covariant action

$$I[x,g] = \frac{T}{2} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \frac{\partial x^\mu}{\partial \sigma^\alpha} \frac{\partial x^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \tag{1.6}$$

(It is also necessary to insert factors $\sqrt{g(\sigma_I)}$ for $I = 1, 2, \ldots N$ to make (1.4) entirely generally covariant.) The field equations for the metric yielded the "gauge" conditions found earlier by Virasoro, and when the fields were assumed to satisfy these equations, the

action (1.6) turns out to reduce to the Nambu form (1.5).

The modern path-integral formulation of string theories uses the action (1.6) in a somewhat different way. Instead of using the field equations for $g_{\alpha\beta}(\sigma)$ to put (1.6) in the Nambu form, one relies on a theorem, that in two dimensions (and only in two dimensions) any metric $g_{\alpha\beta}(\sigma)$ can at least locally by a suitable choice of coordinates be put in the form

$$g_{\alpha\beta}(\sigma) = e^{\varphi(\sigma)}\delta_{\alpha\beta} \tag{1.7}$$

But then, in two dimensions

$$g(\sigma)^{1/2}g^{\alpha\beta}(\sigma) = \delta_{\alpha\beta} \tag{1.8}$$

so (1.6) reduces to the old "analog model" form appearing in (1.4). However, this does not mean that the metric is irrelevant. A given two-dimensional surface will be characterized by various periodicity conditions, which for a metric of the form (1.7) will depend in general on a variety of Teichmüller parameters; alternatively, one can write the periodicity conditions in a fixed parameter-free form, in which case the Teichmüller parameters appear in the metric. The weight functions for these Teichmüller parameters are then to be found by treating $g_{\alpha\beta}(\sigma)$ as a quantum variable in its own right, which must be summed over along with $x^\mu(\sigma)$ in the path integral.

The first to treat $g_{\alpha\beta}(\sigma)$ as well as $x^\mu(\sigma)$ as a quantum variable was Polyakov.[11] In an influential 1981 paper, he emphasized the crucial importance of invariance under the Weyl transformations

$$g_{\alpha\beta}(\sigma) \to f(\sigma)g_{\alpha\beta}(\sigma) \tag{1.9}$$

which in particular leads to (1.8). He also noted the importance of anomalies, quantum effects that can modify classical Weyl transformation properties. One such anomaly occurs in the transformation of the exponential $e^{ip\cdot x(\sigma)}$ in Eq. (1.4); as we will see, this has an

anomalous behaviour under Weyl transformations that for appropriate p^2 values cancels the Weyl non-invariance of the factors $\sqrt{g(\sigma)}$ that have to be inserted in (1.4). The anomaly that particularly concerned Polyakov arises in the path integral of the factor $\exp(-I[x,g])$ itself. As Polyakov showed, this has an anomalous Weyl non-invariance except when there are just 26 $x^\mu(\sigma)$'s, thus offering a new explanation of the old result that bosonic string theories are difficult to interpret except in 26 dimensions. However, the main thrust of Polyakov's paper was the effort to make sense of the extra $\varphi(\sigma)$ degree of freedom in string theories in less than 26 dimensions, rather than the use of the path integral for practical calculations in 26 dimensions.

The use of the Polyakov formalism to calculate the weight functions for the Teichmüller parameters in 26 dimensions seems to have been first proposed in an unpublished 1981 talk of Friedan.[12] The technology for this sort of calculation was developed in a 1983 paper of Alvarez,[13] and put into final form in work of Polchinski[14], Moore and Nelson[15], and D'Hoker and Phong[16]. The first use of this formalism actually to calculate a physical amplitude involving an integral over Teichmüller parameters entirely within the path-integral formalism was that by Polchinski,[14] in just this past year.

2. General Principles

The historical developments outlined in Sec. 1 lead to the following hypothesis for the calculation of general S-matrix elements in the closed bosonic string theory:

$$S(p_1 j_1 \dots p_N j_N) = i \int \frac{dg}{\Omega} \int dx \exp(-I[x,g]) \times V[x,g;p_1 j_1] \quad \dots \quad V[x,g;p_N j_N] \tag{2.1}$$

Here S is the S-matrix element for a reaction involving particles $1, 2, \dots N$ in D spacetime dimensions; the p_I^μ are the D-momenta of these particles, all taken as if incoming, so that $p_I^0 > 0$ or $p_I^0 < 0$ for a particle in the initial or the final state, respectively; the j_I are discrete indices that label all other properties of the particles, including spin; $\int dg$ indicates a sum over all two-dimensional compact closed oriented manifolds, including a path integral over metrics $g_{\alpha\beta}(\sigma)$ on these surfaces; Ω is a group volume, about which more soon; $\int dx$ indicates a path integral over functions $x^\mu(\sigma)$ on the surface; $I[x,g]$ is the action, a functional of $x^\mu(\sigma)$ and $g_{\alpha\beta}(\sigma)$; and $V[x,g;p,j]$ is the vertex functional, a functional of $x^\mu(\sigma)$ and $g_{\alpha\beta}(\sigma)$ that also depends on p^μ and j. We also assume that the action and the vertex functionals are local, that is, that they are integrals of ordinary functions of $x^\mu(\sigma)$ and $g_{\alpha\beta}(\sigma)$ and their derivatives

$$I[x,g] = \int d^2\sigma \mathcal{I}(x^\mu(\sigma), \frac{\partial x^\mu(\sigma)}{\partial \sigma^\alpha}, \dots, g_{\alpha\beta}(\sigma), \dots) \tag{2.2}$$

$$V[x,g;p,j] = \int d^2\sigma \mathcal{V}(x^\mu(\sigma), \frac{\partial x^\mu(\sigma)}{\partial \sigma^\alpha}, \dots, g_{\alpha\beta}(\sigma), \dots; p, j) \tag{2.3}$$

the integral being taken over the whole two-dimensional surfaces.

What determines the action and vertex functionals, and for a given choice of these functionals, how do we know that Eq. (2.1) will give a physically satisfactory S-matrix? Both questions depend for their answers on what we assume for the symmetries of the theory. These are of three types:

1. D-dimensional Poincaré Invariance

We assume that the action is invariant under a group IO(D-1,1) of D-dimensional Poincaré transformations on the coordinates $x^\mu(\sigma)$:

$$I[x', g] = I[x, g] \tag{2.4}$$

where

$$x'^\mu(\sigma) = \Lambda^\mu_\nu x^\nu(\sigma) + a^\mu \tag{2.5}$$

and Λ^μ_ν is an SO(D-1,1) Lorentz transform. The vertex functionals are not invariant, but transform like one-particle states:

$$\begin{aligned} V[x, g; p, j] &= (p^{0'}/p^0)^{1/2} e^{ip'\cdot a} \\ &\times \sum_{j'} D_{jj'}(W(\Lambda, p)) V[x', g; p' j'] \end{aligned} \tag{2.6}$$

where $p'^\mu = \Lambda^\mu_\nu p^\nu$ is the transformed momentum; $W(\Lambda, p)$ is the Wigner rotation, an element of the little group $O(D-1)$ (or $O(D-2)$ for massless particles); and $D_{jj'}$ is the appropriate matrix representation of the little group. This symmetry ensures that (2.1) will have the correct Poincaré transformation properties.

In particular, for the special case of pure translations ($\Lambda^\mu_\nu = \delta^\mu_\nu$), Eq. (2.4) just requires that the integrand $\mathcal{I}$ in (2.2) depends only on $g_{\alpha\beta}(\sigma)$ and its derivatives and on derivatives of $x^\mu(\sigma)$, but not $x^\mu(\sigma)$ itself, while (2.6) requires that the integrand $\mathcal{V}$ in (2.3) equals $e^{ip\cdot x(\sigma)}$ times a function of $g_{\alpha\beta}(\sigma)$ and its derivatives and derivatives of $x^\mu(\sigma)$, but not of $x^\mu(\sigma)$ itself. It follows then that for each connected component of the two-dimensional surfaces, there appears in (2.1) a delta function factor, whose argument is the sum of the momenta for all the vertex functions $\mathcal{V}$ whose σ-arguments are on that component. That is, in order to calculate the connected part of the S-matrix (the part with a single delta function) we should include only *connected* surfaces in (2.1).

2. Two-Dimensional General Covariance

We assume that the action and vertex functionals are invariant under two-dimensional general coordinate transformations

$$\sigma^\alpha \quad \rightarrow \quad \hat{\sigma}^\alpha \quad = \quad \hat{\sigma}^\alpha(\sigma) \tag{2.7}$$

That is,

$$I[\hat{x}, \hat{g}] = I[x, g] \tag{2.8}$$

$$V[\hat{x}, \hat{g}] = V[x, g] \tag{2.9}$$

where

$$\hat{x}^\mu(\hat{\sigma}) = x^\mu(\sigma) \tag{2.10}$$

$$\hat{g}_{\alpha\beta}(\hat{\sigma}) = \frac{\partial \sigma^\lambda}{\partial \hat{\sigma}^\alpha} \frac{\partial \sigma^\delta}{\partial \hat{\sigma}^\beta} \, g_{\lambda\delta}(\sigma) \tag{2.11}$$

This requires that the densities $\mathcal{I}$ and $\mathcal{V}$ in Eqs. (2.2) and (2.3) can be written as $g^{1/2}(\sigma)$ times two-dimensional scalars.

By themselves, D-dimensional Poincaré invariance and two-dimensional general covariance would allow an infinite variety of theories. The path-integral formulation of string theory becomes vastly more rigid with our third symmetry.

3. Weyl Invariance

We assume that the theory is invariant under Weyl transformation of the metric

$$g_{\alpha\beta}(\sigma) \rightarrow \tilde{g}_{\alpha\beta}(\sigma) = f(\sigma) g_{\alpha\beta}(\sigma) \tag{2.12}$$

where $f(\sigma)$ is an arbitrary function on the surface. At first sight, this would seem to require that the action and vertex functionals should be invariant under (2.12):

$$I[x, \tilde{g}] = I[x, g] \tag{2.13}$$

$$V[x,\tilde{g};p,j] = V[x,g;p,j] \tag{2.14}$$

The only local functionals consistent with (2.13) and (2.14) as well as Poincaré invariance and two-dimensional general covariance are

$$\begin{aligned} I_0[x,g] =& \frac{T}{2}\int d^2\sigma\sqrt{g(\sigma)}g^{\alpha\beta}(\sigma)\frac{\partial x^\mu(\sigma)}{\partial\sigma^\alpha}\frac{\partial x^\nu(\sigma)}{\partial\sigma^\beta}\eta_{\mu\nu} \\ &+ \frac{ln\lambda}{4\pi}\int d^2\sigma\sqrt{g(\sigma)}g^{\alpha\beta}(\sigma)R_{\alpha\beta}(\sigma) \end{aligned} \tag{2.15}$$

$$\begin{aligned} V_0[x,g;p,j] =& \int d^2\sigma\sqrt{g(\sigma)}g^{\alpha\beta}(\sigma)\frac{\partial x^\mu(\sigma)}{\partial\sigma^\alpha}\frac{\partial x^\nu(\sigma)}{\partial\sigma^\beta} \\ &\times e_{\mu\nu}(p,j)e^{ip\cdot x(\sigma)} \end{aligned} \tag{2.16}$$

where T and $ln\lambda$ are unknown constants, $e_{\mu\nu}(p,j)$ is a symmetric polarization tensor with suitable Lorentz transformation properties, and $R_{\alpha\beta}(\sigma)$ is the two-dimensional Ricci tensor. (For oriented surfaces there is also a vertex functional with an antisymmetric polarization tensor and the antisymmetric tensor density $\epsilon^{\alpha\beta}$ in the place of $\sqrt{g}g^{\alpha\beta}$.) Note that the density $\sqrt{g}g^{\alpha\beta}R_{\alpha\beta}$ is not invariant under Weyl transformations (except for constant $f(\sigma)$) but its integral is a topological invariant, the Euler characteristic

$$\chi = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}R \tag{2.17}$$

and is therefore Weyl-invariant.

However, this is not the whole story; there are anomalies. Path integrals of products of $x^\mu(\sigma)$ at the same point σ are not well-defined without some sort of regulator, and if the regulator is introduced so as to preserve general covariance, it necessarily spoils Weyl invariance. One consequence is that we need a counterterm in the action:

$$I = I_0 + I_1 \tag{2.18}$$

$$I_1[x,g] = \mu^2\int d^2\sigma\sqrt{g(\sigma)} \tag{2.19}$$

with μ a regulator-dependent constant whose value must be chosen to preserve Weyl invariance despite anomalies. There is another anomaly that cannot be removed by adding a local two-dimensionally generally covariant term to the action, and that (as we shall see in Sec. 5) cancels only in $D = 26$ spacetime dimensions. One more consequence of the anomalies is that the vertex operator (2.16) is actually Weyl invariant only for $p^2 = 0$, while there is an infinite number of other vertex functionals that are not manifestly Weyl invariant, but are made Weyl invariant by the anomalies for specific values of the mass $-p^2$.

With all anomalies cancelling, the integrand of Eq. (2.1) is invariant under the group of two-dimensional coordinate and Weyl transformations. In consequence, the integral over metrics includes as a factor the infinite volume of this group. The factor $1/\Omega$ in Eq. (2.1) is inserted to cancel this volume factor; that is, we take

$$\Omega = \Omega_C \Omega_W \tag{2.20}$$

where Ω_C and Ω_W are the volumes of the groups of two-dimensional coordinate and Weyl transformations, respectively. (This is a slight oversimplification. These volumes are defined in more detail in Section 4. As shown there, the part of the integral over metrics that entails an integration over coordinate transformations yields a factor Ω_C that, although coordinate invariant, is not Weyl invariant. This is why the compensating factor $1/\Omega_C$ in $1/\Omega$ must be taken *inside* the integral over metrics in Eq. (2.1). It is only after the cancellation of these factors of Ω_C, as well as all other Weyl anomalies, that we encounter a factor Ω_W arising from the rest of the integral over metrics, which is then cancelled by the compensating factor $1/\Omega_W$ in $1/\Omega$.)

As we shall see, the final result of all this is to leave us with a *finite-dimensional* integral over the equivalence classes of metrics that are related by coordinate and Weyl

transformations, respectively.

3. Classic Theorems about Riemann Surfaces

Eq. (2.1) sets us the formidable task of integrating over all possible two-dimensional surfaces. Fortunately, there are several remarkable theorems, special to the case of two dimensions, that go a long way toward simplifying our task. I will state these theorems here non-rigorously and without proof; for a more careful treatment, see any of the standard texts on Riemann surfaces.

Theorem 1 Any oriented compact closed connected two-dimensional manifold is topologically equivalent to a sphere with γ handles. The "genus" γ is related to the Euler characteristic (2.17) by

$$\chi = 2 - 2\gamma \tag{3.1}$$

Our integration over two-dimensional surfaces in (2.1) will thus include a sum over values of γ ranging from 0 to ∞.

Theorem 2 By a suitable coordinate transformation, any metric on a two-dimensional manifold can (in a suitable coordinate patch around any point) be put into the "harmonic" form

$$g_{\alpha\beta}(\sigma) = e^{\varphi(\sigma)} c_{\alpha\beta} \tag{3.2}$$

where $c_{\alpha\beta}$ is a constant. We will work with a metric of Euclidean signature, by which is meant that $c_{\alpha\beta}$ is positive-definite, and can therefore (by a linear transformation of the σ^α) be taken as $c_{\alpha\beta} = \delta_{\alpha\beta}$. Then

$$ds^2 \equiv g_{\alpha\beta}(\vec{\sigma}) d\sigma^\alpha d\sigma^\beta = e^{\varphi(\vec{\sigma})} |d\vec{\sigma}|^2 \tag{3.3}$$

By a non-singular Weyl transformation, we can then reduce the metric in any sufficiently small region to the Euclidean form, $ds^2 = |d\vec{\sigma}|^2$. However, for $\gamma \neq 1$ there are topological

obstacles to doing this throughout the surface with a global non-singular Weyl transformation; indeed, for a Euclidean metric $R = 0$, so (2.17) gives $\chi = 0$, which is inconsistent with Eq. (3.1) except for $\gamma = 1$. Nevertheless, as we shall see, it *is* possible by a non-singular global Weyl transformation to adjust the metric so that the curvature scalar $R(\vec{\sigma})$ is constant.

Theorem 3 Any oriented two-dimensional metric space is a complex manifold (and vice versa). In each coordinate patch in which the metric is of the harmonic form (3.2), we introduce complex coordinates

$$z \equiv \sigma_1 + i\sigma_2 \qquad z^* \equiv \sigma_1 - i\sigma_2 \tag{3.4}$$

in terms of which (3.3) is

$$ds^2 = e^{\varphi(z,z^*)}|dz|^2 \tag{3.5}$$

or in other words

$$g_{zz} = g_{z^*z^*} = 0 \quad g_{zz^*} = \frac{1}{2}e^{\varphi(z,z^*)} \tag{3.6}$$

We note that the most general coordinate transformation that preserves the harmonic form (3.5) is either a *conformal transformation*

$$z \to \hat{z} = f(z) \tag{3.7}$$

with f a function of z, not z^*, or else *anticonformal*

$$z \to \hat{z} = g(z^*) \tag{3.8}$$

(In either case, $ds^2 = e^{\hat{\varphi}}|d\hat{z}|^2$, with $\hat{\varphi} = \varphi - |\ell n f'(z)|^2$ or $\hat{\varphi} = \varphi - |\ell n g'(z^*)|^2$.) Now, if the metric has the form (3.5) in two overlapping coordinate patches, with complex coordinates z and $\hat{z}$, then in the overlap between the patches the coordinate transformation $z \to \hat{z}$

must preserve the harmonic form (3.5), and hence must be of the conformal form (3.7) or anticonformal form (3.8). The matrix of coordinate derivatives to these two cases has determinant

$$\mathrm{Det}\left(\frac{\partial\hat{\sigma}^\alpha}{\partial\sigma^\beta}\right) = \begin{vmatrix} \mathrm{Re}f'(z) & \mathrm{Im}f'(z) \\ -\mathrm{Im}f'(z) & \mathrm{Re}f'(z) \end{vmatrix} = |f'(z)|^2$$

or else

$$\mathrm{Det}\left(\frac{\partial\hat{\sigma}^\alpha}{\partial\sigma^\beta}\right) = \begin{vmatrix} \mathrm{Re}g'(z^*) & \mathrm{Im}g'(z^*) \\ \mathrm{Im}g'(z^*) & -\mathrm{Re}g'(z^*) \end{vmatrix} = -|g'(z^*)|^2$$

However, for an *oriented* manifold it must be possible to choose the coordinates in each patch so that in the overlap regions $\mathrm{Det}(\partial\hat{\sigma}/\partial\sigma)$ is *positive*, which means that the transformation $\sigma \rightarrow \hat{\sigma}$ must be conformal, as in (3.7), rather than anticonformal. This is the defining characteristic of a complex manifold.

Theorem 4 By a global singularity-free coordinate and Weyl transformation, we can transform the metric of any closed compact oriented surface of genus γ into the constant curvature forms

$$ds^2 = \frac{|dz|^2}{(1+|z|^2)^2}\ , R = +1 \qquad \gamma = 0 \tag{3.9}$$

$$ds^2 = |dz|^2\ , R = 0 \qquad \gamma = 1 \tag{3.10}$$

$$ds^2 = \frac{|dz|^2}{|Imz|^2}\ , R = -1 \qquad \gamma \geq 2 \tag{3.11}$$

and at the same time map the surface into

$$\hat{C} \qquad \gamma = 0 \tag{3.12}$$

$$C/\Gamma_1 \qquad \gamma = 1 \tag{3.13}$$

$$U/\Gamma_\gamma \qquad \gamma \geq 2 \tag{3.14}$$

Here $\hat{C}$ is the whole complex plane, including the point at infinity as an ordinary point; C is the whole complex plane, *excluding* the point at infinity; and U is the upper half

plane, excluding the real axis and $z = \infty$; and the notation C/Γ_1 and U/Γ_γ means that we identify points z, z' of C or U that are related by some discrete group Γ_γ of conformal transformations that leave the line element (3.10) or (3.11) invariant. (Eq. (3.9) is obtained from the usual metric for a sphere of radius 1/2 in polar coordinates ϑ, φ by setting $z = e^{i\varphi}\cot(\vartheta/2)$.)

To avoid singularities, the group Γ_γ must act freely; that is, around each point z there is a neighborhood, no point of which is carried into z by any element of Γ_γ except the identity. In particular, no element of Γ_γ except the identity has any fixed points. The manifold is to be identified with a "fundamental region" of the group Γ_γ—that is, a region F no point of which is carried into z by any element of Γ_γ (except the unit element), and such that any point in the plane (or upper half plane) can be reached from some point in F by applying an element Γ_γ. Such fundamental regions are said to "tile" C or U. It should be emphasized that for a given Γ_γ, there are in general many ways of choosing a fundamental region, but the manifold is characterized by the group Γ_γ, not by the choice of fundamental region.

Theorem 5 The group Γ_γ is uniquely determined (up to a conformal transformation that maps C or U once into itself) by a finite number N_γ of real "Teichmüller" parameters:

$$N_0 = 0 \text{ (there is no } \Gamma_0) \tag{3.15}$$

$$N_1 = 2 \tag{3.16}$$

$$N_\gamma = 6\gamma - 6 \text{ for } \gamma \geq 2 \tag{3.17}$$

There are no free parameters in the metrics (3.9)-(3.11) so (3.15)-(3.17) give the number of parameters needed to specify completely a surface of given genus, up to coordinate and Weyl transformations.

We are also interested in the *conformal symmetry* group of the metric. This is the group of conformal transformations that map $\hat{C}$ or C or U once into itself and commute with all elements of Γ_γ. It has dimensionality N'_γ, given by

$$N'_0 = 6 \tag{3.18}$$

$$N'_1 = 2 \tag{3.19}$$

$$N'_\gamma = 0 \text{ for } \gamma \geq 2 \tag{3.20}$$

This is then the number of independent infinitesimal coordinate transformations ("conformal Killing vectors") that preserve the form of the metric up to a Weyl transformation. Note that for all γ

$$N_\gamma - N'_\gamma = 6\gamma - 6 \tag{3.21}$$

This is known as the Riemann-Roch theorem.

To describe these Teichmüller parameters and conformal symmetries, it will be convenient now to deal separately with the cases $\gamma = 0, \gamma = 1$, and $\gamma \geq 2$.

$\underline{\gamma = 0}$: The group of conformal transformations that map the whole complex plane C once into itself consists of all Möbius transformation

$$z \rightarrow \hat{z} = \frac{az + b}{cz + d} \tag{3.22}$$

with a, b, c, d arbitrary complex numbers, which by re-scaling numerator and denominator can always be chosen so that

$$ad - bc = 1 \tag{3.23}$$

The product of two Möbius transformations with parameters $[a, b, c, d]$ and $[a', b', c', d']$ is a Möbius transformation with parameters $[a'', b'', c'', d'']$, given by matrix multiplication

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.24}$$

so this group is $SL(2,C)$ (or more precisely, $SL(2,C)/Z_2$, since $[a,b,c,d]$ and $[-a,-b,-c,-d]$ both represent the same Möbius transformation).

Note that every transformation of the form (3.22) has at least one fixed point in $\hat{C}$ (which may be the point at infinity!) so the group $SL(2,C)$ has no fixed-point-free subgroup, and there is no possible periodicity group Γ_0. Thus no parameters are needed to describe the surface up to a Weyl and coordinate transformation, and the group of conformal symmetries is the full group $SL(2,C)$ of all conformal transformations that map $\hat{C}$ once into itself. This group has 3 complex parameters (a,b,c,d, with constraint (3.23)), or 6 real parameters, as stated in (3.18).

$\underline{\gamma = 1}$ The group of conformal transformations that map the finite complex plane C once into itself consists of all Möbius transformations (3.22) that leave $z = \infty$ invariant, and hence have $c = 0$. These are just the linear transformations.

$$z \to \hat{z} = Az + B \tag{3.25}$$

with arbitrary complex coefficients A and B. But Γ_1 must have no fixed points in C, so it contains only linear transformations with $A = 1$, i.e., pure translations. (These do leave $z = \infty$ invariant, but this is not a point in C.) Since we are in two dimensions, we get a compact surface if the group is generated by two independent translations, i.e., it must consist of all transformations.

$$z \to z + n_1 B_1 + n_2 B_2$$

where B_1 and B_2 are complex numbers with B_1/B_2 not real, and n_1 and n_2 are integers. A conformal transformation $z \to \hat{z} = z/B_1$ lets us replace B_1 with unity and B_2 with an arbitrary complex parameter $\tau = B_2/B_1$. The group Γ_1 then consists of all translations of the form

$$z \to z + n_1 + n_2 \tau \tag{3.26}$$

(These also leave the line-element (3.10) invariant.) The torus is represented by a fundamental region of this group, which can for instance be taken as the parallelogram with corners at $z = 0, 1, \tau, 1+\tau$. The surface is specified up to a Weyl and coordinate transformation by the single complex parameter τ, or two real parameters, as stated in (3.16). The group of conformal symmetries consists of all linear transformations (3.25) that commute with Γ_1; these are the translations

$$z \to \hat{z} = z + B \tag{3.27}$$

which have 1 complex or 2 real parameters, as stated in (3.19).

$\underline{\gamma \geq 2}$ The group of conformal transformations that map the upper-half-plane U once into itself consists of all Möbius transformations (3.22) that preserve the sign of Imz. By adjusting a common complex factor, these can be taken to have a, b, c, d *real*, with determinant given by (3.23). They thus form the group $SL(2,R)$ (or strictly speaking, $SL(2,R)/Z_2$, since a sign change in a, b, c, d has no effect in (3.22) or (3.23).) All of these transformations leave the line element (3.11) invariant, so the same is automatically true for any subgroup Γ_γ. Just as we represent the torus with $\gamma = 1$ by a parallelogram, a 4-sided figure bounded by straight lines (geodesics of the metric (3.10)) whose opposite sides are identified, so also we represent a surface with $\gamma \geq 2$ by a 4γ-sided figure, the sides being circular arcs (geodesics of the metric (3.11)) identified in pairs by $SL(2,R)$ transformations $A_1, A_2 \ldots A_{2\gamma}$. The products of these matrices and their inverses generate the group Γ_γ. However, there is a constraint. Consider for instance $\gamma = 2$, label the corners of the fundamental region successively 1, 2, ... 8, and suppose that

A_1 takes side 12 to side 43

A_2 takes side 23 to side 54

A_3 takes side 18 to side 67

A_4 takes side 87 to side 56

The product $A_3^{-1}A_4A_3A_4^{-1}A_2A_1^{-1}A_2^{-1}A_1$ takes z_1 consecutively into $z_4, z_3, z_2, z_5, z_8, z_7, z_6$, and then back to z_1. But no element of Γ_2 except the identity can have any fixed points in U, so this product must be unity

$$A_3^{-1}A_4A_3A_4^{-1}A_2A_1^{-1}A_2^{-1}A_1 = 1 \tag{3.28}$$

In general, each A_n involves 3 real parameters. $(a, b, c, d$, with constraint (3.23)), and there is always one matrix constraint like (3.28), leaving $3(2\gamma - 1)$ free real parameters. However, an $SL(2, R)$ transformation of the upper-half-plane with matrix M yields a description of the same surface with A_n replaced with $M^{-1}A_nM$, so Γ_γ really only depends on $3(2\gamma - 1) - 3 = 6\gamma - 6$ real parameters, as stated in (3.17). Finally, the only $SL(2, R)$ transformation that commutes with all $A_1, A_2 \ldots, A_{2\gamma}$ is the identity, so the group of conformal symmetries is trivial for $\gamma \geq 2$.

We close with two remarks:

(1) Not all different values of the N_γ Teichmüller parameters represent different surfaces. For example, consider the case $\gamma = 1$. First, we can replace z with $-z$, in which case τ in Eq. (3.26) is replaced with $-\tau$, so τ and $-\tau$ represent the same surface. We can therefore always choose τ in the upper half plane

$$Im\tau > 0$$

Next, note that τ and $\tau + n$ represent the same surface for any integer n, because (3.26) is the same as

$$z \rightarrow z + n_1' + n_2(\tau + n)$$

where $n_1' = n_1 - n_2 n$. A little less trivially, τ and $-1/\tau$ represent the same surface, because (3.26) is the same as

$$z' \to z' + n_1' - n_2'/\tau$$

where $z' = z/\tau, n_1' = n_2$, and $n_2' = -n_1$. The group generated by the transformations $\tau \to \tau + n$ and $\tau \to -1/\tau$ is SL(2,Z), consisting of all Möbius transformations (3.22) with real *integer* entries a, b, c, d satisfying (3.23). The space of equivalent Teichmüller parameters τ can be taken as any fundamental region of this group. It is conventional to take this fundamental region as

$$Im\tau > 0, \ |\mathrm{Re}\tau| \leq \frac{1}{2}, \ |\tau| \geq 1 \tag{3.29}$$

(Some points on the boundary are also equivalent.) Unfortunately, the boundaries of the fundamental region are not known for $\gamma \geq 2$.

(2) Although we have focused here on closed oriented surfaces, which are relevant for theories of closed oriented strings, the results quoted here are useful in general string theories, which may involve open and/or unoriented surfaces. This is because the most general such surface can be obtained from a closed oriented surface (its "double") by identifying points related by a suitable anticonformal involution (a transformation with unit square). For instance, the two surfaces with Euler number $\chi = +1$, the disc and real projective plane (respectively open and unoriented) can be obtained from the sphere by identifying points z and $1/z^*$ or $-1/z^*$, respectively.

4. Measures for Metrics

We now consider what is meant by the integral over surfaces, indicated by $\int \mathrm{dg}$ in Eq. (2.1).

First, it will be convenient to introduce a small change[14] in the classic description of Riemann surfaces of genus $\gamma \geq 1$ outlined in Sec. 3. Instead of fixing the metric to

be either (3.10) or (3.11) and then imposing periodicity under a group Γ_γ whose action depends on N_γ Teichmüller parameters, we will instead re-define the coordinates so that the periodicity group is fixed, and the Teichmüller parameters appear in the metric. For instance, for $\gamma = 1$, instead of the group

$$\Gamma_1 : \quad z \to z + n + m\tau \qquad (n, m \varepsilon Z) \tag{4.1}$$

with metric

$$ds^2 = |dz|^2 \tag{4.2}$$

we define a new coordinate z' by

$$z = \mathrm{Re}z' + \tau \mathrm{Im}z' \tag{4.3}$$

so that the periodicity group Γ_1 is τ-independent

$$\Gamma_1 : \quad z' \to z' + n + im \qquad (n, m \varepsilon Z) \tag{4.4}$$

but the metric depends on τ

$$ds^2 = |d\mathrm{Re}z' + \tau d\mathrm{Im}z'|^2 \tag{4.5}$$

This change of formalism can be (and will be supposed to have been) carried out for $\gamma \geq 2$ as well as $\gamma = 1$.

Next, we need to define a volume element in the space of all metrics $g_{\alpha\beta}(\sigma)$. One usually calculates volume elements from metrics, so here we will first need to define the "distance" $\| \delta g \|$ between infinitesimally different metrics $g_{\alpha\beta}(\sigma)$ and $g_{\alpha\beta}(\sigma) + \delta g_{\alpha\beta}(\sigma)$. There is general agreement—though as far as I know no proof—that to avoid introducing new degrees of freedom $\| \delta g \|$ must be "ultralocal," that is, $\| \delta g \|^2$ must be the integral

over σ of a bilinear function of the $\delta g_{\alpha\beta}(\sigma)$, taken at the same point σ, with coefficients that may depend on $g_{\alpha\beta}(\sigma)$, but not on its derivatives. The most general such ultralocal and two-dimensionally generally covariant measure may be written

$$\| \delta g \|^2 = \frac{1}{4} \int d^2\sigma \sqrt{g(\sigma)} \delta g_{\alpha\beta}(\sigma) \delta g_{\gamma\delta}(\sigma) \times \left[B \left(g^{\alpha\gamma}(\sigma) g^{\beta\delta}(\sigma) - \frac{1}{2} g^{\alpha\beta}(\sigma) g^{\gamma\delta}(\sigma) \right) + C g^{\alpha\beta}(\sigma) g^{\gamma\delta}(\sigma) \right] \tag{4.6}$$

with arbitrary constants B and C. Note that although (4.6) is generally covariant, it is *not* Weyl invariant. There is in fact no generally covariant and Weyl-invariant ultralocal form for $\| \delta g \|$. From this fact there arise quantum mechanical anomalies that can destroy Weyl invariance, in the same way that the chiral-noninvariance of the measure for path integrals over fermion fields produces the classic ABJ anomaly.

We use (4.6) in much the same way as in the case of integrals over a finite number of real parameters. That is, if we introduce a parametrization of $g_{\alpha\beta}(\sigma)$ in terms of a discrete and/or continuous set of variables y^ℓ, and write

$$\| \delta g \|^2 = G_{\ell m}[g^{(y)}] \delta y^\ell \delta y^m \tag{4.7}$$

then

$$\int dg \; F[g] = \int \left(\prod_\ell dy^\ell \right) \left(\mathrm{Det} G[g^{(y)}] \right)^{1/2} F[g^{(y)}] \tag{4.8}$$

(This formula has the necessary virtue of being unaffected by a re-definition of the parameters y^ℓ. As we shall see, changes in the definition of G are also inconsequential. An over-all constant factor in (4.8) would matter—it is believed that the over-all normalization of (4.8) is dictated by unitarity, but the argument is obscure to me.)

The parameterization of $g_{\alpha\beta}(\sigma)$ will be adapted to the symmetries of the action. First, choose values for the Teichmüller parameters τ^i, which determines a particular "fiducial metric" $g^{(o)}_{\alpha\beta}(\sigma;\tau)$, such as that given for $\gamma = 1$ by Eq. (4.5). Next, apply

a two-dimensional coordinate transformation $\sigma^a \rightarrow \sigma^\alpha + f^\alpha(\sigma)$. Finally, apply a Weyl transformation $g_{\alpha\beta}(\sigma) \rightarrow g_{\alpha\beta}(\sigma)\exp(\phi(\sigma))$. The metric is thereby specified by the variables τ^i and the functions $f^\alpha(\sigma)$ and $\phi(\sigma)$, which collectively constitute the variables y^ℓ in Eq. (4.7).

The change in the metric when we make an infinitesimal change $\delta\tau, \delta f, \delta\phi$ in these variables is given by

$$\begin{aligned}\delta g_{\alpha\beta}(\sigma) = \delta\tau^i \frac{\partial}{\partial\tau^i} g_{\alpha\beta}(\sigma) + \delta\phi(\sigma) g_{\alpha\beta}(\sigma) \\ + \hat{\delta} f_{\alpha;\beta}(\sigma) + \hat{\delta} f_{\beta;\alpha}(\sigma)\end{aligned} \tag{4.9}$$

Here $g_{\alpha\beta}(\sigma)$ is short for $g_{\alpha\beta}(\sigma;\tau,f,\phi)$; this is the metric used to lower indices and to calculate covariant derivatives in (4.9). Also,

$$\hat{\delta} f \equiv (f + \delta f) * f^{-1} \tag{4.10}$$

where $f_2 * f_1$ denotes the coordinate transformation function obtained when we first transform $\sigma \rightarrow \sigma' = \sigma + f_1(\sigma)$ and then take $\sigma' \rightarrow \sigma' + f_2(\sigma')$. The compound transformation (4.10) appears in (4.9) because performing the transformation $\sigma' \rightarrow \sigma' + \hat{\delta} f\ (\sigma')$ after the transformation $\sigma \rightarrow \sigma' = \sigma + f(\sigma)$ is the same as shifting $f(\sigma)$ by the amount

$$\hat{\delta} f * f - f = (f + \delta f) * f^{-1} * f \ - \ f \ = \ \delta f \tag{4.11}$$

This complication does not arise for the Weyl transformations, because these form an Abelian group.

It is straightforward to insert (4.9) in (4.6), and obtain

$$\begin{aligned}\| \delta g \|^2 = B \int d^2\sigma \sqrt{g} (P_1 \hat{\delta} f + \chi_{.i} \delta\tau^i)_{\alpha\beta} (P_1 \hat{\delta} f + \chi_{.i} \delta\tau^i)^{\alpha\beta} \\ + C \int d^2\sigma \sqrt{g} \left(\delta\phi + g^{\alpha\beta} \hat{\delta} f_{\alpha;\beta} + \frac{1}{2} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial\tau^i} \delta\tau^i \right)^2\end{aligned} \tag{4.12}$$

Here P_1 is a differential operator that maps vector fields $V_\alpha(\sigma)$ into symmetric traceless tensors

$$(P_1 V)_{\alpha\beta} \equiv \frac{1}{2}\left[V_{\alpha;\beta} + V_{\beta;\alpha} - g_{\alpha\beta} g^{\lambda\delta} V_{\lambda;\delta}\right] \tag{4.13}$$

Also, $\chi_{,i}$ is a symmetric traceless tensor

$$(\chi_{,i})_{\alpha\beta} \equiv \frac{1}{2}\left[\frac{\partial g_{\alpha\beta}}{\partial \tau^i} - \frac{1}{2} g_{\alpha\beta} g^{\lambda\delta} \frac{\partial g_{\lambda\delta}}{\partial \tau^i}\right] \tag{4.14}$$

To disentangle $\delta f, \delta\tau$, and $\delta\phi$, we must consider the properties of the differential operator P_1. The equation $P_1\xi = 0$ can be recognized as the condition for a conformal Killing vector, i.e., an infinitesimal coordinate transformation that preserves the form of the metric up to a Weyl transformation. As stated in Section 3, the space of such conformal Killing vectors ξ satisfying $P_1\xi = 0$ (the kernel of P_1) has dimensionality

$$\dim\ker P_1 = \begin{cases} 6 & \gamma = 0 \\ 2 & \gamma = 1 \\ 0 & \gamma \geq 2 \end{cases} \tag{4.15}$$

We can also define the adjoint $P_1^\dagger$ of P_1, which maps traceless tensors $t_{\alpha\beta}$ into vectors:

$$(P_1^\dagger t)_\beta = -t^\alpha_{\beta;\alpha} \tag{4.16}$$

This is the adjoint, in the sense that

$$(P_1^\dagger t, v) = (t, P_1 v) \tag{4.17}$$

where scalar products are defined by

$$(v', v) \equiv \int d^2\sigma \sqrt{g}\; v'^\alpha v_\alpha \tag{4.18}$$

$$(t', t) \equiv \int d^2\sigma \sqrt{g}\; t'^{\alpha\beta} t_{\alpha\beta} \tag{4.19}$$

The traceless symmetric tensors that are orthogonal to all tensors of form $P_1\xi$ thus are just those annihilated by $P_1^\dagger$. The number of such independent tensors is the same as the

number of independent traceless deformations $\delta g_{\alpha\beta}$ of the metric, modulo those of form $(P_1\xi)_{\alpha\beta}$. This number is the same as the number of Teichmüller parameters, given in Section 3:

$$\dim \ker P_1^\dagger = \begin{cases} 0 & \gamma = 0 \\ 2 & \gamma = 1 \\ 6\gamma - 6 & \gamma \geq 2 \end{cases} \tag{4.20}$$

Note incidentally that the Riemann-Roch theorem (3.21) takes the form of an index theorem: For *all* γ,

$$\dim \ker P_1^\dagger - \dim \ker P_1 = 6\gamma - 6 \tag{4.21}$$

As we have seen, the number of Teichmüller parameters and hence the number of the $\chi_{.i}$ in (4.12) is the same as the dimensionality of $\ker P_1^\dagger$. But the $\chi_{.i}$ are not in $\ker P_1^\dagger$. To deal with this complication[15], let ψ_i be a basis for $\ker P_1^\dagger$, i.e., a complete independent set of solutions of

$$P_1^\dagger \psi = 0 \tag{4.22}$$

The symmetric traceless tensors $\chi_{.i}$ in Eq. (4.12) may be decomposed into a term of form $P_1\eta$, plus a traceless tensor perpendicular to all such tensors, and hence a linear combination of the ψ_i:

$$\psi_i = P_1\eta_i + L_{ij}\psi_j \tag{4.23}$$

(For $\gamma = 0$ and $\gamma = 1$ there are vectors annihilated by P_1, so η_i is not unique, but as we shall see this makes no difference.) We solve for L by taking the scalar product with ψ_k; this gives

$$L = N^{-1}M \tag{4.24}$$

where

$$N_{ij} \equiv (\psi_i.\psi_j) \qquad M_{ij} \equiv (\psi_i.\chi_{,j}) \tag{4.25}$$

(Note that N has an inverse, because the ψ_i are linearly independent by definition, and the scalar product (4.19) is positive-definite.) Using (4.22) and (4.23) in (4.12) gives

$$\begin{aligned} \| \delta g \|^2 &= B(N^{-1}MNM^TN^{-1})_{ij}\delta\tau^i\delta\tau^j \\ &+ B\int d^2\sigma\ \sqrt{g}\ \left(P_1(\hat{\delta}f+\eta_i\delta\tau^i)\right)_{\alpha\beta}\left(P_1(\hat{\delta}f+\eta_i\delta\tau^i)\right)^{\alpha\beta} \\ &+ C\int d^2\sigma\ \sqrt{g}\ (\delta\phi + g^{\alpha\beta}\hat{\delta}f_{\alpha;\beta} + \frac{1}{2}g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial\tau^i}\delta\tau^i)^2 \end{aligned} \tag{4.26}$$

Now, the group of coordinate transformations can be parameterized by an infinite set of parameters γ^n. As usual, $\gamma^n = 0$ is taken as the coordinates of the identity. The γ^n can be defined near zero by specifying that for small γ, f^a is

$$\delta f^\alpha(\sigma) = \sum_n u_n^\alpha(\sigma)\delta\gamma^n \tag{4.27}$$

where the $u_n^\alpha(\sigma)$ are a complete orthonormal set that can be taken as the orthonormalized eigenvectors of $P_1^\dagger P_1$:

$$P_1^\dagger P_1 u_n = \lambda_n u_n \tag{4.28}$$

$$\int d^2\sigma\sqrt{g}\ u_n^\alpha u_{m\alpha} = \delta_{nm} \tag{4.29}$$

It follows then that

$$\hat{\delta}f^\alpha(\sigma) \equiv [(f+\delta f)*f^{-1}]^\alpha(\sigma) = \sum_n u_n^\alpha(\sigma)\rho^n{}_m(\gamma)\delta\gamma^m \tag{4.30}$$

where

$$\rho^n{}_m(\gamma) \equiv \left[\frac{\partial}{\partial\gamma'^m}(\gamma'*\gamma^{-1})^n\right]_{\gamma'=\gamma} \tag{4.31}$$

and $(\gamma_2*\gamma_1)^n$ are the γ parameters of the coordinate transformation obtained by first performing the transformation with parameters γ_1 and then the transformation with parameters γ_2. We also expand $\delta\phi(\sigma)$ in a complete orthonormal set of scalars

$$\delta\phi(\sigma) = \sum_k \delta Y^k S_k(\sigma) \tag{4.32}$$

$$\int d^2\sigma \sqrt{g} S_k S_\ell = \delta_{k\ell} \tag{4.33}$$

The parameters y^ℓ in Eq. (4.8) are thus now identified as the τ^i, γ^n, and Y^k.

Using Eqs. (4.27)-(4.33) in (4.26), we have at last

$$\begin{aligned} \| \delta g \|^2 = & B(N^{-1}MNM^TN^{-1})_{ij}\delta\tau^i\delta\tau^j \\ & + \sum_n \lambda_n B(\rho^n{}_m(\gamma)\delta\gamma^m \ + \ \text{terms linear in } \delta\tau)^2 \\ & + \sum_k C(\delta Y^k \ + \ \text{terms linear in } \delta\tau \text{ and } \delta Y)^2. \end{aligned} \tag{4.34}$$

In accordance with Eq. (4.8), our prescription for integrals over $g_{\alpha\beta}(\sigma)$ is then

$$\begin{aligned} \int dgF = & \int \prod_n{}' d\gamma^n \prod_k dY^k \prod_i d\tau^i \mathrm{Det}'\rho(\gamma) \\ & \times \left[\prod_m{}' (B\lambda_n)^{1/2}\right] \left[\prod_k C^{1/2}\right] \mathrm{Det}^{1/2}(BN^{-1}MNM^TN^{-1})F \end{aligned} \tag{4.35}$$

Here the prime on products over n indicates that we are to leave out any "zero mode" (present for $\gamma = 0$ and $\gamma = 1$), for which $\lambda_n = 0$. This omission is necessary because such γ^n do not represent independent degrees of freedom of the metric; Eq. (4.9) shows that the effect on the metric of an infinitesimal coordinate transformation $\delta f^\alpha = u^\alpha_o \delta\gamma^o$ can be compensated by a Weyl transformation $\delta\phi = u^\alpha_{o;\alpha}\delta\gamma^o$, provided that u^α_o is a zero-mode of $P_1^\dagger P_1$, i.e., a conformal Killing vector.

Clearly, we face a touchy problem in interpreting infinite products of constants like $B^{1/2}$ or $C^{1/2}$. Such products can be written as exponentials of infinite sums of constants, and to give these sums a meaning, we use the completeness relation for the normal modes over which we are summing. For instance, for the set of scalars $S_k(\sigma)$, we have a completeness relation

$$\sum_k S_k(\sigma)S_k(\sigma') = \delta^2(\sigma - \sigma') \tag{4.36}$$

It is important to note that here $\delta^2(\sigma - \sigma')$ is the *scalar* delta function, defined so that

$$\int d^2\sigma \sqrt{g(\sigma)} \delta^2(\sigma - \sigma') = 1 \tag{4.37}$$

as required for the consistency of Eqs. (4.36) and (4.33). Setting $\sigma' = \sigma$ in Eq. (4.36), multiplying with $\sqrt{g(\sigma)}$, and integrating over σ, we have then

$$\sum_k 1 = \delta^2(0) \int d^2\sigma \sqrt{g(\sigma)} \tag{4.38}$$

A product over k like that in Eq. (4.35) can thus be written

$$\prod_k C^{1/2} = \exp\left(\frac{1}{2} \ell \mathrm{nC}\, \delta^2(0) \int \mathrm{d}^2\sigma \sqrt{\mathrm{g}(\sigma)}\right) \tag{4.39}$$

(The divergent constant $\delta^2(0)$ has an invariant significance because we defined $\delta^2(\sigma - \sigma')$ as a scalar.) These manipulations can be justified by various regulator techniques, with which $\delta^2(0)$ appears as a constant factor that becomes quadratically divergent as the regulator is removed. (Depending on the regulator method used, it is possible that a logarithmic divergence accompanies the quadratic divergence in the argument of the exponential in Eq. (4.39). Power counting shows[14] that any such logarithmic divergence must multiply the integral of $\sqrt{g}R$, and hence can be dealt with as a renormalization of the constant $\ell n \lambda$, like other such renormalizations encountered below.)

The factor (4.39) is invariant under two-dimensional coordinate transformations (because $\| \delta g \|$ is) but not under Weyl transformations (because $\| \delta g \|$ isn't). What are we to do about this? In practice, the functional $F[g]$ over which we are integrating in Eq. (4.8) will always have a factor exp(-I), where I is the action functional. A factor like (4.39) can therefore always be cancelled by adding to the action a suitable counterterm (2.19), proportional to $\int d^2\sigma \sqrt{g(\sigma)}$.

In much the same way, we can use the completeness relation for the vector eigenmodes of ${P_1}^\dagger P_1$, together with Eq. (4.29), to show that sums of constants over these vector eigenmodes can be expressed as

$$\sum_n 1 = \delta^2(0) \int d^2\sigma \, \sqrt{g(\sigma)} \tag{4.40}$$

However, here we encounter two small further complications: there is a factor B in the elements of the final determinant in Eq. (4.35), and the product over n in (4.35) is not over all n, because it excludes the zero modes of the operator ${P_1}^\dagger P_1$. Now, the final determinant in Eq. (4.35) is of a $N'_\gamma \times N'_\gamma$ matrix, where N'_γ is the number of Teichmüller parameters, while the number of zero modes excluded in $\prod'_n$ is N_γ, the number of conformal Killing vectors. Hence the complete dependence on the constant B is through a factor

$$\begin{aligned} B^{N'_\gamma/2} \prod_n{}' B^{1/2} &= B^{(N'_\gamma - N_\gamma)/2} \prod_n B^{1/2} \\ &= \exp\left[\frac{1}{2}\ell n B \left\{ \frac{3}{4\pi} \int d^2\sigma \sqrt{g} R + \delta^2(0) \int d^2\sigma \sqrt{g} \right\}\right] \end{aligned} \tag{4.41}$$

in which we have used the Riemann-Roch and Gauss-Bonnet theorems

$$N'_\gamma - N_\gamma = 6 - 6\gamma = 3\chi = \frac{3}{4\pi} \int d^2\sigma \sqrt{g} R \tag{4.42}$$

We now see that in order to preserve Weyl invariance at the quantum level, one must add an infinite counterterm in the original action

$$I_1[g] = \frac{1}{2}\ell n(BC)\delta^2(0) \int d^2\sigma \sqrt{g} \tag{4.43}$$

(A further counterterm of this sort, that is made necessary by the Weyl non-invariance of the measure for integrals over $x^\mu(\sigma)$, will be encountered in Section 5.) In addition, the constant $\ell n\lambda$ in Eq. (2.15) receives a finite renormalization

$$\ell n \lambda_{\text{eff}} = \ell n \lambda - \frac{3}{2}\ell n B \tag{4.44}$$

The unknown parameters B and C therefore enter only in the renormalization of the constants $\ell n\lambda$ and μ^2 in Eq. (2.15) and (2.19). Since we do not know the unrenormalized values of these constants anyway, there is no loss of generality in taking B and C to have the convenient values

$$B = C = 1 \tag{4.45}$$

Eq. (4.35) then takes the form

$$\int dgF[g] = \int \left(\prod_n{}' d\gamma^n\right)\left(\prod_k dY^k\right)\left(\prod_i d\tau^i\right) \tag{4.46}$$
$$\mathrm{Det}'\rho(\gamma)\ (\mathrm{Det}'P_1^\dagger P_1)^{1/2}(\mathrm{Det}\ M)\ (\mathrm{Det}\ N)^{-1/2}F[g^{(\tau,\gamma,Y)}]$$

The prime on Det $P_1{}^\dagger P_1$ indicates that we exclude zero modes; that is

$$\mathrm{Det}'P_1{}^\dagger P_1 = \prod_n{}' \lambda_n \tag{4.47}$$

The functions $F[g]$ over which we integrate (as in (2.1)) are generally invariant under two-dimensional coordinate transformations, as is everything else in Eq. (4.46) except $\mathrm{Det}'\rho(\gamma)$. Hence the integral over the variables γ^n that parameterize two-dimensional coordinate transformations merely yields a factor

$$\Omega' = \int \prod_n{}' d\gamma^n \mathrm{Det}'\rho(\gamma) \tag{4.48}$$

If it were not for the primes, this would be the usual definition for the volume of a group, here the group of two-dimensional coordinate transformations. The primes indicate that we are to exclude zero modes of $P_1^\dagger P_1$, which as we have seen correspond to the conformal Killing vectors, which generate the subgroup of coordinate transformations that leave the metric (and all periodicity conditions) invariant up to an over-all σ-dependent scale factor. Because of this exclusion, we can write (4.48) as

$$\Omega' = \Omega_C/\Omega_{CKV} \tag{4.49}$$

where Ω_C is the volume of the full group of two-dimensional coordinate transformations

$$\Omega_C \equiv \int \prod_n d\gamma^n \ \mathrm{Det}\rho(\gamma) \tag{4.50}$$

and Ω_{CKV} is the volume (defined similarly) of the subgroup generated by conformal Killing vectors. For $\gamma \geq 2$ there are no conformal Killing vectors, and we take $\Omega_{CKV} = 1$. We will calculate Ω_{CKV} for $\gamma = 0$ and $\gamma = 1$ in Section 8.

The group volume Ω_C given by Eq. (4.50) is only partly independent of the parameterization of the group of coordinate transformations. Under a reparameterization, $\gamma \to \tilde{\gamma}(\gamma)$, we find from (4.31) that

$$\mathrm{Det}\tilde{\rho}(\tilde{\gamma}) = \left|\frac{\partial\gamma}{\partial\tilde{\gamma}}\right| \quad \mathrm{Det}\rho(\gamma) \quad \left|\frac{\partial\tilde{\gamma}}{\partial\gamma}\right|_{\gamma=0} \tag{4.51}$$

Hence the volume $\tilde{\Omega}_C$ calculated using coordinates $\tilde{\gamma}$ is related to the volume Ω_C calculated using coordinates γ by

$$\tilde{\Omega}_C = \Omega_C \left|\frac{\partial\tilde{\gamma}}{\partial\gamma}\right|_{\gamma=0} \tag{4.52}$$

In the present case we have fixed the definition of the coordinates γ near $\gamma = 0$ by the prescriptions (4.27)–(4.29), and as long as we stick to these prescriptions, Ω_C is parameterization-invariant. However, the defining conditions (4.28) and (4.29) for the eigenfunctions $u^\alpha{}_n(\sigma)$ gives them a dependence on the metric which is not Weyl-invariant, and this introduces via Eq. (4.27) a Weyl non-invariance in the definition of the parameters γ^n near $\gamma^n = 0$, and hence in Ω_C.

This doesn't matter, because Ω_C is cancelled anyway by the factor $1/\Omega_C$ in the $1/\Omega$ in Eq. (2.1). However, the same remarks apply to the factor $1/\Omega_{CKV}$ in Eq. (4.49) that is left over for genus 0 and 1, so this factor is not Weyl invariant.

In addition, for any genus the functional $F(g)$ which is integrated in (4.46) contains another Weyl anomaly, which as we shall see in the next Section arises from the functional

integral over $x^\mu(\sigma)$. Further, as we will see in Section 7, the factors $\mathrm{Det}' P_1{}^\dagger P_1$ and $\mathrm{Det} N$ in Eq. (4.46) are also not Weyl invariant (though $\mathrm{Det} M$ is). Nevertheless, as shown in Section 7, these Weyl non-invariances in Ω_{CKV}, F, $\mathrm{Det}' P_1{}^\dagger P_1$, and N all cancel in the one case of 26 spacetime dimensions.

Anticipating this happy result, we conclude that the integrand of Eq. (4.46) is Y-independent, so that (4.46) contains as a factor the integral over the variables Y^k. This is just the volume of the Weyl group

$$\Omega_W = \int \prod_k dY^k \tag{4.53}$$

and is cancelled by the factor $1/\Omega_W$ in Eq. (2.1) (see (2.20).)

Our final result, after all these cancellations, is

$$\begin{aligned} \int \frac{dgF[g]}{\Omega} = \frac{1}{\Omega_{CKV}} \int \prod_i d\tau^i \ (\mathrm{Det}' P_1^\dagger P_1)^{1/2} \mathrm{Det}\ M \\ \times (\mathrm{Det}\ N)^{-1/2}\ F[g] \end{aligned} \tag{4.54}$$

The functional F and determinants in Eq. (4.54) can be calculated using any metric obtained from the fiducial metric $g^{(0)}_{\alpha\beta}(\sigma;\tau)$ by a coordinate and Weyl transformation—in particular, by using the fiducial metric itself.

We recognize in (4.54) a quantity

$$\prod_i d\tau_i \mathrm{Det} M (\mathrm{Det} N)^{-1/2} \tag{4.55}$$

which for metrics of constant curvature is known in mathematics as the Weil-Peterson measure. However, it must be stressed that although (4.55) is independent of the basis chosen for the zero modes of $P_1^\dagger$, it is not Weyl-invariant, and therefore does not have any particular significance by itself.

5. Integrals over $x^\mu(\sigma)$

There is one more determinant with which we will have to deal. The action $I[x,g]$ is quadratic in $x^\mu(\sigma)$, while the vertex functionals in (2.1) are integrals of exponentials $e^{ip\cdot x}$ times polynomials in the derivatives of $x^\mu(\sigma)$. Hence the functional integral over $x^\mu(\sigma)$ can be done by the usual rules of Gaussian integration:

$$\int dx \exp\left(-\frac{T}{2}\int d^2\sigma\sqrt{g}g^{\alpha\beta}\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}\right)$$
$$\times V[x,g;p_1j_1]\ldots V[x,g;p_nj_n] \tag{5.0}$$
$$= Z[g] < V(p_1j_1)\ldots V(p_nj_n) >_g$$

where $< V\ldots V >_g$ is a correlation function to be calculated by summing over ways of pairing the x's in the vertex functionals, each pairing contributing a product of Greens functions, one for each pair of x's; and $Z[g]$ is the functional integral without vertex operators:

$$Z[g] \equiv \int dx \exp\left(-\frac{T}{2}\int d^2\sigma\sqrt{g}\ g^{\alpha\beta}\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}\right) \tag{5.1}$$

This section will deal with the evaluation of $Z[g]$.

To evaluate this factor, we follow the same procedure as for integrals over $g_{\alpha\beta}(\sigma)$. First, define a measure: the "distance" $||\delta x||$ between two infinitesimally close configurations $x^\mu(\sigma)$ and $x^\mu(\sigma)+\delta x^\mu(\sigma)$ is given as

$$||\delta x||^2 = A\int d^2\sigma\sqrt{g(\sigma)}\delta x^\mu(\sigma)\delta x_\mu(\sigma) \tag{5.2}$$

with A another arbitrary constant like the B and C in Eq. (4.6). This is the most general ultra-local Lorentz-invariant and two-dimensionally generally covariant measure, but it is not Weyl invariant, from which circumstance will follow new conformal anomalies.

Next, choose a discrete parametrization for the integration variable $x^\mu(\sigma)$. It is convenient to take

$$x^\mu(\sigma) = \sum_k S_k(\sigma)a_k^\mu \tag{5.3}$$

where the $S_k(\sigma)$ are now taken to be orthonormalized eigenvectors of the Laplacian operation

$$\nabla S_k(\sigma) = \Lambda_k S_k(\sigma) \tag{5.4}$$

$$\int d^2\sigma \sqrt{g(\sigma)} S_k(\sigma) S_\ell(\sigma) = \delta_{k\ell} \tag{5.5}$$

where

$$\nabla S \equiv \frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g}\, g^{\alpha\beta} \partial_\beta S) \tag{5.6}$$

Inserting (5.3) - (5.6) in Eq. (5.2) gives

$$||\delta x||^2 = A \sum_k \delta a_k^\mu \delta a_{k\mu} \tag{5.7}$$

According to the analog of Eq. (4.8) (applied here to integrals over $x^\mu(\sigma)$ instead of $g_{\alpha\beta}(\sigma)$) Eq. (5.7) tells us that the integral over $x^\mu(\sigma)$ is

$$\int dx \ldots = \int \prod_{k,\mu} \left(A^{1/2} da_k^\mu \right) \ldots \tag{5.8}$$

The factor $Z[g]$ is then

$$Z[g] = \left(\prod_{k,\mu} A^{1/2} \right) \int \left(\prod_{k,\mu} da_k^\mu \right) \exp\left(- \sum_k \Lambda_k a_k^\mu a_{k\mu} \right).$$

For reasons that are not quite clear, it seems to be possible to ignore the complication that $(a_k^o)^2$ appears in the exponent with the "wrong" sign. Then the integrals for all a_k^μ with $\Lambda_k \neq 0$ can be done by the rules of ordinary integration, but we still must reserve special treatment for the a_o^μ corresponding to the zero mode with $\Lambda_o = 0$. That is,

$$Z[g] = \left(\prod_k A \right)^{D/2} \left(\prod_{k \neq 0} \pi/\Lambda_k \right)^{D/2} \int \prod_\mu da_o^\mu$$

where D is the dimensionality, the numbers of values taken by μ. We can re-write this as

$$Z[g] = \left(\prod_k \pi A \right)^{D/2} (\pi \mathrm{Det}' \nabla)^{-D/2} \int \prod_\mu da_o^\mu \tag{5.9}$$

where, in the same notation as in Sec. 4,

$$\mathrm{Det}'\nabla \equiv \prod_{k\neq 0} \Lambda_k \tag{5.10}$$

The infinite product of the constant πA can be evaluated by the same methods as in Sec. 4. Using Eq. (4.38), we have

$$\prod_k \pi A = \exp\left(\ell n(\pi A)\delta^2(0) \int d^2\sigma \sqrt{g(\sigma)}\right) \tag{5.11}$$

This factor must be cancelled by including a term $-\ell n(\pi A)\delta^2(0)$ in the counterterm μ^2 in Eq. (2.19). Since we don't know the value of μ^2 anyway, there is no loss of generality in simply taking A to have the convenient value

$$A = 1/\pi \tag{5.12}$$

What about the integral over a_o^μ? It is a fundamental theorem that the only zero-mode of the Laplacian on a closed compact surface is a constant, so $S_o(\sigma)$ is constant. With the normalization (5.5), we have then

$$S_o = \left[\int d^2\sigma \sqrt{g(\sigma)}\right]^{-1/2} \tag{5.13}$$

It follows that a_o^μ is related to the constant term $x_o^\mu \equiv a_o^\mu S_o$ in $x^\mu(\sigma)$ by

$$a_o^\mu = \left[\int d^2\sigma \sqrt{g(\sigma)}\right]^{1/2} x_o^\mu \tag{5.14}$$

Eq. (5.9) then becomes

$$Z[g] = \left(\frac{\pi \mathrm{Det}'\ \nabla}{\int d^2\sigma \sqrt{g(\sigma)}}\right)^{-D/2} \int \prod_\mu dx_o^\mu \tag{5.15}$$

In vacuum amplitudes this is the complete contribution of the $x^\mu(\sigma)$-integral, and the final integral over x_o^μ simply yields a factor equal to the volume of spacetime. In S-matrix elements, as already mentioned, the integrand contains contributions from the

vertex functionals. However, the dependence of the integrand on a constant term in $x^\mu(\sigma)$ arises solely from the factors $e^{ip\cdot x}$ in each vertex function. The x_o^μ-integral then gives

$$\int \left(\prod_\mu dx_o^\mu\right) \prod_I \exp(ip_I \cdot x_o) = (2\pi)^D \delta^D \left(\sum_I p_I\right) \tag{5.16}$$

which is just the usual momentum-conservation factor. That is, Eq. (5.0) now reads

$$\begin{aligned} &\int dx \exp\left(-\frac{T}{2}\int d^2\sigma \sqrt{g} g^{\alpha\beta} \frac{\partial x^\mu}{\partial \sigma^\alpha}\frac{\partial x_\mu}{\partial \sigma^\beta}\right) \\ &\times V\left[x, g; p_1 j_1\right] \ldots V\left[x, g; p_N j_N\right] \\ &= (2\pi)^D \delta^D \left(\sum_I p_I\right) \left(\frac{\pi \mathrm{Det}' \nabla}{\int d^2\sigma \sqrt{g(\sigma)}}\right)^{-D/2} \\ &\times < V(p_1 j_1) \ldots V(p_N j_N) >_g \end{aligned} \tag{5.17}$$

The calculation of the correlation function $< V \ldots V >$ is discussed in Section 9.

6. Complex Tensor Analysis

In the two previous sections, we encountered determinants of two different second-order differential operators, ${P_1}^\dagger P_1$ and ∇. It will be convenient in examining the conformal anomalies in these determinants to recognize that ${P_1}^\dagger P_1$ and ∇ are special cases of a class of generalized Laplacians, that map two-dimensional tensors of various sorts into tensors of the same sort.

In two dimensions, any tensor can be expressed as a linear combination of symmetric traceless tensors, with coefficients that may include direct products of $g_{\alpha\beta}$ and $\epsilon_{\alpha\beta}\sqrt{g}$. (This is because a tensor that is not symmetric or is symmetric but not traceless in a pair of indices α, β may be expressed as a tensor that is symmetric and traceless in these indices plus $\epsilon_{\alpha\beta}\sqrt{g}$ or $g_{\alpha\beta}$ times a tensor without these indices.) We therefore restrict our attention to totally symmetric covariant traceless tensors. These are completely specified by their rank, N.

It will be convenient to adopt a complex notation. In terms of the coordinates σ^1, σ^2 for which the metric takes the form (3.3), we define $z \equiv \sigma^1 + i\sigma^2$ and $z^* \equiv \sigma^1 - i\sigma^2$, for which the metric takes the form (3.6)

$$g_{zz^*} = \frac{1}{2}\exp\left(\phi(z, z^*)\right) \qquad g_{zz} = g_{z^*z^*} = 0 \tag{6.1}$$

In this notation, a symmetric traceless covariant tensor has only two non-zero components, those with all indices equal to z or equal to z^*. Equivalently, we can take the two independent components of a symmetric traceless tensor $t^{(N)}_{\alpha\beta}\ldots$ of rank N as those with all covariant or all contravariant indices equal to z:

$$t^{(N)}_{+} \equiv t^{(N)zz\ldots z} \tag{6.2}$$

$$t^{(N)}_{-} \equiv t^{(N)}_{zz\ldots z} \tag{6.3}$$

A tensor $t_{\alpha\beta\ldots}$ that is real in Cartesian coordinates σ^1, σ^2 will under complex conjugation have its z and z^* indices interchanged. In particular,

$$\left(t^{(N)}_{zz\ldots z}\right)^* = t^{(N)}_{z^*z^*\ldots z^*} \tag{6.4}$$

and so

$$\left(t^{(N)}_{-}\right)^* = \left(\frac{1}{2}e^{\phi}\right)^N t^{(N)}_{+} \tag{6.5}$$

The invariant scalar product of two symmetric traceless tensors $t^{(N)}$ and $s^{(N)}$ is defined by

$$\begin{aligned}\left(t^{(N)}, s^{(N)}\right) &\equiv \int d^2\sigma\sqrt{g}\, t^{(N)}_{\alpha\beta\ldots} s^{(N)\alpha\beta\ldots} \\ &= \frac{1}{2}\int dz dz^* e^{\phi}\left[t^{(N)}_{+}s^{(N)}_{-} + t^{(N)}_{-}s^{(N)}_{+}\right]\end{aligned} \tag{6.6}$$

and for real tensors

$$\left(t^{(N)}, s^{(N)}\right) = 2^{-N}\int dz dz^* e^{(1+N)\phi} Re\left\{t^{(N)*}_{+}s^{(N)}_{+}\right\} \tag{6.7}$$

or equivalently

$$\left(t^{(N)}, s^{(N)}\right) = 2^N \int dz dz^* e^{(1-N)\phi} Re\left\{t_-^{(N)*} s_-^{(N)}\right\} \tag{6.8}$$

Now we come to the construction of covariant differential operators. For the metric (6.1), the only non-vanishing components of the affine connection are

$$\Gamma^z_{zz} = \partial_z \phi \tag{6.9}$$

and the same with z replaced by z^*. The first-order differential operators that occur naturally in two dimensions are those that map traceless symmetric tensors of rank N into those of rank $N+1$:

$$\begin{aligned}\left(P_N t^{(N)}\right)_- &\equiv \partial_z t^{(N)}_{zz\ldots z} - N\Gamma^z_{zz} t^{(N)}_{zz\ldots z} \\ &= e^{N\phi}\partial_z\left(e^{-N\phi} t_-^{(N)}\right)\end{aligned} \tag{6.10}$$

$$\begin{aligned}\left(P_N t^{(N)}\right)_+ &\equiv g^{zz^*}\partial_z * t^{(N)zz\ldots z} \\ &= 2e^{-\phi}\partial_{z^*} t_+^{(N)}\end{aligned} \tag{6.11}$$

and those that map symmetric traceless tensors of rank $N+1$ into those of rank N:

$$\begin{aligned}\left(P_N{}^\dagger t^{(N+1)}\right)_- &\equiv -g^{zz^*}\partial_{z^*} t^{(N+1)}_{zz\ldots z} \\ &= -2e^{-\phi}\partial_{z^*} t_-^{(N+1)}\end{aligned} \tag{6.12}$$

$$\begin{aligned}\left(P_N^\dagger t^{(N+1)}\right)_+ &\equiv -\left[\partial_z t^{(N+1)zz\ldots z} + (N+1)\Gamma^z_{zz} t^{(N+1)zz\ldots z}\right] \\ &= -e^{-(N+1)\phi}\partial_z\left(e^{(N+1)\phi} t_+^{(N+1)}\right)\end{aligned} \tag{6.13}$$

Note that $P_N{}^\dagger$ is the adjoint of P_N, in the sense that

$$\left(t^{(N+1)}, P_N s^{(N)}\right) = \left(P_N^\dagger t^{(N+1)}, s^{(N)}\right) \tag{6.14}$$

The differential operator P_1 introduced in Section 4 is the same as the P_1 used here.

From the operators P_N and their adjoints we can construct two sets of "Laplacians," self-adjoint second-order differential operators that take symmetric traceless tensors of rank N into others of the same rank. These are defined by

$$\nabla_N \equiv P_N{}^\dagger P_N \tag{6.15}$$

$$\tilde{\nabla}_N \equiv P_{N-1} P_{N-1}^\dagger \tag{6.16}$$

Acting on a tensor $t^{(N)}$, these yield tensors with components

$$\left(\nabla_N t^{(N)}\right)_- = -2e^{-\phi}\partial_z * \left(e^{N\phi}\partial_z \left(e^{-N\phi} t_-^{(N)}\right)\right) \tag{6.17}$$

$$\left(\nabla_N t^{(N)}\right)_+ = -2e^{-(N+1)\phi}\partial_z \left(e^{N\phi}\partial_{z*} t_+^{(N)}\right) \tag{6.18}$$

or

$$\left(\tilde{\nabla}_N t^{(N)}\right)_- = -2e^{(N-1)\phi}\partial_z \left(e^{-N\phi}\partial_z * t_-^{(N)}\right) \tag{6.19}$$

$$\left(\tilde{\nabla}_N t^{(N)}\right)_+ = -2e^{-\phi}\partial_z * \left(e^{-N\phi}\partial_z \left(e^{N\phi} t_+^{(N)}\right)\right) \tag{6.20}$$

For $N = 0$ there is no operator (6.16), while the ordinary Laplacian ∇ is $-\frac{1}{2}\nabla_0$.

7. Conformal Anomalies

As already mentioned several times, the determinants encountered in the previous two sections contain conformal anomalies that would violate Weyl-invariance if not cancelled. It is time now to consider the general form of these anomalies.

Let $\mathcal{O}$ be a general second-order metric-dependent differential operator, like the generalized Laplacians discussed in the previous Section. Then Det$\mathcal{O}$ is the sum of all vacuum diagrams in a boson field theory with Lagrangian $\psi\mathcal{O}\psi$, and ℓnDet$\mathcal{O}$ is the sum of connected vacuum diagrams. In two dimensions ℓnDet$\mathcal{O}$ will be logarithmically divergent, and if we use dimensional regularization, then in n dimensions in place of a divergence we will have a term

$$\ell n \mathrm{Det}\mathcal{O} = \frac{1}{n-2}\int d^n\sigma \mathcal{F} + \ldots \tag{7.1}$$

where $\mathcal{F}$ is a local function of the metric and its derivatives, involving just 2 derivatives, and the omitted terms are finite for $n \to 2$. If the operator $\mathcal{O}$ is generally covariant (that is,

if $\psi\mathcal{O}\psi$ is a scalar) and we are careful to preserve general covariance in our regularization, then $\mathcal{F}$ must be a scalar density, and hence

$$\mathcal{F} = A(\mathcal{O})\sqrt{g}R \tag{7.2}$$

where $A(\mathcal{O})$ is a constant, depending on the operator $\mathcal{O}$. If Det$\mathcal{O}$ is also formally Weyl-invariant in two dimensions, the finite terms in (5.1) are Weyl invariant at $n = 2$. Also, for $n \to 2$ the divergent part of the first term is a Weyl-invariant, proportional to the Euler characteristic. However, the integral of $\mathcal{F}$ contains terms of order $n-2$ that are not Weyl invariant, and when multiplied with $1/n-2$ these yield finite Weyl-noninvariant terms.

To evaluate these terms, recall that in two dimensions, by a coordinate transformation we can in at least a finite coordinate patch take the metric as

$$g_{\alpha\beta}(\sigma) = \delta_{\alpha\beta}e^{\phi(\sigma)} \tag{7.3}$$

For a metric of this form, a direct calculation gives in n dimensions

$$\sqrt{g}R = -\exp\left(\left(\frac{n}{2}-1\right)\phi\right)\left[(n-1)\partial_\alpha\partial_\alpha\phi + \frac{1}{4}(n-1)(n-2)\partial_\alpha\phi\partial_\alpha\phi\right] \tag{7.4}$$

Expanding around $n = 2$ to first order in $n-2$, this gives

$$\sqrt{g}R = -\partial_\alpha\partial_\alpha\phi + (n-2)\left[\frac{1}{2}\partial_\alpha\partial_\alpha\phi + \frac{1}{4}\partial_\alpha\phi\partial_\alpha\phi + \frac{1}{2}\phi\partial_\alpha\partial_\alpha\phi\right] \tag{7.5}$$

Discarding total derivatives (which yield topological invariants, like the Euler characteristic), we find to first order in $n-2$

$$\sqrt{g}R = \frac{1}{4}(n-2)\phi\partial_\alpha\partial_\alpha\phi + \text{derivatives} \tag{7.6}$$

Using this in (7.2) and (7.1) gives then, for $n \to 2$

$$\begin{aligned}\text{Det}\mathcal{O} = \exp\left(\frac{1}{4}A(\mathcal{O})\int d^2\sigma\phi\partial_\alpha\partial_\alpha\phi\right) \\ \times\ \phi\text{−independent factors}\end{aligned} \tag{7.7}$$

It is important to note that the Weyl non-invariant integral here cannot be expressed as a local functional of the metric $g_{\alpha\beta}(\sigma)$, and hence cannot be cancelled by adding a local counterterm to the action, like (2.19). Rather, Weyl invariance will have to be preserved by cancellations of the contributions of different operators $\mathcal{O}$.

For the generalized Laplacians $\nabla_N = P_N{}^\dagger P_N$ with which we have to deal here, there is an important complication arising from the presence of zero modes of P_N and/or $P_N^\dagger$ We denote these respectively by $\xi_a^{(N)}$ and $\psi_i^{(N+1)}$ (the superscript denoting the tensor rank):

$$P_N \xi_a^{(N)} = 0 \tag{7.8}$$

$$P_N{}^\dagger \psi_i^{(N+1)} = 0 \tag{7.9}$$

(For $N = 1$ the indices a and i take N'_γ and N_γ values, respectively.) Using (6.11) and (6.12), we see that these conditions can be written as the requirement that the contravariant z-components of $\xi_a^{(N)}$ and the covariant z-components of $\psi_i^{(N+1)}$ are analytic functions of z:

$$\partial_z * \xi_{a,+}^{(N)} = \partial_z * \psi_{i,-}^{(N+1)} = 0 \tag{7.10}$$

The other components $\xi_{a,-}^{(N)}$ and $\psi_{i,+}^{(N+1)}$ are then given by the reality conditions (6.5). In addition, the eigenmodes must satisfy periodicity conditions specified by the group Γ_γ for $\gamma \geq 1$, and must have finite scalar products (6.7), (6.8). We will not specify the normalization of the eigenmodes, but will simply specify that it is chosen so that $\xi_{a,+}^{(N)}$ and $\psi_{i,-}^{(N+1)}$ are independent of the factor e^ϕ in the metric. (This means that they are *not* orthonormalized.) With this convention, Alvarez[13] shows that for the operators $P_N{}^\dagger P_N$, in place of Eq. (7.7) we have

$$\frac{\mathrm{Det}' P_N{}^\dagger P_N}{\mathrm{Det}H(P_N)\mathrm{Det}H(P_N^\dagger)} = \exp\left(A_N \int d^2\sigma \phi \partial_\alpha \partial_\alpha \phi\right) \times \phi\text{−independent factors} \tag{7.11}$$

where

$$H(P_N)_{ab} \equiv \left(\xi_a^{(N)}, \xi_b^{(N)}\right) \tag{7.12}$$

$$H(P_N{}^\dagger)_{ij} \equiv (\psi_i^{(N+1)}, \psi_j^{(N+1)}) \tag{7.13}$$

the scalar products being defined as in Section 6. (Note that the remaining ambiguity in the normalization of $\xi_a^{(N)}$ and $\psi_i^{(N+1)}$ affects only the ϕ-independent factor in (7.11).) In addition, Alvarez gives the constant A_N in (7.11) as

$$A_N = \frac{1+6N(N+1)}{12\pi} \tag{7.14}$$

except for that for $N = 0$

$$A_0 = \frac{1}{24\pi} \tag{7.14}'$$

(This difference arises because $P_N^\dagger P_N$ for $N \geq 1$ acts on tensors with two independent non-vanishing components, while $P_0{}^\dagger P_0$ acts on one-component scalars).

Now let's see how this works for the operators of interest here.

$\underline{N=0}$. Here $P_0{}^\dagger P_0$ is (up to a constant) the Laplacian ∇. Also, P_0 has one zero mode, a constant $\xi^{(0)}$ that can be taken to have any ϕ-independent value, say $\xi^{(0)} = 1$, so (7.12) gives

$$\mathrm{Det}H(P_0) = \int d^2\sigma\sqrt{g(\sigma)} \tag{7.15}$$

On the other hand, any zero modes $\psi^{(1)}$ of $P_0{}^\dagger$ are defined with $\psi_{i,-}^{(1)}$ independent of ϕ, so that the scalar product (6.8) is also independent of ϕ:

$$H(P_1{}^\dagger)_{ij} = 2\int dzdz^* Re\left(\psi_{i,-}^{(1)*}\psi_{j,-}^{(1)}\right) \tag{7.16}$$

For $N = 0$, Eq. (7.11) then yields

$$\frac{\mathrm{Det}'\nabla}{\int d^2\sigma\sqrt{g(\sigma)}} = \exp\left(\frac{1}{24\pi}\int d^2\sigma\phi\partial_\alpha\partial_\alpha\phi\right) \times \phi\text{−independent factors} \tag{7.17}$$

$\underline{N=1}$ In Section 4 we introduced symmetric traceless tensor zero modes ψ_i of the operator ${P_1}^\dagger$, but did not specify their normalization. (The dependence of the matrices M and N on this normalization cancels in Eq. (4.54).) Let us now suppose that the ψ_i are normalized according to the prescription adopted here for $\psi^{(2)}$ (i.e., that the covariant z-components $\psi_{i\,zz}$ should be ϕ-independent), in which case the two sets of zero modes can be identified. In particular, it follows that the matrix N_{ij} in Section 4 can be identified as

$$N_{ij} = H({P_1}^\dagger)_{ij} \tag{7.18}$$

Now, what about $H(P_1)$? This factor is present in Eq. (7.11) only for genus 0 and 1, where P_1 has zero modes, and in these cases we also encounter a ϕ-dependent factor Ω_{CKV} in Eq. (4.54). This group volume is defined as in (4.50), with the group parameterization near the identity fixed by the condition that a transformation with parameters $\delta\gamma^a$ shifts the coordinate by

$$\delta\sigma^\alpha = {u_a}^\alpha(\sigma)\delta\gamma^a$$

where the u_a are orthonormalized zero-modes of P_1:

$$P_1 u_a = 0$$

$$\int {u_a}^\alpha {u_b}^\beta \sqrt{g} g_{\alpha\beta} d^2\sigma = \delta_{ab}$$

We can define a new set of coordinates $\tilde{\gamma}$ by using the ϕ-independent non-orthonormalized zero modes $\xi_a^{(1)}$ in place of u_a:

$$\delta\sigma^\alpha = \xi_a^{(1)\alpha}(\sigma)\delta\tilde{\gamma}^a$$

If we set $\xi_a^{(1)} = {C_a}^b u_b$ then $H(P_1) = CC^T$ and $\delta\gamma^b = {C_a}^b\delta\tilde{\gamma}^a$, so the Jacobian of the transformation $\gamma \to \tilde{\gamma}$ is given by

$$\left|\frac{\partial\tilde{\gamma}}{\partial\gamma}\right|_{\gamma=0} = (\mathrm{Det}H(P_1))^{-1/2}$$

The group volume $\tilde{\Omega}_{CKV}$ calculated using coordinates $\tilde{\gamma}$ is ϕ-independent, so the original group volume Ω_{CKV} has a ϕ-dependence given by

$$\begin{aligned}\Omega_{CKV} &= \tilde{\Omega}_{CKV} \Big/ \left|\frac{\partial\tilde{\gamma}}{\partial\gamma}\right|_{\gamma=0} \\ &= \tilde{\Omega}_{CKV}\,(\mathrm{Det}H(P_1))^{1/2}\end{aligned} \tag{7.19}$$

Putting together (7.18), (7.19), and (7.14), we see that Eq. (7.11) reads here

$$\begin{aligned}\frac{\mathrm{Det}'P_1^\dagger P_1}{\mathrm{Det}N\ (\Omega_{CKV})^2} &= \exp\left(\frac{13}{12\pi}\int d^2\sigma\phi\partial_\alpha\partial_\alpha\phi\right) \\ &\times \phi-\text{independent factors}\end{aligned} \tag{7.20}$$

Finally, we note that the covariant components of the tensor $\chi_{.i}$ defined by Eq. (4.14) are proportional to e^ϕ, while the covariant components of ψ_i are ϕ-independent, and the factor $\sqrt{g}g^{\alpha\beta}g^{\gamma\delta}$ in the scalar product of ψ_i and $\chi_{.j}$ is proportional to $e^{-\phi}$. It follows that the matrix M_{ij} defined by (4.25) is ϕ-independent.

We can now pull everything together. Using Eqs. (5.17) and (4.54) in (2.1), we find that the S-matrix is

$$\begin{aligned}S(p_1j_1\ldots p_Nj_N) &= i(2\pi)^D\delta^D\left(\sum_I p_I\right)\sum_\gamma\lambda^{-\chi} \\ &\times\int d\mu(\tau) < V(p_1j_1)\ldots V(p_Nj_N) >_g\end{aligned} \tag{7.21}$$

where $d\mu(\tau)$ is the measure

$$d\mu(\tau) = \left(\mathrm{Det}'P_1^\dagger P_1\right)^{1/2}\mathrm{Det}\ M\,(\mathrm{Det}\ N)^{-1/2}\,\Omega_{CKV}^{-1}\left(\frac{\pi\mathrm{Det}'\nabla}{\int d^2\sigma\sqrt{g(\sigma)}}\right)^{-D/2}\prod_i d\tau_i \tag{7.22}$$

(The factor $\lambda^{-\chi}$ comes from the second term in the action (2.15).) From (7.17) and (7.20), we see that the measure $d\mu(\tau)$ is ϕ-independent if and only if $D = 26$, while it is 2-dimensionally generally covariant for all D. Hence, in 26 dimensions $d\mu(\tau)$ can be calculated using any metric $g_{\alpha\beta}(\sigma)$ of genus γ and Teichmüller parameters τ_i. Section 9 shows (for $\gamma = 0$) that the same is true of the correlation functions $< VV\ldots V >_g$.

8. Calculation of Ω_{CKV}

We now turn to a calculation of the volume of Ω_{CKV} of the group of conformal symmetries, which consists of all coordinate transformations that leave the metric $g_{\alpha\beta}(\sigma)$ (and all periodicity conditions) invariant up to a Weyl transformation. This group is trivial except for the lowest orders, genus zero and one, and it will be convenient to treat these two cases separately.

Genus 0

The conformal symmetry group here is $SL(2,C)$. This is non-compact, so Ω_{CKV} here is actually infinite. To put this quantity in a useful form, it is necessary to consider the way that it appears in calculating scattering amplitudes. As in Section 3, we represent the genus zero surface by the whole complex plane $\hat{C}$, including the point at infinity. The N-particle amplitude then takes the form of a multiple integral

$$A_N \equiv \int d^2z_1 \dots d^2z_N \mathcal{F}(z_1 \dots z_N) \tag{8.1}$$

The group $SL(2,C)$ is represented by Möbius transformations

$$z \to z_\gamma = \frac{az+b}{cz+d} \tag{8.2}$$

with three complex parameters γ^n given by any three of the complex quantities a, b, c, d, the fourth being given by the condition that $ad - bc = 1$. Under such a transformation, we have

$$d^2z_{1\gamma} \dots d^2z_{N\gamma} \mathcal{F}(z_{1\gamma} \dots z_{N\gamma}) = d^2z_1 \dots d^2z_N \mathcal{F}(z_1 \dots z_N) \tag{8.3}$$

It follows that A_N is infinite, because it involves an integral over the infinite volume of $SL(2,C)$. Following Faddeev and Popov, we introduce 3 complex "gauge-fixing" functions

$f_\ell(z_1 \dots z_N)$ (with $\ell = 1, 2, 3$), and consider the gauge-fixed integral

$$A_N[f] \equiv \int d^2 z_1 \dots d^2 z_N \mathcal{F}(z_1 \dots z_N) \mathcal{D}(z_1 \dots z_N) \prod_{\ell=1}^{3} |\delta\left(f_\ell(z_1 \dots z_N)\right)|^2 \tag{8.4}$$

where:

$$\mathcal{D}(z_1 \dots z_N) \equiv \left|\mathrm{Det}\left(\partial f_\ell(z_{1\gamma} \dots z_{N\gamma})/\partial\gamma^n\right)_{\gamma=0}\right|^2 \tag{8.5}$$

By a change of variables, we can change z_i to $z_{i\gamma}$ everywhere in (8.4), and then use the Möbius symmetry (8.3) to write this as

$$\begin{aligned} A_N[f] = \int d^2 z_1 \dots d^2 z_N \mathcal{F}(z_1 \dots z_N) \mathcal{D}(z_{1\gamma} \dots z_{N\gamma}) \\ \times \prod_\ell \left|\delta\left(f_\ell(z_{1\gamma} \dots z_{N\gamma})\right)\right|^2 \end{aligned} \tag{8.6}$$

valid for arbitrary Möbius transformation parameters γ^n. We note that the determinant here may be written

$$\begin{aligned} \mathcal{D}(z_{1\gamma} \dots z_{N\gamma}) = \left|\mathrm{Det}\left(\partial f_\ell(z_{1\gamma} \dots z_{N\gamma})/\partial\gamma^n\right)\right|^2 \\ \times \left|\mathrm{Det}\left(\partial(\gamma' * \gamma)^n/\partial\gamma'^m\right)_{\gamma'=0}\right|^2 \end{aligned}$$

By setting $\gamma' = \gamma'' * \gamma^{-1}$ in the second factor, we can write this as

$$\begin{aligned} \mathcal{D}(z_{1\gamma} \dots z_{N\gamma}) = \left|\mathrm{Det}\left(\partial f_\ell(z_{1\gamma} \dots z_{N\gamma})/\partial\gamma^n\right)\right|^2 \\ \times |\mathrm{Det}\rho(\gamma)|^{-2} \end{aligned} \tag{8.7}$$

where $\rho(\gamma)$ is the matrix encountered in Section 4:

$$\rho(\gamma)^m{}_n \equiv \left(\partial(\gamma'' * \gamma^{-1})^m/\partial\gamma''^n\right)_{\gamma''=\gamma}$$

Multiplying Eq. (8.6) with $|\mathrm{Det}\rho(\gamma)|$ and integrating over γ, we have then

$$A_N[f]\Omega_{CKV} = A_N \tag{8.8}$$

where Ω_{CKV} is the group volume

$$\Omega_{CKV} = \int |\mathrm{Det}\rho(\gamma)|^2 \prod_n d\gamma^n d\gamma^{n*} \tag{8.9}$$

(We have a $|\mathrm{Det}\rho|^2$ here, because in a real notation ρ would be a 6×6 matrix, the direct sum of the complex 3×3 ρ and $\rho*$.) Thus the effect of dividing by A_N by Ω_{CKV} is just to convert it to the gauge-fixed form (8.4).

As we can see from (8.5), the gauge-fixed integral (8.4) obtained by dividing by Ω_{CKV} depends on the normalization of the Möbius parameters γ^nnear the identity $\gamma^n = 0$. We saw in Section 4 that these parameters should be chosen so that the change in the coordinate $z = \sigma^1 + i\sigma^2$ produced by a Möbius transformation near the identity is

$$\delta z = \sum_{n=1}^{3} u_n(z)\delta\gamma^n$$

with orthonormalized functions $u_n(z)$. The orthonormalization depends on the metric, which we take in the standard constant curvature form

$$g_{zz} = g_{z*z*} = 0 \qquad g_{zz*} = \frac{1}{2}(1+|z|^2)^{-2} \tag{8.10}$$

(See Eq. (3.9).) The orthonormalization condition is then

$$\begin{aligned}\delta_{nm} &= \int g_{zz*}^2 \left[u_n(z)u_m^*(z) + u_n^*(z)u_m(z)\right] dzdz^* \\ &= \int (1+|z|^2)^{-4}\mathrm{Re}\left[u_n(z)u_m^*(z)\right] d\mathrm{Re}z dImz\end{aligned}$$

(The second factor of g_{zz*} is just the square root of the determinant of the metric.) Infinitesimal Möbius transformations shift z by quadratic functions of z; the three orthonormalized functions of z that span the space of quadratic functions may be taken as

$$\begin{aligned}u_1(z) &= \sqrt{\frac{3}{\pi}} \\ u_2(z) &= \sqrt{\frac{6}{\pi}}z \\ u_3(z) &= \sqrt{\frac{3}{\pi}}z^2\end{aligned}$$

That is, we normalize the γ^n so that near the identity

$$z_\gamma = z + \sqrt{\frac{3}{\pi}}\left[\delta\gamma^1 + \sqrt{2}\delta\gamma^2 z + \delta\gamma^3 z^2\right] \tag{8.11}$$

It is convenient to take the three gauge-fixing functions simply as linear expressions in any three of the z's, say the first three

$$f_\ell(z_1 \dots z_N) = z_\ell - c_\ell \tag{8.12}$$

with arbitrary constants c_ℓ. A straightforward calculation then gives the determinant (8.5) as

$$\mathcal{D} = \frac{54}{\pi^3} |(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|^2 \tag{8.13}$$

We conclude that *the effect of dividing by the Möbius group volume Ω_{CKV} is just to fix any 3 of the z's, say z_1, z_2, z_3, at arbitrary complex values, and to multiply the intregal by the factor (8.13).* However, this result depends on our choice of the metric (8.10), and is valid only if we use this metric everywhere else. Also, although we have taken pains to obtain the factor $54/\pi^3$ in Eq. (8.13), this appears together with another factor λ^{-2}, so it is of no importance until we carry our calculations up to genus-two amplitudes, which also depend on λ.

As an incidental application of this result, consider the integral

$$\int \frac{d^2z_1 d^2z_2 d^2z_3}{|z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2}$$

The integrand here satisfies the Möbius invariance condition (8.3), so this integral can be expressed as the group volume Ω_{CKV}, times the value of the integrand at arbitrary fixed values of z_1, z_3, and z_3, times the factor (8.13). The z-dependent factors cancel, leaving us with the result that[17]

$$\Omega_{CKV} = \frac{\pi^3}{54} \int \frac{d^2z_1 d^2z_2 d^2z_3}{|z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2} \tag{8.14}$$

The integral here diverges, so this formula should be regarded as merely a mnemonic, that reminds us how to interpret the factor $1/\Omega_{CKV}$ in (7.22).

Genus 1

The conformal symmetry group here is T^2/Γ_1; here T^2 is the group of translations in two dimensions; Γ_1 is the periodicity group for genus 1; and T^2/Γ_1 means that we identify translations that differ by elements of Γ_1. This is a compact group, so in contrast to the case of genus zero, Ω_{CKV} is here a finite function of the Teichmüller parameter.

Since Ω_{CKV} is coordinate invariant (though not Weyl invariant), we will get the same result for Ω_{CKV} whether we use a metric of form (4.2) or (4.5). Let us take the metric as (4.2), for which the periodicity group is (4.1). The conformal symmetry group then consists of translations, which can be written as

$$z \to z + N(\gamma^1 + i\gamma^2) \tag{8.15}$$

with real parameters γ^n, and two translations γ^n and γ'^n identified if

$$N(\gamma^{1\prime} + i\gamma^{2\prime}) = N(\gamma^1 + i\gamma^2) + n + m\tau$$

with n and m arbitrary integers. The quantity N is chosen so that the coefficients of γ^1 and γ^2 in (8.15) are orthonormalized, using the Euclidean metric (4.2) and taking z to run over the parallelogram with corners $0, 1, \tau, 1+\tau$. This gives

$$N = (Im\tau)^{-1/2}$$

Since the group of translations is Abelian, the matrix $\rho(\gamma)$ here is unity, so Ω_{CKV} is just the integral

$$\Omega_{CKV} = \int d\gamma^1 d\gamma^2$$

with $N(\gamma^1 + i\gamma^2)$ taken to run over the same parallelogram. This gives

$$\Omega_{CKV} = N^{-2} Im\tau = (Im\tau)^2 \tag{8.16}$$

This is equivalent to the result obtained by Polchinski,[14] using the metric in the form (4.5).

9. Vertex Functionals

We now turn to the structure of the vertex functionals $V[x,g;p,j]$ that appear in the functional integrals (2.1) for S-matrix elements. These vertex functionals are to be determined from the requirements (2.6), (2.9), and (2.14), imposed by invariance under Lorentz transformations, 2-dimensional coordinate transformations, and Weyl transformations, respectively[18]. The requirements of Lorentz invariance and 2-dimensional general covariance together tell us that the vertex functionals take the form

$$V[x,g;p,j] = \int d^2\sigma \sqrt{g(\sigma)} e^{ip\cdot x} \mathcal{S}(\sigma;p,j) \tag{9.1}$$

where $\mathcal{S}(\sigma)$ is a 2-scalar formed out of $g_{\alpha\beta}(\sigma)$, $\epsilon_{\alpha\beta}$, the 2-curvature $R^{(2)}(\sigma)$, and covariant derivatives of $x^\mu(\sigma)$ [but not $x^\mu(\sigma)$ itself], chosen to give V the Lorentz transformation property (2.6). For instance for scalar particles, $\mathcal{S}(\sigma)$ is a factor $(2p^o)^{-1/2}$ times a linear combination of the terms

$$1, \qquad R^{(2)}, \qquad \frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta} g^{\alpha\beta}\eta_{\mu\nu},$$

$$\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta}\frac{\partial x^\rho}{\partial\sigma^\gamma}\frac{\partial x^\tau}{\partial\sigma^\delta} g^{\alpha\beta}g^{\gamma\delta}\eta_{\mu\nu}\eta_{\rho\tau},$$

$$\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta}\frac{\partial x^\rho}{\partial\sigma^\gamma}\frac{\partial x^\tau}{\partial\sigma^\delta} g^{\alpha\beta}g^{\gamma\delta}\left(\eta_{\mu\nu}\eta_{\nu\tau}+\eta_{\mu\tau}\eta_{\nu\rho}\right),$$

and so on. At first sight, it looks like only the third term satisfies the requirement of Weyl invariance (because $\sqrt{g}g^{\alpha\beta}$ is Weyl-invariant in two dimensions), but as we shall see, this conclusion is quite changed by the presence of anomalies.

Consider how to calculate correlation functions of the vertex functionals (9.1). As discussed in Section 2, the relevant portion of the action is quadratic in $x^\mu(\sigma)$, so the correlation functions are to be calculated by the usual rules of Gaussian integration; we

sum over all ways of pairing $x^\mu(\sigma)$'s from the various vertex functionals, each pairing contributing a product of factors, one from each pair. The factor $< x^\mu(\sigma)x^\nu(\sigma') >$ for a pair of x's is the "reciprocal" of the "matrix" that multiplies $\frac{-1}{2}x^\mu(\sigma)x^\nu(\sigma')$ in the action. That is,

$$< x^\mu(\sigma)x^\nu(\sigma') >= T^{-1}\eta^{\mu\nu}G(\sigma,\sigma') \tag{9.2}$$

where G is the Greens function of the Laplacian operator, defined by

$$\begin{aligned} -\frac{\partial}{\partial\sigma^\alpha}\left(\sqrt{g(\sigma)}g^{\alpha\beta}(\sigma)\frac{\partial}{\partial\sigma^\beta}G(\sigma,\sigma')\right) \\ = \delta_0^2(\sigma-\sigma') - A^{-1}\sqrt{g(\sigma)} \end{aligned} \tag{9.3}$$

Here δ_0^2 is the ordinary (scalar-*density*) delta function, with

$$\int d^2\sigma\delta_0^2(\sigma-\sigma') = 1 \tag{9.4}$$

and A is the area of the manifold

$$A = \int d^2\sigma\sqrt{g(\sigma)} \tag{9.5}$$

(The second term on the right-hand side of Eq. (9.3) is needed in order that this side of the equation should give zero when integrated over the manifold, as is automatically the case for the left-hand side of Eq. (9.3). As we shall see, this term has no significant effect on the Weyl transformation properties of the Greens function.) The Greens function is defined (up to an immaterial additive constant) by Eq. (9.3), plus the various periodicity conditions incorporated in the group Γ_γ.

Now let us consider the Weyl transformation properties of the Greens function. The quantities $\sqrt{g}g^{\alpha\beta}$ and $\delta_0^2(\sigma-\sigma')$ are Weyl-invariant, so under an infinitesimal Weyl transformation $\delta g_{\alpha\beta} = g_{\alpha\beta}\delta\phi$, the change δG in the Greens function is given by

$$\begin{aligned} -\frac{\partial}{\partial\sigma^\alpha}\left(\sqrt{g(\sigma)}g^{\alpha\beta}\frac{\partial}{\partial\sigma^\beta}\delta G(\sigma,\sigma'')\right) \\ = \sqrt{g(\sigma)}\left(-A^{-1}\delta\phi(\sigma) + A^{-2}\int d^2\sigma''\sqrt{g(\sigma'')}\delta\phi(\sigma'')\right) \end{aligned}$$

The important point is that this has a solution of the form

$$\delta G(\sigma,\sigma') = f(\sigma) + h(\sigma')$$

Because of the momentum-conservation condition $\sum_I p_I = 0$, such a change in G has no effect on any physical amplitude. The change in the contraction of two x's, both from the factors e^{ipx}, is proportional to

$$\sum_{I,J} p_I.p_J(f(\sigma_I) + h(\sigma_J)) = 0$$

The change in the contraction of x's from the factors $e^{ip\cdot x}$ with an $x(\sigma)$ from the function S, which appears with a differential operator $\mathcal{D}_\sigma$ acting on it, is proportional to

$$\sum_I p_I \mathcal{D}_\sigma(f(\sigma_I) + h(\sigma)) = \sum_I p_I \mathcal{D}_\sigma h(\sigma) = 0$$

Finally, the change in the contraction of two derivatives $\mathcal{D}_\sigma x^\mu(\sigma)$ and $\mathcal{D}'_\sigma x^\nu(\sigma')$ from the function S is proportional to

$$\mathcal{D}_\sigma \mathcal{D}'_\sigma (f(\sigma) + h(\sigma')) = 0$$

Although the Greens function itself is Weyl-invariant, a violation of Weyl invariance creeps in again when we regulate it. A regulator is necessary, because we are going to have to contract x's in the same vertex functional at the same point σ, and the solution of Eq. (9.3) has a logarithmic singularity at $\sigma = \sigma'$. From now on, we will work in a coordinate system, in which in at least a finite patch around σ and σ', the metric is $g_{\alpha\beta}(\sigma) = \exp(\phi(\sigma))\,\delta_{\alpha\beta}$. In such a coordinate system, the differential operator on the left side of Eq. (9.2) is the ordinary Laplacian, and the solution takes the form

$$G(\vec{\sigma},\vec{\sigma}') = \frac{-1}{4\pi}\ell n|\vec{\sigma} - \vec{\sigma}'|^2 + \ldots \tag{9.6}$$

where the extra terms in (9.6) are non-singular as $\vec{\sigma}$ approaches $\vec{\sigma}'$. This is Weyl-invariant, but infinite for $\vec{\sigma} = \vec{\sigma}'$. Any coordinate-invariant regulator will in effect set a minimum value ϵ on the *proper* distance between $\vec{\sigma}$ and $\vec{\sigma}'$, so that in the limit $\vec{\sigma} \to \vec{\sigma}'$, we have

$$e^{\phi(\vec{\sigma})}|\vec{\sigma} - \vec{\sigma}'|^2 \to \epsilon^2$$

That is, the regulated Greens function has the limiting value

$$G(\vec{\sigma}, \vec{\sigma}) = \frac{1}{4\pi}\left[\phi(\vec{\sigma}) - \ell n \epsilon^2\right] + \ldots \tag{9.7}$$

the omitted terms being finite and ϕ-independent. This result is valid for any *fixed* manifold of arbitrary topology—the importance of the qualification "fixed" will be taken up a little later.

We are now in a position to consider the implications of invariance under *constant* Weyl transformations, i.e., those in which $\phi(\sigma)$ changes by an additive constant. These affect only the contractions of the undifferentiated x's in $e^{ip\cdot x}$ with each other. The complete Weyl dependence arising from contractions within a single $e^{ip\cdot x}$ factor may be calculated as

$$\begin{aligned} e^{ip\cdot x(\vec{\sigma})} &=: e^{ip\cdot x(\vec{\sigma})}: \\ &\times \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!}\frac{(2n)!}{n!2^{2n}}\left[\frac{p^2}{T}G(\vec{\sigma},\vec{\sigma})\right]^n \end{aligned}$$

where colons indicate that in path integrals we are not to include contractions of x's within the colons with each other. (The factor $(2n)!/n!2^{2n}$ is the number of ways that $2n$ x's can be paired with each other.) Using (9.7), this is

$$\begin{aligned} e^{ip\cdot x(\vec{\sigma})} &=: e^{ip\cdot x(\vec{\sigma})}: \\ &\times \exp\left(\frac{-p^2}{8\pi T}\left(\phi(\vec{\sigma}) - \ell n\epsilon^2\right)\right) \times \ldots \end{aligned} \tag{9.8}$$

the omitted factors being finite and Weyl-invariant. We see that there are 3 separate sources of dependence of the vertex functional on a constant term in $\phi(\sigma)$:

a. The Weyl anomaly, which contributes a factor $\exp(-p^2\phi/8\pi T)$.

b. The volume factor $\sqrt{g}$ in (9.1), which contributes a factor e^{ϕ}.

c. The factors $g^{\alpha\beta}$ and $\epsilon^{\alpha\beta}/\sqrt{g}$ in the 2-scalar S in (9.1) (including the $g^{\alpha\beta}$ in the 2-curvature scalar $R^{(2)}$), which gives a factor $\exp(-N\phi)$, where N is the number of $g^{\alpha\beta}$ and $\epsilon^{\alpha\beta}/\sqrt{g}$ factors. (That is, $2N$ is the number of derivatives acting on $x^{\mu}(\sigma)$ and $g_{\alpha\beta}(\sigma)$ in S.) The condition for invariance under constant Weyl transformations is then that these factors should cancel:

$$-p^2 = m^2 = 8\pi T(N-1) \tag{9.9}$$

We see that the lowest mass states with $N = 0$ are tachyons, with $m^2 = -8\pi T$. Since S contains no derivatives, the particle is a scalar, with vertex functional

$$V_{tach}\,[x, g; p, N = 0] = \frac{\kappa_T}{\sqrt{2p^o}} \int d^2\sigma : e^{ip\cdot x(\sigma)} : \tag{9.10}$$

The next mass level with $N = 1$ contains massless particles.

In order to sort out the vertex functions and particle types for $N \geq 1$ it is necessary to demand also the cancellation of all dependence on derivatives of $\phi(\vec{\sigma})$. These derivatives enter from four separate sources:

a. Weyl anomalies in the contraction of derivatives of $x^{\mu}(\sigma)$ in $S(\sigma)$ with $x^{\nu}(\sigma)$ in $e^{ip\cdot x}$.

b. Weyl anomalies in the contraction of derivatives of $x^{\mu}(\sigma)$ in $S(\sigma)$ with each other.

c. The affine connections appearing in covariant derivatives of $x^{\mu}(\sigma)$ in $S(\sigma)$.

d. The curvature scalars $R^{(2)}(\sigma)$ in $S(\sigma)$.

For instance, consider the $N = 1$ vertices for massless particles. These involve just two derivatives of $x^{\mu}(\sigma)$ and/or $g_{\alpha\beta}(\sigma)$. Lorentz invariance and 2-dimensional coordinate invariance allow us to classify these under three categories:

Gravitons: Suppose we take a symmetric product of two derivatives

$$V_{grav} = e_{\mu\nu} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \frac{\partial x^\mu}{\partial \sigma^\alpha} \frac{\partial x^\nu}{\partial \sigma^\beta} e^{ip\cdot x} \tag{9.11}$$

but take the polarization tensor to be traceless as well as symmetric

$$e^\lambda_\lambda = 0 \qquad\qquad e_{\mu\nu} = e_{\nu\mu} \tag{9.12}$$

Here derivatives of $\phi(\sigma)$ arise only from the Weyl anomaly in the contraction of $\partial_\alpha x^\mu$ or $\partial_\beta x^\nu$ with one of the x's in $\exp(ip \cdot x)$. Without evaluating these terms explicitly, we can easily see that their contribution is proportional respectively to $p^\mu e_{\mu\nu}$ or $p^\nu e_{\mu\nu}$, so Weyl invariance requires that this vanish:

$$p^\mu e_{\mu\nu} = p^\nu e_{\mu\nu} = 0 \tag{9.13}$$

The polarization tensor $e_{\mu\nu}$ is thus symmetric, traceless, and transverse, and therefore in D dimensions would have $(D+1)(D-2)/2$ independent components. However, there is also a gauge symmetry. Under a change in the polarization tensor of form

$$\delta e_{\mu\nu} = e_\mu p_\nu + e_\nu p_\mu \tag{9.14}$$

the vertex functional changes by

$$\begin{aligned} \delta V &= -2ie_\mu \int d^2\sigma \sqrt{g} g^{\alpha\beta} \frac{\partial x^\mu}{\partial \sigma^\alpha} \frac{\partial}{\partial \sigma^\beta} e^{ip\cdot x} \\ &= +2ie_\mu \int d^2\sigma \sqrt{g} e^{ip\cdot x} \nabla x^\mu \\ &= \frac{2ie_\mu}{T} \int d^2\sigma \sqrt{g} e^{ip\cdot x} \frac{\delta I}{\delta x^\mu} \end{aligned}$$

If we suppose also that e^μ is transverse

$$p^\mu e_\mu = 0 \tag{9.15}$$

then we can re-write this as

$$\delta V = \frac{2ie_\mu}{T} \int d^2\sigma \sqrt{g} \frac{\delta}{\delta x^\mu} (e^{ip\cdot x} I)$$

and so the effect of this change in x^μ vanishes when we do a functional integral over x^μ. There are $D-1$ independent gauge transformations, so the number of independent gauge-invariant components of the polarization tensor is

$$\frac{(D+1)(D-2)}{2} - (D-1) = \frac{(D-2)(D-1)}{2} - 1$$

This is the correct number of components for the symmetric traceless tensor representation of the little group $O(D-2)$; indeed, by a choice of gauge, we can eliminate the time and longitudinal components of $e_{\mu\nu}$, leaving just a traceless symmetric tensor with indices running over $D-2$ values. This identifies these particles as gravitons.

Dilatons:

Next consider the vertex obtained by taking the $e_{\mu\nu}$ above proportional to $\eta_{\mu\nu}$:

$$V \quad \propto \quad \eta_{\mu\nu} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \frac{\partial x^\mu}{\partial \sigma^\alpha} \frac{\partial x^\mu}{\partial \sigma^\beta} e^{ip\cdot x} \tag{9.16}$$

This has Weyl anomalies arising from both contractions of $\partial_\alpha x^\mu$ or $\partial_\beta x^\nu$ with x's from $e^{ip\cdot x}$, and also from contractions of $\partial_\alpha x^\mu$ and $\partial_\beta x^\nu$ with each other. The first is eliminated if we replace $\eta_{\mu\nu}$ with a transverse tensor

$$\eta_{\mu\nu} \to \eta_{\mu\nu} - n_\mu p_\nu - n_\nu p_\mu$$

where n_μ is any D-vector satisfying

$$n_\mu p^\mu = 1 \tag{9.17}$$

Note that a change in n_μ that preserves the value of $n \cdot p$ would have no effect, by the same argument used above to verify gauge invariance for gravitons. Thus, the presence of

this new vector n_μ does not indicate the presence of new degrees of freedom. We are not finished, however, because we have also to deal with the Weyl anomaly in the contraction of $\partial_\alpha x^\mu$ with $\partial_\beta x^\nu$. (This was incorrectly assumed to be traceless in my previous work[18] on vertex functionals. Its importance was pointed out and its implications analyzed by de Alwis[19].) The trace of this anomaly takes the value

$$g^{\alpha\beta}\langle \partial_\alpha x^\mu \partial_\beta x^\nu \rangle = \frac{\eta^{\mu\nu}}{2\pi T}\nabla\phi = -\frac{\eta^{\mu\nu}}{8\pi T}R^{(2)}$$

The dilaton vertex functional must therefore be taken as

$$V_{dil} = e\int d^2\sigma\sqrt{g}e^{ip\cdot x}\left[(\eta_{\mu\nu} - n_\mu p_\nu - n_\nu p_\mu)\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta}g^{\alpha\beta} + \frac{(D-2)}{8\pi T}R^{(2)}\right] \tag{9.18}$$

with as yet undetermined normalization constant e. This particle is of course a scalar.

2-Form

Finally, we can contract the two derivatives antisymmetrically

$$V_{2f} = f_{\mu\nu}\int d^2\sigma\epsilon^{\alpha\beta}\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta}e^{ip\cdot x} \tag{9.19}$$

where $\epsilon^{\alpha\beta}$ is the Levi-Civita tensor density, with $\epsilon^{12} = -\epsilon^{21} = 1, \epsilon^{11} = \epsilon^{22} = 0$, and $f_{\mu\nu}$ is now an antisymmetric polarization tensor:

$$f_{\mu\nu} = -f_{\nu\mu} \tag{9.20}$$

The only Weyl anomaly we have to deal with here appears in the contraction of $\partial_\alpha x^\mu$ or $\partial_\beta x^\nu$ with one of the x's in $\exp(ip\cdot x)$. To eliminate this, we must take $f_{\mu\nu}$ transverse

$$p^\mu f_{\mu\nu} = 0 \tag{9.21}$$

There is also a gauge invariance: for the same reason as for gravitons; physical amplitudes are unchanged if we change $f_{\mu\nu}$ by

$$\delta f_{\mu\nu} = f_\mu p_\nu - f_\nu p_\mu \tag{9.22}$$

with f_μ transverse

$$f_\mu p^\mu = 0 \tag{9.23}$$

Changing f_μ by $\delta f^\mu = f p_\mu$ gives no change in the gauge transformation, so there are only $D-2$ independent gauge transformations, and the number of degrees of freedom here is

$$\frac{1}{2}D(D-1) - (D-1) - (D-2) = \frac{1}{2}(D-2)(D-3),$$

which is just right for the antisymmetric tensor representation of the little group $O(D-2)$.

(We might consider one other possible class of $N=1$ vertex functionals

$$V = b_\mu \int d^2\sigma \sqrt{g} g^{\alpha\beta} x^\mu_{;\alpha;\beta} e^{ip\cdot x}$$

In order to avoid a Weyl anomaly from the contraction of $x^\mu_{;\alpha;\beta}$ with x's in $\exp(ip\cdot x)$, it is necessary to take b_μ transverse

$$b_\mu p^\mu = 0$$

But then V can be re-written as

$$V = b_\mu \int d^2\sigma \sqrt{g} \frac{\delta}{\delta x^\mu} \left(e^{ip\cdot x} I\right)$$

and vanishes when we carry out functional integrals over x^μ.)

This analysis has been extended to higher mass levels in Ref. 18. This was covered in my Jerusalem lectures, but I have nothing to add to what is in Ref. 18, and will just refer the reader to that paper. A more systematic analysis of the higher mass levels has more recently been provided by Ref. 20.

I want now to take up one problem that may have bothered the reader. The foregoing analysis nowhere mentioned the nature of the Riemann surface on which the vertex functionals are to be placed. This may seem reasonable, because we have been dealing with anomalies that arise from the regulators introduced to deal with singularities that occur when we let the arguments of various x's approach each other, and this would seem to involve only local properties of the manifold, independent of the genus and Teichmüller parameters. Yet this cannot be correct. From the condition of invariance under constant Weyl transformations we have derived a mass formula, Eq. (9.9), that clearly can't be exact to all orders of perturbation theory. Indeed, one-loop corrections to this mass formula have been explicity calculated in Ref. 21. Apparently the Weyl anomalies are affected by the Riemann surface being a torus rather than a sphere, but how can this be? An answer has been given by a recent paper[22] of Fischler and Susskind. They show that the Weyl anomalies arising from singularities in $\langle x^\mu(\sigma)x^\nu(\sigma')\rangle$ when σ approaches σ' are affected by *small* handles, with dimensions of the order of $|\sigma - \sigma'|$. It is believed that the full mass formula, with all radiative corrections, is determined by the condition of Weyl invariance in the presence of small topological fixtures.[23]

10. Tree Amplitudes

We now return briefly to our starting point, and indicate briefly how the path-integral formalism is used to calculate scattering amplitudes in the tree approximation. This is usually done by "completing the square" in the Gaussian integral, but if only for the sake of variety here we will use a different method.

Every vertex functional contains in the integrand a factor $:\exp(ip\cdot x):$. The colons indicate that we are not to pair x's in one such exponential (such pairings having already been used to cancel powers of e^{ϕ}), but of course we must include all pairings of x's from different vertex functionals. In pairing x's from two different exponentials, we encounter the factor

$$\sum_{n=0}^{\infty}\left(\frac{i^n}{n!}\right)^2\frac{n!^2}{n!}\left(p_{1\mu}p_{2\nu}\langle x^{\mu}(\sigma_1)x^{\nu}(\sigma_2)\rangle\right)^n$$

(The factor $n!^2/n!$ is the number of ways of choosing n pairs from two *different* collections.) Using (9.2) and (9.6), our result may be written

$$\begin{aligned}&:\exp\left(ip_1\cdot x(\sigma_1)\right):\times:\exp\left(ip_2\cdot x(\sigma_2)\right):\\ &\qquad =:\exp\left(ip_1\cdot x(\sigma_1)+ip_2\cdot x(\sigma_2)\right):|\vec{\sigma}_1-\vec{\sigma}_2|^{p_1\cdot p_2/2\pi T}\end{aligned}\tag{10.1}$$

Continuing in this way, we find

$$\begin{aligned}&\prod_I:\exp\left(ip_I\cdot x(\sigma)\right):\\ &\qquad =:\exp\left(i\sum_I p_I\cdot x(\sigma_I)\right):\prod_{I<J}|\sigma_I-\sigma_J|^{p_I\cdot p_J/2\pi T}\end{aligned}\tag{10.2}$$

This formula can be used in calculating correlation functions of any sort of vertex functionals. In the case of the tachyonic vertex functionals (9.10) there are no other x's, and we wind up with a correlation function in which there is nothing left to pair, so that only the constant term remains

$$\langle:\exp\left(i\sum_I p_I\cdot x(\sigma_I)\right):\rangle=1\tag{10.3}$$

The correlation function of the tachyonic vertex functionals is thus given simply by (now using complex notation)

$$\begin{aligned}\langle V_{tach}(p_1)\ldots V_{tach}(p_N)\rangle &= \kappa_T^n \left(2p_1^0\ldots 2p_N^0\right)^{-1/2} \\ &\times \int d^2z_1\ldots d^2z_N \prod_{I<J} |z_I - z_J|^{p_I\cdot p_J/2\pi T}\end{aligned} \tag{10.4}$$

As discussed in Section 8, the effect of dividing by the group volume Ω_{CKV} is to fix three of the z_J, say z_1, z_2, z_3, and to multiply Eq. (10.4) with the factor (8.13). The S-matrix (7.21) for a purely tachyonic reaction in the tree ($\chi = 2$) approximation is then

$$\begin{aligned}S\left(p_1\ldots p_N\right) &= i(2\pi)^{26}\delta^{26}\left(\sum_I p_I\right) \\ &\times 54\pi^{-3}\lambda^{-2}\left(2p_1^0\ldots 2p_N^0\right)^{-1/2}\kappa_T^n \\ &\times |z_1 - z_2|^2\,|z_2 - z_3|^2\,|z_3 - z_1|^2 \\ &\times \int d^2z_4\ldots d^2z_N \prod_{I<J} |z_I - z_J|^{p_I\cdot p_J/2\pi T}\end{aligned} \tag{10.5}$$

Aside from normalization, this is the result obtained by Virasoro[5] and Shapiro[6] in 1969-70. It is consistent with unitarity (i.e., factorization) only if κ_T is given a specific value in terms of λ:

$$\kappa_T^2 = (2\pi)^{-25} 4\pi^5 T\lambda^2/27 \tag{10.6}$$

There are similar results, obtained from unitarity, for the normalization of the vertex functionals for arbitrary particles[18-20].

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