INTRODUCTION TO STRING THEORY*

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Gerard 't Hooft

Institute for Theoretical Physics
Utrecht University, Leuvenlaan 4
3584 CC Utrecht, the Netherlands
and
Spinoza Institute
Postbox 80.195
3508 TD Utrecht, the Netherlands

e-mail: `g.thooft@phys.uu.nl`
`internet: http://www.phys.uu.nl/~thooft/`

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1. Strings in QCD.

1.1. The linear trajectories.

In the '50's, mesons and baryons were found to have many excited states, called resonances, and in the '60's, their scattering amplitudes were found to be related to the so-called Regge trajectories: $J=\alpha(s)$, where J is the angular momentum and $s=M^2$, the square of the energy in the center of mass frame. A resonance occurs at those s values where $\alpha(s)$ is a nonnegative integer (mesons) or a nonnegative integer plus $\frac{1}{2}$ (baryons). The largest J values at given s formed the so-called 'leading trajectory'. Experimentally, it was discovered that the leading trajectories were almost linear in s:

$$\alpha(s)=\alpha(0)+\alpha' s\ . \tag{1.1}$$

Furthermore, there were 'daughter trajectories':

$$\alpha(s)=\alpha(0)-n+\alpha' s\ . \tag{1.2}$$

where n appeared to be an integer. $\alpha(0)$ depends on the quantum numbers such as strangeness and baryon number, but α' appeared to be universal, approximately 1 GeV^{-2}.

It took some time before the simple question was asked: suppose a meson consists of two quarks rotating around a center of mass. What force law could reproduce the simple behavior of Eq. (1.1)? Assume that the quarks move highly relativistically (which is reasonable, because most of the resonances are much heavier than the lightest, the pion). Let the distance between the quarks be r. Each has a transverse momentum p. Then, if we allow ourselves to ignore the energy of the force fields themselves (and put $c=1$),

$$s=M^2=(2p)^2\ . \tag{1.3}$$

The angular momentum is

$$J=2\,p\,\frac{r}{2}=p\,r\ . \tag{1.4}$$

The centripetal force must be

$$F=\frac{p\,c}{r/2}=\frac{2p}{r}\ . \tag{1.5}$$

For the leading trajectory, at large s (so that $\alpha(0)$ can be ignored), we find:

$$r=\frac{2J}{\sqrt{s}}=2\alpha'\sqrt{s}\quad ;\qquad F=\frac{s}{2\,J}=\frac{1}{2\,\alpha'}\ , \tag{1.6}$$

or: the force is a constant, and the potential between two quarks is a linearly rising one.

But it is not quite correct to ignore the energy of the force field, and, furthermore, the above argument does not explain the daughter trajectories. A more satisfactory model of the mesons is the *vortex model*: a narrow tube of field lines connects the two quarks. This linelike structure carries all the energy. It indeed generates a force that is of a universal, constant strength: $F = \mathrm{d}E/\mathrm{d}r$. Although the quarks move relativistically, we now ignore their contribution to the energy (a small, negative value for $\alpha(0)$ will later be attributed to the quarks). A stationary vortex carries an energy T per unit of length, and we take this quantity as a constant of Nature. Assume this vortex, with the quarks at its end points, to rotate such that the end points move practically with the speed of light, c. At a point x between $-r/2$ and $r/2$, the angular velocity is $v(x) = c\,x/(r/2)$. The total energy is then (putting $c = 1$):

$$E \;=\; \int_{-r/2}^{r/2} \frac{T\,\mathrm{d}x}{\sqrt{1-v^2}} \;=\; T\,r\int_0^1 (1-x^2)^{-1/2}\,\mathrm{d}x \;=\; \tfrac{1}{2}\pi\,T\,r\;, \tag{1.7}$$

while the angular momentum is

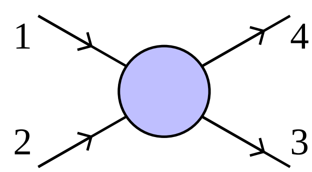
$$J \;=\; \int_{-r/2}^{r/2} \frac{T\,v\,x\,\mathrm{d}x}{\sqrt{1-v^2}} \;=\; \tfrac{1}{2}T\,r^2\int_0^1 \frac{x^2\,\mathrm{d}x}{\sqrt{1-x^2}} \;=\; \frac{T\,r^2\pi}{8}\;. \tag{1.8}$$

Thus, in this model also,

$$\frac{J}{E^2} = \frac{1}{2\pi T} = \alpha' \quad ; \qquad \alpha(0) = 0\,, \tag{1.9}$$

but the force, or *string tension*, T, is a factor π smaller than in Eq. (1.6).

1.2. The Veneziano formula.



Consider elastic scattering of two mesons, (1) and (2), forming two other mesons (3) and (4). Elastic here means that no other particles are formed in the process. The ingoing 4-momenta are $p_\mu^{(1)}$ and $p_\mu^{(2)}$. The outgoing 4-momenta are $p_\mu^{(3)}$ and $p_\mu^{(4)}$. The c.m. energy squared is

$$s = -(p_\mu^{(1)} + p_\mu^{(2)})^2\,. \tag{1.10}$$

An independent kinematical variable is

$$t = -(p_\mu^{(1)} - p_\mu^{(4)})^2\,. \tag{1.11}$$

Similarly, one defines

$$u = -(p_\mu^{(1)} - p_\mu^{(3)})^2\,, \tag{1.12}$$

but that is not independent:

$$s + t + u = \sum_{i=1}^{4} m_{(i)}^2 \ . \tag{1.13}$$

G. Veneziano asked the following question: What is the simplest model amplitude that shows poles where the resonances of Eqs. (1.1) and (1.2) are, either in the s-channel or in the t-channel? We do not need such poles in the u-channel since these are often forbidden by the quantum numbers, and we must avoid the occurrence of double poles.

The Gamma function, $\Gamma(x)$, has poles at negative integer values of x, or, $x = 0,\ -1,\ -2,\ \cdots$. Therefore, Veneziano tried the amplitude

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \ . \tag{1.14}$$

Here, the denominator was planted so as to avoid double poles when both $\alpha(s)$ and $\alpha(t)$ are nonnegative integers. This formula is physically acceptable only if the trajectories $\alpha(s)$ and $\alpha(t)$ are linear, for the following reason. Consider the *residue* of one of the poles in s. Using $\Gamma(x) \to \frac{(-1)^n}{n!}\frac{1}{x+n}$ when $x \to -n$, we see that

$$\alpha(s) \to n \geq 0 \quad : \qquad A(s,t) \to \frac{(-1)^n}{n!}\frac{1}{n-\alpha(s)}\,\frac{\Gamma(-\alpha(t))}{\Gamma(-\alpha(t)-n)} \ . \tag{1.15}$$

Here, the $\alpha(t)$ dependence is the polynomial

$$\Gamma(a+n)/\Gamma(a) = (a+n-1)\cdots(a+1)a \ ; \qquad a = -\alpha(t) - n \ , \tag{1.16}$$

called the Pochhammer polynomial. Only if $\alpha(t)$ is linear in t, this will be a polynomial of degree n in t. Notice that, in the c.m. frame,

$$t \ = \ -(p_\mu^{(1)} - p_\mu^{(4)})^2 \ = \ m_{(1)}^2 + m_{(4)}^2 - 2E_{(1)}E_{(4)} + 2|p_{(1)}||p_{(4)}|\cos\theta \ . \tag{1.17}$$

Here, θ is the scattering angle. In the case of a linear trajectory in t, we have a polynomial of degree n in $\cos\theta$. From group representation theory, we know that, therefore, the intermediate state is a superposition of states with angular momentum J maximally equal to n. We conclude that the n^{th} resonance in the s channel consists of states whose angular momentum is maximally equal to n. So, the leading trajectory has $J = \alpha(s)$, and there are daughter trajectories with lower angular momentum. Notice that this would not be true if we had forgotten to put the denominator in Eq. (1.14), or if the trajectory in t were not linear. Since the Pochhammer polynomials are not the same as the Legendre polynomials, superimposed resonances appear with J lower than n, the daughters. An important question concerns the *sign* of these contributions. A negative sign could indicate intermediate states with indefinite metric, which would be physically unrealistic. In the early '70s, such questions were investigated purely mathematically. Presently, we know that it is more fruitful to study the *physical interpretation* of Veneziano's amplitude (as well as generalizations thereof, which were soon discovered).

The Veneziano amplitude $A(s,t)$ of Eq. (1.14) is the beta function:

$$A(s,t)=B(-\alpha(s),-\alpha(t))=\int_0^1 x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1}\mathrm{d}x\,. \tag{1.18}$$

The fact that the poles of this amplitude, at the leading values of the angular momentum, obey exactly the same energy-angular momentum relation as the rotating string of Eq. (1.9), is no coincidence, as will be seen in what follows (section 6, Eq. (6.22)).

2. The classical string.

Consider a kind of material that is *linelike*, being evenly distributed over a line. Let it have a tension force T. If we stretch this material, the energy we add to it is exactly T per unit of length. Assume that this is the *only* way to add energy to it. This is typical for a vortex line of a field. Then, if the material is at rest, it carries a mass that (up to a factor c^2, which we put equal to one) is also T per unit of length. In the simplest conceivable case, there is no further structure in this string. It then does not alter if we Lorentz transform it in the longitudinal direction. So, we assume that the energy contained in the string only depends on its velocity in the transverse direction. This dependence is dictated by relativity theory: if $u_\perp^\mu$ is the 4-velocity in the transverse direction, and if both the 4-momentum density p^μ and u^μ transform the same way under transverse Lorentz transformations, then the energy density $\mathrm{d}U/\mathrm{d}\ell$ must be just like the energy of a particle in 2+1 dimensions, or

$$\frac{\mathrm{d}U}{\mathrm{d}\ell}=\frac{T}{\sqrt{1-v_\perp^2/c^2}}\,. \tag{2.1}$$

In a region where the transverse velocity $v_\perp$ is non-relativistic, this simply reads as

$$U=U^{\mathrm{kin}}+V \quad ; \qquad U^{\mathrm{kin}}=\int \tfrac{1}{2}T\,v_\perp^2\,\mathrm{d}\ell\,, \quad V=\int T\,\mathrm{d}\ell\,, \tag{2.2}$$

which is exactly the energy of a non-relativistic string with mass density T and a tension T, responsible for the potential energy. Indeed, if we have a string stretching in the z-direction, with a tiny deviation $\tilde{x}(z)$, where $\tilde{x}$ is a vector in the (xy)-direction, then

$$\frac{\mathrm{d}\ell}{\mathrm{d}z}=\sqrt{1+\Big(\frac{\partial\tilde{x}}{\partial z}\Big)^2}\approx 1+\tfrac{1}{2}\Big(\frac{\partial\tilde{x}}{\partial z}\Big)^2\;; \tag{2.3}$$

$$U\approx\int\mathrm{d}z\bigg(T+\tfrac{1}{2}T\Big(\frac{\partial\tilde{x}}{\partial z}\Big)^2+\tfrac{1}{2}T(\dot{\tilde{x}})^2\bigg)\,. \tag{2.4}$$

We recognize a 'field theory' for a two-component scalar field in one space-, one time-dimension.

In the non-relativistic case, the Lagrangian is then

$$L=U^{\mathrm{kin}}-V=-\int T(1-\tfrac{1}{2}v_\perp^2)\,\mathrm{d}\ell=-\int T\sqrt{1-v_\perp^2}\,\mathrm{d}\ell\,. \tag{2.5}$$

Since the eigen time $\mathrm{d}\tau$ for a point moving in the transverse direction along with the string, is given by $\mathrm{d}t\sqrt{1-v_\perp^2}$, we can write the action S as

$$S = \int L\,\mathrm{d}t = -\int T\,\mathrm{d}\ell\,\mathrm{d}\tau\,. \tag{2.6}$$

Now observe that this expression is Lorentz covariant. Therefore, if it holds for describing the motion of a piece of string in a frame where it is non-relativistic, it must describe the same motion in all lorentz frames. Therefore, this is the action of a string. The 'surface element' $\mathrm{d}\ell\,\mathrm{d}\tau$ is the covariant measure of a piece of a 2-surface in Minkowski space.

To understand hadronic particles as excited states of strings, we have to study the dynamical properties of these strings, and then quantize the theory. At first sight, this seems to be straightforward. We have a string with mass per unit of length T and a tension force which is also T (in units where $c=1$). Think of an infinite string stretching in the z direction. The transverse excitation is described by a vector $x^{\mathrm{tr}}(z,t)$ in the $x\,y$ direction, and the excitations move with the speed of sound, here equal to the speed of light, in the positive and negative z-direction. This is nothing but a two-component massless field theory in one space-, one time-dimension. Quantizing that should not be a problem.

Yet it is a non-linear field theory; if the string is strongly excited, it no longer stretches in the z-direction, and other tiny excitations then move in the z-direction slower. Strings could indeed reorient themselves in any direction; to handle that case, a more powerful scheme is needed. This would have been a hopeless task, if a fortunate accident would not have occurred: the classical theory is exactly soluble. But, as we shall see, the quantization of that exact solution is much more involved than just a renormalizable massless field theory.

In Minkowski space-time, a string sweeps out a 2-dimensional surface called the "world sheet". Introduce two coordinates to describe this sheet: σ is a coordinate along the string, and τ a timelike coordinate. The world sheet is described by the functions $X^\mu(\sigma,\tau)$, where μ runs from 0 to d, the number of space dimensions[1]. We could put $\tau = X^0 = t$, but we don't have to. The surface element $\mathrm{d}\ell\,\mathrm{d}\tau$ of Eq. (2.6) will in general be the absolute value of

$$\Sigma^{\mu\nu} = \frac{\partial X^\mu}{\partial\sigma}\frac{\partial X^\nu}{\partial\tau} - \frac{\partial X^\nu}{\partial\sigma}\frac{\partial X^\mu}{\partial\tau}\ , \tag{2.7}$$

We have

$$\tfrac{1}{2}\Sigma^{\mu\nu}\Sigma^{\mu\nu} = (\partial_\sigma X^\mu)^2(\partial_\tau X^\nu)^2 - (\partial_\sigma X^\mu \partial_\tau X^\mu)^2\ . \tag{2.8}$$

The surface element on the world sheet of a string is timelike. Note that we are assuming the sign convention $(-\,+\,+\,+)$ for the Minkowski metric; throughout these notes, a repeated index from the middle of the Greek alphabet is read as follows:

$$X^{\mu 2} \equiv \eta_{\mu\nu}X^\mu X^\nu = {X^1}^2 + {X^2}^2 + \cdots + (X^{D-1})^2 - {X^0}^2\ ,$$

[1]We use D to denote the total number of spacetime dimensions: $d = D-1$.

where D stands for the number of space-time dimensions, usually (but not always) $D = 4$. We must write the Lorentz invariant timelike surface element that figures in the action as

$$S = -T \int \mathrm{d}\sigma\,\mathrm{d}\tau \sqrt{(\partial_\sigma X^\mu \partial_\tau X^\mu)^2 - (\partial_\sigma X^\mu)^2 (\partial_\tau X^\nu)^2}\ . \tag{2.9}$$

This action, Eq. (2.9), is called the Nambu-Goto action. One way to proceed now is to take the coordinates σ and τ to be light-cone coordinates on the string world sheet. In order to avoid confusion later, we refer to such coordinates as σ^+ and σ^- instead of σ and τ. These coordinates are defined in such a way that

$$(\partial_+ X^\mu)^2 = (\partial_- X^\mu)^2 = 0\ . \tag{2.10}$$

The second term inside the square root is then a double zero, which implies that it also vanishes to lowest order if we consider an infinitesimal variation of the variables $X^\mu(\sigma^+,\,\sigma^-)$. Thus, keeping the constraint (2.10) in mind, we can use as our action

$$S = T \int \partial_+ X^\mu \partial_- X^\mu\, \mathrm{d}\sigma^+ \mathrm{d}\sigma^-\ . \tag{2.11}$$

With this action being a bilinear one, the associated Euler-Lagrange equations are linear, and easy to solve:

$$\partial_+\,\partial_-\, X^\mu = 0\ ; \qquad X^\mu = a^\mu(\sigma^+) + b^\mu(\sigma^-)\ . \tag{2.12}$$

The conditions (2.10) simply imply that the functions $a^\mu(\sigma^+)$ and $b^\mu(\sigma^-)$, which would otherwise be arbitrary, now have to satisfy one constraint each:

$$(\partial_+ a^\mu(\sigma^+))^2 = 0\ ; \qquad (\partial_- b^\mu(\sigma^-))^2 = 0\ . \tag{2.13}$$

It is not hard to solve these equations: since $\partial_+ a^0 = \sqrt{(\partial_+ \vec{a})^2}$, we have

$$a^0(\sigma^+) = \int^{\sigma^+} \sqrt{(\partial_+ \vec{a}(\sigma_1))^2}\mathrm{d}\sigma_1\ ; \qquad b^0(\sigma^-) = \int^{\sigma^-} \sqrt{(\partial_+ \vec{b}(\sigma_1))^2}\mathrm{d}\sigma_1\ , \tag{2.14}$$

which gives us $a^0(\sigma^+)$ and $b^0(\sigma^-)$, given $\vec{a}(\sigma^+)$ and $\vec{b}(\sigma^-)$. This completes the classical solution of the string equations.

Note that Eq. (2.11) can only be used if the *sign* of this quantity remains the same everywhere.

Exercise: Show that $\partial_+ X^\mu \partial_- X^\mu$ can switch sign only at a point $(\sigma_0^+,\,\sigma_0^-)$ where $\partial_+ a^\mu(\sigma_0^+) = C \cdot \partial_- b^\mu(\sigma_0^-)$. In a generic case, such points will not exist.
This justifies our sign assumption.

For future use, we define the *induced metric* $h_{\alpha\beta}(\sigma, \tau)$ as

$$h_{\alpha\beta} = \partial_\alpha X^\mu\, \partial_\beta X^\mu\ , \tag{2.15}$$

where indices at the beginning of the Greek alphabet, running from 1 to 2, refer to the two world sheet coordinates, for instance:

$$\sigma^1 = \sigma \ , \quad \sigma^2 = \tau \ , \qquad \text{or, as the case may be,} \qquad \sigma^{1,2} = \sigma^{\pm} \ , \tag{2.16}$$

the distances between points on the string world sheet being defined by $\mathrm{d}s^2 = h_{\alpha\beta}\,\mathrm{d}\sigma^\alpha \mathrm{d}\sigma^\beta$. The Nambu-Goto action is then

$$S = -T\int \mathrm{d}^2\sigma\sqrt{h} \quad ; \qquad h = -\det_{\alpha\beta}(h_{\alpha\beta}) \quad , \qquad \mathrm{d}^2\sigma = \mathrm{d}\sigma\,\mathrm{d}\tau \ . \tag{2.17}$$

We can actually treat $h_{\alpha\beta}$ as an independent variable when we replace the action (2.9) by the so-called *Polyakov action*:

$$S = -\frac{T}{2}\int \mathrm{d}^2\sigma\sqrt{h}\,h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\mu \ , \tag{2.18}$$

where, of course, $h^{\alpha\beta}$ stands for the inverse of $h_{\alpha\beta}$. Varying this action with respect to $h^{\alpha\beta}$ gives

$$h^{\alpha\beta} \to h^{\alpha\beta} + \delta h^{\alpha\beta} \ ; \quad \delta S = T\int \mathrm{d}^2\sigma\ \delta h^{\alpha\beta}\,\sqrt{h}\,(\partial_\alpha X^\mu \partial_\beta X^\mu - \tfrac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\mu) \ . \tag{2.19}$$

Requiring δS in Eq. (2.19) to vanish for all $\delta h^{\alpha\beta}(\sigma,\tau)$ does *not* give Eq. (2.15), but instead

$$h_{\alpha\beta} = C(\sigma,\tau)\partial_\alpha X^\mu \partial_\beta X^\mu \ . \tag{2.20}$$

Notice, however, that the conformal factor $C(\sigma,\tau)$ cancels out in Eq. (2.18), so that varying it with respect to $X^\mu(\sigma,\tau)$ still gives the correct string equations. C is not fixed by the Euler-Lagrange equations at all.

So-far, all our equations were invariant under coordinate redefinitions for σ and τ. In any two-dimensional surface with a metric $h_{\alpha\beta}$, one can rearrange the coordinates such that

$$h_{12} = h_{21} = 0 \ ; \quad h_{11} = -h_{22} \quad , \qquad \text{or:} \qquad h_{\alpha\beta} = \eta_{\alpha\beta}e^\phi \ , \tag{2.21}$$

where $\eta_{\alpha\beta}$ is the flat Minkowski metric $\mathrm{diag}(-1,1)$ on the surface, and e^ϕ some conformal factor. Since this factor cancels out in Eq. (2.18), the action in this gauge is the bilinear expression

$$S = -\tfrac{1}{2}T\int \mathrm{d}^2\sigma\,(\partial_\alpha X^\mu)^2 \ . \tag{2.22}$$

Notice that in the light-cone coordinates $\sigma^\pm = \frac{1}{\sqrt{2}}(\tau \pm \sigma)$, where the flat metric $\eta_{\alpha\beta}$ takes the form

$$\eta_{\alpha\beta} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \tag{2.23}$$

this action takes the form of Eq. (2.11). Now we still have to impose the constraints (2.10). How do we explain these here? Well, it is important to note that the gauge condition (2.21) does *not* fix the coordinates completely: we still have invariance under the group of conformal transformations. They replace $h_{\alpha\beta}$ by a different world sheet metric of the same form (2.21). We must insist that these transformations leave the action (2.18) stationary as well. Checking the Euler-Lagrange equations $\delta S/\delta h_{\alpha\beta} = 0$, we find the remaining constraints. Keeping the notation of Green, Schwarz and Witten, we define the world-sheet energy-momentum tensor $T_{\alpha\beta}$ as

$$T_{\alpha\beta} = -\frac{2\pi}{\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}} \ . \tag{2.24}$$

In units where $T = \frac{1}{\pi}$, we have

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\mu - \tfrac{1}{2} h_{\alpha\beta}(\partial X)^2 \ . \tag{2.25}$$

In light-cone coordinates, where $h_{\alpha\beta}$ is proportional to Eq. (2.23), we have

$$T_{++} = (\partial_+ X^\mu)^2 \quad , \qquad T_{--} = (\partial_- X^\mu)^2 \quad , \qquad T_{+-} = T_{-+} = 0 \ . \tag{2.26}$$

Demanding these to vanish is now seen as the constraint on our solutions that stems from the field equations we had *before* requiring conformal invariance. They should be seen as a boundary condition.

The solutions to the Euler-Lagrange equations generated by the Polyakov action (2.18) is again (2.12), including the constraints (2.13).

3. Open and closed strings.

What has now been established is the local, classical equations of motion for a string. What are the boundary conditions?

3.1. The Open string.

To describe the open string we use a spacelike coordinate σ that runs from 0 to π, and a timelike coordinate τ. If we impose the conformal gauge condition, Eq. (2.21), we might end up with a coordinate σ that runs from some value $\sigma_0(\tau)$ to another, $\sigma_1(\tau)$. Now, however, consider the light-cone coordinates $\sigma^\pm = \frac{1}{\sqrt{2}}(\sigma \pm \tau)$. A transformation of the form

$$\sigma^+ \to f^+(\sigma^+) \ , \tag{3.1}$$

leaves the metric $h_{\alpha\beta}$ of the same form (2.21) with $\eta_{\alpha\beta}$ of the form (2.23). It is not difficult to convince oneself that this transformation, together with such a transformation for σ^-, can be exploited to enforce the condition $\sigma_0(\tau) = 0$ and $\sigma_1(\tau) = \pi$.

In principle we now have two possibilities: either we consider the functions $X^\mu(\sigma,\tau)$ at the edges to be fixed (*Dirichlet* boundary condition), so that also the variation $\delta X^\mu(\sigma,\tau)$ is constrained to be zero there, or we leave these functions to be free (*Neumann* boundary condition). An end point obeying the Dirichlet boundary condition cannot move. It could be tied onto an infinitely heavy quark, for instance. An end point obeying the Neumann boundary condition can move freely, like a light quark. For the time being, this is the more relevant case.

Take the action (2.22), and take an arbitrary infinitesimal variation $\delta X^\mu(\sigma,\tau)$. The variation of the action is

$$\delta S = T\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma \left(-\partial_\sigma X^\mu\,\partial_\sigma \delta X^\mu + \partial_\tau X^\mu\,\partial_\tau \delta X^\mu\right)\ . \tag{3.2}$$

By partial integration, this is

$$\begin{aligned} \delta S &= T\int \mathrm{d}\tau \int_0^\pi \mathrm{d}\sigma\, \delta X^\mu(\partial_\sigma^2 - \partial_\tau^2)X^\mu \\ &+ T\int \mathrm{d}\tau\Big(\delta X^\mu(0,\tau)\partial_\sigma X^\mu(0,\tau) - \delta X^\mu(\pi,\tau)\partial_\sigma X^\mu(\pi,\tau)\Big)\ . \end{aligned} \tag{3.3}$$

Since this has to vanish for all choices of $\delta X^\mu(\sigma,\tau)$, we read off the equation of motion for $X^\mu(\sigma,\tau)$ from the first term, whereas the second term tells us that $\partial_\sigma X^\mu$ vanishes on the edges $\sigma = 0$ and $\sigma = \pi$. This can be seen to imply that no *momentum* can flow in or out at the edges, so that there is no force acting on them: the edges are free end points.

3.2. The closed string.

In the case of a closed string, we choose as our boundary condition:

$$X^\mu(\sigma,\,\tau) = X^\mu(\sigma+\pi,\,\tau)\ . \tag{3.4}$$

Again, we must use transformations of the form (3.1) to guarantee that this condition is kept after fixing the conformal gauge. The period π is in accordance with the usual convention in string theory.

Exercise: Assuming the string world sheet to be timelike, check that we can impose the boundary condition (3.4) on any closed string, while keeping the coordinate condition (2.21), or, by using coordinate transformations exclusively of the form

$$\sigma^+ \to \tilde\sigma^+(\sigma^+) \quad , \qquad \sigma^- \to \tilde\sigma^-(\sigma^-)\ . \tag{3.5}$$

3.3. Solutions.

3.3.1. The open string.

For the open string, we write the solution (2.12) the following way:

$$X^\mu(\sigma,\tau) = X_L^\mu(\sigma+\tau) + X_R^\mu(\tau-\sigma)\ . \tag{3.6}$$

In Sect. 3.1, we saw that at the boundaries $\sigma = 0$ and $\sigma = \pi$ the boundary condition is $\partial_\sigma X^\mu = 0$. Therefore, we have

$$\partial_\tau X^\mu_L(\tau) - \partial_\tau X^\mu_R(\tau) = 0 \; ; \tag{3.7}$$

$$\partial_\tau X^\mu_L(\tau+\pi) - \partial_\tau X^\mu_R(\tau-\pi) = 0 \; . \tag{3.8}$$

The first of these implies that X^μ_L and X^μ_R must be equal up to a constant, but no generality is lost if we put that constant equal to zero:

$$X^\mu_R(\tau) = X^\mu_L(\tau) \; . \tag{3.9}$$

Similarly, the second equation relates $X^\mu_L(\tau+\pi)$ to $X^\mu_R(\tau-\pi)$. Here, we cannot remove the constant anymore:

$$X^\mu_L(\tau+\pi) = X^\mu_L(\tau-\pi) + \pi u^\mu \tag{3.10}$$

where u^μ is just a constant 4-vector. This implies that, apart from a linear term, $X^\mu_L(\tau)$ must be periodic:

$$X^\mu_L(\tau) = \tfrac{1}{2} X^\mu_0 + \tfrac{1}{2}\tau u^\mu + \sum_n a^\mu_n \, e^{-in\tau} \; , \tag{3.11}$$

and so we write the complete solution as

$$X^\mu(\sigma,\tau) = X^\mu_0 + \tau u^\mu + \sum_{n\neq 0} \, a^\mu_n \, e^{-in\tau} \, 2\cos(n\sigma) \; . \tag{3.12}$$

In Green, Schwarz and Witten, the coordinates

$$\sigma^\pm = \tau \pm \sigma \tag{3.13}$$

are used, and the conversion factor

$$\ell = \sqrt{2\alpha'} = 1/\sqrt{\pi T} \; . \tag{3.14}$$

They also write the coefficients slightly differently. Let us adopt their notation:

$$X^\mu_R(\tau) = X^\mu_L(\tau) = \tfrac{1}{2} x^\mu + \tfrac{1}{2}\ell^2 p^\mu \tau + \frac{i}{2}\ell \sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n \, e^{-in\tau} \; ; \tag{3.15}$$

$$X^\mu(\sigma,\tau) = x^\mu + \ell^2 p^\mu \tau + i\ell \sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n \, e^{-in\tau} \cos n\sigma \, . \tag{3.16}$$

3.3.2. The closed string.

The closed string boundary condition (3.4) is read as

$$\begin{aligned} X^\mu(\sigma,\tau) \;&=\; X^\mu(\sigma+\pi,\tau) \;= \\ X^\mu_L(\sigma+\tau) + X^\mu_R(\tau-\sigma) \;&=\; X^\mu_L(\sigma+\pi+\tau) + X^\mu_R(\tau-\sigma-\pi) \; . \end{aligned} \tag{3.17}$$

We deduce from this that the function

$$X_R^\mu(\tau) - X_R^\mu(\tau - \pi) = X_L^\mu(\tau + \pi + 2\sigma) - X_L^\mu(\tau + 2\sigma) = C\, u^\mu \tag{3.18}$$

must be independent of σ and τ. Choosing the coefficient $C = \frac{1}{2}\pi$, we find that, apart from a linear term, $X_R^\mu(\tau)$ and $X_L^\mu(\tau)$ are periodic, so that they can be written as

$$\begin{aligned} X_R^\mu(\tau) &= \tfrac{1}{2} u^\mu \tau + \sum_n a_n^\mu\, e^{-2in\tau} \ ; \\ X_L^\mu(\tau) &= \tfrac{1}{2} u^\mu \tau + \sum_n b_n^\mu\, e^{-2in\tau} \ . \end{aligned} \tag{3.19}$$

So we have

$$X^\mu(\sigma, \tau) = X_0^\mu + u^\mu\, \tau + \sum_{n \neq 0} e^{-2in\tau} (a_n^\mu\, e^{-2in\sigma} + b_n^\mu\, e^{2in\sigma}) \ , \tag{3.20}$$

where reality of X^μ requires

$$(a_n^\mu)^* = a_{-n}^\mu \quad ; \qquad (b_n^\mu)^* = b_{-n}^\mu \ . \tag{3.21}$$

Here, as in Eq. (3.12), the constant vector u^μ is now seen to describe the total 4-velocity (with respect to the τ coordinate), and X_0^μ the c.m. position at $t = 0$. We shall use Green-Schwarz-Witten notation:

$$X^\mu = x^\mu + \ell^2 p^\mu \tau + \frac{i}{2} \ell \sum_{n \neq 0} \frac{1}{n} e^{-2in\tau} (\alpha_n^\mu\, e^{2in\sigma} + \tilde{\alpha}_n^\mu\, e^{-2in\sigma}) \ . \tag{3.22}$$

It is important not to forget that the functions X_R^μ and X_L^μ must also obey the constraint equations (2.10), which is equivalent to demanding that the energy-momentum tensor $T_{\mu\nu}$ in Eq. (2.26) vanishes.

From now on, we choose our units of time and space such that

$$\ell = 1 \ . \tag{3.23}$$

3.4. The light-cone gauge.

The gauge conditions that we have imposed, Eqs.(2.10), still leave us with one freedom, which is to reparametrize the coordinates σ^+ and σ^-:

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+) \quad ; \qquad \sigma^- \to \tilde{\sigma}^-(\sigma^-) \ . \tag{3.24}$$

For the closed string, these new coordinates may be chosen independently, as long as they keep the same periodicity conditions (3.17). For the open string, we have to remember that the boundary conditions mandate that the functions X_L^μ and X_R^μ are identical functions, see Eq. (3.9); therefore, if $\tilde{\sigma}^+ = f(\sigma^+)$ then $\tilde{\sigma}^-$ must be $f(\sigma^-)$ with the same function f. The functions $\tau = \frac{1}{2}(\sigma^+ + \sigma^-)$ and $\sigma = \frac{1}{2}(\sigma^+ - \sigma^-)$ therefore transform into

$$\begin{aligned} \tilde{\tau} &= \tfrac{1}{2}\Big(f(\tau + \sigma) + f(\tau - \sigma)\Big) \ ; \\ \tilde{\sigma} &= \tfrac{1}{2}\Big(f(\tau + \sigma) - f(\tau - \sigma)\Big) \ . \end{aligned} \tag{3.25}$$

Requiring the boundary conditions for $\sigma = 0$ and for $\sigma = \pi$ not to change under this transformation implies that the function $f(\tau) - \tau$ must be periodic in τ with period 2π, analogously to the variables X^μ_L, see Equ. (3.10). Comparing Eq. (2.12) with (3.25), we see that we can choose $\tilde\tau$ to be one of the X^μ variables. It is advisable to choose a lightlike coordinate, which is one whose square in Minkowski space vanishes:

$$\tilde\tau = X^+/u^+ \ + \ \text{constant} \qquad (u^+ = p^+ \ , \text{since } \ell = 1) \ . \tag{3.26}$$

In a space-time with D dimensions in total, one defines

$$X^\pm = (X^0 \pm X^{D-1})/\sqrt{2} \ . \tag{3.27}$$

We usually express this as

$$X^+(\sigma, \tau) = x^+ + p^+\tau \, , \tag{3.28}$$

which means that, in this direction, all higher harmonics α^+_n vanish.

For the closed string, the left- and right moving components can be gauged separately. Choosing the new coordinates $\tilde\sigma$ and $\tilde\tau$ as follows:

$$\tilde\sigma^+ = \tilde\tau + \tilde\sigma = \frac{2}{p^+} X^+_L \ + \ \text{constant} \quad , \qquad \tilde\sigma^- = \tilde\tau - \tilde\sigma = \frac{2}{p^+} X^+_R \ + \ \text{constant} \ , \tag{3.29}$$

so that (3.26) again holds, implies Eq. (3.28), and therefore,

$$\alpha^+_n = \tilde\alpha^+_n = 0 \qquad (n \neq 0) \ . \tag{3.30}$$

3.5. Constraints.

In this gauge choice, we can handle the constraints (2.10) quite elegantly. Write the *transverse parts* of the X variables as

$$X^{\rm tr} = (X^1, \, X^2, \, \cdots, X^{D-2}) \, . \tag{3.31}$$

Then the constraints (2.10) read as

$$2\partial_+ X^+ \, \partial_+ X^- = (\partial_+ X^{\rm tr})^2 \quad ; \qquad 2\partial_- X^+ \, \partial_- X^- = (\partial_- X^{\rm tr})^2 \ . \tag{3.32}$$

Now in the (τ, σ) frame, we have

$$\begin{aligned} \partial_+ X^+ \ &= \ \partial_+\tau \, \partial_\tau X^+ + \partial_+\sigma \, \partial_\sigma X^+ \ = \ \tfrac12(\partial_\tau + \partial_\sigma) X^+ \ = \ \tfrac12 p^+ \, ; \\ \partial_- X^+ \ &= \ \tfrac12(\partial_\tau - \partial_\sigma) X^+ \ = \ \tfrac12 p^+ \ , \end{aligned} \tag{3.33}$$

so that

$$\begin{aligned} p^+\partial_+ X^- \ &= \ (\partial_+ X^{\rm tr})^2 \ = \ \tfrac14 \Big((\partial_\tau + \partial_\sigma) X^{\rm tr} \Big)^2 \ ; \\ p^+\partial_- X^- \ &= \ (\partial_- X^{\rm tr})^2 \ = \ \tfrac14 \Big((\partial_\tau - \partial_\sigma) X^{\rm tr} \Big)^2 \ ; \\ \partial_\tau X^- \ &= \ \frac{1}{2p^+} \Big((\partial_\tau X^{\rm tr})^2 + (\partial_\sigma X^{\rm tr})^2 \Big) \ ; \\ \partial_\sigma X^- \ &= \ \frac{1}{p^+} \partial_\sigma X^{\rm tr} \, \partial_\tau X^{\rm tr} \ . \end{aligned} \tag{3.34}$$

3.5.1. for open strings:

Let us define the coefficients $\alpha_0^\mu = p^\mu$. Then we can write, see Eqs. (3.15) and (3.16),

$$\partial_\tau X^\mu = \partial_+ X^\mu + \partial_- X^\mu \quad ; \qquad \partial_\sigma X^\mu = \partial_+ X^\mu - \partial_- X^\mu \ ; \tag{3.35}$$

$$\begin{aligned} \partial_+ X^\mu = \partial_+ X^\mu_L &= \tfrac{1}{2}\sum_n \alpha_n^\mu\, e^{-in(\tau+\sigma)} \ ; \\ \partial_- X^\mu = \partial_- X^\mu_R &= \tfrac{1}{2}\sum_n \alpha_n^\mu\, e^{-in(\tau-\sigma)} \ , \end{aligned} \tag{3.36}$$

and the constraints (3.34) read as

$$\begin{aligned} \partial_+ X^- &= \tfrac{1}{2}\sum_n \alpha_n^-\, e^{-in\sigma^+} = \tfrac{1}{4p^+}\sum_{n,m} \alpha_n^{\rm tr}\alpha_m^{\rm tr}\, e^{-i(n+m)\sigma^+} \ ; \\ \partial_- X^- &= \tfrac{1}{2}\sum_n \alpha_n^-\, e^{-in\sigma^-} = \tfrac{1}{4p^+}\sum_{n,m} \alpha_n^{\rm tr}\alpha_m^{\rm tr}\, e^{-i(n+m)\sigma^-} \ . \end{aligned} \tag{3.37}$$

Both these equations lead to the same result for the α^- coefficients:

$$\alpha_n^- = \frac{1}{2p^+}\sum_k \alpha_k^{\rm tr}\,\alpha_{n-k}^{\rm tr} = \frac{1}{2p^+}\sum_{k=-\infty}^{\infty}\sum_{i=1}^{D-2} \alpha_k^i\,\alpha_{n-k}^i \ . \tag{3.38}$$

Here we see the advantage of the factors $1/n$ in the definitions (3.16). One concludes that (up to an irrelevant constant) $X^-(\sigma,\tau)$ is completely fixed by the constraints. The complete solution is generated by the series of numbers α_n^i, where $i = 1, \cdots, D-2$, for the transverse string excitations, including α_0^i, the transverse momenta. There is no further constraint to be required for these coefficients.

3.5.2. for closed strings:

In the case of the closed string, we define $\alpha_0^\mu = \tilde\alpha_0^\mu = \frac{1}{2}p^\mu$. Then Eq. (3.22) gives

$$\begin{aligned} \partial_+ X^\mu = \partial_+ X^\mu_L &= \textstyle\sum_n \alpha_n^\mu\, e^{-2in(\tau+\sigma)} \ ; \\ \partial_- X^\mu = \partial_- X^\mu_R &= \textstyle\sum_n \tilde\alpha_n^\mu\, e^{-2in(\tau-\sigma)} \ . \end{aligned} \tag{3.39}$$

Eq. (3.34) becomes

$$\begin{aligned} \partial_+ X^- &= \tfrac{1}{p^+}(\partial_+ X^{\rm tr})^2 = \tfrac{1}{p^+}\textstyle\sum_{n,m} \alpha_n^{\rm tr}\alpha_m^{\rm tr}\, e^{-2i(n+m)\sigma^+} \\ \partial_- X^- &= \tfrac{1}{p^+}(\partial_- X^{\rm tr})^2 = \tfrac{1}{p^+}\textstyle\sum_{n,m} \tilde\alpha_n^{\rm tr}\tilde\alpha_m^{\rm tr}\, e^{-2i(n+m)\sigma^-} \ . \end{aligned} \tag{3.40}$$

Thus, we get

$$\alpha_n^- = \frac{1}{p^+}\sum_k \alpha_k^{\rm tr}\alpha_{n-k}^{\rm tr} \quad ; \qquad \tilde\alpha_n^- = \frac{1}{p^+}\sum_k \tilde\alpha_k^{\rm tr}\tilde\alpha_{n-k}^{\rm tr} \ . \tag{3.41}$$

3.6. Energy, momentum, angular momentum.

What are the total energy and momentum of a specific string solution? Consider a piece of string, during some short time interval, where we have conformal coordinates σ and τ. For a stationary string, at a point where the induced metric is given by $\mathrm{d}s^2 = C(\sigma,\tau)^2(\mathrm{d}\sigma^2 - \mathrm{d}\tau^2)$, the energy per unit of length is

$$P^0 = \frac{\partial p^0}{C\partial\sigma} = T = T\frac{\partial X^0}{C\partial\tau}. \tag{3.42}$$

Quite generally, one has

$$P^\mu = T\frac{\partial X^\mu}{\partial\tau} . \tag{3.43}$$

Although this reasoning would be conceptually easier to understand if we imposed a "time gauge", $X^0 = \mathrm{Const}\cdot\tau$, all remains the same in the light-cone gauge. In chapter 4, subsection 4.1 , we derive the energy-momentum density more precisely from the Lagrange formalism.

We have

$$P^\mu_{\mathrm{tot}} = \int_0^\pi P^\mu \mathrm{d}\sigma = T\int_0^\pi \frac{\partial X^\mu}{\partial\tau}\mathrm{d}\sigma = \pi T\ell^2 p^\mu , \tag{3.44}$$

see Eq. (3.22). With the convention (3.14), this is indeed the 4-vector p^μ.

We will also need the total angular momentum. For a set of free particles, counted by a number $A = 1,\cdots, N$, the covariant tensor is

$$J^{\mu\nu} = \sum_{A=1}^N (x^\mu_A p^\nu_A - x^\nu_A p^\mu_A) . \tag{3.45}$$

In the usual 4 dimensional world, the spacelike components are easily recognized to be $\varepsilon^{ijk}J_k$. The space-time components are the conserved quantities

$$J^{i0} = \sum_A (x^i_A E_A - t p^i_A) . \tag{3.46}$$

For the string, we have

$$J^{\mu\nu} = \int_0^\pi \mathrm{d}\sigma\,(X^\mu P^\nu - X^\nu P^\mu) = T\int_0^\pi \mathrm{d}\sigma\left(X^\mu\frac{\partial X^\nu}{\partial\tau} - X^\nu\frac{\partial X^\mu}{\partial\tau}\right) , \tag{3.47}$$

and if here we substitute the solution (3.16) for the open string, we get

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^\infty \frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n) . \tag{3.48}$$

The first part here describes orbital angular momentum. The second part describes the spin of the string.

The importance of the momentum and angular momentum is that, in a quantum theory, these will have to be associated to operators that generate translations and rotations, and as such they will have to be absolutely conserved quantities.

4. Quantization.

Quantization is not at all a straightforward procedure. The question one asks is, does a Hilbert space of states $|\psi\rangle$ exist such that one can define operators $X^\mu(\sigma,\tau)$ that allow reparametrization transformations for the (σ,τ) coordinates. It should always be possible to transform $X^0(\sigma,\tau)$ to become the c-number τ itself, because time is not supposed to be an operator, and this should be possible starting from any Lorentz frame, so as to ensure lorentz invariance. It is *not* self-evident that such a procedure should always be possible, and indeed, we shall see that often it is not.

There are different procedures that can be followed, all of which are equivalent. Here, we do the *light-cone quantization*, starting from the light-cone gauge.

4.1. Commutation rules.

After fixing the gauge, our classical action was Eq. (2.22). Write

$$S=\int \mathrm{d}\tau L(\tau) \quad ; \qquad L(\tau)=\frac{T}{2}\int \mathrm{d}\sigma\left((\dot X^\mu)^2-(X^{\mu\prime})^2\right)=U^{\mathrm{kin}}-V\ , \tag{4.1}$$

where $\dot X$ stands for $\partial X/\partial\tau$ and $X'=\partial X/\partial\sigma$. This is the Lagrange function, and it is standard procedure to define the momentum as its derivative with respect to $\dot X^\mu$. Here:

$$P^\mu=T\dot X^\mu\ . \tag{4.2}$$

In analogy to conventional Quantum Mechanics, we now try the following commutation rules:

$$\begin{aligned}\Big[X^\mu(\sigma),\,X^\nu(\sigma')\Big] &= \Big[P^\mu(\sigma),\,P^\nu(\sigma')\Big]\ =\ 0\\ \Big[X^\mu(\sigma),\,P^\nu(\sigma')\Big] &= i\eta^{\mu\nu}\delta(\sigma-\sigma')\ ,\end{aligned} \tag{4.3}$$

where $\eta^{\mu\nu}=\mathrm{diag}(-1,1,\cdots,1)$. These should imply commutation rules for the parameters x^μ, p^μ, α_n^μ and $\tilde\alpha_n^\mu$ in our string solutions. Integrating over σ, and using

$$\int_0^\pi \cos m\sigma\,\cos n\sigma\,\mathrm{d}\sigma=\tfrac12\pi\,\delta_{mn}\quad ,\qquad m,n>0\ , \tag{4.4}$$

we derive for the open string:

$$\begin{aligned}x^\mu &= \frac{1}{\pi}\int_0^\pi \mathrm{d}\sigma\,X^\mu(\sigma)\quad ;\qquad p^\mu\ =\ \frac{1}{T\ell^2\pi}\int_0^\pi \mathrm{d}\sigma\,P^\mu\ ;\\ \alpha_n^\mu &= \frac{1}{\pi\ell}\int_0^\pi \mathrm{d}\sigma\,\cos n\sigma\left(\frac{P^\mu}{T}-in\,X^\mu(\sigma)\right)\ ;\\ \alpha_{-n}^\mu &= (\alpha_n^\mu)^\dagger\ .\end{aligned} \tag{4.5}$$

For these coefficients then, Eqs. (4.3) yields the following commutation rules (assuming ℓ to be chosen as in (3.14)):

$$[\,x^\mu,\,x^\nu]\ =\ [\,p^\mu,\,p^\nu]\ =\ 0\quad ,\qquad [\,x^\mu,\,p^\nu]\ =\ i\eta^{\mu\nu}\ ; \tag{4.6}$$

$$[\,\alpha^\mu_m,\,\alpha^\nu_n] \;=\; \frac{1}{\pi^2\ell^2 T}\int_0^\pi \mathrm{d}\sigma\cos m\sigma\,\cos n\sigma\,(m-n)\,\eta^{\mu\nu} \;=\; 0 \quad \text{if} \quad n,\,m>0\;; \tag{4.7}$$
$$[\,\alpha^\mu_m,\,\alpha^\nu_{-n}] \;=\; \frac{1}{\pi^2\ell^2 T}\int_0^\pi \mathrm{d}\sigma\cos m\sigma\,\cos n\sigma\,(m+n)\,\eta^{\mu\nu} \;=\; n\,\delta_{mn}\eta^{\mu\nu}\;. \tag{4.8}$$

The equation (4.8) shows that (the space components of) α^μ_n are annihilation operators:

$$[\,\alpha^i_m,\,(\alpha^j_n)^\dagger] = n\,\delta_{mn}\delta^{\,ij} \tag{4.9}$$

(note the unusual factor n here, which means that these operators contain extra normalization factors $\sqrt{n}$, and that the operator $(\alpha^i_n)^\dagger\alpha^i_n = nN_{i,n}$, where $N_{i,n}$ counts the number of excitations)

It may seem to be a reason for concern that Eqs. (4.6) include an unusual commutation relation between *time* and *energy*. This however must be regarded in combination with our constraint equations: starting with arbitrary wave functions in space and time, the constraints will impose equations that correspond to the usual wave equations. This is further illustrated for point particles in Green-Schwarz-Witten p. 19.

Thus, *prior to imposing the constraints*, we work with a *Hilbert space* of the following form. There is a single (open or closed) string (at a later stage, one might compose states with multiple strings). This single string has a center of mass described by a wave function in space and time, using all D operators x^μ (with p^μ being the canonically associated operators $-i\eta^{\mu\nu}\partial/\partial x^\nu$). Then we have the string excitations. The non-excited string mode is usually referred to as the 'vacuum state' $|\,0\rangle$ (not to be confused with the space-time vacuum, where no string is present at all). All string excited states are then obtained by letting the creation operators $(\alpha^i_n)^\dagger = \alpha^i_{-n}$, $n>0$ act a finite number of times on this vacuum. If we also denote explicitly the total momentum of the string, we get states $|p^\mu, N_{1,1}, N_{1,2},\dots\rangle$. It is in this Hilbert space that *all* x^μ and p^μ are operators, acting on wave functions that can be any function of x^μ.

4.2. The constraints in the quantum theory.

Now return to the constraint equations (3.38) for the open string and (3.41) for the closed string in the light-cone gauge. In the classical theory, for $n=0$, this is a constraint for p^-:

$$p^- = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left((p^i)^2 + \sum_{m\neq 0}\alpha^i_{-m}\alpha^i_m\right)\;. \tag{4.10}$$

This we write as

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2}(p^i)^2 = 2\sum_{i=1}^{D-2}\sum_{m=1}^{\infty}(\alpha^i_m)^\dagger\alpha^i_m + \,? \tag{4.11}$$

As we impose these constraints, we have to reconsider the commutation rules (4.6) — (4.8). *The constrained operators obey different commutation rules*; compare ordinary quantum mechanics: as soon as we impose the Schrödinger equation, $\partial\psi/\partial t = -i\hat{H}\psi$, the coordinate t must be seen as a c-number, and the Hamiltonian as some function of the other operators of the theory, whose commutation rules it inherits. The commutation rules (4.6) — (4.8) from now on *only hold for the transverse parts of these operators*, not for the $+$ and $-$ components, the latter will have to be computed using the constraints.

Up to this point, we were not concerned about the *order* of the operators. However, Eqs. (4.10) and (4.11) have really only been derived classically, where the order between α^i_m and $(\alpha^i_m)^\dagger$ was irrelevant. Here, on the other hand, switching the order would produce a constant, comparable to a 'vacuum energy'. What should this constant here be? String theorists decided to put here an arbitrary coefficient $-2a$:

$$M^2 = 2\left(\sum_{i,n} n\, N_{i,n} - a\right) \ . \tag{4.12}$$

Observe that: (i) the quantity $\alpha(M^2) = \frac{1}{2}M^2 + a$ is a non-negative integer. So, a is the 'intercept' $\alpha(0)$ of the trajectories (1.1) and (1.2) mentioned at the beginning of these lectures. And (ii): $\frac{1}{2}M^2$ increases by at least one unit whenever an operator $(\alpha^i_n)^\dagger$ acts. An operator $(\alpha^i_n)^\dagger$ can increase the angular momentum of a state by at most one unit (Wigner-Eckart theorem). Apparently, $\alpha' = \frac{1}{2}$ in our units, as anticipated in Eq. (3.14), as we had put $\ell = 1$. It is now clear why the daughter trajectories are separated from the leading trajectories by integer spacings.

At this point, a mysterious feature shows up. The lowest mass state, referred to as $|\,0\rangle$, has $\frac{1}{2}M^2 = -a$, and appears to be non-degenerate: there is just one such state. Let us now count all first-excited states. They have $\frac{1}{2}M^2 = 1 - a$. The only way to get such states is:

$$|i\rangle = \alpha^i_{-1}|\,0\rangle\ ; \qquad i = 1, \cdots, D-2\ . \tag{4.13}$$

Because of the space-index i, these states transform as a vector in space-time. They describe a 'particle' with spin 1. Yet they have only $D-2$ components, while spin one particles have $D-1$ components (3 if space-time is 4 dimensional: if $\ell = 1$, $m = \pm 1$ or 0) The only way to get a spin one particle with $D-2$ components, is if this state has mass zero, like a photon. Gauge-invariance can then remove one physical degree of freedom. Apparently, consistency of the theory requires $a = 1$. This however gives a ground state of negative mass-squared: $\frac{1}{2}M^2 = -a = -1$. The theory therefore has a tachyon. We will have to live with this tachyon for the time being. Only *super symmetry* can remove the tachyon, as we shall see in Chapter 12.

The *closed* string is quantized in subsection 4.4.

4.3. The Virasoro Algebra.

In view of the above, we use as a starting point the quantum version of the constraint. For the open string:

$$\alpha^-_n = \frac{1}{2p^+}\left(\sum_{i,m} :\alpha^i_{n-m}\alpha^i_m: -2a\delta_n\right)\ , \tag{4.14}$$

where the sum is over all m (including $m=0$) and $i=1,\cdots,D-2$. The symbols :: stand for *normal ordering*: c-numbers are added in such a way that the vacuum expectation value of the operators in between is zero, which is achieved by switching the order of the two terms if necessary (here: if m is negative and $n-m$ positive). The symbol δ_n is defined by

$$\delta_n = 0 \quad \text{if} \quad n \neq 0 \; ; \qquad \delta_0 = 1 \; . \tag{4.15}$$

Eqs. (4.7) and (4.8) are written as

$$[\,\alpha^i_m,\,\alpha^j_n] = m\,\delta^{\,ij}\delta_{m+n} \; . \tag{4.16}$$

Using the rule

$$[AB,\,C] = [A,\,C]B + A[B,\,C] \; , \tag{4.17}$$

we can find the commutation rules for α^-_n:

$$[\,\alpha^i_m,\,\alpha^-_n] = m\,\alpha^i_{m+n}/p^+ \; . \tag{4.18}$$

More subtle is the derivation of the commutator of two α^-. Let us first consider the commutators of the quantity

$$L^1_m = \tfrac{1}{2}\sum_k :\alpha^1_{m-k}\alpha^1_k: \quad ; \qquad [\,\alpha^1_m,\,L^1_n] = m\,\alpha^1_{m+n} \; . \tag{4.19}$$

What is the commutator $[L^1_m,\,L^1_n]$? Note that: since the L^1_m are normal-ordered, their action on any physical state is completely finite and well-defined, and so their commutators should be finite and well-defined as well. In some treatises one sees infinite and divergent summations coming from infinite subtraction due to normal-ordering, typically if one has an infinite series of terms that were not properly ordered to begin with. We should avoid such divergent expressions. Indeed, the calculation of the commutator can be done completely rigorously, but to do this, we have to keep the order of the terms in mind. What follows now is the explicit calculation. It could be done faster and more elegantly if we allowed ourselves more magic, but here we give priority to understanding the physics of the argument.

Give the definition with the right ordering:

$$L^1_m = \tfrac{1}{2}\Big(\sum_{k\geq 0}\alpha^1_{m-k}\,\alpha^1_k + \sum_{k<0}\alpha^1_k\,\alpha^1_{m-k}\Big) \; ; \tag{4.20}$$

$$\begin{aligned}
\Big[L^1_m,\,L^1_n\Big] &= \tfrac{1}{2}\Big(\sum_{k\geq 0}\Big[\,\alpha^1_{m-k},\,L^1_n\Big]\,\alpha^1_k + \sum_{k\geq 0}\alpha^1_{m-k}\,\Big[\,\alpha^1_k,\,L^1_n\Big] \\
&+ \sum_{k<0}\Big[\,\alpha^1_k,\,L^1_n\Big]\,\alpha^1_{m-k} + \sum_{k<0}\alpha^1_k\,\Big[\,\alpha^1_{m-k},\,L^1_n\Big]\Big) = \\
&= \tfrac{1}{2}\Big(\sum_{k\geq 0}(m-k)\,\alpha^1_{m+n-k}\alpha^1_k + \sum_{k\geq 0}k\,\alpha^1_{m-k}\,\alpha^1_{n+k} \\
&+ \sum_{k<0}k\,\alpha^1_{k+n}\,\alpha^1_{m-k} + \sum_{k<0}(m-k)\,\alpha^1_k\,\alpha^1_{m+n-k}\Big) \; .
\end{aligned} \tag{4.21}$$

If $n+m\neq 0$, the two α's in each term commute, and so their order is irrelevant. In that case, we can switch the order in the last two terms, and replace the variable k by $k-n$ in terms # 2 and 3, to obtain

$$(4.21) = \tfrac{1}{2}\sum_{\text{all } k}(m-n)\,\alpha^1_{m+n-k}\,\alpha^1_k = (m-n)L^1_{m+n} \qquad \text{if} \qquad m+n\neq 0\ . \tag{4.22}$$

If, however, $m=-n$, an extra contribution arises since we insist to have normal ordering. Let us take $m>0$ (in the other case, the argument goes the same way). Only in the second term, the order has to be switched, for the values $0\le k\le m$. From (4.16), we see that this give a factor $m-k$. Thus, we get an extra term:

$$+\tfrac{1}{2}\sum_{k=1}^{m} k(m-k)\,\delta_{m+n}\ . \tag{4.23}$$

Now use

$$\sum_1^m k = \tfrac{1}{2}m(m+1) \quad , \qquad \sum_1^m k^2 = \tfrac{1}{6}m(m+1)(2m+1)\ , \tag{4.24}$$

to obtain

$$(4.23) = \tfrac{1}{4}m^2(m+1) - \tfrac{1}{12}m(m+1)(2m+1) = \tfrac{1}{12}m(m+1)(m-1)\ . \tag{4.25}$$

Thus, one obtains the *Virasoro algebra*:

$$[L^1_m,\, L^1_n] = (m-n)L^1_{m+n} + \tfrac{1}{12}m(m^2-1)\,\delta_{m+n}\ , \tag{4.26}$$

a very important equation for field theories in a two-dimensional base space. Now, since $\alpha^-_n = (\sum_i L^i_n - a\delta_n)/p^+$, where i takes $D-2$ values, their commutator is

$$\left[\,\alpha^-_m,\, \alpha^-_n\right] = \frac{m-n}{p^+}\,\alpha^-_{m+n} + \frac{\delta_{m+n}}{{p^+}^2}\,\left(\tfrac{D-2}{12}\,m(m^2-1)+2m\,a\right)\ . \tag{4.27}$$

To facilitate further calculations, let me give here the complete table for the commutators of the coefficients α^μ_n, x^μ and p^μ (as far as will be needed):

$[X,\,Y]$	x^i	x^-	p^i	p^+	p^-	α^i_m	α^-_m	$\overset{X}{\rightarrow}$
x^j	0	0	$-i\delta^{ij}$	0	$-i\,p^j/p^+$	$-i\,\delta^{ij}\delta_m$	$-i\,\alpha^j_m/p^+$	
x^-	0	0	0	i	$-ip^-/p^+$	0	$-i\,\alpha^-_m/p^+$	
p^j	$i\,\delta^{ij}$	0	0	0	0	0	0	
p^+	0	$-i$	0	0	0	0	0	
p^-	ip^i/p^+	ip^-/p^+	0	0	0	$m\,\alpha^i_m/p^+$	$m\,\alpha^-_m/p^+$	
α^j_n	$i\,\delta^{ij}\delta_n$	0	0	0	$-n\,\alpha^j_n/p^+$	$m\,\delta^{ij}\delta_{m+n}$	$-n\,\alpha^j_{m+n}/p^+$	
α^-_n	$i\,\alpha^i_n/p^+$	$i\,\alpha^-_n/p^+$	0	0	$-n\,\alpha^-_n/p^+$	$m\,\alpha^i_{m+n}/p^+$	$\star\star$	

$Y\downarrow$

$$\star\star \;=\; \frac{m-n}{p^+}\,\alpha^-_{m+n} + \Big(\tfrac{D-2}{12}\,m(m^2-1) + 2a\,m\Big)\frac{\delta_{m+n}}{(p^+)^2}$$

One may wonder why p^- does not commute with x^i and x^-. This is because we first impose the constraints and then consider the action of p^-, which now plays the role of a Hamiltonian in Quantum mechanics. x^i and x^- are time dependent, and so they do not commute with the Hamiltonian.

4.4. Quantization of the closed string

The closed string is described by Eq. (3.22), and here we have *two* constraints of the form (3.36), one for the left-movers and one for the right movers. The classical alpha coefficients (with $\alpha^\mu_0 = \tilde\alpha^\mu_0 = \frac{1}{2}p^\mu$), obey Eqs. (3.41). In the quantum theory, we have to pay special attention to the order in which the coefficients are multiplied; however, if $n \neq 0$, the expression for α^-_n only contains terms in which the two alphas commute, so we can copy the classical expressions without ambiguity to obtain the operators:

$$\left.\begin{aligned}\alpha^-_n &= \frac{1}{p^+}\sum_{k=-\infty}^{+\infty}\sum_{i=1}^{D-2}\alpha^i_k\alpha^i_{n-k}\\ \tilde\alpha^-_n &= \frac{1}{p^+}\sum_{k=-\infty}^{+\infty}\sum_{i=1}^{D-2}\tilde\alpha^i_k\tilde\alpha^i_{n-k}\end{aligned}\right\}\qquad \text{if}\quad n\neq 0\ . \tag{4.28}$$

A similar quantization procedure as for open strings yields the commutation relations

$$[x^\mu,\,p^\nu] = -\eta^{\mu\nu}\ ;\qquad [\alpha^i_m,\,\alpha^j_n] = [\tilde\alpha^i_m,\,\tilde\alpha^j_n] = m\,\delta_{m+n}\delta^{ij}\ ;\qquad [\tilde\alpha^\mu_n,\,\alpha^\nu_m] = 0\ . \tag{4.29}$$

As for the zero modes, it is important to watch the order in which the α's are written. Our expressions will only be meaningful if, in the infinite sum, creation operators appear at the left and annihilation operators at the right, otherwise all terms in the sum give contributions, adding up to infinity. As in Eq. (4.12), we assume that, after normal ordering of the α's, finite c-numbers a and $\tilde{a}$ remain:

$$\begin{aligned} \alpha_0^- = \tfrac{1}{2}p^- = \frac{1}{p^+}\sum_{i=1}^{D-2}\Big(\alpha_0^i\alpha_0^i + \sum_{k>0}\alpha_{-k}^i\alpha_k^i + \sum_{k<0}\alpha_k^i\alpha_{-k}^i - 2a\Big) = \\ = \frac{1}{p^+}\sum_{i=1}^{D-2}\Big(\alpha_0^i\alpha_0^i + 2\sum_{k>0}\alpha_{-k}^i\alpha_k^i - 2a\Big)\ , \end{aligned} \tag{4.30}$$

and similarly for the right-movers. So now we have:

$$M^2 = 2p^+p^- - (p^{\rm tr})^2 = 8\left(\sum_{i=1}^{D-2}\sum_{m=1}^{\infty}(\alpha_m^i)^\dagger\alpha_m^i \ - a\right) = 8\left(\sum_{i=1}^{D-2}\sum_{m=1}^{\infty}(\tilde\alpha_m^i)^\dagger\tilde\alpha_m^i \ - \tilde a\right)\ . \tag{4.31}$$

4.5. The closed string spectrum

We start by constructing a Hilbert space using a vacuum $|\,0\rangle$ that satisfies

$$\alpha_m^i|\,0\rangle = \tilde\alpha_m^i|\,0\rangle = 0\ , \qquad \forall m > 0\ . \tag{4.32}$$

The mass of such a state is

$$M^2|\,0\rangle = 8(-a)|\,0\rangle = 8(-\tilde a)|\,0\rangle\ , \tag{4.33}$$

so we must require: $a = \tilde a$. Let us try to construct the first excited state:

$$|\,i\rangle \equiv \alpha_{-1}^i|\,0\rangle\ . \tag{4.34}$$

Its mass is found as follows:

$$\begin{aligned} M^2|\,i\rangle &= 8\Big(\sum_{j=1}^{D-2}\sum_{m=1}^{\infty}(\alpha_m^j)^\dagger\alpha_m^j - a\Big)\alpha_{-1}^i|\,0\rangle \\ &= 8\sum_{j=1}^{D-2}(\alpha_1^j)^\dagger[\alpha_1^j,\,\alpha_{-1}^i]|\,0\rangle - a|\,i\rangle \\ &= 8|\,i\rangle - 8a|\,i\rangle = 8(1-a)|\,i\rangle \quad \to \qquad M^2 = 8(1-a)\ . \end{aligned} \tag{4.35}$$

However, there is also the constraint for the *right*-going modes:

$$\begin{aligned} M^2|\,i\rangle = 8\Big(\sum_{j=1}^{D-2}\sum_{m=1}^{\infty}(\tilde\alpha_m^j)^\dagger\tilde\alpha_m^j - a\Big)|\,i\rangle \\ = -8a|\,i\rangle \quad \to \qquad M^2 = -8a\ . \end{aligned} \tag{4.36}$$

This is a contradiction, and so our vector state does not obey the constraints; it is not an element of our Hilbert space.

The next state we try is the tensor state $|\,i,j\rangle \equiv \alpha^i_{-1}\tilde\alpha^j_{-1}|\,0\rangle$. We now find that it does obey both constraints, which both give:

$$M^2 = 8(1-a)\ . \tag{4.37}$$

However, it transforms as a $(D-2)\times(D-2)$ representation of the *little group*, being the group of only rotations in $D-2$ dimensions. For the open string, we found that this was a reason for the ensuing vector particle to be a photon, with mass equal to zero. Here, also, consistency requires that this tensor-particle is massless. The state $|\,ij\rangle$ falls apart in three irreducible representations:

- an antisymmetric state: $|[ij]\rangle \ = \ -|[j\,i]\rangle \ = \ |\,ij\rangle - |\,j\,i\rangle$,
- a traceless symmetric state: $|\,\{ij\}\rangle \ = \ |\,ij\rangle + |\,j\,i\rangle - \frac{2}{D-2}\,\delta_{ij}|\,kk\rangle$,
- and a trace part: $|\,s\rangle \ = \ |\,kk\rangle$.

The dimensionality of these states is:
$\frac{1}{2}(D-2)(D-3)$ for the antisymmetric state (a rank 2 form),
$\frac{1}{2}(D-2)(D-1)-1$ for the symmetric part (the "graviton" field), and
1 for the trace part (a scalar particle, called the "dilaton").
There exist no massive particles that could transform this way, so, again, we must impose $M=0$, implying $a=1$ for the closed string. The massless antisymmetric state would be a pseudoscalar particle in $D=4$; the symmetric state can only describe something like the graviton field, the only spin 2 tensor field that is massless and has $\frac{1}{2}\cdot 2\cdot 3-1=2$ polarizations. We return to this later.

5. Lorentz invariance.

An alternative way to quantize the theory is the so-called covariant quantization, which is a scheme in which Lorentz covariance is evident at all steps. Then, however, one finds many states which are 'unphysical'; for instance, there appear to be $D-1$ vector states whereas we know that there are only $D-2$ of them. Quantizing the system in the light-cone gauge has the advantage that all physically relevant states are easy to identify, but the price we pay is that Lorentz invariance is not easy to establish, since the τ coordinate was identified with X^+. Given a particular string state, what will it be after a Lorentz transformation?

Just as the components of the angular momentum vector are the operators that generate an infinitesimal rotation, so we also have operators that generate a Lorentz boost. Together, they form the tensor $J^{\mu\nu}$ that we derived in Eq. (3.48). The string states, with all their properties that we derived, should be a *representation* of the Lorentz group. What this means is the following. If we compute the *commutators* of the operators (3.48), we should get the same operators at the right hand side as what is dictated by group theory:

$$[\,p^\mu,\,p^\nu] \ = \ 0\ ; \tag{5.1}$$

$$
\begin{aligned}
\left[p^{\mu},\, J^{\nu\varrho}\right] &= -i\eta^{\mu\nu}p^{\varrho} + i\eta^{\mu\varrho}p^{\nu}\ ; &(5.2)\\
\left[J^{\mu\nu},\, J^{\varrho\lambda}\right] &= -i\eta^{\nu\varrho}J^{\mu\lambda} + i\eta^{\mu\varrho}J^{\nu\lambda} + i\eta^{\nu\lambda}J^{\mu\varrho} - i\eta^{\mu\lambda}J^{\nu\varrho} &(5.3)
\end{aligned}
$$

(where we included the momentum operators, so this is really the Poincaré group). For most of these equations it is evident that these equations are right, but for the generators that generate a transformation that affects x^+, it is much less obvious. This is because such transformations will be associated by σ, τ transformations. The equations that require explicit study are the ones involving J^{i-}. Writing

$$
\begin{aligned}
J^{\mu\nu} = \ell^{\mu\nu} + E^{\mu\nu} \qquad &; \qquad \ell^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu}\ ;\\
E^{\mu\nu} &= -i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^{\mu}_{-n}\alpha^{\nu}_{n} - \alpha^{\nu}_{-n}\alpha^{\mu}_{n})\ , \qquad (5.4)
\end{aligned}
$$

enables us to check these commutation relations using the Commutator Table of Section 4.3. If the theory is Lorentz invariant, these operators, which generate infinitesimal Lorentz rotations, automatically obey the commutation rules (5.2) and (5.3). However, since we introduced the c-numbers in the commutation rules by hand, it is far from obvious whether this is indeed the case. In particular, we should check whether

$$
\left[p^{-},\, J^{j-}\right] \stackrel{?}{=} 0\ , \qquad (5.5)
$$

$$
\left[J^{i-},\, J^{j-}\right] \stackrel{?}{=} 0\ . \qquad (5.6)
$$

Before doing this, one important remark: The definition of J^{i-} contains products of terms that do not commute, such as x^ip^-. The operator *must* be Hermitian, and that implies that we must correct the classical expression (3.48). We remove its anti-Hermitean part, or, we choose the *symmetric* product, writing $\frac{1}{2}(x^ip^- + p^-x^i)$, instead of x^ip^-. Note, furthermore, that x^+ is always a c-number, so it commutes with everything.

Using the Table, one now verifies that the $\ell^{\mu\nu}$ among themselves obey the same commutation rules as the $J^{\mu\nu}$. Using (4.17), one also verifies easily that

$$
[p^-,\, \ell^{j-}] = 0\ , \quad \text{and} \quad [p^-,\, E^{j-}] = 0\,. \qquad (5.7)
$$

We strongly advise the reader to do this exercise, bearing the above symmetrization procedure in mind. Remains to prove (5.6). This one will turn out to give complications.

Finding that

$$
\begin{aligned}
&[x^i,\, E^{j-}] = -iE^{ij}/p^+ \quad , \quad [x^-,\, E^{j-}] = iE^{j-}/p^+\ , &(5.8)\\
\text{and}\quad &[p^-,\, E^{j-}] = 0 \quad , \quad [p^i,\, E^{j-}] = 0\ , &(5.9)
\end{aligned}
$$

we get

$$
[\ell^{i-},\, E^{i-}] = -iE^{ij}p^-/p^+ - iE^{j-}p^i/p^+\ , \qquad (5.10)
$$

and so a short calculation gives

$$
[J^{i-},\, J^{j-}] = (-2iE^{ij}p^- + iE^{i-}p^j - iE^{j-}p^i)/p^+ + [E^{i-},\, E^{j-}]\ . \qquad (5.11)
$$

To check whether this vanishes, we have to calculate the commutator $[E^{i-},\, E^{j-}]$, which is more cumbersome. The calculations that follow now are done exactly in the same way as the ones of the previous chapter: we have to keep the operators in the right order, otherwise we might encounter intermediate results with infinite c-numbers. We can use the result we had before, Eq. (4.27). An explicit calculation, though straightforward, is a bit too bulky to be reproduced here in all detail, so we leave that as an exercise. An intermediate result is:

$$\begin{aligned}[E^{i-},\, \alpha_m^-] \;&=\; \frac{i}{p^+}(p^i\alpha_m^- - p^-\alpha_m^i) + \frac{i}{p^+}\sum_{k>0}\frac{m}{k}\left(\alpha_{-k}^i\alpha_{m+k}^- - \alpha_{m-k}^-\alpha_k^i\right) + \\ &+\; \frac{i}{(p^+)^2}\left(\frac{m(m-1)}{2} - \frac{R(m)}{m}\right)\alpha_m^- \ , \qquad \text{if} \quad m\neq 0\ , \end{aligned} \tag{5.12}$$

where $R(m)$ is the expression occurring in the commutator (4.27):

$$\frac{R(m)}{m} = \tfrac{D-2}{12}(m^2-1) + 2a\ . \tag{5.13}$$

Finally, one finds that nearly everything in the commutator (5.11) cancels out, but not quite. It is the c-numbers in the commutator list of subsection 4.3 that give rise to a residual term:

$$\begin{aligned}[J^{i-},\, J^{j-}] \;&=\; -\frac{1}{(p^+)^2}\sum_{m=1}^{\infty}\Delta_m\left(\alpha_{-m}^i\alpha_m^j - \alpha_{-m}^j\alpha_m^i\right)\ , \\ \Delta_m \;&=\; -\frac{R(m)}{m^2} + 2m \;=\; \tfrac{26-D}{12}\,m + \left(\tfrac{D-2}{12} - 2a\right)\frac{1}{m}\ . \end{aligned} \tag{5.14}$$

Insisting that this should vanish implies that this theory only appears to work if the number D of space-time dimensions is 26, and $a=1$; the latter condition we already noticed earlier.

6. Interactions and vertex operators.

The simplest string interaction is the process of splitting an open string in two open strings and the reverse, joining two open strings at their end points. With the machinery we have now, a complete procedure is not yet possible, but a first attempt can be made. We consider one open string that is being manipulated at one end point, say the point $\sigma=0$. In the light-cone gauge, the Hamiltonian p^- receives a small perturbation

$$H^{\rm int} = \sum_{k^\mu}\varepsilon(k)\,e^{-ik^-X^+-ik^+X^-+ik^{\rm tr}X^{\rm tr}}\ , \tag{6.1}$$

where X^μ stands for $X^\mu(\tau,0)$, $\tau = X^+/p^+$. What will be the transitions caused by such a perturbation?

For this, one uses standard perturbation theory. The first order correction is written as

$$\langle{\rm out}|\int H^{\rm int}(\tau){\rm d}\tau|\,{\rm in}\rangle\ , \tag{6.2}$$

but this amplitude does not teach us very much — only those matrix elements where k^- in $H^{\rm int}$ matches the energy difference between the in-state and the out-state, contribute. Of more interest is the second order correction, because this shows a calculable dependence on the total momentum exchanged.

The second order coefficient describing a transition from a state $|\,{\rm in}\rangle$ to a state $|\,{\rm out}\rangle$ is, up to some kinematical factors, the amplitude

$$\int_{-\infty}^{\infty}{\rm d}\tau^1\int_0^{\infty}{\rm d}\tau\langle{\rm out}|\,H^{\rm int}(\tau^1+\tau)\,H^{\rm int}(\tau^1)|\,{\rm in}\rangle\ , \tag{6.3}$$

where the Heisenberg notation is used in expressing the time dependence of the interaction Hamiltonian. We are interested in the particular contribution where the initial state has momentum k_4^μ, the first insertion of $H^{\rm int}$ goes with the Fourier coefficient $e^{ik_3^\mu X^\mu}$, the second insertion with Fourier coefficient $e^{ik_2^\mu X^\mu}$, and the final state has momentum $-k_1^\mu$ (we flipped the sign here so that all momenta k_i^μ now will refer to *ingoing* amounts of 4-momentum, as will become evident shortly). For simplicity, we take the case that the initial string and the final string are in their ground state. Thus, what we decide to compute is the amplitude

$$\mathcal{A}=\int_{-\infty}^{\infty}{\rm d}\tau^1\int_0^{\infty}{\rm d}\tau\ {}_{\rm out}\langle 0,\,-k_1|\,e^{ik_2^\mu X^\mu(\tau^1+\tau,\,0)}\,e^{ik_3^\mu X^\mu(\tau^1,\,0)}|\,0,k_4\rangle_{\rm in}\ . \tag{6.4}$$

It is to be understood that, eventually, one has to do the integral $\int{\rm d}^4k_2\int{\rm d}^4k_3\,\varepsilon(k_2)\varepsilon(k_3)\,\mathcal{A}$.

We now substitute the formula for $X^\mu(\tau,0)$, using Eq. (3.16):

$$X^\mu(\tau,0)=x^\mu+p^\mu\tau+\sum_{n\neq 0}\frac{i}{n}\alpha_n^\mu\,e^{-in\tau}\ . \tag{6.5}$$

However, there is a problem: X^μ contains pieces that do not commute. In particular, the expressions for X^- give problems, since it contains α_n^-, which is quadratic in the α_m^i, and as such obeys the more complicated commutation rules. We simplify our problem by limiting ourselves to the case $k^\pm=0$. Thus, k_2^μ and k_3^μ only contain $k^{\rm tr}$ components. It simplifies our problem in another way as well: $H^{\rm int}$ now does not depend on x^-, so that p^+ is conserved. Therefore, we may continue to treat p^+ as a c-number.

$X^{\rm tr}$ contain parts that do not commute:

$$[\,\alpha_m^i,\,\alpha_n^j]=m\delta^{ij}\delta_{m+n}\ , \tag{6.6}$$

but what we have at the right hand side is just a c-number. Now, since we insist we want only finite, meaningful expressions, we wish to work with sequences of α_n^i operators that have the annihilation operators $(n>0)$ to the right and creation operators $(n<0)$ to the left. We can write

$$X^i(\tau,0)=x^i+p^i\tau+A^i+A^{i\dagger}\ , \tag{6.7}$$

where A^i contains the annihilation operators $(n>0)$ and $A^{i\dagger}$ the creation operators. Within each of these four terms there are quantities that all commute, but one term does not commute with all others. The commutators are all c-numbers.

Operators A and B whose commutator is a c-number, obey the following equations:

$$e^{A+B}=e^A\,e^B\,e^{\frac{1}{2}[B,A]}=e^B\,e^A\,e^{-\frac{1}{2}[B,A]} \tag{6.8}$$

One can prove this formula by using the Campbell-Baker Hausdorff formula, which expresses the remainder as an infinite series of commutators; here the series terminates because the first commutator is a c-number, so that all subsequent commutators in the series vanish. One can also prove the formula in several other ways, for instance by series expansions. Thus, we can write the transverse contributions to our exponentials as

$$e^{ik^{\rm tr}X^{\rm tr}}=e^{ik^{\rm tr}(A^{\rm tr})^\dagger}\cdot e^{ik^{\rm tr}A^{\rm tr}}\cdot e^{ik^{\rm tr}x^{\rm tr}-\frac{1}{2}(k^{\rm tr})^2\,[A,\,A^\dagger]}\ . \tag{6.9}$$

The last exponent is just a c-number, and the first two are now in the correct order. The point to be stressed here is that the c-number diverges:

$$[A,\,A^\dagger]=\sum_{n,m>0}[\frac{i}{m}\alpha_m,\,\frac{-i}{n}\alpha_{-n}]=\sum_{n=1}^\infty\frac{1}{n}\ , \tag{6.10}$$

so we should not have started with the Hamiltonian (6.1), but with one where the exponentials are normal-ordered from the start:

$$H^{\rm int}=\sum_{k^\mu}\varepsilon(k)\,e^{-ik^\mu x^\mu+ik^{\rm tr}A^{\rm tr\,\dagger}}\,e^{ik^{\rm tr}A^{\rm tr}}\ ; \tag{6.11}$$

we simply absorb the divergent c-number in the definition of $\varepsilon(k)$. Finally, we use the same formula (6.8) to write

$$e^{ik^{\rm tr}(x^{\rm tr}+p^{\rm tr}\tau)}=e^{ik^{\rm tr}p^{\rm tr}\tau}\,e^{ik^{\rm tr}x^{\rm tr}}\,e^{-\frac{1}{2}i(k^{\rm tr})^2\tau} \tag{6.12}$$

(Note that, here, $k^{\rm tr}$ are c-numbers, whereas $x^{\rm tr}$ and $p^{\rm tr}$ are operators)

Using the fact that

$$\langle 0|\,A^\dagger=0\ \rightarrow\ \langle 0|\,e^{ik^{\rm tr}A^{\rm tr\,\dagger}}=\langle 0|\qquad\text{and}\qquad A\,|\,0\rangle=0\ \rightarrow\ e^{ik^{\rm tr}A^{\rm tr}}|\,0\rangle=|\,0\rangle\ , \tag{6.13}$$

Eq. (6.4) now becomes

$$\begin{aligned}\mathcal{A}=\int_{-\infty}^\infty {\rm d}\tau^1\int_0^\infty {\rm d}\tau\ {}_{\rm out}\langle 0,\,-k_1|\,&e^{ik_2^{\rm tr}p^{\rm tr}(\tau^1+\tau)}\,e^{ik_2^{\rm tr}x^{\rm tr}}\,e^{-\frac{1}{2}i(k_2^{\rm tr})^2(\tau^1+\tau)}\,e^{ik_2^{\rm tr}A^{\rm tr}(\tau^1+\tau)}\\ &e^{ik_3^{\rm tr}A^{\rm tr}(\tau^1)^\dagger}\,e^{\frac{1}{2}i(k_3^{\rm tr})^2\tau^1}\,e^{ik_3^{\rm tr}x^{\rm tr}}\,e^{ik_3^{\rm tr}p^{\rm tr}\tau^1}|\,0,k_4\rangle_{\rm in}\ .\end{aligned} \tag{6.14}$$

Again, using (6.13), together with (6.8), we can write

$$\langle 0|\,e^{ik_2^{\rm tr}A^{\rm tr}(\tau^1+\tau)}\,e^{ik_3^{\rm tr}A^{\rm tr}(\tau^1)^\dagger}\,|\,0\rangle=e^{-(k_2^ik_3^j)[A^i(\tau^1+\tau),\,A^j(\tau^1)^\dagger]}\ , \tag{6.15}$$

where the commutator is

$$[A^i(\tau),\,A^j(0)^\dagger]\ =\ \sum_{n=1}^\infty e^{-in\tau}\tfrac{1}{n}\,\delta^{ij}\ =\ -\ln(1-e^{-i\tau})\delta^{ij}\ , \tag{6.16}$$

so that (6.15) becomes

$$(1-e^{-i\tau})^{\,k_2^{\rm tr}k_3^{\rm tr}}\ . \tag{6.17}$$

Since the initial state is a momentum eigen state, the operator $p^{\rm tr}$ just gives the momentum $-k_1^{\rm tr}$, whereas the operator e^{ik_3x} replaces k_4 by k_4+k_3. We end up with

$$\begin{aligned}\mathcal{A} \;=\; & \delta^{D-2}(k_1+k_2+k_3+k_4)\int_{-\infty}^{\infty}{\rm d}\tau^1\int_0^{\infty}{\rm d}\tau \\ & e^{-i(k_1^{\rm tr}k_2^{\rm tr})(\tau^1+\tau)-\frac12 i(k_2^{\rm tr})^2(\tau^1+\tau)}(1-e^{-i\tau})^{\,k_2^{\rm tr}k_3^{\rm tr}}\,e^{\frac12 i(k_3^{\rm tr})^2\tau^1+ik_3^{\rm tr}k_4^{\rm tr}\tau^1}\ .\end{aligned} \tag{6.18}$$

The integral over τ^1 (in a previous version of the notes it was conveniently ignored, putting τ^1 equal to zero) actually gives an extra Dirac delta:

$$\begin{aligned}&\delta\Big(-k_1^{\rm tr}k_2^{\rm tr}-\tfrac12(k_2^{\rm tr})^2+\tfrac12(k_3^{\rm tr})^2+k_3^{\rm tr}k_4^{\rm tr}\Big)=\\ &\delta\Big(\tfrac12(k_1^{\rm tr})^2-\tfrac12(k_1^{\rm tr}+k_2^{\rm tr})^2+\tfrac12(k_3^{\rm tr}+k_4^{\rm tr})^2-\tfrac12(k_4^{\rm tr})^2\Big)\ .\end{aligned} \tag{6.19}$$

Since states (1) and (4) are ground states, $M_1^2=M_4^2$, and momentum conservation implies that the entry of the delta function reduces to $p^+(k_1^-+k_4^-)$. This is the delta function enforcing momentum conservation in the $-$-direction.

The remaining integral,

$$\int_0^{\infty}{\rm d}\tau\ e^{-\frac12 i(k_2^{\rm tr})^2\tau-i(k_1^{\rm tr}k_2^{\rm tr})\tau}(1-e^{-i\tau})^{\,k_2^{\rm tr}k_3^{\rm tr}}\ , \tag{6.20}$$

does not change if we let τ run from 0 to $-i\infty$ instead of ∞, and writing

$$e^{-i\tau}=x\qquad,\qquad {\rm d}\tau=\frac{i{\rm d}x}{x}\ , \tag{6.21}$$

we see that the integral in (6.18) is

$$i\int_0^1{\rm d}x\ x^{k_1^{\rm tr}k_2^{\rm tr}+\frac12(k_2^{\rm tr})^2-1}(1-x)^{k_2^{\rm tr}k_3^{\rm tr}}\ . \tag{6.22}$$

Let us use the Mandelstam variables[2] s and t of Eqs. (1.10) and (1.11), noting that

$$\begin{aligned}k_1k_2 \;&=\; \tfrac12\Big((k_1+k_2)^2-k_1^2-k_2^2\Big) \;=\; \tfrac12(-s-k_1^2-k_2^2)\ ;\\ k_2k_3 \;&=\; \tfrac12(-t-k_2^2-k_3^2)\ ,\end{aligned} \tag{6.23}$$

to write (6.22) as

$$iB\left(\tfrac12(-s-k_1^2),\,\tfrac12(-t-k_2^2-k_3^2+1)\right)\ , \tag{6.24}$$

where B is the beta function, Eq. (1.18). This is exactly the Veneziano formula (1.14), *provided* that $\alpha(s)=\frac12 s+a$, $\frac12 k_1^2=a$, and $\frac12(k_2^2+k_3^2)=1+a$. If we put $a=1$ and all external momenta in the same ground state, we recover Veneziano's formula exactly. The derivation may seem to be lengthy, but this is because we carefully went through all the details. It is somewhat awkward that we had to put $k_2^+=k_3^+=0$, but, since the final answer is expected to be Lorentz invariant, it is reasonable to expect it to be more generally valid. It is a very important feature of string theory that the answers only make sense as long as the external states are kept on mass shell.

[2] Note that the signs of the momenta are defined differently.

7. BRST quantization.

Modern quantization techniques often start with the functional integral. When setting this up, it usually looks extremely formal, but upon deeper studies the methods turn out to be extremely powerful, enabling one to find many different, but completely equivalent quantum mechanical expressions. In these proofs, one now uses Becchi-Rouet-Stora-Tyutin symmetry, which is a super symmetry. We here give a brief summary.

The action S of a theory is assumed to contain a piece quadratic in the field variables $A_i(x)$, together with complicated interaction terms. In principle, any quantum mechanical amplitude can be written as a functional integral over 'field configurations' $A_i(\vec{x},t)$ and an initial and a final wave function:

$$\mathcal{A} = \int \mathcal{D}A_i(\vec{x},t)\langle A_i(\vec{x},\,T)\,|\,e^{-iS\left(A(\vec{x},t)\right)}\,|\,A_i(\vec{x},\,0)\rangle\ , \tag{7.1}$$

which is written as $\int \mathcal{D}A\, e^{-iS}$ for short. However, if there is any kind of local gauge symmetry for which the action is invariant (such as in QED, Yang Mills theory or General Relativity), which in short-hand looks like

$$A(x) \rightarrow \Omega(x)A(x)\ , \tag{7.2}$$

then there are *gauge orbits*, large collections of field configurations $A_i(\vec{x},t)$, for which the total action does not change. Along these orbits, obviously the functional integral (7.1) does not converge. In fact, we are not interested in doing the integrals along such orbits, we only want to integrate over states which are physically distinct. This is why one needs to *fix the gauge*.

The simplest way to fix the gauge is by imposing a constraint on the field configurations. Suppose that the set of *infinitesimal* gauge transformations is described by 'generators' $\Lambda_a(x)$, where the index a can take a number of values (in YM theories: the dimensionality of the gauge group; in gravity: the dimensionality D of space-time, in string theory: 2 for the two dimensions of the string world sheet, plus one for the Weyl invariance). One chooses functions $f_a(x)$, such that the condition

$$f_a(x) = 0 \tag{7.3}$$

fixes the choice of gauge — assuming that all configurations can be gauge transformed such that this condition is obeyed. Usually this implies that the index a must run over as many values as the index of the gauge generators.

In perturbation expansion, we assume $f_a(x)$ at first order to be a linear function of the fields $A_i(x)$ (and possibly its derivatives). Also the gauge transformation is linear at lowest order:

$$A_i \rightarrow A_i + \hat{T}_i^a\Lambda_a(x)\ , \tag{7.4}$$

where $\Lambda_a(x)$ is the generator of infinitesimal gauge transformations and $\hat{T}_i^a$ may be an operator containing partial derivatives. If the gauge transformations are also linear in first order, then what one requires is that the combined action,

$$f_a(A,x) \rightarrow f_a(A+\hat{T}\,\Lambda,\,x) = f_a(x) + \hat{m}_a^b\,\Lambda_b(x)\ , \tag{7.5}$$

is such that the operator $\hat{m}_a^b$ has an inverse, $(\hat{m}^{-1})_a^b$. This guarantees that, for all A, one can find a Λ that forces f to vanish.

However, $\hat{m}$ might have zero modes. These are known as the Faddeev-Popov ghosts. Subtracting a tiny complex number, $i\varepsilon$ from $\hat{m}$, removes the zero modes, and turns the Faddeev Popov ghosts into things that look like fields associated to particles, hence the name. Now let us be more precise.

What is needed is a formalism that yields the same physical amplitudes if one replaces one function $f_a(x)$ by any other one that obeys the general requirements outlined above. In the functional integral, one would like to impose the constraint $f_a(x) = 0$. The amplitude (7.1) would then read

$$\mathcal{A} \stackrel{?}{=} \int \mathcal{D}A\, \varepsilon^{-iS}\, \delta(f_a(x)) \ , \tag{7.6}$$

but this would *not* be insensitive to transitions to other choices of f_a. Compare a simple ordinary integral where there is invariance under a rotation of a plane:

$$\mathcal{A} \stackrel{?}{=} \int \mathrm{d}^2\vec{x}\, \mathrm{d}^2\vec{y}\, F(|\,x|, |\,y|)\, \delta(x^1) \ . \tag{7.7}$$

Here, the 'gauge-fixing function' $f(\vec{x}, \vec{y}) = x^1$ removes the rotational invariance. Of course this would yield something else if we replaced f by y^1. To remove this failure, one must add a Jacobian factor:

$$\mathcal{A} = \int \mathrm{d}^2\vec{x}\, \mathrm{d}^2\vec{y}\, F(|\,x|, |\,y|)\, |\,x|\, \delta(x^1) \ . \tag{7.8}$$

The factor $|\,x|$ arises from the consideration of rotations over an infinitesimal angle θ:

$$x^1 \rightarrow x^1 - x^2\theta \ , \qquad x^1 = 0 \ ; \quad |\,x^2| = |\,x| \ . \tag{7.9}$$

This way, one can easily prove that such integrals yield the same value if the gauge constraint were replaced, for instance, by $|\,y|\,\delta(y^1)$. The Jacobian factors are absolutely necessary.

Quite generally, in a functional integral with gauge invariance, one must include the Jacobian factor

$$\Delta = \det\left(\partial f_a(x)/\partial\Lambda_b(y)\right) \ . \tag{7.10}$$

If all operators in here were completely linear, this would be a harmless multiplicative constant, but usually, there are interaction terms, or Δ may depend on crucial parameters in some other way. How does one compute this Jacobian?

Consider an integral over complex variables ϕ_a:

$$\int \mathrm{d}\phi_a \mathrm{d}\phi^{a*} \int e^{-\phi^{a*} M_a^b \phi_b} \ . \tag{7.11}$$

The outcome of this integral should not depend on unitary rotations of the integrand ϕ_a. Therefore, we may diagonalize the matrix M:

$$M_a^b \phi_b^{(i)} = \lambda^{(i)} \phi_a^{(i)} \ . \tag{7.12}$$

One then reads off the result:

$$\text{Eq. (7.11)} = \prod_{(i)} \Big(\frac{2\pi}{\lambda^{(i)}}\Big) = C\,(\det(M))^{-1} = C\,\exp(\mathrm{Tr}\,\log M)\ , \tag{7.13}$$

where C now is a constant that only depends on the dimensionality of M and this usually does not depend on external factors, so it can be ignored (Note that the real part and the imaginary part of ϕ each contribute a square root of the eigenvalue λ). The advantage of the expression (7.11) is that it has exactly the same form as other expressions in the action, so computing it in practice goes just like the computation of the other terms. We obtained the *inverse* of the determinant, but that causes no difficulty: we add a minus sign for every contribution of this form whenever it appears in an exponential form:

$$\det(M)^{-1} = \exp(-\mathrm{Tr}\,\log M)\ . \tag{7.14}$$

Alternatively, one can observe that such minus signs emerge if we replace the bosonic 'field' $\phi_a(x)$ by a *fermionic* field $\eta_a(x)$:

$$\det(M) = \int \mathcal{D}\eta\mathcal{D}\bar\eta\,\exp(\bar\eta^a M_a^b \eta_b)\ . \tag{7.15}$$

Indeed, this identity can be understood directly if one knows how to integrate over anticommuting variables (called Grassmann variables) η_i, which are postulated to obey $\eta_i\,\eta_j = -\eta_j\,\eta_i$:

$$\int \mathrm{d}\eta\,1 = 0\ ; \qquad \int \mathrm{d}\eta\,\eta = 1\ . \tag{7.16}$$

For a single set of such anticommuting variables η, $\bar\eta$, one has

$$\eta^2 = 0\ ; \quad \bar\eta^2 = 0\ ; \qquad \exp(\bar\eta M\eta) = 1 + \bar\eta M\eta\ ; \qquad \int \mathrm{d}\eta\mathrm{d}\bar\eta(1+\bar\eta M\eta) = M\ . \tag{7.17}$$

In a gauge theory, for example, one has

$$A_\mu^a \ \rightarrow\ A_\mu^a + D_\mu\Lambda_a\ ; \qquad D_\mu X^a = \partial_\mu X^a + g\varepsilon_{abc}A_\mu^b X^c\ ; \tag{7.18}$$

$$f_a(x) \ =\ \partial_\mu A_\mu^a(x)\ ;$$

$$f_a \ \rightarrow\ f_a + \partial_\mu D_\mu \Lambda_a\ ; \tag{7.19}$$

$$\Delta \ =\ \int \mathcal{D}\eta(x)\mathcal{D}\bar\eta(x)\ \exp(\bar\eta^a\,\partial_\mu D_\mu \eta_a(x))\ . \tag{7.20}$$

The last exponential forms an addition to the action of the theory, called the *Faddeev-Popov action*. Let us formally write

$$S = S^{\mathrm{inv}}(A) + \lambda^a(x) f_a(x) + \bar\eta\frac{\partial f}{\partial\Lambda}\eta\ . \tag{7.21}$$

Here, $\lambda^a(x)$ is a *Lagrange multiplier field*, which, when integrated over, enforces $f^a(x) \rightarrow 0$. In gauge theories, infinities may occur that require renormalization. In that case, it is important to check whether the renormalization respects the gauge structure of the theory.

By Becchi, Rouet and Stora, and independently by Tyutin, this structure was discovered to be a symmetry property relating the anticommuting ghost field to the commuting gauge fields: a super symmetry. It is the symmetry that has to be respected at all times:

$$\delta A^a(x) = \bar{\varepsilon}\frac{\partial A^a(x)}{\partial \Lambda^b(x')}\eta^b(x')\,; \tag{7.22}$$

$$\begin{aligned} \delta\eta^a(x) &= \tfrac{1}{2}\bar{\varepsilon} f^{abc}\eta^b(x)\,\eta^c(x)\,; \\ \delta\bar{\eta}^a(x) &= -\bar{\varepsilon}\lambda^a(x)\;; \\ \delta\lambda^a(x) &= 0\,. \end{aligned} \tag{7.23}$$

Here, f^{abc} are the *structure constants* of the gauge group:

$$[\Lambda^a,\,\Lambda^b](A) = f^{abc}\Lambda^c(A) \tag{7.24}$$

$\bar{\varepsilon}$ is infinitesimal. Eq. (7.22) is in fact a gauge transformation generated by the infinitesimal field $\bar{\varepsilon}\eta$.

8. The Polyakov path integral. Interactions with closed strings.

Two closed strings can meet at one point, where they rearrange to form a single closed string, which later again splits into two closed strings. This whole process can be seen as a single sheet of a complicated form, living in space-time. The two initial closed strings, and the two final ones, form holes in a sheet which otherwise would have the topology of a sphere. If we assume these initial and final states to be far separated from the interaction region, we may shrink these closed loops to points. Thus, the amplitude of this scattering process may be handled as a string world sheet in the form of a sphere with four points removed. These four points are called 'vertex insertions'.

More complicated interactions however may also take place. Strings could split and rejoin several times, in a process that would be analogous to a multi-loop Feynman diagram in Quantum Field Theory. The associated string world sheets then take the form of a torus or sheets with more complicated topology: there could be g splittings and rejoinings, and the associated world sheet is found to be a closed surface of *genus* g.

The Polyakov action is

$$S(h,X) = -\frac{T}{2}\int \mathrm{d}^2\sigma\sqrt{h}\,h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\mu\ . \tag{8.1}$$

To calculate amplitudes, we want to use the partition function

$$Z = \int \mathcal{D}h\mathcal{D}X\, e^{-S(h,X)}\,, \tag{8.2}$$

The conformal factor (the overall factor $e^{\phi(\sigma_1,\sigma_2)}$), is immaterial as it cancels out in the action (8.1). A constraint such as

$$\det(h) = 1\ , \tag{8.3}$$

can be imposed without further consequences (we simply limit ourselves to variables $h_{\alpha\beta}$ with this property, regardless the choice of coordinates).

But we do want to fix the reparametrization gauge, for instance by using coordinates σ^+ and σ^-, and imposing $h_{++} = h_{--} = 0$. It is here that we can use the BRST procedure. If we would insert (as was done in the previous sections)

$$1 = \int \mathcal{D}h_{++}\, \mathcal{D}h_{--}\, \delta(h_{++})\, \delta(h_{--}) \ , \tag{8.4}$$

we would find it difficult to check equivalence with other gauge choices, particularly if the topology is more complicated than the sphere.

To check the equivalence with other gauge choices, one should check the contribution of the Faddeev-Popov ghost, see the previous section. The general, infinitesimal local coordinate transformation on the world sheet is

$$\sigma^\alpha \to \sigma^\alpha + \xi^\alpha \ , \qquad f(\sigma) \to f(\sigma) + \xi^+ \partial_+ f + \xi^- \partial_- f \ . \tag{8.5}$$

As in General relativity, a *co-vector* $\phi_\alpha(\sigma)$ is defined to be an object that transforms just like the derivative of a *scalar* function $f(\sigma)$:

$$\begin{aligned} \partial_\beta f(\sigma) \quad &\to \quad \partial_\beta f(\sigma) + \xi^\lambda \partial_\lambda\, \partial_\beta f + (\partial_\beta \xi^\lambda) \partial_\lambda f \ ; \\ \phi_\beta(\sigma) \quad &\to \quad \phi_\beta(\sigma) + \xi^\lambda \partial_\lambda \phi_\beta + (\partial_\beta \xi^\lambda) \phi_\lambda \ . \end{aligned} \tag{8.6}$$

The metric $h_{\alpha\beta}$ is a *co-tensor*, which means that it transforms as the *product* of two co-vectors, which we find to be

$$h_{\alpha\beta}(\sigma) \to h_{\alpha\beta}(\sigma) + \xi^\lambda \partial_\lambda h_{\alpha\beta} + (\partial_\alpha \xi^\lambda) h_{\lambda\beta} + (\partial_\beta \xi^\lambda) h_{\alpha\lambda} \ . \tag{8.7}$$

As in General Relativity, the *co-variant derivative* of a vector field $\phi_\alpha(\sigma)$ is defined as

$$\nabla_\alpha \phi_\beta \equiv \partial_\alpha \phi_\beta - \Gamma^\gamma_{\alpha\beta}\, \phi_\gamma \ ; \qquad \Gamma^\gamma_{\alpha\beta} = \tfrac{1}{2} h^{\gamma\delta} \{ \partial_\alpha h_{\beta\delta} + \partial_\beta h_{\alpha\delta} - \partial_\delta h_{\alpha\beta} \} \ . \tag{8.8}$$

This definition is carefully arranged in such a way that $\nabla_\alpha \phi_\beta$ also transforms as a co-tensor, as defined in Eq. (8.7).

The metric $h_{\alpha\beta}$ is a special co-tensor. It so happens that the derivatives in Eq. (8.7), together with the other terms, can be rearranged in such a way that they are themselves written as covariant derivatives of $\xi_\alpha = h_{\alpha\beta} \xi^\beta$:

$$h_{\alpha\beta} \to h_{\alpha\beta} + \delta h_{\alpha\beta} \ ; \qquad \delta h_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha \ . \tag{8.9}$$

The determinant h of $h_{\alpha\beta}$ transforms as

$$\delta h = 2h\, h^{\alpha\beta} \nabla_\alpha \xi_\beta \ , \tag{8.10}$$

and since we restrict ourselves to tensors with determinant one, Eq. (8.3), we have to divide $h_{\alpha\beta}$ by $\sqrt{h}$. This turns the transformation rule (8.9) into

$$\delta h_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - h_{\alpha\beta} h^{\gamma\kappa} \nabla_\gamma \xi_\kappa \ . \tag{8.11}$$

Since the gauge fixing functions in Eq. (8.4) are h_{++} and h_{--}, our Faddeev-Popov ghosts will be the ones associated to the determinant of $\nabla_\pm$ in

$$\delta h_{++} = 2\nabla_+\xi_+ \qquad \delta h_{--} = 2\nabla_-\xi_- \ , \tag{8.12}$$

while $\delta h_{+-} = 0$. So, we should have fermionic fields $c_\pm,\ b_{++},\ b_{--}$, described by an extra term in the action:

$$\mathcal{L}^{\text{F.-P}} = -\frac{T}{2}\int \mathrm{d}^2\sigma\{\partial_\alpha X^\mu\,\partial^\alpha X^\mu + 2b^{++}\nabla_+c_+ + 2b^{--}\nabla_-c_-\ \}\ . \tag{8.13}$$

In Green, Schwarz & Witten, the indices for the c ghost is raised, and those for the b ghost are lowered, after which they interchange positions[3] So we write

$$\mathcal{L}^{\text{F.-P}} = -\frac{T}{2}\int \mathrm{d}^2\sigma\{\partial_\alpha X^\mu\,\partial^\alpha X^\mu + 2c^-\nabla_+b_{--} + 2c^+\nabla_-b_{++}\}\ . \tag{8.14}$$

The addition of this ghost field improves our formalism. Consider for instance the constraints (3.32), which can be read as $T_{++} = T_{--} = 0$. Since the α coefficients are the Fourier coeeficients of $\partial_\pm X^\mu$, we can write these conditions as

$$\Big(\sum_{\mu=1}^{D} L_m^\mu\Big)|\psi\rangle \stackrel{?}{=} 0\ , \tag{8.15}$$

where the coefficients L_m^μ are defined in Eq. (4.19). States $|\psi\rangle$ obeying this are then "physical states". Now, suppose that we did *not* impose the light-cone gauge restriction, but assume the unconstrained commutation rules (4.9) for *all* α coefficients. Then the commutation rules (4.26) would have to hold for all L_m^μ, and this would lead to contradictions unless the c-number term somehow cancels out. It is here that we have to enter the ghost contribution to the energy-momentum tensors $T_{\mu\nu}$.

8.1. The energy-momentum tensor for the ghost fields.

We shall now go through the calculation of the *ghost energy-momentum tensor* $T^{\text{gh}}_{\alpha\beta}$ a bit more carefully than in Green-Schwarz-Witten, page 127. Rewrite the ghost part of the Lagrangian (8.14) as

$$S_{\text{gh}} \stackrel{?}{=}\ -\frac{T}{2}\int \mathrm{d}^2\sigma\sqrt{h}\,h^{\alpha\beta}(\nabla_\alpha c^\gamma)b_{\beta\gamma}\ , \tag{8.16}$$

where, by partial integration, we brought Eq. (3.1.31) of Green-Schwarz-Witten in a slightly more convenient form. However, we must impose as a further constraint the fact that $b_{\alpha\beta}$ is traceless. For what comes next, it is imperative that this condition is also

[3] Since the determinant of a matrix is equal to that of its mirror, and since raising and lowering indices does not affect the action of the covariant derivative ∇, these changes are basically just notational, and they will not affect the final results. One reason for replacing the indices this way is that now the ghosts transform in a more convenient way under a Weyl rescaling.

expressed in Lagrange form. The best way to do this is by adding an extra Lagrange multiplier field c:

$$S_{\rm gh}=-\frac{T}{2}\int{\rm d}^2\sigma\sqrt{h}\,\Big(h^{\alpha\beta}(\nabla_\alpha c^\gamma)b_{\beta\gamma}+h^{\alpha\beta}\,c\,b_{\alpha\beta}\Big)\ . \tag{8.17}$$

This way, we can accept all variations of $b_{\alpha\beta}$ that leave it symmetric ($b_{\alpha\beta}=b_{\beta\alpha}$). Integration over c will guarantee that $b_{\alpha\beta}$ will eventually be traceless.

After this reparation, we can compute $T^{\rm gh}_{\alpha\beta}$. The energy-momentum tensor is normally defined by performing infinitesimal variations of $h_{\alpha\beta}$:

$$h_{\alpha\beta}\rightarrow h_{\alpha\beta}+\delta h_{\alpha\beta}\ ;\qquad S\rightarrow S+\frac{T}{2}\int{\rm d}^2\sigma\,\sqrt{h}\,T^{\alpha\beta}\delta h_{\alpha\beta}\ . \tag{8.18}$$

Since we vary $h^{\alpha\beta}$ but not $b_{\alpha\beta}$, this may only be done in the complete ghost Lagrangian (8.17). Using the following rules for the variations:

$$\begin{aligned}\delta(h^{\alpha\beta})&=-\delta h^{\alpha\beta}\ ;\qquad \delta\sqrt{h}=\tfrac12\sqrt{h}\,\delta h^\alpha_\alpha\ ;\\ \delta\Gamma^\gamma_{\alpha\beta}&=\tfrac12(\nabla_\alpha\delta h^\gamma_\beta+\nabla_\beta\delta h^\gamma_\alpha-\nabla^\gamma\delta h_{\alpha\beta})\ ,\end{aligned} \tag{8.19}$$

so that

$$\delta(\nabla_\alpha c^\gamma)=(\delta\Gamma^\gamma_{\alpha\kappa})c^\kappa\ , \tag{8.20}$$

one finds:

$$\begin{aligned}\delta S=\frac{-T}{2}\int{\rm d}^2\sigma\,\sqrt{h}\,\Big(\delta h_{\kappa\lambda}\Big\{\tfrac12 h^{\kappa\lambda}\Big((\nabla_\alpha c^\gamma)\,b^\alpha_\gamma+c\,b^\alpha_\alpha\Big)-(\nabla^\kappa c^\gamma)b^\lambda_\gamma-c\,b^{\kappa\lambda}\Big\}&\\ +\tfrac12(\nabla_\lambda\delta h_{\gamma\alpha})c^\lambda\,b^{\alpha\gamma}\Big)&\ ,\end{aligned} \tag{8.21}$$

where indices were raised and lowered in order to compactify the expression. For the last term, we use partial integration, writing it as $-\frac12\delta h_{\gamma\alpha}\nabla_\lambda(c^\lambda\,b^{\alpha\gamma})$, to obtain[4]

$$T^{\rm gh}_{\alpha\beta}=\tfrac12 c^\gamma\nabla_\gamma b_{\alpha\beta}+\tfrac12(\nabla_\alpha c^\gamma)b_{\beta\gamma}+\tfrac12(\nabla_\beta c^\gamma)b_{\alpha\gamma}-\tfrac12 h_{\alpha\beta}(\nabla_\kappa c^\gamma)b^\kappa_\gamma\ . \tag{8.22}$$

Here, to simplify the expression, we made use of the equations of motion for the ghosts:

$$\begin{aligned}b^\alpha_\alpha=0\ ,&\qquad \nabla^\alpha b_{\alpha\beta}=0\ ,\\ \nabla_\alpha c_\beta+\nabla_\beta c_\alpha+2c\,h_{\alpha\beta}=0\ ,&\qquad c=-\tfrac12\nabla_\lambda c^\lambda\ .\end{aligned} \tag{8.23}$$

Then last term in Eq. (8.22) is just what is needed to make $T^{\rm gh}_{\alpha\beta}$ traceless. The fact that it turns out to be traceless only *after* inserting the equations of motion (8.23) has to do with the fact that conformal invariance of the ghost Lagrangian is a rather subtle feature that does not follow directly from its Lagrangian (8.17).

[4]Note that the covariant derivative ∇ has the convenient property that $\int{\rm d}^2\sigma\,\sqrt{h}\,\nabla_\alpha F^\alpha(\sigma)=$ boundary terms, if F^α transforms as a contra-vector, as it normally does.

Since, according to its equation of motion, $\nabla_- b_{++} = 0$ and $b_{+-} = 0$, we read off:

$$T_{++} = \tfrac{1}{2} c^+ \nabla_+ b_{++} + (\nabla_+ c^+) b_{++} \ , \tag{8.24}$$

and similarly for T_{--}, while $T_{+-} = 0$.

The Fourier coefficients for the energy momentum tensor ghost contribution, L_m^{ghost}, can now be considered. In the quantum theory, one then has to promote the ghost fields into operators obeying the anti-commutation rules of fermionic operator fields. When these are inserted into the expression for L_m^{ghost}, they *must be put in the correct order*, that is, creation operator at the left, annihilation operator at the right. Only this way, one can assure that, when acting on the lost energy states, these operators are finite.

Subsequently, the commutation rules are then derived. They are found to be

$$[L_m^{\text{ghost}},\, L_n^{\text{ghost}}] = (m-n) L_{m+n}^{\text{ghost}} + \Big(-\frac{26}{12}\, m^3 + \frac{2}{12} m \Big) \delta_{m+n} \ . \tag{8.25}$$

The L_m^μ associated to X^μ all obey the commutation rules (4.26). The total constraint operator associated with energy-momentum is generated by *three* contributions:

$$L_m^{\text{tot}} = \sum_{\mu=1}^{D} L_m^\mu + L_m^{\text{ghost}} - a\delta_{m0} \ , \tag{8.26}$$

and since for different μ these L_m^μ obviously commute, we find the algebra

$$[L_m^{\text{tot}},\, L_n^{\text{tot}}] = (m-n) L_{m+n}^{\text{tot}} + \Big(\frac{D-26}{12}\, m^3 - \frac{D-2-24a}{12}\, m \Big)\, \delta_{m+n} \ , \tag{8.27}$$

so that the constraint

$$L_m^{\text{tot}} |\psi\rangle = 0 \tag{8.28}$$

can be obeyed by a set of "physical states" only if $D = 26$ and $a = 1$. The ground state $|0\rangle$ has $L_0^{\text{tot}}|0\rangle = 0 = L_0^{\text{ghost}}|0\rangle$, so, for instance in the open string,

$$-\tfrac{1}{8} M^2 = \tfrac{1}{8} p_\mu^2 = \sum_{\mu=1}^{D} L_0^\mu = a = 1 \ , \tag{8.29}$$

which leads to the same result as in light-cone quantization: the ground state is a tachyon since $a = 1$, and the number of dimensions D must be 26.

9. T-Duality.

(this chapter was copied from Amsterdam lecture notes on string theory)
Duality is an invertible map between two theories sending states into states, while preserving the interactions, amplitudes and symmetries. Two theories that are dual to one and another can in some sense be viewed as being physically identical. In some special cases a theory can be dual to itself. An important kind of duality is called T-duality, where 'T' stands for 'Target space', the D-dimensional space-time. We can map one target space into a different target space.

9.1. Compactifying closed string theory on a circle.

To rid ourselves of the 22 surplus dimensions, we imagine that these extra dimensions are 'compactified': they form a compact space, typically a torus, but other possibilities are often also considered. To study what happens in string theory, we now compactify one dimension, say the last spacelike dimension, X^{25}. Let it form a circle with circumference $2\pi R$. This means that a displacement of all coordinates X^{25} into

$$X^{25\prime} \to X^{25} + 2\pi R \ , \tag{9.1}$$

sends all states into the same states. R is a free parameter here. Since the quantum wave function has to return to itself, the displacement operator $U = \exp(ip_{25} 2\pi R)$ must have the value 1 on these states. Therefore,

$$p_{25} = n/R \ , \tag{9.2}$$

where n is any integer. For a closed string, $\alpha_0^\mu + \tilde{\alpha}_0^\mu$ is usually identified with p^μ (when ℓ is normalized to one), therefore, we have in Eq. (3.22),

$$p^{25} = \alpha_0^{25} + \tilde{\alpha}_0^{25} = n/R \ . \tag{9.3}$$

This would have been the end of the story if we had been dealing with particle physics: p^{25} is quantized. But, in string theory, we now have other modes besides these. Let us limit ourselves first to closed strings. A closed string can now also wind around the periodic dimension. If the function X^{25} is assumed to be a continuous function of the string coordinate σ, then we may have

$$X^{25}(\sigma + \pi, \tau) = X^{25}(\sigma, \tau) + 2\pi m R \ , \tag{9.4}$$

where m is any integer. Therefore, the general closed string solution, Eq. (3.22) must be replaced by

$$X^{25} = x^{25} + p^{25}\tau + 2mR\,\sigma + \text{oscillators} \ . \tag{9.5}$$

Since we had split the solution in right- and left movers, $X^\mu = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma)$, and

$$\begin{aligned} \frac{\partial X_R^\mu(\tau)}{\partial \tau} &= \sum_n \alpha_n^\mu \, e^{-2in\tau} \ , \\ \frac{\partial X_L^\mu(\tau)}{\partial \tau} &= \sum_n \tilde{\alpha}_n^\mu \, e^{-2in\tau} \ , \end{aligned} \tag{9.6}$$

we find that

$$\begin{aligned} \alpha_0^{25} &= \tfrac{1}{2} p^{25} - mR &&= \frac{n}{2R} - mR \ , \\ \tilde{a}_0^{25} &= \tfrac{1}{2} p^{25} + mR &&= \frac{n}{2R} + mR \ , \end{aligned} \tag{9.7}$$

where both m and n are integers.

The constraint equations, Eqs. (4.30) for the closed string for $n = 0$ now read

$$\begin{aligned} \tfrac{1}{2}p^- &= \frac{1}{p^+}\Big(\tfrac{1}{4}(p^{\rm tr})^2 + (\alpha_0^{25})^2 + 2\sum_{i,\,k>0} \alpha_k^{i\,\dagger}\alpha_k^i - 2a\Big) \\ &= \frac{1}{p^+}\Big(\tfrac{1}{4}(p^{\rm tr})^2 + (\tilde\alpha_0^{25})^2 + 2\sum_{i,\,k>0} \tilde\alpha_k^{i\,\dagger}\tilde\alpha_k^i - 2a\Big)\ . \end{aligned} \tag{9.8}$$

If we define M to be the mass in the non-compact dimensions, then

$$\begin{aligned} M^2 = -\sum_{n=0}^{24} p^\mu p^\mu &= \frac{n^2}{R^2} + 4\,m^2R^2 - 4nm + 8\sum \alpha_k^{i\,\dagger}\alpha_k^i - 8a \\ &= \frac{n^2}{R^2} + 4\,m^2R^2 + 4nm + 8\sum \tilde\alpha_k^{i\,\dagger}\tilde\alpha_k^i - 8a\ . \end{aligned} \tag{9.9}$$

Note that the occupation numbers of the left-oscillators will differ from those of the right-oscillators by an amount nm, fortunately an integer.

If we wish, we can now repeat the procedure for any other number of dimensions, to achieve a compactification over multi-dimensional tori.

9.2. *T*-duality of closed strings.

Eq. (9.7) exhibits a peculiar feature. If we make the following replacements:

$$\begin{aligned} &R \leftrightarrow 1/2R\ ; \qquad && n \leftrightarrow m\ ; \\ &\alpha_0^{25} \leftrightarrow -\alpha_0^{25}\ ; \qquad && \tilde\alpha_0^{25} \leftrightarrow \tilde\alpha_0^{25}\ , \end{aligned} \tag{9.10}$$

the equations continue to hold. The factor 2 is an artifact due to the somewhat awkward convention in Green, Schwarz & Witten to take π instead of 2π as the period of the σ coordinate. This is 'T-duality'. The theories dual to one another are:

- Bosonic string theory, compactified on a circle of radius R, describing a string of momentum quantum number n and winding number m, and
- Bosonic string theory, compactified on a circle of radius $1/2R$, describing a string of momentum quantum number m and winding number n.

The precise prescription for the map is deduced from the relations (9.10): it requires the 'one-sided parity transformation'

$$X_R^{25} \leftrightarrow -X_R^{25}\ ; \qquad X_L^{25} \leftrightarrow X_L^{25}\ . \tag{9.11}$$

Not only do the theory and its dual have exactly the same mass spectrum, as can be deduced from Eq. (9.9), but all other conceivable properties match.

It is generally argued, that, because of this duality, theories with compactification radius $R < 1/\sqrt{2}$ make no sense physically; they are identical to theories with R greater than that. Thus, there is a minimal value for the compactification radius.

9.3. *T*-duality for open strings.

Consider now a theory with open strings. Although higher order interactions were not yet discussed in these lectures, we can see how they may give rise to the emergence of closed strings in a theory of open strings. If strings can join their end points, they can also join both ends to form a closed loop. This is indeed what the intermediate states look like when higher order effects are calculated. The closed string sector now allows for the application of T-duality there. So the question can be asked: can we identify a duality transformation for open strings? At first sight, the answer seems to be no. But we can ask what kind of 'theory' we do get when applying a T-duality transformation on an open string.

The open string also contains left- and right going modes, but they are described by one and the same function, $X_L^\mu(\tau) = X_R^\mu(\tau)$, see Eq. (3.9) Let us now write instead:

$$\begin{aligned} X_L^{25}(\tau) &= \tfrac{1}{2}(x^{25}+c) + \tfrac{1}{2}p^{25}\tau + \tfrac{1}{2}i\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}\,e^{-in\tau}\ ; \\ X_R^{25}(\tau) &= \tfrac{1}{2}(x^{25}-c) + \tfrac{1}{2}p^{25}\tau + \tfrac{1}{2}i\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}\,e^{-in\tau}\ , \end{aligned} \tag{9.12}$$

where the number c appears to be immaterial, as it drops out of the sum

$$X^{25}(\sigma,\tau) = X_L^{25}(\tau+\sigma) + X_R^{25}(\tau-\sigma)\ , \tag{9.13}$$

and x^{25} is still the center of mass in the 25-direction.

Let us assume X^{25} to be periodic with period $2\pi R$. Thus, p^{25} is quantized:

$$p^{25} = n/R\ , \tag{9.14}$$

Let us again perform the transformation that appears to be necessary to get the 'dual theory': $X_R^{25} \leftrightarrow -X_R^{25}$. The resulting field X^{25} is

$$\tilde{X}^{25}(\sigma,\tau) = X_L^{25}(\tau+\sigma) - X_R^{25}(\tau-\sigma) = c + p^{25}\sigma + \sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-in\tau}\sin n\sigma\ . \tag{9.15}$$

of course all other coordinates remain the same as before. We see two things:

i There is no momentum. One of the end points stays fixed at position c. Furthermore,

ii The oscillating part has sines where we previously had cosines. This means that the boundary conditions have changed.

What were Neumann boundary conditions before, now turned into Dirichlet boundary conditions: one end point stays fixed at $X^{25} = c$, the other at $X^{25} = c + \pi n/R$. This, we interpret as a string that is connected to a fixed plane $X^{25} = c$, while wrapping n times around a circle with radius $\frac{1}{2R}$.

Thus, we encountered a new kind of object: *Dirichlet branes*, or D-branes for short. They will play an important role. We found a sheet with $D-2=24$ internal space dimensions and one time dimension. We call this a D24-brane. The duality transformation can be carried out in other directions as well, so quite generally, we get Dp-branes, where $0 \le p \le 25$ is the number of internal dimensions for which there is a Neumann boundary condition (meaning that, in those directions, the end points move freely. In the other, $D-p-1$ directions, there is a Dirichlet boundary condition: those end point coordinates are fixed. T-duality interchanges Neumann and Dirichlet boundary conditions, hence, T-duality in one dimension replaces a Dp-brane into a $D\ p \pm 1$ brane. These D-branes at first sight may appear to be rather artificial objects, but when it is realized that one of the closed string solutions acts as a graviton, so that it causes curvature of space and time, allowing also for any kind of target space coordinate transformation, we may suspect that D-branes may also obtain curvature, and with that, thay might become interesting, dynamical objects, worth studying.

9.4. Multiple branes.

The two end points of an open string may be regarded as living in a D25-brane. We now consider a generalization of string theory that we could have started off from right at the beginning: consider a set of N D25-branes. The end points of a string can sit in any one of them. Thus, we get an extra quantum number $i=1,\cdots,N$, associated to each end point of a string. In QCD, this was done right from the beginning, in order to identify the quantum numbers of the quarks at the end points. The quantum numbers i were called *Chan-Paton* factors. All open strings are now regarded as $N\times N$ matrices, transforming as the N^2 dimensional adjoint representation of $U(N)$. We write these states as $|\,\psi\rangle\otimes\lambda_{ij}$.

A three point scattering amplitude, with λ^a_{ij}, $a=1,2,3$ describing the asymptotic states, obtains a factor that indicates the fact that the end points each remain in their own D-brane:

$$\delta_{i\,i'}\delta^{j\,j'}\delta^{kk'}\,\lambda^1_{ij}\lambda^2_{j'k}\lambda^3_{k'i'} = \mathrm{Tr}\,\lambda^1\lambda^2\lambda^3\ . \tag{9.16}$$

in the target space theory, the $U(N)$ has the interpretation of a gauge symmetry; the massless vectors in the string spectrum will be interpreted as gauge bosons for the non-Abelian gauge symmetry $U(N)$.

9.5. Phase factors and non-coinciding D-branes.

The previous subsection described 'stacks' of D-branes. However, we might wish to describe D-branes that do not coincide in target space. Take for instance several D24 branes that are separated in the X^{25} direction. Let the D-brane with index i sit at the spot $X^{25}=\tilde R\theta_i$. An open string with labels i at its end points must have these end points fixed at these locations with a Dirichlet boundary condition. Let us now 'T-dualize' back. In Eq. (9.15), we now have to substitute

$$c=\tilde R\,\theta_i\ ;\qquad p^{25}=\frac{\tilde R}{\pi}(\theta_j-\theta_i+2\pi n)\ , \tag{9.17}$$

where n counts the number of integral windings. The T-dual of this is Eq. (9.13), where x^{25} is arbitrary and

$$p^{25} = (\theta_j - \theta_i + 2\pi n)/(2\pi R) \ , \tag{9.18}$$

where $R = 1/2\tilde{R}$. here, we see that we do not have the usual periodicity boundary condition as $X^{25} \to X^{25} + 2\pi R$, but instead, what is called a 'twisted boundary condition':

$$\psi(X^{25} + 2\pi R) \ = \ e^{ip^{25}(2\pi R)}\,\psi(X^{25}) \ = \ e^{i(\theta_j - \theta_i)}\psi(X^{25}) \ . \tag{9.19}$$

This boundary condition may arise in the presence of a gauge vector potential $A_{25,\,j}^{\ \ i}$.

A twisted boundary condition (9.19) could also occur for a single open string. In that case, the dual string will be an *oriented* string, whose two end points sit on different D-branes, a distance $\theta\tilde{R}$ apart.

10. Complex coordinates.

In modern field theory, we often perform the *Wick rotation*: write time t $(= x^0)$ as $t = ix^4$, and choose x^4 to be real instead of imaginary. This turns the Lorentz group $SO(D-1,1)$ into a more convenient $SO(D)$. We do the same thing for the τ coordinate of the string world sheet: $\sigma = \sigma^1$, $\tau = i\sigma^2$. The coordinates $\sigma^\pm$ then become complex. Define

$$\omega = \sigma^+ = \sigma^1 + i\sigma^2 \ ; \qquad \bar{\omega} = -\sigma^- = \sigma^1 - i\sigma^2 \ . \tag{10.1}$$

The reasons for the sign choices will become clear in a moment (Eqs. (10.4) and (10.6)) — but in the literature, one will discover that authors are sloppy and frequently switch their notation; there is no consensus. Since $-e^\phi \mathrm{d}\sigma^+\mathrm{d}\sigma^- = e^\phi \mathrm{d}\omega\mathrm{d}\bar{\omega}$, the conformal gauge, $h^{\alpha\beta} = -\frac{1}{2}e^\phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, now reads

$$\mathrm{d}s^2 = e^\phi \, \mathrm{d}\omega \, \mathrm{d}\bar{\omega} \ . \tag{10.2}$$

Instead of the light-cone gauge in Minkowski space, we can now map the string world sheet onto a circle in Euclidean space. *This can still be done if we consider the complete world sheet of strings joining and splitting at their end point* (as long as the resulting diagram remains simply connected; loop diagrams will require further refinements that we have not yet discussed).

In the case of the open string, we make the transition to the variables

$$z = e^{i\omega} \ , \qquad \bar{z} = e^{-i\bar{\omega}} \ , \tag{10.3}$$

so that the solution (3.15) reads

$$\frac{\mathrm{d}X_L}{\mathrm{d}\tau} = \tfrac{1}{2}\sum_n \alpha_n^\mu \Big(\frac{1}{z}\Big)^n \ , \qquad \frac{\mathrm{d}X_R}{\mathrm{d}\tau} = \tfrac{1}{2}\sum_n \alpha_n^\mu \Big(\frac{1}{\bar{z}}\Big)^n \ . \tag{10.4}$$

Eq. (10.3) is a *conformal* transformation (see below).

In the case of the closed string, we have periodicity with period π in $\sigma^{\pm}$. Therefore, it is advised to make the transition to the variables

$$z = e^{2i\omega} \ , \qquad \bar{z} = e^{-2i\bar{\omega}} \ . \tag{10.5}$$

The closed solution, Eq. (3.22), then looks like

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \sum_n \tilde{\alpha}_n^\mu \Big(\frac{1}{z}\Big)^n + \sum_n \alpha_n^\mu \Big(\frac{1}{\bar{z}}\Big)^n \tag{10.6}$$

(usually, α_n^μ and $\tilde{\alpha}_n^\mu$ are switched here).

Indices are raised and lowered as in General Relativity: $A_\mu = g_{\mu\nu} A^\nu$. Here, the metric is $h_{\alpha\beta}$, and we keep the notation $\pm$ for the indices, which now refer to z and $\bar{z}$. To stay as close as possible to the (somewhat erratic) notation of Green, Schwarz and Witten, we make the sign switch of Eq. (10.1) so,

$$t_+ = \tfrac{1}{2}\, e^\phi t^- \ ; \qquad t_- = \tfrac{1}{2}\, e^\phi t^+ \ . \tag{10.7}$$

A conformal transformation is one that keeps the form of the metric, Eq. (10.2). The most general[5] conformal transformation is

$$z \rightarrow z' = f(z) \ , \qquad \bar{z} \rightarrow \bar{z}' = \bar{f}(\bar{z}) \ , \tag{10.8}$$

or: $\partial z'/\partial \bar{z} = 0$ and $\partial \bar{z}'/\partial z = 0$. The transformation (10.5) is an example. The relation $\bar{z} = z^*$ relates the transformation of z to that of $\bar{z}$. From (10.2) we read off that the conformal factor $\varrho = e^\phi$ transforms as

$$\varrho \rightarrow \varrho' = \left| \frac{\mathrm{d}z'}{\mathrm{d}z} \right|^{-2} \varrho \ . \tag{10.9}$$

As we defined the transformation rules (8.6) for vectors and (8.7) for tensors, to be just like those for gradients, we find that an object with n_u upper and n_ℓ lower holomorphic indices $(+)$, and $\bar{n}_u$ upper and $\bar{n}_\ell$ lower anti-holomorphic $(-)$ indices, transforms as

$$t \rightarrow t' = \Big(\frac{\mathrm{d}z'}{\mathrm{d}z}\Big)^{n_u - n_\ell} \Big(\frac{\mathrm{d}\bar{z}'}{\mathrm{d}\bar{z}}\Big)^{\bar{n}_u - \bar{n}_\ell} t \ . \tag{10.10}$$

The number $n = n_\ell - n_u$ $(\bar{n} = \bar{n}_\ell - \bar{n}_u)$ is called the holomorphic (antiholomorphic) *dimension* of a tensor t.

Now let us consider the covariant derivative ∇. The Christoffel symbol Γ, Eq. (8.8), in the conformal gauge (10.2) is easily seen to have only two non vanishing components:

$$\Gamma^+_{++} = \partial_+ \phi \ , \qquad \Gamma^-_{--} = \partial_- \phi \ . \tag{10.11}$$

[5]This is not quite true; we could interchange the role of z and $\bar{z}$. Such 'antiholomorphic' transformations are rarely used.

Consequently, the covariant derivatives of a holomorphic tensor with only $+$ indices are

$$\begin{aligned}\nabla_+ t^{\cdots}_{++\cdots} &= (\partial_+ - n\,\partial_+\phi)\, t^{\cdots}_{++\cdots} \\ \nabla_- t^{\cdots}_{++\cdots} &= \partial_- t^{\cdots}_{++\cdots}\ ,\end{aligned} \tag{10.12}$$

and similarly for an antiholomorphic tensor.

We define the quantities $h = n_u - n_\ell$ and $\bar h = \bar n_u - \bar n_\ell$ as the *weights* of a tensor field $t(z,\bar z)$. Consider an infinitesimal conformal transformation:

$$(z',\, \bar z') = (z,\, \bar z) + (\xi(z),\, \bar\xi(\bar z))\ , \qquad \frac{\partial\xi}{\partial\bar z} = 0 = \frac{\partial\bar\xi}{\partial z}\ . \tag{10.13}$$

Then a field t transforms as

$$t \to t' = t + \delta_\xi t + \delta_{\bar\xi} t\ ; \qquad \delta_\xi t = \Big(\xi(z)\frac{\partial}{\partial z} + h\Big(\frac{\partial\xi(z)}{\partial z}\Big)\Big) t(z,\, \bar z)\ . \tag{10.14}$$

Since not all fields transform in this elementary way, we call such a field a *primary* field. The quantity $\Delta = h + \bar h$ is called the (holomorphic) *scaling dimension* of a field, since under a scaling transformation $z \to e^\lambda \cdot z$, a field t transforms as $t \to e^{\lambda\Delta} t$. The quantity $s = h - \bar h$ is called *spin*.

The stress tensor $T_{\alpha\beta}$ is traceless: $T_{+-} = 0$, and it obeys $\partial_\alpha T_{\alpha\beta} = 0$, or more precisely:

$$\eta^{\alpha\gamma}\partial_\alpha T_{\gamma\beta} = 0 \ \to\ \partial_- T_{++} = 0 = \partial_+ T_{--}\ . \tag{10.15}$$

Therefore, T_{++} is holomorphic. We write its Laurent expansion as a function of z :

$$T_{++} = T(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2}\ , \qquad L_n^\dagger = L_{-n}\ . \tag{10.16}$$

Conversely, using Cauchy's formula, we can find the coefficients L_n out of $T(z)$:

$$L_n = \oint \frac{\mathrm{d}z}{2\pi i} z^{n+1} T(z)\ . \tag{10.17}$$

The T field will turn out not to transform as a primary field.

11. Fermions in strings.

11.1. Spinning point particles.

For a better understanding of strings it is useful to handle conventional particle theories in a similar manner. So let us concentrate on a point particle on its 'world line", the geodesic. The analogue of the Nambu-Goto action (2.9) is the geodesic action (2.6), or

$$S = -\int \mathrm{d}\tau\, m\sqrt{-(\partial_\tau x^\mu)^2} \tag{11.1}$$

(Note that $(\partial_\tau x^\mu)^2 = -\partial_\tau x^{0\,2} + \partial_\tau x^{i\,2} < 0$). The equivalent of the Polyakov action (2.18):

$$S = \int \mathrm{d}\tau \tfrac{1}{2}\Big(\frac{(\partial_\tau x^\mu)^2}{e} - e\, m^2\Big) \ . \tag{11.2}$$

In this action, $e(\tau)$ is a degree of freedom equivalent to $\sqrt{h}$ or $-h_{00}/\sqrt{h}$. Its equation of motion is

$$e = \frac{1}{m}\sqrt{-(\partial_\tau x^\mu)^2} \ , \tag{11.3}$$

where the sign was chosen such that it matches (2.6) and (11.1)

When there are several kinds of particles in different quantum states, we simply replace m^2 in (11.2) by an operator-valued quantity M^2. Its eigenvalues are then the mass-squared of the various kinds of particles.

If these particle have spin, it just means that M^2 is built out of operators that transform non-trivially under the rotation group. However, if we want our theory to be Lorentz invariant, these operators should be built from Lorentz vectors and tensors. They will contain Lorentz indices μ that range from 1 to 4. Now, the Lorentz group is non-compact, and its complete representations are therefore infinite-dimensional. Our particle states should only be *finite* representations of the *little group*, which is the subset of Lorentz transformations that leave $\partial_\tau x^\mu$ unchanged. So, if we have operators A_μ, then in order to restrict ourselves to the physically acceptable states, we must impose constraints on the values of $(\partial_\tau x^\mu)A_\mu$.

In the case of spin $\frac{1}{2}$, the vector operators one wants to use are the γ^μ, and, since $p^\mu = (m/e)\,\partial_\tau x^\mu$, the constraint equation for the states $|\,\psi\rangle$ is

$$(i\,\partial_\tau x^\mu\,\gamma^\mu + m\,e)|\,\psi\rangle = 0 \ . \tag{11.4}$$

We make contact to string theory if, here, $m\,e$ is replaced by operators that also obey (anti-)commutation rules with the gamma's. We'll see how this happens; there will be more than one set of gamma's. The conclusion from this subsection is that modes with spin, either integer or half-odd integer, can be introduced in string theory if we add *anticommuting variables* γ^μ to the system, which then will have to be subject to constraints.

11.2. The fermionic Lagrangian.

This part of the lecture notes is far from complete. The student is advised to read the different views displayed in the literature.

In principle, there could be various ways in which we can add fermions to string theory. A natural attempt would be to put them at the end points of a string, just the way strings are expected to emerge in QCD: strings with quarks and antiquarks at their end points. Apart from anomalies that are then encountered in higher order loop corrections (not yet discussed here), one problem is quite clear from what we have seen already: the tachyonic mode. There is an ingenious way to get rid of this tachyon by using 'world sheet fermions' that exist also in the bulk of the string world sheet. As we shall see, it is super symmetry on the world sheet that will then save the day. The resulting space-time picture will also exhibit supersymmetry, so, if it is QCD that one might try to reproduce, it will be QCD with gaugino's present: fermions in the adjoint representation. In QCD, such fermions are attached to two string end points, so, these fermions will indeed reside between the boundaries, not at the boundaries, of the string world sheet.

We shall introduce them in a very pedestrian way. The gamma matrices mentioned above will be treated as operators, and they will act just as fermionic fields in four dimensional space time, accept that they carry a Lorentz index: $\psi^\mu(\tau,\sigma)$.

Apart from that, the world sheet fermion thinks it lives only in one space-, one time dimension. Its description differs a bit from the usual four dimensional case. Our fermion fields $\psi^\mu(\tau,\sigma)$ are two-dimensional Majorana spinors, that is, *real, two component spinors*, $\begin{pmatrix}\psi_1^\mu\\ \psi_2^\mu\end{pmatrix}$. In the usual path-integral picture, these two components are anticommuting numbers: $\{\psi_i^\mu,\,\psi_j^\nu\}=0$. In two dimensions, we use just two gamma matrices instead of four. Since they differ from the usual four-dimensional ones, we call them $\varrho^\alpha,\ \alpha=1,2$. Sticking to the Green-Schwarz-Witten notation, we choose

$$\varrho^1=\begin{pmatrix}0&i\\ i&0\end{pmatrix}\ ,\qquad \varrho^2=\begin{pmatrix}0&-1\\ 1&0\end{pmatrix}\ ,\qquad \varrho^0=i\varrho^2\ , \tag{11.5}$$

obeying

$$\{\varrho^\alpha,\,\varrho^\beta\}=-2\eta^{\alpha\beta}\ . \tag{11.6}$$

Note that summation convention in Minkowski space means

$$\overline{\psi}^\mu\chi^\mu=\sum_{i=1}^{D-1}\overline{\psi}^i\chi^i-\overline{\psi}^0\chi^0\ . \tag{11.7}$$

Usually, one defines $\overline{\psi}^\mu=\psi^{\mu\dagger}\gamma^0$. Now, we leave out the $\dagger$, since ψ is real[6] (provided, of course, that the Minkowski time component is taken to be ψ^0, not ψ^4). So, we take

$$\overline{\psi}^\mu=\psi^{\mu\mathrm{T}}\varrho^0\ , \tag{11.8}$$

where $\psi^{\mu\mathrm{T}}$ stands for $(\psi_1^\mu,\,\psi_2^\mu)$ instead of $\begin{pmatrix}\psi_1^\mu\\ \psi_2^\mu\end{pmatrix}$. The Dirac Lagrangian for massless, two-dimensional Majorana fermions is then

$$\mathcal{L}=\frac{i}{2\pi}\overline{\psi}^\mu\varrho^\alpha\partial_\alpha\psi^\mu\ . \tag{11.9}$$

The factor $1/2\pi$ is only added here for later convenience; we do not have that in conventional field theory. Since ϱ^0 is antisymmetric, two different spinors ψ and χ, obey $\overline{\chi}\psi=-\psi^{\mathrm{T}}(\overline{\chi}^{\mathrm{T}})$, and since fermion fields anticommute,

$$\overline{\chi}\psi=\overline{\psi}\chi\ . \tag{11.10}$$

[6]The notation differs from other conventions. For instance, in my own field theory courses, I define the $\gamma^\mu,\ \mu=1,\cdots,4$ to be hermitean, not antihermitean like here. But then we also take $\overline{\psi}=\psi^\dagger\gamma^4$, instead of (11.8), so that, in Eq. (11.9), factors i will cancel.

Also, we have

$$\overline{\chi}\varrho^\mu\psi = -\overline{\psi}\varrho^\mu\chi \ . \tag{11.11}$$

Eq. (11.9) actually already assumes that we are in the conformal gauge. We can avoid that by writing the Lagrangian in a reparametrization-invariant way. To do this, we need the square root of the metric $h_{\alpha\beta}$ of the Polyakov action (2.18). This is the so-called 'Vierbein field' e^A_α (here, 'Zweibein' would be more appropriate):

$$h_{\alpha\beta} = \eta_{AB}\, e^A_\alpha\, e^B_\beta \ ; \qquad h^{\alpha\beta} = \eta_{AB}\, e^{A\alpha} e^{B\beta} \ . \tag{11.12}$$

As we have with the metric tensor $h_{\alpha\beta}$, we define $e^{A\alpha}$ to be the inverse of e^A_α, and we can use this matrix to turn the 'internal Lorentz indices' A, B into lower or upper external indices α, β (the internal indices are not raised or lowered, as they are always contracted with η_{AB}). We then define ϱ^A, $A = 1, 2$, or $A = 0, 1$, as in (11.5),

$$\varrho^\mu = e^{A\mu}\varrho^A \ . \tag{11.13}$$

The index 0 in ϱ^0 in Eq. (11.8) is an internal one, and we must write (11.9) as

$$\mathcal{L} = \frac{i}{2\pi}\sqrt{h}\ \overline{\psi}^\mu\, e^{A\alpha}\varrho^A\partial_\alpha\psi^\mu. \tag{11.14}$$

At first sight, this seems not to be conformally invariant, but we can require ψ to transform in a special way under conformal transformations, such that Eq. (11.9) re-emerges if we return to the conformal gauge condition [7].

There are considerable simplifications due to the fact that these 'fermions' only live in one space dimension. The Dirac equation $\varrho^\alpha\partial_\alpha\psi = 0$ can be rewritten as

$$(\varrho_+\partial_- + \varrho_-\partial_+)\psi = 0 \ , \tag{11.15}$$

where, in the basis of (11.5),

$$\begin{aligned} \varrho^+ &= \varrho^0 + \varrho^1 \ , & \varrho^- &= \varrho^0 - \varrho^1 \ ; \\ \varrho_+ &= -\tfrac{1}{2}\varrho^- = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \ , & \varrho_- &= -\tfrac{1}{2}\varrho^+ = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} \ . \end{aligned} \tag{11.16}$$

Because of these expressions, ψ^μ_1 and ψ^μ_2 are renamed as ψ^μ_- and ψ^μ_+, and (11.15) turns into

$$\partial_-\psi^\mu_+ = 0 \ , \qquad \partial_+\psi^\mu_- = 0 \ , \tag{11.17}$$

or, ψ_+ is holomorphic and ψ_- anti-holomorphic.

The energy-momentum tensor is obtained by considering a variation of $h_{\alpha\beta}$ and the associated 'Zweibein' e^A_α in the action

$$S = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma\Big(\partial_\alpha X^\mu\partial^\alpha X^\mu - i\overline{\psi}^\mu\, \varrho^\alpha\partial_\alpha\psi^\mu\Big) \ , \tag{11.18}$$

[7] The effect of this is that, in Eq. (11.14), we must replace $\sqrt{h}$ by $\sqrt[4]{h}$.

after inserting the metric tensor and the zweibein as in Eqs. (2.18) and (11.14):

$$e^A_{\ \alpha} \ \rightarrow \ e^A_{\ \alpha} + \delta e^A_{\ \alpha} \ ; \tag{11.19}$$

$$e^{A\alpha}\, e^A_{\ \beta} = \delta^\alpha_\beta \ , \quad \text{so,} \quad e^{A\alpha} \ \rightarrow \ e^{A\alpha} + \delta(e^{A\alpha}) \ ; \quad \delta(e^{A\alpha}) = -e^{B\alpha}\delta e^B_{\ \beta} e^{A\beta} \ . \tag{11.20}$$

This first leads to

$$\delta S = \frac{1}{\pi}\int \mathrm{d}^2\sigma\sqrt{h}\, T^{A\alpha}\delta e^A_{\ \alpha} \ , \qquad T^{A\gamma} = e^{A\beta}h^{\gamma\alpha}T_{\alpha\beta}$$
$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\mu - \tfrac{1}{2} i\overline{\psi}^\mu \varrho_\beta \partial_\alpha \psi^\mu \ - \tfrac{1}{2}\eta_{\alpha\beta}(\mathrm{Trace}) \tag{11.21}$$

This is not yet the symmetric $T_{\alpha\beta}$ of Eq. (4.1.14) in Green, Schwarz and Witten[8] The antisymmetric part of $T_{\alpha\beta}$, however, is the generator of internal lorentz transformations, and since the theory is invariant under those, it should vanish after inserting the equations of motion. Indeed, from Eqs. (11.17) and (11.16) we see that $\varrho_+\partial_-\psi^\mu = 0$ and $\varrho_-\partial_+\psi^\mu = 0$, separately. So, Eq. (11.21) can be rewritten as[9]

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\mu - \tfrac{1}{4} i\overline{\psi}^\mu \varrho_\alpha \partial_\beta \psi^\mu - \tfrac{1}{4} i\overline{\psi}^\mu \varrho_\beta \partial_\alpha \psi^\mu \ - \tfrac{1}{2}\eta_{\alpha\beta}(\mathrm{Trace}) \tag{11.22}$$

Plugging in the expressions (11.5) and (11.16) for the ϱ-matrices, we get[10]

$$\begin{aligned} T_{++} &= \partial_+ X^\mu \partial_+ X^\mu + \tfrac{1}{2} i\, \psi^\mu_+ \partial_+ \psi^\mu_+ \ ; \\ T_{--} &= \partial_- X^\mu \partial_- X^\mu + \tfrac{1}{2} i\, \psi^\mu_- \partial_- \psi^\mu_- \ . \end{aligned} \tag{11.23}$$

These are exactly the energy and momentum as one expects for a fermionic field, analogous to the 3+1 dimensional case.

11.3. Boundary conditions.

According to the variation principle, the total action should be stationary under infinitesimal variations $\delta\psi^\mu_\pm$ of $\psi^\mu_\pm$. The fermionic action (11.14) varies according to

$$\begin{aligned} \delta S &= \frac{i}{2\pi}\int \mathrm{d}^2\sigma\Big(\overline{\delta\psi}^\mu(\varrho\partial)\psi^\mu + \overline{\psi}^\mu(\varrho\partial)\delta\psi^\mu\Big) \\ &= \frac{i}{2\pi}\int \mathrm{d}^2\sigma\Big(\overline{\delta\psi}^\mu(\varrho\partial)\psi_\mu + \partial_\alpha(\overline{\psi}^\mu\varrho^\alpha\delta\psi^\mu) - \overline{\partial_\alpha\psi}^\mu\varrho^\alpha\delta\psi^\mu\Big) \\ &= \frac{i}{2\pi}\int \mathrm{d}^2\sigma\Big(2\overline{\delta\psi}^\mu(\varrho\partial)\psi^\mu + \partial_\alpha(\overline{\psi}^\mu\varrho^\alpha\delta\psi^\mu)\Big) \ , \end{aligned} \tag{11.24}$$

where (11.11) was used. The equation of motion, $\varrho\partial\psi^\mu = 0$, follows, as expected. But, in the case of an open string, the total integral gives a boundary contribution at the end points of the integration over the σ coordinate (the τ integration gives no boundary effects, as the variations vanish at $\tau \rightarrow \pm\infty$). Therefore, we must require

$$\overline{\psi}^\mu\varrho^1\delta\psi^\mu\Big|^{\sigma=\pi}_{\sigma=o} = (\psi^\mu_-\delta\psi^\mu_- - \psi^\mu_+\delta\psi^\mu_+)\Big|^{\sigma=\pi}_{\sigma=0} = 0 \ . \tag{11.25}$$

We have the following possibilities:

[8] Which also appears to contain a sign error.

[9] To see that the trace of the fermionic $T_{\alpha\beta}$ vanishes, the remark in the footnote under Eq1. (11.14) must be observed.

[10] The sign error mentioned in an earlier footnote disappears here.

- $\psi^\mu_+ = \pm\psi^\mu_-$ and $\delta\psi^\mu_+ = \pm\delta\psi^\mu_-$ at $\sigma = 0,\ \pi$.
- $\delta\psi^\mu_\pm(\tau,\ \sigma = 0, \pi) = 0$.

In the last case, the end points are fixed. This is the analogue of the bosonic Dirichlet condition. It would however be too restrictive in combination with the equations (11.17).

In the first case, with which we proceed, we decide to choose $\psi^\mu_+(\tau, 0) = \psi^\mu_-(\tau, 0)$, which, due to some freedom of defining the sign of the wave functions, is no loss of generality. But then, at the point $\sigma = \pi$, there are two possibilities:

- $\psi^\mu_+(\tau, \pi) = \psi^\mu_-(\tau, \pi)$ (Ramond)
- $\psi^\mu_+(\tau, \pi) = -\psi^\mu_-(\tau, \pi)$ (Neveu-Schwarz)

The equations (11.17) are now solved using exactly the same techniques as in section 3.3.1. In the Ramond case, we find periodicity in the coordinates $\sigma_\pm = \tau \pm \sigma$, but in the Neveu-Schwarz case, there is 'anti-periodicity', which is easy to accommodate:

Ramond:
$$\begin{aligned} \psi^\mu_- &= \tfrac{1}{\sqrt{2}}\sum_n d^\mu_n\, e^{-in(\tau-\sigma)}\ , \\ \psi^\mu_+ &= \tfrac{1}{\sqrt{2}}\sum_n d^\mu_n\, e^{-in(\tau+\sigma)}\ , \qquad \text{with } n \text{ integer, and } d^{\mu\dagger}_n = d^\mu_{-n}\ , \end{aligned} \tag{11.26}$$

Neveu-Schwarz:
$$\begin{aligned} \psi^\mu_- &= \tfrac{1}{\sqrt{2}}\sum_r b^\mu_r\, e^{-ir(\tau-\sigma)}\ , \\ \psi^\mu_+ &= \tfrac{1}{\sqrt{2}}\sum_r b^\mu_r\, e^{-ir(\tau+\sigma)}\ , \qquad \text{with } r+\tfrac{1}{2} \text{ integer, and } b^{\mu\dagger}_r = b^\mu_{-r}\ . \end{aligned} \tag{11.27}$$

The factors $1/\sqrt{2}$ are for later convenience.

We can invert these equations:

$$d^\mu_n = \frac{1}{\pi\sqrt{2}}\int_0^\pi \mathrm{d}\sigma\Big(\psi_+(\tau,\sigma)\, e^{in(\tau+\sigma)} + \psi_-(\tau,\sigma)\, e^{in(\tau-\sigma)}\Big) \qquad \text{(Ramond)}\ ; \tag{11.28}$$

$$b^\mu_r = \frac{1}{\pi\sqrt{2}}\int_0^\pi \mathrm{d}\sigma\Big(\psi_+(\tau,\sigma)\, e^{ir(\tau+\sigma)} + \psi_-(\tau,\sigma)\, e^{ir(\tau-\sigma)}\Big) \quad \text{(Neveu-Schwarz)}\ . \tag{11.29}$$

For *closed* strings, we have no end points but only periodicity conditions. Since the left moving modes are here independent of the right moving modes. Using the letter n for integers and r for integers plus $\frac{1}{2}$, we have for the right movers:

$$\begin{aligned} &\psi^\mu_-(\tau,\sigma) = \psi^\mu_-(\tau,\ \sigma+\pi) \ \rightarrow\ \psi^\mu_- = \tfrac{1}{\sqrt{2}}\sum_n d^\mu_n e^{-2in(\tau-\sigma)}\ , \\ \text{or}\quad &\psi^\mu_-(\tau,\sigma) = -\psi^\mu_-(\tau,\ \sigma+\pi) \ \rightarrow\ \psi^\mu_- = \tfrac{1}{\sqrt{2}}\sum_r b^\mu_r e^{-2ir(\tau-\sigma)}\ . \end{aligned} \tag{11.30}$$

Again, choosing the sign is the only freedom we have, since ψ^μ is real. Similarly for the left movers:

$$\begin{array}{ll} & \psi_+^\mu(\tau,\sigma)=\psi_+^\mu(\tau,\ \sigma+\pi) \quad\to\quad \psi_+^\mu=\frac{1}{\sqrt{2}}\sum_n \tilde{d}_n^\mu e^{-2in(\tau+\sigma)}\ , \\ \text{or} & \psi_+^\mu(\tau,\sigma)=-\psi_+^\mu(\tau,\ \sigma+\pi) \quad\to\quad \psi_+^\mu=\frac{1}{\sqrt{2}}\sum_r \tilde{b}_r^\mu e^{-2ir(\tau+\sigma)}\ . \end{array} \tag{11.31}$$

We refer to these four cases as: R-R, R-NS, NS-R and NS-NS.

11.4. Anticommutation rules

We choose a prefactor in the fermionic action:

$$\mathcal{L}=\frac{1}{2\pi}i\overline{\psi}^\mu\,\varrho\partial\,\psi^\mu\ . \tag{11.32}$$

If we quantize such a theory, the anticommutation rules will be:

$$\{\psi_A^\mu(\tau,\sigma),\,\psi_B^\nu(\tau,\sigma')\}=\pi\delta(\sigma-\sigma')\delta_{AB}\delta^{\mu\nu}\ , \tag{11.33}$$

where A and B are the spin indices, $\pm$. To see that these are the correct quantization conditions for Majorana fields, let us split up a conventional Dirac field into two Majorana fields:

$$\begin{aligned} \psi_{\rm Dirac} &= \tfrac{1}{\sqrt{2}}(\psi_1+i\psi_2)\ , \\ \overline{\psi}_{\rm Dirac} &= \tfrac{1}{\sqrt{2}}(\overline{\psi}_1-i\overline{\psi}_2)\ . \end{aligned} \tag{11.34}$$

We then usually have

$$\mathcal{L}_{\rm Dirac}=i\overline{\psi}\,\varrho\partial\,\psi\ , \tag{11.35}$$

and the usual canonical arguments lead to the anticommutation rules

$$\begin{aligned} \{\psi_A(x,t),\,\psi_B(y,t)\} &= 0\ , \\ \{\psi_A^\dagger(x,t),\,\psi_B^\dagger(y,t)\} &= 0\ ; \\ \{\psi_A(x,t),\,\psi_B^\dagger(y,t)\} &= \delta(x-y)\delta_{AB}\ . \end{aligned} \tag{11.36}$$

Now substituting (11.34), we find

$$\mathcal{L}_{\rm Majorana}=\tfrac{1}{2}i(\overline{\psi}_1\,\varrho\partial\,\psi_1+\overline{\psi}_2\,\varrho\partial\,\psi_2)\ , \tag{11.37}$$

whereas, from (11.36),

$$\begin{aligned} \{\psi_{1A}(x,t),\,\psi_{1B}(y,t)\} &= \{\psi_{2A}(x,t),\,\psi_{2B}(y,t)\} = \delta(x-y)\delta_{AB}\ ; \\ \{\psi_{1A}(x,t),\,\psi_{2B}(y,t)\} &= 0\ . \end{aligned} \tag{11.38}$$

If ψ is multiplied with a constant, then we must use the same constant in Eqs. (11.38) as in (11.37). This explains how the canonical formalism leads to Eq. (11.33) from (11.32).

We can now use Eq. (11.33) to derive the commutation rules for the coefficients d_n^μ and b_r^μ from Eqs. (11.28) and (11.29):

$$\begin{aligned}\{d_m^\mu,\, d_n^\nu\} &= \frac{\delta^{\mu\nu}}{2\pi^2}\int_0^\pi \mathrm{d}\sigma\Big(\pi\, e^{i(\tau+\sigma)(m+n)} + \pi\, e^{i(\tau-\sigma)(m+n)}\Big) \\ &= \delta^{\mu\nu}\delta_{m+n}\ ; \qquad (11.39)\\ \{b_r^\mu,\, b_s^\nu\} &= \delta^{\mu\nu}\delta_{r+s}\ . \qquad (11.40)\end{aligned}$$

Just like the coefficients of the bosonic oscillators, the d_n^μ with positive n lower the world sheet energy by an amount n, and the b_r^μ by an amount r, so if n and r are positive, they are annihilation operators. If n and r are negative, they raise the energy, so they are creation operators.

Note that the anti-commutation rules given here are the ones *prior* to imposing local gauge conditions, just like in the bosonic sector of the theory (see Section 11.8).

Note also, that the operators b_r in the Neveu-Schwarz case, raise and lower the energy (read: the mass-squared of the string modes) by *half*-integer units. Thus, we get bosons with half-integer spacings, while the fermions are all integer units apart. This strange asymmetry between fermions and bosons will be removed in Chapter 12.

11.5. Spin.

Like always with fermions, the occupation numbers can only be 0 or 1. The *zero modes*, d_0^μ, give rise to a degeneracy in the spectrum: spin. In the case of the half-integer modes, the Neveu-Schwarz sector, these zero modes do not occur. The ground state is unique: a particle with spin zero. All other modes are obtained by having operators α_n^μ and/or b_r^μ act on them. Since these are vector operators, we only get integer spin states this way. Therefore, the Neveu-Schwarz sector only describes bosonic, integer spin modes.

In the Ramond sector, however, the ground state is degenerate. There is a unique set of operators d_0^μ, obeying

$$\{d_0^\mu,\, d_0^\nu\} = \eta^{\mu\nu}\ . \qquad (11.41)$$

They do not raise or lower the energy, so they leave M^2 invariant. They will be identified with the Dirac matrices. If the Dirac matrices are chosen to obey the anticommutation rule

$$\{\Gamma^\mu,\, \Gamma^\nu\} = -2\eta^{\mu\nu}\ , \qquad (11.42)$$

then we have

$$\Gamma^\mu = i\sqrt{2}\, d_0^\mu\ . \qquad (11.43)$$

Thus, for the open string, it is the Ramond sector where the ground state is a fermion. All excited states are then fermions also, because the mode operators α_n^μ and d_n^μ all transform as vectors in Minkowski space.

For the closed string, we must have an *odd* number of such zero modes, to see fermionic states arise. Therefore, we expect the R-R and NS-NS sectors both to be bosonic, whereas the R-NS and NS-R sectors are fermionic.

11.6. Supersymmetry.

To enable us to impose the constraints, we need world sheet supersymmetry. The supersymmetry transformations are generated by a *real* anticommuting generator ε:

$$\begin{aligned} \delta X^\mu &= \bar\varepsilon\psi^\mu = \overline{\psi}^\mu\varepsilon , \\ \delta\psi^\mu &= -i(\varrho\partial)X^\mu\varepsilon \qquad \text{or equivalently,} \qquad \delta\overline{\psi}^\mu = i\bar\varepsilon\,(\varrho\partial)X^\mu . \end{aligned} \tag{11.44}$$

One quickly verifies that the total action,

$$S = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma\Big(\partial_\alpha X^\mu\partial^\alpha X^\mu - i\overline{\psi}^\mu\,\varrho^\alpha\partial_\alpha\psi^\mu\Big) , \tag{11.45}$$

is left invariant (use partial integration, and $\varrho^\alpha\partial_\alpha\,\varrho^\beta\partial_\beta = -\eta^{\alpha\beta}\partial_\alpha\partial_\beta$, see Eq. (11.6)).

This supersymmetry is a global symmetry. If ε were τ,σ-dependent, we would have

$$\delta S = \frac{-2}{\pi}\int \mathrm{d}^2\sigma(\partial_\alpha\bar\varepsilon)J^\alpha , \tag{11.46}$$

with

$$J_\alpha = -\tfrac{1}{2}\varrho^\beta\varrho_\alpha\psi^\mu\partial_\beta X^\mu \tag{11.47}$$

(the factor $-\frac{1}{2}$ being chosen for future convenience). We have

$$\partial_\alpha J^\alpha = 0 \qquad \text{and} \qquad \varrho_\alpha J^\alpha = 0 . \tag{11.48}$$

The first of these can easily be derived from the equations of motion, and the second follows from

$$\varrho^\alpha\varrho^\beta\varrho_\alpha = 0 . \tag{11.49}$$

Plugging in the expressions (11.5) and (11.16) for the ϱ-matrices, we get

$$J_+ = \psi^\mu_+\partial_+X^\mu ; \qquad J_- = \psi^\mu_-\partial_-X^\mu . \tag{11.50}$$

All other components vanish.

The supercurrent (11.47), (11.50) is closely related to the energy-momentum tensor $T_{\alpha\beta}$. Later, we shall see that they actually are super partners. So, since we already have the constraint $T_{\alpha\beta} = 0$, supersymmetry will require $J^\alpha = 0$ as well. From the discussion in subsection 11.1, we derive that this is exactly the kind of constraint needed to get a finite fermionic mass spectrum. Such a constraint, however, can only be imposed if we turn our supersymmetric theory into a *locally* supersymmetric theory. After all, the constraint $T_{\alpha\beta} = 0$ came from invariance under *local* reparametrization invariance. A detailed discussion of this can be found in Green-Schwarz-Witten, section 4.3.5.

11.7. The super current.

As for the bosonic string, we can now summarize the constraints for a string with fermions:

$$\begin{aligned} L_n &= \frac{1}{\pi}\int_0^\pi \mathrm{d}\sigma\Big(e^{in\sigma}T_{++}+e^{-in\sigma}T_{--}\Big) \ , \\ F_n &= \frac{\sqrt{2}}{\pi}\int_0^\pi \mathrm{d}\sigma\Big(e^{in\sigma}J_{+}+e^{-in\sigma}J_{-}\Big) \ , \qquad \text{(Ramond)} \\ G_r &= \frac{\sqrt{2}}{\pi}\int_0^\pi \mathrm{d}\sigma\Big(e^{ir\sigma}J_{+}+e^{-ir\sigma}J_{-}\Big) \ . \quad \text{(Neveu-Schwarz)} \end{aligned} \tag{11.51}$$

Working out the form of L_n, we find that it now also contains contributions from the fermions (by plugging the Fourier coefficients of Eqs. (11.26) and (11.27) in the fermionic part of (11.23)):

$$\begin{aligned} L_n &= L_n^{(\alpha)}+L_n^{(d)} \ , \qquad \text{(Ramond)} \\ L_n &= L_n^{(\alpha)}+L_n^{(b)} \ , \qquad \text{(Neveu-Schwarz)} \end{aligned} \tag{11.52}$$

where

$$\begin{aligned} L_n^{(\alpha)} &= \tfrac{1}{2}\sum_m :\alpha^\mu_{-m}\alpha^\mu_{n+m}: \quad , \\ L_n^{(d)} &= \tfrac{1}{2}\sum_m (m+n) :d^\mu_{-m}d^\mu_{n+m}: \qquad \text{with } m \text{ integer,} \\ L_n^{(b)} &= \tfrac{1}{2}\sum_r (r+n) :b^\mu_{-r}b^\mu_{n+r}: \qquad \text{with } r \text{ integer } +\tfrac{1}{2} \ . \end{aligned} \tag{11.53}$$

Here, we again normal-ordered the expressions (that is, removed the vacuum contributions). In the last two expressions, the terms $+n$ can be dropped, because the d's and the b's anticommute. In Green-Schwarz-Witten, due to a more symmetric expression for $T_{\alpha\beta}$, there still is the term $+\frac{1}{2}n$.

Next, we get

$$\begin{aligned} F_n &= \sum_m \alpha^\mu_{-m}d^\mu_{n+m} \ ; \\ G_r &= \sum_m \alpha^\mu_{-m}b^\mu_{r+m} \ . \end{aligned} \tag{11.54}$$

Normal ordering was not necessary here.

Let us compute the commutators. The easiest is $[L_m,\, F_n]$. We see that

$$\begin{aligned} [L_m^{(\alpha)},\, \alpha^\nu_k] &= -k\,\alpha^\nu_{k+m} \ , \\ [L_m^{(d)},\, d^\nu_k] &= -(\tfrac{1}{2}m+k)d^\nu_{k+m} \ . \end{aligned} \tag{11.55}$$

From this,

$$[L_m,\, F_n] = (\tfrac{1}{2}m-n)F_{m+n} \ . \tag{11.56}$$

Similarly,

$$[L_m,\, G_r] = (\tfrac{1}{2}m - r)G_{m+r}\ . \tag{11.57}$$

Next, we consider $[G_r,\, G_s]$. We take the case $r \geq 0$. If A and B are bosonic operators, and ψ and η fermionic, and if the bosonic operators commute with the fermionic ones, one has

$$\begin{aligned}\{A\psi,\, B\eta\} &= [A,\, B]\,\psi\eta + BA\,\{\psi,\, \eta\} \\ &= -[A,\, B]\,\eta\psi + AB\,\{\eta,\, \psi\}\ ,\end{aligned} \tag{11.58}$$

which one very easily verifies by writing the (anti-)commutators in full. Using Eq. (11.54) for the G_r coefficients, we write

$$\{G_r,\, G_s\} = \sum_{m,k}\left(\begin{array}{ll}\text{either} & [\alpha^\mu_{-m},\, \alpha^\nu_{-k}]\, b^\mu_{r+m} b^\nu_{s+k} + \alpha^\nu_{-k}\alpha^\mu_{-m}\,\{b^\mu_{r+m},\, b^\nu_{s+k}\} \\ \text{or} & -[\alpha^\mu_{-m},\, \alpha^\nu_{-k}]\, b^\nu_{s+k} b^\mu_{r+m} + \alpha^\mu_{-m}\alpha^\nu_{-k}\,\{b^\nu_{s+k},\, b^\mu_{r+m}\}\end{array}\right). \tag{11.59}$$

Substituting the values for the commutators, we find that the summation over k can be written as

$$= \sum_k\left(\begin{array}{ll}\text{either} & k\, b^\mu_{r-k} b^\mu_{s+k} + \alpha^\mu_{-k}\alpha^\mu_{r+s+k} \\ \text{or} & -k\, b^\mu_{s+k} b^\mu_{r-k} + \alpha^\mu_{r+s+k}\alpha^\mu_{-k}\end{array}\right)\ . \tag{11.60}$$

If $r + s \neq 0$, this unambiguously leads to $2L_{r+s}$. But, if $r + s = 0$, we have to look at the ordering.

Take the case that $r > 0,\ s < 0$ (the other case goes just the same way). Notice that, for $k > r$, the top line has vanishing expectation value, so it leads directly to the corresponding contributions in $2L_{r+s}$. The same is true for the bottom line, if $k \leq 0$. Only for the values of k between 0 and r, both of these lines give the same extra contributions: $k\,\delta^{\mu\mu} = k\,D$, if D is the number of dimensions included in the sum. We have to add these contributions for $0 \leq k \leq r - \frac{1}{2}$. This gives (note that k is an integer):

$$\sum_{k=0}^{r-\frac{1}{2}} k\,\delta_{r+s}\, D = \tfrac{1}{2}(r - \tfrac{1}{2})(r + \tfrac{1}{2})\delta_{r+s}\, D\ . \tag{11.61}$$

Thus we find

$$\{G_r,\, G_s\} = 2L_{r+s} + B(r)\delta_{r+s}\ , \qquad B(r) = \tfrac{1}{2}D(r^2 - \tfrac{1}{4})\ . \tag{11.62}$$

The calculation of $[F_m,\, F_n]$ goes exactly the same way, except for one complication: we get a contribution from $\pm m\, d^\mu_0 d^\mu_0$. Here, we have to realize that $(d^\mu_0)^2 = \frac{1}{2}\{d^\mu_0,\, d^\mu_0\} = \frac{1}{2}$. Thus, Eq. (11.61) is then replaced by

$$(\tfrac{1}{2}m + \sum_{k=0}^{m-1} k)\,\delta_{m+n}\, D = \tfrac{1}{2}m^2\delta_{m+n}\, D\ , \tag{11.63}$$

so we get

$$\{F_m,\, F_n\} = 2L_{m+n} + B(m)\delta_{m+n}\ , \qquad B(m) = \tfrac{1}{2}Dm^2\ . \tag{11.64}$$

Finally, the commutator $[L_m,\, L_n]$ can be computed. One finds,

$$[L_m,\, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n}\ , \tag{11.65}$$

with

$$\begin{aligned} A(m) &= \tfrac{1}{8}Dm^3\ , && \text{(Ramond)}\ , \\ A(m) &= \tfrac{1}{8}D(m^3-m)\ , && \text{(Neveu-Schwarz)}\ . \end{aligned} \tag{11.66}$$

Again, this is *before* imposing gauge constraints, such as in the next section. Eq. (11.65) is to be compared to the expression (4.26) for the bosonic case.

To reproduce Eqs. (11.66) is a useful exercise: if we split L_m into a Bose part and a Fermi part: $L_m = L_m^{\mathrm{B}} + L_m^{\mathrm{F}}$, then the bosonic part gets a contribution as in Eq. (4.26) for each of the contributing dimensions:

$$[L_m^{\mathrm{B}},\, L_n^{\mathrm{B}}] = (m-n)L_{m+n}^{\mathrm{B}} + \tfrac{D}{12}m(m^2-1)\delta_{m+n}\ , \tag{11.67}$$

the fermionic contribution is then found in a similar way to obey

$$[L_m^{\mathrm{F}},\, L_n^{\mathrm{F}}] = (m-n)L_{m+n}^{\mathrm{F}} + \tfrac{D}{24}m(m^2-1)\delta_{m+n} \qquad \text{(Neveu-Schwarz)}, \tag{11.68}$$

$$[L_m^{\mathrm{F}},\, L_n^{\mathrm{F}}] = (m-n)L_{m+n}^{\mathrm{F}} + \tfrac{D}{24}m(m^2+2)\delta_{m+n} \qquad \text{(Ramond)}, \tag{11.69}$$

Again, in the summations, a limited number of terms have to be switched into the normal order position, and this gives rise to a finite contribution to the central charge term. The first result comes from adding half-odd-integer contributions, while in the second case, as before, we have had to take into account that $d_0^2 = \frac{1}{2}$.

Note, that Eq. (11.64) implies that $F_0^2 = L_0$, so if we have a state with $F_0|\,\psi\rangle = \mu|\,\psi\rangle$, then also $L_0|\,\psi\rangle = \mu^2|\,\psi\rangle$, and so also the vacuum values for L_0 and F_0 are linked.

Often, we will first only take the transverse modes, in which case D must be replaced by $D-2$.

11.8. The light-cone gauge for fermions

On top of the gauge fixing condition $J_\alpha(\tau,\sigma) = 0$, which is analogous to $T_{\alpha\beta} = 0$ for the bosonic case, there is the fermionic counterpart of the coordinate fixing condition $X^+ = \tau$, which we referred to previously as the light-cone gauge. Since ψ^+ is the superpartner of X^+, one imposes the extra condition

$$\psi_A^+ = \psi_A^0 + \psi_A^{D-1} = 0 \tag{11.70}$$

(We can omit the superfluous index A, if we define $\psi(\tau,\sigma) = \psi_+(\tau,\sigma)$ if $\sigma > 0$ and $\psi_-(\tau,-\sigma)$ if $\sigma < 0$).

The subsidiary conditions implied by the vanishing of J_α and $T_{\alpha\beta}$ take the form

$$\begin{aligned} \psi^\mu \partial_+ X^\mu &= 0 , \\ (\partial_+ X^\mu)^2 + \frac{i}{2}\psi^\mu \partial_+ \psi^\mu &= 0 . \end{aligned} \tag{11.71}$$

Given the gauge choices $\partial_+ X^+ = \frac{1}{2}p^+$ and $\psi^+ = 0$, these equations can be solved for the light-cone components ψ^- and $\partial_+ X^-$:

$$\begin{aligned} \partial_+ X^- &= \frac{1}{p^+}\Big((\partial_+ X^{\rm tr})^2 + \tfrac{1}{2} i \psi^{\rm tr}\partial_+ \psi^{\rm tr}\Big) , \\ \psi^- &= \frac{2}{p^+}\psi^{\rm tr}\partial_+ X^{\rm tr} . \end{aligned} \tag{11.72}$$

In terms of the Fourier modes, this gives (in the NS case)

$$\begin{aligned} \alpha_n^- &= \frac{1}{2p^+}\sum_{i=1}^{D-2}\Bigg(\sum_{m=-\infty}^{\infty} : \alpha^i_{n-m}\alpha^i_m : \\ &\quad + \sum_{r=-\infty}^{\infty} r \, : b^i_{n-r} b^i_r : \Bigg) - \frac{a\delta_n}{p^+} ; \end{aligned} \tag{11.73}$$

$$b_r^- = \frac{1}{p^+}\sum_{i=1}^{D-2}\sum_{s=-\infty}^{\infty} \alpha^i_{r-s} b^i_s . \tag{11.74}$$

As in Section (4.3), Eq. (4.14)

As in Section 5, we can again construct the generators of Lorentz transformations, $J^{\mu\nu}$. Manipulations identical to the ones described in Section 5, give[11]

$$[J^{i-},\, J^{j-}] = -\frac{1}{(p^+)^2}\sum_{m=1}^{\infty}\Delta_m(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m) , \tag{11.75}$$

with

$$\Delta_m = m\left(\frac{D-2}{8} - 1\right) + \frac{1}{m}\left(2a - \sigma\frac{D-2}{8}\right) . \tag{11.76}$$

Here, the parameter $\sigma = 1$ for the Neveu-Schwarz case and $\sigma = 0$ for Ramond, as it derives directly from the commutator (11.66). Since the commutator (11.76) must vanish for all m, we must have

$$D = 10 \qquad \text{and} \qquad a = \tfrac{1}{2}\sigma . \tag{11.77}$$

[11] A way to check Eq. (11.75) is to observe that J^{i-} vanishes on the vacuum, so that it suffices to compute expressions such as $[[[J^{1-}, J^{2-}], \alpha^1_m], \alpha^2_{-m}]$.

12. The GSO Projection.

We see from the last subsection that the fermionic Ramond sector has no tachyon, and that there is one tachyon in the bosonic Neveu-Schwarz sector. Also, we found that the fermionic spectrum is at integer spacings while the bosons are half-integer spaced.

The discovery to be discussed in this section is that one can impose a further constraint on the states. The constraint is often 'explained' or 'justified' in the literature in strange ways. Here, I follow my present preference. Our discussion begins in the vacuum sector of the Ramond sector. These states are degenerate because we have the d_0 operators, which commute with L_0. As was explained in subsection 11.5, these are just the gamma matrices Γ^μ (apart from a factor $i\sqrt{2}$).

12.1. The open string.

We established that the intercept $a = 0$ in the Ramond sector. This means that the fermions in the ground state are massless. The question is, how degenerate is this ground state? What is the degeneracy of all fermionic states?

It so happens that in $D = 10$ dimensions, massless fermions allow for two constraints. One is that we can choose them to be Majorana fermions. This means that the gamma matrices, if normalized as in (11.42), are purely imaginary. Secondly, one can use a Weyl projection. Like with neutrinos, we can project out one of two chiral modes. The spinors of this chiral mode are only 8 dimensional. To understand these, we begin with the generic construction of (real, positively normalized) gamma matrices in D dimensions:

- In 3 dimensions, we can use the 3 Pauli matrices τ^1, τ^2, and τ^3, which are 2×2 matrices and obey $\{\tau^i, \tau^j\} = 2\delta^{ij}$.
- If we have matrices γ^α in d dimensions, obeying $\{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta}$, then we can construct two more, to serve in $\tilde{d} = d + 2$ dimensions, by choosing a Hilbert space twice as big:

$$\tilde{\gamma}^\alpha = \begin{pmatrix} 0 & \gamma^\alpha \\ \gamma^\alpha & 0 \end{pmatrix} , \qquad \tilde{\gamma}^{d+1} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \tilde{\gamma}^{d+2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{12.1}$$

  which we often write in a more compact form:

$$\tilde{\gamma}^\alpha = \gamma^\alpha \times \tau^1 , \qquad \tilde{\gamma}^{d+1} = \mathbf{1} \times \tau^2 , \qquad \tilde{\gamma}^{d+2} = \mathbf{1} \times \tau^3 \tag{12.2}$$

  (of course, the τ matrices may be permuted here).

Thus we see that in $d = 2k$ and $d = 2k+1$ dimensions, the gamma's are matrices acting on spinors with $2^k = 2^{d/2}$ or $2^{(d-1)/2}$ components. How can we understand this in terms of the d_n^μ operators of the string modes?

The situation with the d_n^μ for $n \neq 0$ is straightforward: for each μ and each $n > 0$, the operators d_{-n}^μ and d_n^μ together form the creation and annihilation operators for a fermionic Hilbert space of two states:

$$d^{\mu}_{-n} = d^{\mu\,\dagger}_{n} \ ; \qquad \{d^{\mu}_{n},\, d^{\nu\,\dagger}_{n}\} = \delta^{\mu\nu} \ . \tag{12.3}$$

These however are independent operators *only* for the $D-2$ *transverse* dimensions in the light-cone formalism: we have $d^{+}_{n} = 0$ and d^{-}_{n} are all determined by the supergauge constraint. Thus for each pair $(n,\,-n)$ we have $D-2$ factors of 2 in our Hilbert space. Naturally, for the single case $n = 0$, there will be $\frac{D-2}{2}$ factors of two. Since $D = 10$, the zero mode spinors should be $2^4 = 16$ dimensional.

Indeed, formally one can construct the gamma's out of the d^{μ}_{0}'s pairwise. Write $\mu = 2i$ or $\mu = 2i-1$, where $i = 1,\,\cdots,\,4$. Then

$$\begin{aligned} b^i &= \tfrac12(\gamma^{2i} + i\gamma^{2i-1}) \quad , & & \\ b^{i\dagger} &= \tfrac12(\gamma^{2i} - i\gamma^{2i-1}) \quad , & \gamma^\mu &= \sqrt{2}\, d^{\mu}_{0} \ , \\ \{b^i,\, b^j\} = 0 \ , \qquad \{b^i,\, b^{j\dagger}\} &= \delta^{ij} \quad , & i,j &= 1,\cdots,4 \ . \end{aligned} \tag{12.4}$$

The 4 operators b^i and $b^{i\dagger}$ are fermionic annihilation and creation operators, and hence each of them demands a factor 2 degeneracy in the spectrum of states. The fact that we must limit ourselves to *real* spinors (Majorana spinors), is a consequence of supersymmetry (after all, the X^{μ} operators are real also). Limiting oneself to real spinors is not so easy. The gamma matrices constructed along the lines of the argument given earlier would have real and imaginary components. A slightly different arrangement, however, enables us to find a representation where they are all real. If τ^1 and τ^3 are chosen real and τ^2 imaginary (as is usually done), we can choose the 8 gamma's as follows:

$$\begin{array}{lcll} \gamma^1 &=& \tau^2 \times \tau^2 \times \tau^2 \times \tau^2 \ , \qquad & \gamma^2 = \mathbf{1} \times \tau^1 \times \tau^2 \times \tau^2 \ , \\ \gamma^3 &=& \mathbf{1} \times \tau^3 \times \tau^2 \times \tau^2 \ , & \gamma^4 = \tau^1 \times \tau^2 \times \mathbf{1} \times \tau^2 \ , \\ \gamma^5 &=& \tau^3 \times \tau^2 \times \mathbf{1} \times \tau^2 \ , & \gamma^6 = \tau^2 \times \mathbf{1} \times \tau^1 \times \tau^2 \ , \\ \gamma^7 &=& \tau^2 \times \mathbf{1} \times \tau^3 \times \tau^2 \ , & \gamma^8 = \mathbf{1} \times \mathbf{1} \times \mathbf{1} \times \tau^1 \ , \end{array} \tag{12.5}$$

We see that here, differently from the construction in Eq. (12.2), all gamma's contain an even number of τ^2 matrices, so they are all real. The matrices Γ^{μ}, used earlier, obey the commutation rule $\{\Gamma^{\mu},\ \Gamma^{\nu}\} = -2\eta^{\mu\nu}$, so, because of the minus sign, they then become imaginary, for all spacelike μ.

The representation (12.5) is unitarily equivalent to the construction following (12.2), because all representations of gamma matrices in an even number of dimensions are equivalent.

To obtain the full Dirac equation, we first must add two more gamma's. Following a procedure such as in (12.2), $\gamma^9 = 1\times1\times1\times1\times\tau^1$ and $\gamma^0 = 1\times1\times1\times1\times i\tau^2$ (if the others are multiplied with τ^3). They are still real, but they force us to use a 32 dimensional spinor space. The Dirac equation, $(m + \Gamma^{\mu}p_{\mu})\psi = 0$, with $p^2 = -m^2$, then projects out half of these, which is why the allowed spinors form the previously constructed 16 dimensional set.

The Dirac equation can be understood to arise in the zero mode sector in the following way. Note that

$$\gamma^+ = \gamma^0 + \gamma^9 = \tau^1 + i\tau^2 = \tau^1(1 - \tau^3) \ . \tag{12.6}$$

If we limit ourselves to the sector $\tau^3|\,\psi\rangle = +|\,\psi\rangle$, then $\gamma^+ = 0$, in accordance with our supersymmetry gauge condition (11.70). The constraint equation (11.74) in the Ramond sector now reads

$$p^+\gamma^- = p^i\gamma^i + \sqrt{2}\sum_{n\neq 0}\alpha^i_{-n}d^i_n \ . \tag{12.7}$$

On the zero state, the right hand side is zero, and so we see that this state obeys

$$(-\gamma^+p^- - \gamma^-p^+ + \gamma^ip^i)|\,\psi\rangle = 0 \ .$$

For the non-zero modes, the right hand side will generate mass terms.

The analogue of γ^5 in 4 dimensions is here

$$\Gamma^{11} = \prod_{\mu=0}^{9}\Gamma^\mu \ . \tag{12.8}$$

It, too, is real. As in four dimensions, the massless Dirac equation allows us to use $\frac{1}{2}(1\pm\Gamma^{11})$ as projection operators. But, unlike the 4 dimensional case, this chiral projector is real, so it projects out *real, chiral Majorana* states. From (12.8) it is clear that Γ^{11} acts within our 16 dimensional space, where it is[12]

$$\Gamma^{11} = \prod_{\mu=1}^{8}\gamma^\mu \ = \ \mathbf{1} \ \times\mathbf{1} \ \times\mathbf{1} \ \times\tau^3 \ . \tag{12.9}$$

Thus, the massless fermionic string modes in the Ramond sector have two, conserved, helicities, depending on the Lorentz-invariant value of $\Gamma^{11} = \pm 1$.

It was the discovery of Gliozzi, Scherk and Olive (GSO) that one can impose a further constraint on the superstring. In the zero mode sector of the Ramond case, it amounts to keeping just one of the two helicities: $\Gamma^{11}|\,\psi\rangle = +|\,\psi\rangle$. These $|\,\psi\rangle$ form an 8 dimensional real spinor.

What will this constraint imply for the massive sectors and for the Neveu-Schwarz sector? To answer this, let us look at the condition on the $(\sigma\,\tau)$ world sheet. Γ^{11} anticommutes with all d^μ_0. We generalize this into an operator called $\overline{\Gamma}$ in Green, Schwarz and Witten, which anticommutes with all $\psi^\mu(\sigma,\,\tau)$. In the light-cone gauge, this operator can be interpreted as a formal product over all of σ space: using the fact that the transverse components of $\psi^\mu(\sigma,\,0)$ essentially commute as gamma matrices do, $\{\psi^i(\sigma),\,\psi^j(\sigma')\} = \delta^{ij}\delta(\sigma - \sigma')$, we define

$$\overline{\Gamma} = C\prod_{\sigma,\ \mu=1}^{8}\psi^\mu(\sigma,\,0) \ , \tag{12.10}$$

[12] times a factor τ^3, but in our 16-dimensional subspace, that is one.

where C is a (divergent) constant adjusted to make $\overline{\Gamma}^2 = 1$. We find that this operator, made from the existing operators of the theory, anticommutes with all $\psi^\mu(\sigma,\,\tau)$, including indeed also ψ^+ (which is zero) and ψ^-, which, due to the light-cone gauge condition, depends linearly on all transverse fermionic fields.

The property $\{\overline{\Gamma},\, d_0^\mu\} = 0$ and the condition $\overline{\Gamma}|\,\psi\rangle = |\,\psi\rangle$ selects out the chiral Majorana mode described by the 8-component spinor in the Ramond zero sector. If we impose $\overline{\Gamma} = 1$ on the other, massive, modes of the Ramond sector, we find that the surviving modes are

$$d^{\mu_1}_{-n_1} \cdots d^{\mu_N}_{-n_N} |\, 0\rangle^\pm \ , \tag{12.11}$$

where the $\pm$ sign refers to the helicity eigenstate, and it is plus if N even and minus if N odd.

How does $\overline{\Gamma}$ act in the Neveu-Schwarz sector? Since $\overline{\Gamma}$ commutes with L_0, and since the ground state is non-degenerate, the ground state is an eigenstate of $\overline{\Gamma}$. If it has eigenvalue $+1$, then all states created by an operator b^μ_{-r} exactly once, have eigenvalue -1, and so on. GSO projection would then select out all states created by an *even* number of the fermionic b^μ_{-r} operators. But we can also assume that the ground state has eigenvalue -1. In that case, we only keep the states created by an odd number of b^μ_{-r}.

This latter choice has two important features. One is, that the only remaining tachyonic state, the Neveu-Schwarz ground state, is now eliminated. The other important aspect is that now the spectrum of bosonic Neveu-Schwarz states precisely matches the fermionic Ramond states: since, in Neveu-Schwarz, $a = \frac{1}{2}$, its first excited state is massless, like the Ramond fermions, and furthermore, since μ takes on $D-2$ values, this state is *also* 8-fold degenerate. All other surviving Neveu-Schwarz states now have integer values for $\frac{1}{2}M^2$, as the Ramond states do. The fact that, in the light-cone gauge, all massive Ramond states match with states in the Neveu-Schwarz sector, is related to a curious mathematical theorem that is not easy to prove. A way to calculate the degeneracy of the spectrum is briefly discussed in the next subsection. A curious feature is that the **8** representation of $SO(8)$ is vectorlike in the Neveu-Schwarz sector, while it is a spinor in the Ramond sector. If $D-2$ were different from 8, these representations would not have matched. This situation is further explained in the Green-Schwarz-Witten book, Appendix 5A and B.

The GSO projection enables us to have a tachyon-free string spectrum. It could have been brought forward that the Ramond sector was also tachyon-free, but that sector only contains fermionic states. Anyway, we still have to check the closed string.

12.2. Computing the spectrum of states.

The general method to compute the number of states consists of calculating, for the entire Hilbert space,

$$G(q) = \sum_{n=0}^{\infty} W_n q^n = \mathrm{Tr}\, q^N \ , \tag{12.12}$$

where q is a complex number corresponding to $1/z = e^{-i\tau}$, as in Eq. (10.3), W_n is the degree of degeneracy of the n^{th} level, and N is the number operator,

$$N = \sum_{\mu=1}^{D-2} \Big(\sum_{n=1}^{\infty} \alpha_{-n}\alpha_n + \sum_{r>0} r d_{-r} d_r \Big) = \sum_{\mu=1}^{D-2} \Big(\sum_{n=1}^{\infty} n\, N^{\mathrm{Bos}}_{\mu,\,n} + \sum_{r>0} r\, N^{\mathrm{Ferm}}_{\mu,\,r} \Big) \ , \tag{12.13}$$

where the sum over the fermionic operators is either over integers (Ramond) or integers $+\frac{1}{2}$ (Neveu-Schwarz). Since N receives its contributions independently from each mode, we can write $G(q)$ as a product:

$$G(q) = \prod_{\mu=1}^{D-2} \prod_{n=1}^{\infty} \prod_{r>0} f_n(q) g_r(q) \ , \tag{12.14}$$

with

$$f_n(q) = \sum_{m=0}^{\infty} q^{nm} = \frac{1}{1-q^n} \ , \tag{12.15}$$

while

$$g_r(q) = \sum_{m=0}^{1} q^{rm} = 1 + q^r \ . \tag{12.16}$$

We find that, for the purely bosonic string in 24 transverse dimensions:

$$G(q) = \prod_{n=1}^{\infty} (1-q^n)^{-24} \ . \tag{12.17}$$

The Taylor expansion of this function gives us the level density functions W_n. There are also many mathematical theorems concerning functions of this sort.

For the superstring in 8 transverse dimensions, we have

$$\begin{aligned} G(q) &= \prod_{n=1}^{\infty} \left(\frac{1+q^{n-\frac{1}{2}}}{1-q^n} \right)^8 \quad \text{(NS)} \ ; \\ G(q) &= 16 \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^8 \quad \text{(Ramond)} \ , \end{aligned} \tag{12.18}$$

where, in the Ramond case, the overall factor 16 comes from the 16 spinor elements of the ground state.

Now let us impose the GSO projection. In the Ramond case, it simply divides the result by 2, since we start with an 8 component spinor in the ground state. In the NS case, we have to remove the states with even fermion number. This amounts to

$$G(q) = \tfrac{1}{2} \mathrm{Tr} \left(q^N - (-1)^F q^N \right) \ , \tag{12.19}$$

where F is the fermion number. Multiplying with $(-1)^F$ implies that we replace $g(r)$ in Eq. (12.16) by

$$\tilde g(r)=\sum_{m=0}^{1}(-q)^{rm}=1-q^r\ . \tag{12.20}$$

This way, Eq. (12.18) turns into

$$\begin{aligned} G_{\rm NS}(q) &= \frac{1}{2\sqrt q}\left[\prod_{n=1}^{\infty}\left(\frac{1+q^{n-\frac12}}{1-q^n}\right)^8-\prod_{n=1}^{\infty}\left(\frac{1-q^{n-\frac12}}{1-q^n}\right)^8\right] \qquad \text{(NS)}\ ; \\ G_{\rm R}(q) &= 8\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^8 \qquad\qquad \text{(Ramond)}\ . \end{aligned} \tag{12.21}$$

Here, in the NS case, we divided by $\sqrt q$ because the ground state can now be situated at $N=-\frac12$, and it cancels out.

The mathematical theorem alluded to in the previous subsection says that, in Eq. (12.21), $G_{\rm NS}(q)$ and $G_{\rm R}(q)$ are equal. Mathematica gives for both:

$$\begin{aligned} G(q) &= 8+128\,q+1152\,q^2+7680\,q^3+42112\,q^4+200448\,q^5 \\ &\quad +855552\,q^6+3345408\,q^7+12166272\,q^8+\cdots\ . \end{aligned} \tag{12.22}$$

12.3. String types.

In closed strings, we have the same situation. We add fermions both to the left-movers and the right-movers, and first impose super symmetry in the world sheet. In the open string, left- and right movers were identical, since they reflect into one another at the end points. In closed strings, we only have the periodicity conditions. As stated earlier, this means that there are four sectors, the R-R, R-NS, NS-R and NS-NS sectors, depending on whether there is a twist in the boundary conditions or not.

In all of these sectors, we impose GSO projection, both to the left-movers and the right movers. Imposing $\bar\Gamma|\,\psi\rangle=+|\,\psi\rangle$, implies a certain *chirality* to the resulting state:

$$\tfrac{1}{8!}\varepsilon_{\mu_1\mu_2\cdots\mu_8}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_8}|\,\psi\rangle=+|\,\psi\rangle\ . \tag{12.23}$$

The absolute sign here depends on the definition of ε and the orientation of the coordinates. But the *relative* sign for the left- and the right-movers is physically relevant. In the open string, it has to be the same. In the closed string, we can also choose the sign the same, or we may decide to have opposite chiralities.

The theory with the same chirality left and right, where open string solutions are allowed (but closed ones also) is called 'type I'. As we will see later, we will have the option of attaching quantum numbers at the end points of the open string. Imagining 'quarks' at the end points, we attach two indices to the open string states: $|\,\psi\rangle\to|\,\psi,ab\rangle$. If this were a quark theory, this would lead to an overall symmetry of the form $U(N)\times U(N)$. Also, the zero mass state would now be a vector particle described by a field in the adjoint representation of dimension N^2 of $U(N)$. According to Quantum Field Theory, such objects can only be understood as Yang-Mills fields, but this means that the quantum numbers of this field must be those of the generators of a local gauge group. Besides the group $U(N)$, one may have one of the orthogonal groups $SO(N)$. In that case, the adjoint representation that the field is in, is a *real, antisymmetric* tensor: $A^\mu_{ab} = -A^\mu_{ba}$.

For a string, this means that interchanging the end points is to be used as a symmetry transformation Ω, and we should restrict ourselves to those states that have $\Omega|\,\psi\rangle = -|\,\psi\rangle$. Now Ω replaces the coordinate σ by $\pi-\sigma$, which means that all modes that go with $\cos n\sigma$ where n is odd, switch signs. Consequently, all states that have an odd value for $\alpha' M^2$, switch sign under Ω. Therefore, the fields of the massive states transform as tensors A_{ab} in $SO(N)$ such that $A_{ab} = \pm A_{ba}$, with the + sign when $\alpha' M^2$ is odd, and the minus sign when $\alpha' M^2$ is even.

The above was concluded merely by looking at the massless states and interpreting them as gauge fields. Actually, one must check the consistency of the entire string theory. This leads to more constraints: the open string only allows $SO(32)$ as a gauge group. The symmetry under the parity operator Ω implies that this string is non-orientable. It can undergo transitions into closed strings (by having end points merge), provided these closed strings are also non-orientable. This constitutes the 'type I' superstring theory.

If the string does not have end points, it may be orientable. This will be called a string theory of 'type II'. We can choose the sign of the chirality for left- and right movers opposite. This will be called a string theory of 'type IIA'. Because of the opposite chiralities, the IIA theory as a whole is left-right symmetric. If the chiralities of left and right movers are equal, the string is of 'type IIB'. In this theory, the string modes are left-right asymmetric.

Finally, we have the so-called 'heterotic string theories'. One chooses the fermions such that they move to the right only. In the left-moving sector, we must choose extra degrees of freedom in order to saturate the algebraic conditions that lead to Lorentz-invariance. There are different ways to do this. One possibility is to put there 16 more bosonic fields, but this gives a problem with the zero modes. If our 16 bosonic fields were unconstrained, these zero modes would lead to an infinite degeneracy that we do not want in our theory. Therefore, one must compactify these bosonic dimensions. Assume for the moment that we have a 16-dimensional torus. The bosonic part of the theory, in these 16 compactified dimensions (now labled by the index I), then looks as follows:

$$X^I(\sigma,\tau) = q^I + p^I\tau + w^I\sigma + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^I_n e^{2in(\sigma-\tau)} \ . \tag{12.24}$$

Here, p^I is the c.m. momentum, and w^I are the winding distances (see Chapter 9 on T-duality).

The reason why we need compactification on a torus of a fixed size R, is that the excitations (12.24) may only depend on one coordinate $\sigma^- = \tau - \sigma$. Therefore,

$$w^I = -p^I \ . \tag{12.25}$$

Here, the allowed momenta p^I form a lattice, and we must insist that the winding distances w^I form the *same* lattice. This led physicists to the study of *self-dual lattices*. The appropriate lattices are said to exist only in 16 dimensions, leading to either the group $SO(32)$ or $E_8 \times E_8$ for the heterotic string (to be explained later). Also, a procedure akin to the GSO projection is needed here to remove the tachyon.

13. Zero modes

In the low-energy limit, only the massless modes are seen. They interact with one another in a special way, and it is important to learn about the properties and interactions of these states. Let us enumerate the different 'theories' and their distinct sectors.

- *The open superstring (type I).* Here, we have two sectors:
  - *Ramond:* There is one chiral spinor $|\,\psi\rangle$ with $2^{D/2-2} = 2^3 = \mathbf{8}$ real components: $\psi(x)$.
  - *Neveu-Schwarz:* There is one vector field (the "photon"), obtained from the GSO-excluded vacuum state $|\,0\rangle$ as follows: $b^i_{-1/2}|\,0\rangle = A^i(x)$. It has $D-2 = \mathbf{8}$ components.

  These states are then each given their Chan-Paton coefficients, so that their degeneracy is multiplied by $\frac{1}{2}N \times (N-1) = \mathbf{496}$ (to be discussed later).

- *Closed Superstring Type IIA*. Here, the left- and right modes (to be referred to by the letters $\scriptstyle L$ and $\scriptstyle R$) each have their R and NS sectors, and each of these sectors requires a GSO projection:
  - NS-NS: The only massless states are

$$b^{i,L}_{-1/2}\, b^{j,R}_{-1/2}|\,0,0\rangle = A^{ij}(x)\ . \tag{13.1}$$

    As a representation of the transverse $SO(8)$, it splits into
    * a *scalar* $\phi(x)\delta^{ij} \to \mathbf{1}$ (the "dilaton"),
    * an *antisymmetric* tensor field $B^{ij}(x) = -B^{ij}(x)$ with dimension $\frac{1}{2}(D-2)(D-3) = \mathbf{28}$ (the "axion"), and
    * a *traceless symmetric* tensor field $G^{ij}(x) = G^{ji}(x)$ with $G^{ii} = 0$, having dimension $\frac{1}{2}(D-2)(D-1) - 1 = \mathbf{35}$ (the "graviton").
  - NS-R: A vector times a chiral spinor: $b^{i,L}_{-1/2}|\,0,\psi\rangle = \psi^i(x)$. It splits into
    * a chiral spin $\frac{1}{2}$ spinor: $\psi(x) = \Gamma^i\psi^i(x)$ of dimension $\mathbf{8}$ (the "dilatino"), and
    * a *spin* $\frac{3}{2}$ field $\psi^i(x)$ with $\Gamma^i\psi^i = 0$ (the "gravitino"). It has dimension $(D-2)^2 - 8 = \mathbf{56}$.
  - R-NS: Same as above:

- ∗ a dilatino of dimension $\mathbf{8}$, and
- ∗ a gravitino of dimension $\mathbf{56}$.

The chiralities of these states are opposite to those of the NS-R states.

– R-R: A left chiral spinor times a right chiral spinor, $\mathbf{8}_L \times \mathbf{8}_R$. The way it splits up in representations of $SO(8)$ can be derived by writing it as $|\,\psi_R\rangle\langle\psi_L|$. The components of this double-spinor can be determined by sandwiching it with strings of gamma matrices: $\langle\psi_L|\,\Gamma^{i_1}\cdots\Gamma^{i_k}|\,\psi_R\rangle$. Now, the GSO projector $\bar{\Gamma}$ gives

$$\bar{\Gamma}|\,\psi_L\rangle = +|\,\psi_L\rangle \ ; \qquad \bar{\Gamma}|\,\psi_R\rangle = -|\,\psi_R\rangle \ , \tag{13.2}$$

since, in type IIA, the left- and right-chiralities are opposite. This implies that

$$\overline{\Gamma}\Gamma^{i_1}\cdots\Gamma^{i_k}|\,\psi_L\rangle = (-1)^k\Gamma^{i_1}\cdots\Gamma^{i_k}|\,\psi_L\rangle \ , \tag{13.3}$$

and since all inner products between states with opposite chiralities vanish, only odd series of gammas (having k odd), contribute. Also, these states have

$$\overline{\Gamma}\Gamma^1\cdots\Gamma^k = \Gamma^8\cdots\Gamma^{k+1} \ , \tag{13.4}$$

so that, if we have series with more than 4 gammas, they can be expressed as series with fewer gammas. From this, it is concluded that these 64 states split into:

- ∗ one vector state $\langle\psi_L|\,\Gamma^i|\,\psi_R\rangle = C^i(x) \to \mathbf{8}$, and
- ∗ an antisymmetric 3-tensor field $C^{ijk} = \langle\psi_L|\,\Gamma^i\Gamma^j\Gamma^k|\,\psi_R\rangle$ of dimension $\frac{8\cdot 7\cdot 6}{3!} = \mathbf{56}$.

• *Closed superstring of type IIB*: The first three sectors are as in the type IIA superstring:

– NS-NS:

- ∗ One scalar field, $\mathbf{1}$;
- ∗ an antisymmetric field $B^{ij} = -B^{ji} \to \mathbf{28}$, and
- ∗ a symmetric traceless graviton $G^{ij} = G^{ji}\,;\ G^{ii} = 0 \ \to\ \mathbf{35}$.

– NS-R and R-NS (here each having the *same* chirality) :

- ∗ A spin $\frac{1}{2}$ chiral Majorana field ψ of dimension $\mathbf{8}$ (dilatino), and
- ∗ a gravitino field ψ^i of spin $\frac{3}{2}$, of dimension $\mathbf{56}$.

But the RR state is different:

– RR: The $\mathbf{8}_L \times \mathbf{8}_L$ state only admits even series of gamma matrices, so we have

- ∗ a scalar field, $C(x) = \langle\psi_L|\,\psi_L\rangle$, dimension $\mathbf{1}$,
- ∗ an antisymmetric rank 2 tensor $C^{ij} = -C^{ji} = \langle\psi_L|\,\Gamma^i\Gamma^j|\,\psi_L\rangle \to \mathbf{28}$, and
- ∗ a *self-dual antisymmetric 4-tensor field* $C^{i_1\cdots i_4} = \langle\psi_L|\,\Gamma^{i_1}\cdots\Gamma^{i_4}|\,\psi_L\rangle \to \frac{1}{2}\left(\frac{8\cdot 7\cdot 6\cdot 5}{24}\right) = \mathbf{35}$.

- *The closed type I superstring* is as type IIB, except for the fact that these strings are non-orientable — there is a symmetry $\sigma \leftrightarrow \pi - \sigma$. this means that we have half as many states. Thus, there is only one of each of the representations $\mathbf{1}, \mathbf{8}, \mathbf{28}, \mathbf{35}$ and $\mathbf{56}$. To be precise, in the NS-NS sector the axion field B^{ij}, and in the R-R sector the scalar $C(x)$ and the 4-tensor $C^{i_1 i_2 i_3 i_4}$ are odd under Ω so that they disappear from the spectrum. Under Ω, the R-NS sector transforms into the NS-R sector, so that only the even superpositions of these states survive.

We see that the type I theory has one gravitino, the type IIA theory has two gravitino modes with opposite chirality, and the type IIB theory has two gravitinos with equal chirality. These things imply that the target space has a local supersymmetry, since only theories with local supersymmetry (supergravity theories) allow for the existence of an elementary spin $\frac{3}{2}$ gravitino field. As type I superstrings have only one gravitino, they have a target space $N = 1$ local supersymmetry. The other theories have local supersymmetry with $N = 2$, since they have two gravitinos. In view of the chiralities of the gravitinos, the associated supergravity theories are different, and they are also labeled as type IIA and type IIB.

- *The Heterotic Strings.* Here, we distinguish the zero modes due to excitations in the 10 physical dimensions from the ones on the self-dual, compactified 16-torus. Since the fermions only move to the right, we have single NS and R modes. From the 10 physical dimensions:
  - The NS sector contains the bosons:
    $\alpha^{i\,L}_{-1}\, b^{j\,R}_{-1/2} |\, 0, 0\rangle = A^{ij} \rightarrow \mathbf{8} \times \mathbf{8}$, which, as before, splits up into
    - a scalar $\mathbf{1}$,
    - an antisymmetric tensor $B^{ij} \rightarrow \mathbf{28}$, and
    - a symmetric, traceless graviton field $G^{ij} \rightarrow \mathbf{35}$
  - The R sector contains the fermions:
    $\alpha^{i\,R}_{-1} |\, 0, \psi\rangle_R = \psi^i(x)$, which splits up into
    - a chiral $\mathbf{8}$, and
    - a gravitino $\mathbf{56}$.

  Both sectors have 64 states in total.
  From the compactified dimensions:

  - Bosons in the NS sector:
    - $\alpha^{I,L}_{-1}\, b^{i\,R}_{-1/2} |\, 0, 0\rangle \rightarrow \mathbf{16} \times \mathbf{8}$;
    - $b^{i\,R}_{-1/2} |\, \vec{p}_L^{\,2} = 2,\, 0\rangle \rightarrow \mathbf{480} \times \mathbf{8}$.

    Together, they form an 8 component vector in the $\frac{1}{2} \cdot 32 \cdot 31$ representation of either $SO(32)$ or $E_8 \times E_8$.
  - Fermions in the R sector:
    - $\alpha^{I\,L}_{-1} |\, 0, \psi\rangle \rightarrow \mathbf{16} \times \mathbf{8}$, and
    - $|\, \vec{p}_L^{\,2} = 2,\, \psi\rangle \rightarrow \mathbf{480} \times \mathbf{8}$.

13.1. Field theories associated to the zero modes.

As we suggested already at several locations, the zero mass solutions must be associated to fields that correspond to a very specific dynamical system in the low-energy domain. This domain is also what we are left over with if we take the limit where the slope parameter α' tends to zero. This is called the *zero-slope limit*. We are now interested in the field theories that we get in this limit. In particular, the question may be asked what kind of couplings will be allowed between these fields.

The open type I string theory was already discussed in this limit. We have a chiral Majorana field, obeying the Dirac equation,

$$\Gamma^\mu \partial_\mu \psi = 0 \ . \tag{13.5}$$

Apart from the demands that this field is chiral and Majorana, there are no symmetries to be imposed, so interactions with this field can easily be described by terms in an effective Lagrangian. The spin-one mode, however, must be described by a vector field $A^\mu(x)$, where $\mu = 1, \cdots 10$, of which only the 8 transverse components represent physical degrees of freedom. In particle physics, this situation can only be handled if this vector field is a *gauge field*. In the most general case, we have N components of such fields, A_μ^a, $a = 1, \cdots, N$ and the index a is associated to the generators of a Lie-group:

$$[T^a,\, T^b] = i f_{abc} T^c \ , \tag{13.6}$$

where f_{abc} are the structure constants of the group. All interactions must be invariant under the Yang-Mills gauge transformation, its infinitesimal form being

$$A_\mu^a \rightarrow A_\mu^{a\prime} = A_\mu^a - \partial_\mu \Lambda^a + f_{abc} \Lambda^b A_\mu^c \ , \tag{13.7}$$

while other fields may transform as

$$\phi \rightarrow \phi' = \phi + i \Lambda^a T^a \phi \ . \tag{13.8}$$

Here, $\Lambda^a = \Lambda^a(\mathbf{x}, t)$ depends on space and time.

The Lagrangian must be such that the kinetic terms for the physical components are as usual: $(\partial_0 A^{\mathrm{tr}})^2 - (\partial_i A^{\mathrm{tr}})^2$. This is achieved by the Yang-Mills Lagrangian:

$$\begin{aligned} &\mathcal{L} = -\tfrac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a \ , \\ \text{with} \qquad &G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{abc} A_\mu^b A_\nu^c \ . \end{aligned} \tag{13.9}$$

The bilinear part of this Lagrangian is, after partial integration,

$$\mathcal{L} = -\tfrac{1}{2} (\partial_\mu A_\nu^a)^2 + \tfrac{1}{2} (\partial_\mu A_\mu^a)^2 \ . \tag{13.10}$$

We can fix the gauge by demanding Λ^a to be such that, for instance, $\partial_i A_i^a = 0$, so that, in momentum space, the component of A in the spacelike $\vec{p}$ direction vanishes. In momentum space, we then have

$$\mathcal{L} = -\tfrac{1}{2} (p_\mu A_\nu^a)^2 + \tfrac{1}{2} (p_\mu A_\mu^a)^2 = -\tfrac{1}{2} (\vec{p}^{\,2} - p_0^2) A_{\mathrm{tr}}^{a\,2} + \tfrac{1}{2} (\vec{p}^{\,2} - p_0^2) A_0^{a\,2} + \tfrac{1}{2} p_0^2 A_0^{a\,2} \ . \tag{13.11}$$

For the A_0 term, we see that the p_0 dependence cancels out, so that A_0^a cannot propagate in time. It just generates the Coulomb interaction. Only the 8 transverse components survive.

Now, let us concentrate on the massless modes of the closed string. They all have a symmetric, two-index tensor, $G^{\mu\nu}$. As was the case for the vector fields, only the transverse components of this vector field propagate. An 'ordinary' massive spin-two particle would have a symmetric field of which $\frac{1}{2}D(D-1)-1 = 44$ components propagate. To reduce this number to $\frac{1}{2}(D-1)(D-2)-1 = 35$, we again need a kind of gauge-invariance. The gauge transformation must be

$$G^{\mu\nu} \rightarrow G^{\mu\nu\,\prime} = G^{\mu\nu} + \partial_\mu U^\nu + \partial_\nu U^\mu + \text{ higher order terms.} \tag{13.12}$$

This was also seen before: it is the infinitesimal coordinate transformation in General Relativity,

$$\begin{aligned} g'_{\mu\nu}(x) &= \frac{\partial(x+u)^\alpha}{\partial x^\mu}\,\frac{\partial(x+u)^\beta}{\partial x^\nu}\,g_{\alpha\beta}(x+u(x)) \\ &= g_{\mu\nu}(x) + \partial_\mu u_\nu + \partial_\nu u_\mu + \text{ higher orders.} \end{aligned} \tag{13.13}$$

Indeed, in GR, one can choose the gauge condition $g_{\mu 0} = -\delta_{\mu 0}$, after which all modes with one or two indices either in the time direction or in the momentum direction, all become non-propagating modes. Only the transverse helicities of the graviton fields survive. The massless, symmetric, spin two modes of the superstring evidently describe the gravitational field in target space.

In the NS-R and the R-NS modes, we now also have spin $\frac{3}{2}$ modes. They have 56 components each, and even if we would add the 8 spin $\frac{1}{2}$ fields, we would have less than the $9.16-16$ components of a massive spin $\frac{3}{2}$ field (which cannot be chiral), so, here also, there must be some gauge invariance. This time, it is local target space supersymmetry. We have the different type I, type IIA and type IIB 10-dimensional supergravity theories. The gauge transformation is

$$\psi_\mu \rightarrow \psi'_\mu = \psi_\mu + \partial_\mu\varepsilon\ , \tag{13.14}$$

where $\varepsilon(x)$ is an 8 component Majorana supersymmetry generator field.

According to supergravity theory, the fermionic and kinetic part of the action can be written as

$$-i\overline{\psi}_\mu\,\Gamma^{\mu\varrho\sigma}\partial_\varrho\psi_\sigma\ , \tag{13.15}$$

where

$$\Gamma^{\mu\varrho\sigma} = \begin{cases} \Gamma^\mu\Gamma^\varrho\Gamma^\sigma & \text{if} \quad \mu,\ \varrho \text{ and } \sigma \text{ are all different,} \\ 0 & \text{if} \quad \mu=\varrho\ , \quad \mu=\sigma\ , \quad \text{or} \quad \varrho=\sigma\ . \end{cases} \tag{13.16}$$

Because of the antisymmetry of $\Gamma^{\mu\varrho\sigma}$, Eq. (13.15) is invariant under the space-time dependent transformation (13.14). One can *fix the gauge* by choosing

$$\Gamma^\mu\psi_\mu = 0\ , \tag{13.17}$$

as this fixes the right hand side of $\Gamma^\mu \partial_\mu \varepsilon$ in (13.14). Writing

$$\Gamma^{\mu\varrho\sigma} = \Gamma^\mu \Gamma^\varrho \Gamma^\sigma - \delta^{\mu\varrho}\Gamma^\sigma - \delta^{\varrho\sigma}\Gamma^\mu + \delta^{\mu\sigma}\Gamma^\varrho \tag{13.18}$$

(which one simply proves by checking the cases where some of the indices coincide), we find that, in this gauge, the wave equation following from (13.15), $\Gamma^{\mu\varrho\sigma}\partial_\varrho\psi_\sigma = 0$, simplifies into

$$-\Gamma^\mu \partial_\varrho \psi_\varrho + \Gamma^\varrho \partial_\varrho \psi_\mu = 0 \; . \tag{13.19}$$

Multiplying this with Γ^μ then gives, in addition to (13.17),

$$D\partial^\varrho \psi_\varrho - (2\delta^{\varrho\mu} - \Gamma^\varrho\Gamma^\mu)\partial_\varrho\psi_\mu = 0 \quad \rightarrow \quad \partial_\mu\psi_\mu = 0 \tag{13.20}$$

(since $D > 2$). This turns (13.19) into the ordinary Dirac equation $\Gamma\partial\,\psi_\mu = 0$.

The mass shell condition is then $p^2 = 0$. In momentum space, at a given value p^μ of the 4-momentum, the wave function can be split up as follows:

$$\psi_\mu = \varepsilon^i_\mu \psi_i + p_\mu \psi^{(1)} + \bar p_\mu \psi^{(2)} \; , \tag{13.21}$$

where $\bar p_\mu = (-1)^{\delta^{\mu 0}} p_\mu$ and ε^i_μ is defined to be orthogonal to both p_μ and $\bar p_\mu$, so that this part of the wave function is purely transversal. The mass shell condition gives $p^2 = 0\,,\ p_\mu \bar p_\mu > 0$. Thus, Eq. (13.20) implies

$$\psi^{(2)} = 0 \; . \tag{13.22}$$

Furthermore, the contribution of $\psi^{(1)}$ can be eliminated by performing an on-shell gauge transformation of the form (13.14). So, we see that indeed only the completely transverse fields survive. The gauge condition (13.17) also selects out only those fields that have $\Gamma^i\psi_i = 0$. This way we indeed get on-shell the 56-component gravitino field. The type I theory requires only one chiral supersymmetry generator ε, the type II theories require two.

An interesting case is the massless scalar field $\phi(x)$. We see that all closed string models produce at least one of such modes. Of course, there is no mathematical difficulty in admitting such fields, although in the real world there is no evidence of their presence. Since this scalar appears to be associated with the trace of the $G^{\mu\nu}$ field, its interactions often appear to come in the combination $e^\phi\sqrt{g}\,R$. It is therefore referred to as the 'dilaton field'. Dilatons also arise whenever one or more dimensions in a generally relativistic theory are compactified:

$$g_{\mu\nu} \rightarrow \begin{pmatrix} \tilde g_{\mu\nu} & A_\mu \\ A_\mu & \phi \end{pmatrix} \; . \tag{13.23}$$

The fact that our theories are generating massless scalar fields might suggest that they are related to a system in 11 or 12 dimensions. Indeed, supergravity can be formulated in up to 11 dimensions, and one finds that the dilaton emerges as part of the $N = 2$ supermultiplet after Kaluza-Klein reduction to 10 dimensions.

13.2. Tensor fields and D-branes.

So-far, the fields we saw were recognizable from other field theories. But what do the completely antisymmetric tensor fields $B^{\mu\nu}$, $C^{\mu\nu}$, $C^{\lambda\mu\nu}$ and the self-dual $C^{\mu_1\cdots\mu_4}$ correspond to?

In conventional, 4 dimensional field theories, such fields do not generate anything new. The antisymmetric tensor $B^{\mu\nu}$ has only one component in the transverse direction: $\frac{1}{2}(D-2)(D-3)=1$. This is a spinless particle, and such particles are more conveniently described using scalar fields. In higher dimensions, however, they represent higher spin massless particles. If we use fields $C^{\mu_1\cdots\mu_k}$ to describe such particles in a Lorentz-covariant way, we again encounter the difficulty that the timelike components should not represent physical particles. Again, local gauge-invariance is needed to remove them all. We demonstrate this for the case $k=2$, the higher tensors can be treated in a completely analogous way. Let the dynamical field be $B^{\mu\nu}=-B^{\nu\mu}$. Introduce the 'field strength'

$$H^{\lambda\mu\nu}=\partial_\lambda B^{\mu\nu}+\partial_\mu B^{\nu\lambda}+\partial_\nu B^{\lambda\mu}\ . \tag{13.24}$$

H is completely antisymmetric in its indices. The kinetic part of our Lagrangian will be

$$\mathcal{L}=-\tfrac{1}{12}H^{\lambda\mu\nu}H^{\lambda\mu\nu}\ . \tag{13.25}$$

This 'theory' has a local gauge invariance:

$$B^{\mu\nu}\rightarrow B^{\mu\nu\,\prime}=B^{\mu\nu}+\partial_\mu\zeta^\nu-\partial_\nu\zeta^\mu\ , \tag{13.26}$$

where $\zeta^\mu(x)$ may depend on space-time. Note that $H^{\lambda\mu\nu}$ is invariant under this gauge transformation. Note also, that ζ^μ itself has no effect if it is a pure derivative: $\zeta^\mu=\partial_\mu\zeta(x)$. As we did in the Lagrangian (13.9), we choose a 'Coulomb gauge':

$$p_iB^{i\mu}=0\ , \tag{13.27}$$

where i only runs over the spacelike components (this gives a condition on $\partial_i^2\zeta^\mu$, and since we can impose $\partial_i\zeta^i=0$, it can always be fulfilled). In momentum space, take $\vec p$ in the $(D-1)$-direction. Then we split $B^{\mu\nu}$ as follows:

$$\begin{aligned} B_{(\mathrm{tr})}&=B^{ij}\ ,\quad i,j=1,\cdots D-2\ , & B_{(1)}&=B^{i\,D-1}\ ,\\ B_{(2)}&=B^{i0}\ , & B_{(3)}&=B^{0\,D-1}\ . \end{aligned} \tag{13.28}$$

$B_{(1)}$ and $B_{(3)}$ are put equal to zero by our gauge condition. The Lagrangian is then

$$\mathcal{L}=-\tfrac{1}{4}p_\mu^2\,(B^{ij}_{(\mathrm{tr})})^2+\tfrac{1}{2}\vec p^{\,2}\,(B^{i0}_{(2)})^2\ . \tag{13.29}$$

Exactly as in the vector case, described in Eq. (13.11), the time component $B_{(2)}$, does not propagate in time, since it only carries the spacelike part of the momentum in its Lagrangian. it generates a force resembling the Coulomb force.

Curiously, the gauge group (13.26) is Abelian for all higher tensors — only in the vector case, we can have non-Abelian Yang-Mills gauge transformations. It is not known how to turn (13.26) into a non-Abelian version, and, actually, this is considered unlikely to be possible at all, and neither is it needed. The tensor fields do have a remarkable physical implication.

String theory does produce interactions between strings. We have seen in Chapter 6 how to compute some of these, but tree diagrams as well as loop corrections showing all sorts of interactions of merging and splitting strings can all be computed unambiguously. So we do have interactions between these fields. The equations of motion for the anti-symmetric tensor fields therefore contain nonlinear terms to be added into our linearized equations such as $\partial_\lambda H^{\lambda\mu\nu}=0$. So, the zero modes, as well as the massive string modes, will all add contributions to the source $J^{\mu\nu}$ in

$$\partial_\lambda H^{\lambda\mu\nu}=J^{\mu\nu}\ . \tag{13.30}$$

Now notice that these sources are conserved:

$$\partial_\mu J^{\mu\nu}=0\ . \tag{13.31}$$

This follows from the antisymmetry of $H^{\lambda\mu\nu}$, and violation of this conservation law would lead to violation of the gauge symmetry (13.26), which we cannot allow. What kind of charge is it that is conserved here?

In the Maxwell case. the source is J^μ, and its time component is the electric charge density, $Q=\int \mathrm{d}^{D-1}\vec x\, J^0(\vec x,t)$. Current conservation implies charge conservation: $\mathrm{d}Q(t)/\mathrm{d}t=0$. In the case of tensor fields, we have conserved vector charges:

$$Q^\nu(t)=\int \mathrm{d}^{D-1}\vec x J^{0\nu}(\vec x,t)\ . \tag{13.32}$$

Next, we want to see the analogue of charge quantization, $Q=Ne$, for vector charges. Consider the component in the 3-direction. The fields $B^{\mu3}$ and $H^{\lambda\mu3}$ obey the same equations as A^μ and $F^{\lambda\mu}$ of the Maxwell field, except that $\partial/\partial x^3$ is left out: $B^{33}=H^{3\mu3}=0$. So, the continuity equations not only hold in the $D-1$ dimensional space, but also on the $D-2$ dimensional plane $x^3=$ Constant. Furthermore, we have

$$\frac{\mathrm{d}Q^3}{\mathrm{d}x^3}\ =\ \int \mathrm{d}^{D-2}\vec x\,\partial_3 J^{03}(\vec x)\ =\ -\int \mathrm{d}^{D-2}\vec x\sum_{\substack{i\neq3\\ i\neq0}}\partial_i J^{0i}\ =\ 0\ , \tag{13.33}$$

so the charge does not change if we move the plane around. We interpret this as a *flux* going through the plane. The tensor source describes conserved fluxes.

If these fluxes are quantized then, indeed, we are describing strings going through the plane $x^3=$ Constant. Thus, the source function suggests that we have ‘charged strings’.

In case of higher tensor fields, this generalizes directly into ‘charged 2-branes’, ‘3-branes’, etc. This is how we get a first indication that D-branes may be more than mathematical singularities in string theory — they may emerge as regular solutions of the string field equations. However, since not all tensor fields emerge in the various types of string theories, we expect only 1-branes (solitonic strings) if there are 2-tensor fields, 2-branes if there are 3-tensor fields, etc.

13.3. S-duality.

If we take the massless states *but ignore all their interactions*, we note that the 'free' field theories that they generate in space-time allow for dual transformations. Consider a theory described by the interaction (13.25). The field equations for $H^{\lambda\mu\nu}$ are obtained if we extremize the Lagrangian (13.25) for all those fields $H^{\lambda\mu\nu}$ that can be written as a curl, Eq. (13.24). Also, the *functional integral* is obtained by integrating $\exp(i\int\mathcal{L}\mathrm{d}^D x)$ over all fields H coming from a B field as written in (13.24). A general theorem from mathematics (very easy to prove) says that the B field exists if and only if $H^{\lambda\mu\nu}$ obeys a Bianchi equation:

$$\partial_\kappa H^{\lambda\mu\nu} \quad \pm \quad \text{cyclic permutations} \quad = \quad 0 \tag{13.34}$$

(the sign being determined by the permutations).

We can view this as a *constraint* on the functional integral. Such a constraint, however, can also be imposed in a different way. We add a *Lagrange multiplier field* $K_{\kappa\lambda\mu\nu}$ to our degrees of freedom. Just as the previous fields, K is completely antisymmetric in all its indices. Thus, the total Lagrangian becomes[13]

$$\mathcal{L} = -\tfrac{1}{12}H^{\lambda\mu\nu}H^{\lambda\mu\nu} + \tfrac{1}{6}iK_{\kappa\lambda\mu\nu}\partial_\kappa H^{\lambda\mu\nu}\ . \tag{13.35}$$

The factor i here allows us to do functional integrals — the functional integral over the K field then produces a Dirac delta in function space, forcing Eq. (13.34) to hold.

By partial integration, we can let ∂_κ in Eq.(13.35) act in the other direction. The Gaussian integral over the H field can now be done. We split off a pure square:

$$\mathcal{L} = -\tfrac{1}{12}(H^{\lambda\mu\nu} - i\partial_\kappa K_{\kappa\lambda\mu\nu})^2 - \tfrac{1}{12}(\partial_\kappa K_{\kappa\lambda\mu\nu})^2\ . \tag{13.36}$$

The square just disappears when we integrate over the H field (with a contour shift in a complex direction), so only the last term survives. It is actually more convenient now to replace the K field by its 'dual',

$$A^{\mu_1\cdots\mu_{D-4}} = \tfrac{1}{4!}\varepsilon^{\mu_1\cdots\mu_{D-4}\,\kappa\lambda\mu\nu}K_{\kappa\lambda\mu\nu}\ , \tag{13.37}$$

so that the Lagrangian (13.36) becomes[14]

$$\mathcal{L} = -\tfrac{1}{2\cdot(D-4)!}\tilde{H}^2_{\mu_1\cdots\mu_{D-3}}\ , \tag{13.38}$$

with

$$\tilde{H}_{\mu_1\cdots\mu_{D-3}} = \tfrac{1}{D-3}(\partial_{\mu_1}A_{\mu_2\cdots\mu_{D-3}} \pm \text{cyclic permutations}). \tag{13.39}$$

[13] In these notes, we do not distinguish upper indices from lower indices in the space-time fields, since space-time is assumed to be flat. Of course, we do include the minus signs in the summations when a time component occurs.

[14] The numerical coefficients, such as $\frac{1}{12}, 1/(D-4)$, etc., are chosen for convenience only. Generally, one divides by the number of possible symmetry permutations of the term in question, which makes the outcome of simple combinatorial calculations very predictable.

We note, that this Lagrangian is exactly like the one we started off with, except that the fields have a different number of indices, in general. The independent field variable $B^{\mu\nu}$ is replaced by a field $A_{\mu_1\cdots\mu_{D-4}}$; in general the number of indices of the dual A field is $D-2$ minus the number of indices of the B field. Notice that, in 4 dimensions, the field dual to a two-index field $B^{\mu\nu}$ is a scalar. Notice also that, by eliminating the B field, we no longer have the possibility to add a mass term for the B field, or even interactions. Interactions, as well as mass terms, for the A field have entirely different effects. This duality, called S-duality (S standing for 'space-time'), holds rigorously only in the absence of such terms.

In 4 dimensions, the Maxwell field is self-dual, in the absence of currents. This is the magnetic - electric duality. Electric charges in one formalism, behave as magnetic charges in the other and *vice versa*. The Dirac relation,

$$e \cdot g_m = 2\pi n \,, \tag{13.40}$$

where n must be an integer, tells us that, if the coupling constant in an interaction theory is e, the dual theory, if it exists at all, will have interactions with strength $g_m = 2\pi n/e$.

Now, return to the zero modes of string theory. We wrote the fields of these modes as fields with 4 indices or less, because the dual transformation would allow us to transform the higher fields into one of these anyhow. S-duality pervades the zero mode sector of string theory.

There are, however, interactions, and how to deal with these is another matter. We do note that the sources of $C^{\mu_1\cdots\mu_k}$ fields are $k-1$-branes. The sources of their S-duals are $D-k-1$-branes. The relation between the strengths by which they couple to the C fields must be as in the Dirac relation (13.40).

Type IIB strings generate fields C with all even numbers of indices. If we S-dualize the fields, the sources of strength q of the two-index fields are replaced by sources with coupling strengths $2\pi/q$. Thus, S-duality transforms weakly coupled strings to strongly couples strings. This, at least, is what string theoreticians believe. Actually, for strings, the S duality transformation can be turned into a larger group of discrete transformations.

Now, a note about T-duality. This kind of duality can be treated in a manner very similar to S-duality. Here, we perform the transformation on the X^μ fields on the string world sheet. Write the Polyakov action as

$$S = -\tfrac{1}{2}T\int \mathrm{d}^2\sigma (A^\mu_\alpha)^2 \,, \tag{13.41}$$

where A^μ_α is *constrained* to be the gradient of a field $X_\mu(\sigma)$. This constraint can also be imposed by demanding

$$\partial_\alpha A^\mu_\beta - \partial_\beta A^\mu_\alpha = 0 \,. \tag{13.42}$$

So, we introduce the Lagrange multiplier $Y^\mu(\sigma)$, a world-sheet scalar because the world sheet has only two dimensions:

$$S = \int \mathrm{d}^2\sigma \Big(-\tfrac{1}{2}T(A^\mu_\alpha)^2 - i\varepsilon^{\alpha\beta}Y^\mu\partial_\alpha A^\mu_\beta \Big) \,. \tag{13.43}$$

We turn to the dual field $\tilde{A}^\mu_\alpha = \varepsilon^{\alpha\beta} A^\mu_\beta$, in terms of which we get

$$S = \int \mathrm{d}^2\sigma\Big(-\tfrac{1}{2}T(\tilde{A}^\mu_\alpha - \tfrac{i}{T}\partial_\alpha Y^\mu)^2 - \tfrac{1}{2T}(\partial_\alpha Y^\mu)^2\Big)\ , \tag{13.44}$$

where, again, the quadratic term vanishes upon integrating over the $\tilde{A}$ field. Note that the string constant T turned into $1/T$. This is because the expression (13.43) can be seen as a Fourier transform. The exponent of the original action was a Gaussian expression; the new exponentiated action is the Fourier transform of that Gaussian, which is again a Gaussian, where the constant in the exponent is replaced by its inverse. If we would choose the zero mode of X^μ to be periodic, the zero mode of the Y^μ field would be on the *Fourier dual* space of that, which is a discrete lattice. This is how this zero mode received a Neumann boundary condition where it had been Dirichlet before. It also explains why it is said that T duality replaces the string constant T into $1/T$.

14. Miscelaneous and Outlook.

String Theory has grown into a vast research discipline in a very short time. There are many interesting features that can be calculated in detail. However, enthusiasm sometimes carries its supporters too far. Promises are made concerning the 'ultimate theory of time and matter' that could not be fulfilled, and I do not expect that they will, unless a much wider viewpoint is admitted. String theory appears to provide for a new framework allowing one to numerous hitherto unsuspected structures in field theories. Possibly strings, together with higher dimensional D-branes, are here to stay as mathematical entities in the description of particles at the Planck length, but I for one expect that more will be needed before a satisfactory insight in the dynamical laws of our world is achieved.

In these notes, I restrict myself as much as possible to real calculations that can be done, while trying to avoid wild speculations. The speculations frequently discussed in the literature do however inspire researchers to carry on, imagine, and speculate further. A surprisingly coherent picture emerges, but we still do not know how to turn these ideas into workable models.

Let me briefly mention the things one can calculate.

14.1. String diagrams

The functional integrals can be generalized to include an arbitrary number of strings in the *asymptotic states*. In section 6, we mentioned how to compute interactions with strings as if they were due to perturbations in the Hamiltonian. One might have wondered why interactions with 'external' strings, being extended objects, nevertheless have to be represented as point interactions. The reason for this is that string theory does not allow any of the sources considered to be 'off mass shell', that is, disobey their own equations of motion. So, the external strings have to be on mass shell. In a setting where the interactions are handled perturbatively, this means that the initial and final strings in an interaction are stable objects, entering at time $= -\infty$ and leaving at time $= +\infty$. The string world sheet shows these asymptotic states as sheets extended infinitely far in one dimension τ, with another coordinate σ essentially staying finite. If we now turn to the conformal gauge, and map this world sheet on a compact complex surface such as the interior of a circle, then the initial and final strings always show up as *singular points* on this surface. Take for instance a single string stretching from $\sigma^2 = -\infty$ to $\sigma^2 = +\infty$, while $0 \le \sigma^1 \le \pi$ (see Section 10). This can be mapped on a circle if we use a coordinate

$$z = \frac{ie^\omega}{1 + ie^\omega}\ , \qquad \omega = \sigma^2 + i\sigma^1\ . \tag{14.1}$$

Notice that

$$z - \tfrac{1}{2} = \frac{ie^\omega - 1}{2(ie^\omega + 1)}\ , \tag{14.2}$$

so that the end points $\sigma^1 = 0$ and $\sigma^1 = \pi$, where $e^\omega = a\ =$ real, have

$$|\, z - \tfrac{1}{2}| = \tfrac{1}{2}\ , \tag{14.3}$$

so this is a circle. The points $\sigma^2 = \pm\infty$ are the points 0 and 1.

Similarly, if we have a *closed* string at an end point, this shows up in a conformal gauge as a single singular point in the bulk of a string world sheet (rather than at the boundary, as with open strings). In a conformal treatment of the theory, these singular points are handled as 'vertex insertions', and the mathematics ends up being essentially as what we did in section 6.

14.2. Zero slope limit

In particular, one can now compute the strength of the interactions of the zero modes. There is one 'coupling parameter' g_s, which is the overall coefficient that goes with this diagram, or the normalization of the functional integral. This coefficient is not fixed. The interactions can be represented as a *low energy effective action.* We here give the results to be found in the literature, limiting ourselves to the bosonic parts only. Note that the results listed below could be modified by redefinitions of the fields. The field normalization are usually defined by fixing the kinetic parts in the Lagrangian, but in theories containing gravity, this could be done in different ways.

14.2.1. Type II theories

The bosonic part for type IIA is:

$$S^{IIA}_{\rm eff} = \frac{1}{(2\pi)^7}\int {\rm d}^{10}x\sqrt{g}\quad \Big\{R - \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{12}e^{-\phi}H^2 - \tfrac{1}{4}e^{3\phi/2}F^2 - \tfrac{1}{48}e^{\phi/2}G^2$$
$$- \tfrac{1}{2304}\tfrac{1}{\sqrt{g}}\varepsilon^{\mu_0\cdots\mu_9}B_{\mu_0\mu_1}G_{\mu_2\cdots\mu_5}G_{\mu_6\cdots\mu_9}\Big\}\ . \tag{14.4}$$

Here, R is the Riemann scalar. This part is conventional gravity. ϕ is the dilaton scalar, and we do notice that it appears exponentially in many of the interactions. It has no mass term, which means that its *vacuum value*, $\langle\phi\rangle_0$ is not fixed. This value does determine the interaction strengths of the other fields, relative to gravity. H^2 is the kinetic part of the B field, Eq. (13.25); $F^2 = F^{\mu\nu}F_{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$, and C_μ is the vector field from the R-R sector. $G_{\mu\nu\kappa\lambda} = \partial_\mu C_{\nu\kappa\lambda}$ + cyclic permutations; it is the covariant curl belonging to the $C_{\mu\nu\lambda}$ field in the R-R sector. The last term is an interaction between the B and the C fields. Notice that it is gauge-invariant upon integration, because the C fields obey the Bianchi identities: $\varepsilon^{\mu_0\cdots\mu_9}\partial_{\mu_1}G_{\mu_2\cdots\mu_5} = 0$.

In the type IIB theory, things are more complicated. It has a rank 4 tensor with self-dual rank 5 curvature fields. There is no simple action for this, but the equations of motion can be written down. The same difficulty occurs in type IIB supergrvity, to which this theory is related.

14.2.2. Type I theory

The effective action for the type I theory is

$$S^I_{\text{eff}} = \frac{1}{(2\pi)^7}\int \mathrm{d}^{10}x\sqrt{g}\Big\{R - \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{4}e^{\phi/2}(F^a_{\mu\nu}F^{a\,\mu\nu}) - \tfrac{1}{12}e^{\phi}H^2\Big\}\ . \tag{14.5}$$

Here, $F^a_{\mu\nu}$ is the Yang-Mills field strength associated to the gauge field A^a_μ.

14.2.3. The heterotic theories

The effective action is here:

$$S^H_{\text{eff}} = \frac{1}{(2\pi)^7}\int \mathrm{d}^{10}x\sqrt{g}\Big\{R - \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{4}e^{-\phi/2}(F^a_{\mu\nu}F^{a\,\mu\nu}) - \tfrac{1}{12}e^{-\phi}H^2\Big\}\ , \tag{14.6}$$

where the Yang-Mills field is either that of $E_8 \times E_8$ or $SO(32)$. Note the difference with (14.5): the dilaton field occurs with a different sign.

14.3. Strings on backgrounds

Strings that propagate in a curved space-time can be described by the world sheet action

$$S = -\frac{T}{2}\int \mathrm{d}^2\sigma\sqrt{h}\,h^{\alpha\beta}\partial_\alpha X^\mu\,\partial_\beta X^\nu G_{\mu\nu}(X(\sigma))\ , \tag{14.7}$$

where now $G_{\mu\nu}$ is a given function of the coordinates X^μ. It turns out, however, that this only works if, right from the start, $G_{\mu\nu}$ obeys equations that tell us that these fields, also, are 'on mass shell'. This means that they must obey Einstein's equations. Studying the conformal anomalies (which are required to cancel out) not only gives us these equations, but also adds higher order corrections to them. The antisymmetric B tensor field can also be added to the background, yielding an additive term to the Lagrangian (14.7) of the form:

$$+\,\tfrac{1}{2}\int \mathrm{d}^2\sigma \varepsilon^{\alpha\beta} B^{\mu\nu}(X)\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\ , \tag{14.8}$$

where ε is the usual antisymmetric tensor in the world sheet. Note, that this term transforms covariantly under transformations of the σ coordinates. It is in fact analogous to electric charge for an elementary particle:

$$q\int \mathrm{d}\tau A_\mu(x)\,\partial_\tau x^\mu\ . \tag{14.9}$$

14.4. Coordinates on D-branes. Matrix theory.

For a string with Dirichlet boundary conditions on a D-brane, one may impose the more general boundary condition

$$X^\mu(0,\tau)=Y^\mu(u)\ , \tag{14.10}$$

where u is a set of $p+1$ coordinates for the brane. In a flat D-brane, u is just the set of X coordinates for which we have a Neumann boundary: these X components move along the brane. Further study of the consistency of this system eventually leads for an expression for the action of the Y coordinates of the p dimensional D-brane. Strings with both ends on the same D-brane can move freely on the brane. Their bosonic zero modes, which are associated to the u coordinates, describe particles living on the D-brane. Strings with both ends on different D-branes are more complicated to describe, unless the D-branes coincide, or coincide approximately. In that case, we have coordinates u carrying the indices $i,\ j$ of two different branes.

Take the case $p=9$, where the D-brane covers all of space-time, and u is just the general coordinate X^μ itself. Now we see that these coordinates may carry two indices. this lead to speculations concerning a theory where the coordinates X^μ are promoted to $N\times N$ matrices $X^{i\,\mu}_j$. We can describe a set of N free particles by taking the eigenvalues of the matrix X^μ. The theory is not (yet?) well understood, but its name has already been discovered: M-theory.

14.5. Orbifolds

Manifolds M, in particular flat manifolds, allow for symmetries under various groups of discrete coordinate transformations. If we call the group of such a symmetry transformation S, then requiring this symmetry for all string states implies that the strings are really living in the space M/S. If S has one or more fixed points on M, then these remain special points for the string. The presence of these special points reduces the large groups of continuous symmetries of the system, and indeed may give it more structure. Orbifolds play an important role in string theory, in particular the heterotic string.

14.6. Dualities

More and more examples of duality are found. it appears that dualities relate all string theories to one another. This is sometimes claimed to imply that we really do have only one theory. That is not quite correct; duality may link the mathematical equations of one theory to another, but this does not mean that the theories themselves are equal, since our essentially infinite, four dimensional, space-time is represented in a different way in these different theories.

14.7. Black holes

Black holes are (also) not the subject of these notes. D-branes wrapped over compactified dimensions are found to carry gravitational fields, because of their large mass. In special cases, where the D-brane charge is set equal to the mass, the approximately classical solution for a collection of a large number of such branes can be described. It appears to behave much like a black hole with electric charge equal to its mass (called the BPS limit, after Bogomolnyi, Prasad and Sommerfield). According to the theory of black holes, these objects must carry *entropy*:

$$S = \tfrac{1}{4}(\text{area}) \ . \tag{14.11}$$

According to statistical physics, entropy also equals the logarithm of the total number of quantum states such a system can be in. The D-brane sector of string theory appears to generate the correct number of quantum states.

14.8. Outlook

String theory clearly appears to be strikingly coherent. What seems to be missing presently, however, is a clear description of the *local* nature of its underlying physical laws. In all circumstances encountered until now, it has been imperative that external fields, in- and outgoing strings and D-branes are required to obey their respective field equations, or lie on their respective mass shells. Thus, only effects due to external perturbations can be computed when these external perturbations obey equations of motion. To me, this implies that we do not understand what the *independent* degrees of freedom are, and there seems to be no indication that these can be identified. String theoreticians are right in not allowing themselves to be disturbed by this drawback.

String Theory Exercises*

March 19, 2004

Exercise 1

Let us consider the action describing a point particle of mass m moving freely in a $d+1$ dimensional Minkowski space–time. It will be expressed as the invariant length of the world–line

$$S = \int_{\tau_0}^{\tau_1} d\tau\,\mathcal{L} = -m\int_{\tau_0}^{\tau_1} ds = -m\int_{\tau_0}^{\tau_1} d\tau\,\sqrt{-\dot{x}^\mu \dot{x}_\mu}. \tag{1.1}$$

where $x^\mu(\tau)$ is the space–time position of the particle at the proper time τ and $\dot{x}^\mu = dx^\mu/d\tau$. The space–time metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$.

- Show that S is invariant with respect to *reparametrization* of the world line: $\tau \to \tau' = \tau'(\tau)$.
- Compute the momentum of the particle $p^\mu = \eta^{\mu\nu}\,\delta\mathcal{L}/\delta\dot{x}^\nu$ and show that it describes a particle of mass m.
- Find the equation of motion for $x^\mu(\tau)$ by minimizing the action with respect to a variation $x^\mu \to x^\mu + \delta x^\mu$ and find the most general solution. In particular show that, if we give a physical interpretation to τ as being the time, namely if we set $x^0(\tau) \propto \tau$, the solution has $v^\mu = \dot{x}^\mu = \mathrm{const}$. Show that on the solutions the action can be written in the form:

$$S = -m\,|x_1 - x_0|. \tag{1.2}$$

$$\begin{aligned}
|x_1 - x_0| &= \sqrt{-(x_1^\mu - x_0^\mu)(x_{1\mu} - x_{0\mu})} \\
x_0^\mu &= x^\mu(\tau_0)\ ;\ \ x_1^\mu = x^\mu(\tau_1) \\
x_1^\mu - x_0^\mu &= v^\mu\,(\tau_1 - \tau_0).
\end{aligned} \tag{1.3}$$

*References to equations in the lecture notes refer to the 10/02/04 version of the notes.

and that the momentum can be written as:

$$p^\mu \;=\; m\,\frac{(x_1^\mu - x_0^\mu)}{|x_1 - x_0|}. \tag{1.4}$$

(for later convenience we allow p^0 to have also negative values and we interpret the energy of the particle to be $E = |p^0|$. According to this notation, in a scattering process, the incoming particles will have $p^0 = E > 0$ while the outgoing will have $p^0 = -E < 0$.)

- Now consider a different action on the same world line of the particle, but this time function of two independent quantities, namely $e(\tau)$ and $x^\mu(\tau)$:

$$S' \;=\; \frac{1}{2}\int d\tau\;(e^{-1}\,\dot{x}^\mu\dot{x}_\mu - e m^2) \tag{1.5}$$

Show that S' is invariant with respect to the reparametrization transformation in τ which, in its infinitesimal form is:

$$\begin{aligned} \delta x^\mu &= \epsilon(\tau)\,\dot{x}^\mu \\ \delta e &= \frac{d}{d\tau}\,(\epsilon(\tau)\,e) \\ \epsilon(\tau_0) &= \epsilon(\tau_1) \;=\; 0. \end{aligned} \tag{1.6}$$

[**Hint:** *To show invariance of the action under the above infinitesimal transformations it suffices to show that the corresponding variation of the action is a total derivative of the form written below:*

$$\mathcal{L}(e+\delta e, \dot{x}^\mu + \delta\dot{x}^\mu) \;=\; \mathcal{L}(e, \dot{x}^\mu) + \frac{d}{d\tau}\,(\epsilon\mathcal{L})\,. \tag{1.7}$$

for these transformations the above result follows without the use of the equations of motion. Since by hypothesis ϵ vanishes at the extrema of integration the total derivative in eq. (1.7) does not contribute.]

Compute the field equations from S' corresponding to the fields $e(\tau)$ and $x^\mu(\tau)$ and show that S' is classically equivalent to S.

- Consider a scattering process of N particles described by the world lines of the particles which start at different points x_i^μ ($i = 1,\ldots,N$) and intersect in the same scattering point y^μ. Let the process be described by the total action $S = \sum_{i=1}^N S_i$, S_i being the world line actions of the single particles, by minimizing S with respect to the position y^μ of the scattering point deduce the momentum conservation condition for the process: $\sum_{i=1}^N p_i^\mu = 0$ (recall that in our notation the incoming particles have $p^0 > 0$ and the outgoing have $p^0 < 0$). [**Hint:** *Use eq. (1.2) to express the single actions as $S_i = -m_i\,|y - x_i|$. Then from the condition $\delta S/\delta y^\mu = 0$ and the expression (1.4) for the momenta deduce* $\sum_{i=1}^N p_i^\mu = 0$.]

Exercise 2

A generic surface $\mathcal{S}$ spanned by affine parameters τ, σ and embedded in a higher dimensional space (ambient space) with coordinates X^μ, can be described by the parametric equations $X^\mu = X^\mu(\tau, \sigma)$. An infinitesimal element of its surface $d\mathcal{S}$ can be represented by a tensor $d\mathcal{S}^{\mu\nu}(\tau, \sigma) = (dX^\mu \wedge dX^\nu)_{|\mathcal{S}} = \Sigma^{\mu\nu}(\tau, \sigma)d\tau d\sigma$ which defines the projection of $d\mathcal{S}$ on the plane (X^μ, X^ν), the tensor $\Sigma^{\mu\nu}(\tau, \sigma)$ defines the plane tangent to the surface $\mathcal{S}$ at the point (τ, σ) and is defined as follows:

$$\Sigma^{\mu\nu} = \partial_\tau X^\mu \partial_\sigma X^\nu - \partial_\tau X^\nu \partial_\sigma X^\mu \,.$$

The area of the surface $A(\mathcal{S})$ is then defined by the integral:

$$A(\mathcal{S}) = \int_{\mathcal{S}} \sqrt{\frac{1}{2}\, d\mathcal{S}_{\mu\nu} d\mathcal{S}^{\mu\nu}} = \int_{\mathcal{S}} d\tau d\sigma \sqrt{\frac{1}{2}\, \Sigma_{\mu\nu}\Sigma^{\mu\nu}} \tag{2.1}$$

Compute the Σ tensor and the area of the surface given by:

$$\begin{aligned} X^0 &= A\tau \\ X^1 &= -B\tau \\ X^2 &= \sigma \end{aligned}$$

where A and B are constants, $\tau \in (0, 1)$ and $\sigma \in (0, 1)$. You may consider the metric of the ambient space to be Euclidean. Compare the result of the integral with what you would expect.

Exercise 3

Consider the Nambu–Goto action described by equation (2.9) of the lecture notes. Show that the expression in the square root can be written as the determinant of a 2×2 matrix (induced metric) $h_{\alpha\beta}$ defined as

$$h_{\alpha\beta} \;=\; \partial_\alpha X^\mu \, \partial_\beta X_\mu \tag{3.1}$$

In the Nambu–Goto action the only independent function is $X^\mu(\sigma, \tau)$. It is possible to reformulate the theory in a classically equivalent way using the Polyakov action which describes $h_{\alpha\beta}(\sigma, \tau)$ and $X^\mu(\sigma, \tau)$ as independent fields and has the advantage of not having the square root in the integral (see next exercise)

Exercise 4

Consider the Polyakov action of a string moving on a D–dimensional Minkowski background (with metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$):

$$S = -\frac{T}{2} \int_\Sigma d\sigma^2 \sqrt{h(\sigma)}\, h^{\alpha\beta}(\sigma)\, \partial_\alpha X^\mu\, \partial_\beta X_\mu \tag{4.1}$$

$$\begin{aligned} \sigma^\alpha &= \{\sigma, \tau\} \;\; ; \;\; X^\mu \;=\; X^\mu(\sigma^\alpha)\,;\, \sigma \in [0, \pi]\,;\, \tau \in (-\infty, \infty) \\ \Sigma &= \text{world sheet} \;\; ; \;\; h_{\alpha\beta}(\sigma^\gamma) \;=\; \text{metric on } \Sigma \\ h(\sigma^\gamma) &= -\det(h_{\alpha\beta}) \;\; ; \;\; \alpha = 1, 2\,;\, \mu \;=\; 0, \ldots, \mathrm{D}-1 \end{aligned}$$

where we have used the following short hand notation for partial derivatives: $\partial_\gamma = \frac{\partial}{\partial \sigma^\gamma}$. Moreover whenever repeated upper and lower indices occur summation is understood: $v^\alpha w_\alpha = \sum_\alpha v^\alpha w_\alpha$.

- Show that the *local*, i.e. σ^α–dependent, symmetry transformations are:

$$\underline{\text{reparametrization}}: \qquad \sigma^\alpha \;\rightarrow\; \sigma^{\alpha\prime} \;=\; \sigma^{\alpha\prime}(\sigma) \tag{4.2}$$

$$\underline{\text{Weyl transformations}}: \qquad h_{\alpha\beta} \;\rightarrow\; \Omega^2(\sigma^\alpha)\, h_{\alpha\beta} \tag{4.3}$$

- Compute the energy momentum tensor $T_{\alpha\beta}$ defined by:

$$\begin{aligned} h_{\alpha\beta} &\;\rightarrow\; h_{\alpha\beta} + \delta h_{\alpha\beta} \;\Rightarrow\; S \;\rightarrow\; S + \delta S \\ \delta S &= -\frac{T}{2} \int_\Sigma d\sigma^2 \sqrt{h(\sigma)}\, \delta h_{\alpha\beta}\, T^{\alpha\beta} \end{aligned} \tag{4.4}$$

  which condition on $T_{\alpha\beta}$ does invariance under Weyl transformations imply?

- Show that the *global*, i.e. σ^α–independent, transformations on the X^μ fields which leave S invariant are the Poincaré transformations:

$$\begin{aligned} X^\mu &\;\rightarrow\; X'^\mu = \Lambda^\mu{}_\nu\, X^\nu + a^\mu \\ \eta_{\mu\nu}\, \Lambda^\nu{}_\rho\, \Lambda^\mu{}_\gamma &= \eta_{\rho\gamma} \end{aligned} \tag{4.5}$$

  where both the Lorentz transformation Λ and the translation parameter a^μ do not depend on σ^α.

- Write the equations of motion for the fields $h_{\alpha\beta}(\sigma)$ and $X^\mu(\sigma)$ and show that S is *classically* equivalent to the Nambu–Goto action S':

$$S' = -T \int_\Sigma d\sigma^2 \sqrt{-\det\left(\partial_\alpha X^\mu\, \partial_\beta X_\mu\right)} \tag{4.6}$$

Exercise 5

By fixing reparametrization invariance let us reduce the world sheet metric to the form:

$$h_{\alpha\beta}(\sigma) \;=\; \lambda(\sigma)\,\eta_{\alpha\beta} \;=\; \lambda(\sigma)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.1}$$

- write the action S with this metric
- consider an infinitesimal coordinate transformation on the world sheet:

$$\sigma^\alpha \;\rightarrow\; \sigma^{\alpha\prime} \;=\; \sigma^\alpha + \epsilon^\alpha(\sigma) \tag{5.2}$$

  which implies that we transform X^μ as follows:

$$X^\mu \;\rightarrow\; X^\mu + \delta X^\mu \;=\; X^\mu + \epsilon^\gamma\,\partial_\gamma X^\mu$$

  As it can be deduced from eq. (3.1), the metric $h_{\alpha\beta}$ will transform as follows:

$$\delta h_{\alpha\beta} \;=\; (\partial_\alpha \epsilon^\gamma)\,h_{\gamma\beta} + (\partial_\beta \epsilon^\gamma)\,h_{\gamma\alpha} + \epsilon^\gamma \partial_\gamma h_{\alpha\beta} \tag{5.3}$$

  After fixing the metric to the form (5.1) there is still a residual invariance of the action under *conformal* transformations. A conformal transformation is defined as a coordinate transformation whose only effect is to rescale the metric, namely such that the corresponding infinitesimal variation of the metric has the general form:

$$\delta h_{\alpha\beta} \;=\; C(\sigma)\,h_{\alpha\beta} + \epsilon^\gamma \partial_\gamma h_{\alpha\beta} \tag{5.4}$$

  where $C(\sigma)$ is an infinitesimal function of σ^α which depends on the infinitesimal coordinate shift ϵ. Using equ. (5.3) show that the conformal transformations are generated by an infinitesimal parameter ϵ fulfilling the following condition:

$$(\partial_\alpha \epsilon^\gamma)\,h_{\gamma\beta} + (\partial_\beta \epsilon^\gamma)\,h_{\gamma\alpha} \;=\; (\partial_\gamma\,\epsilon^\gamma)\;h_{\alpha\beta} \tag{5.5}$$

  recall the convention on repeated indices and partial derivation: $(\partial_\gamma\,\epsilon^\gamma = \sum_\gamma \frac{\partial}{\partial\sigma^\gamma}\,\epsilon^\gamma)$ and find the expression of $C(\sigma)$ in equ. (5.4) in terms of ϵ.
- using the light–cone coordinates:

$$\sigma^\pm \;=\; \frac{1}{\sqrt{2}}(\tau \pm \sigma) \tag{5.6}$$

  show that conformal transformations are characterized by $\epsilon^+ = \epsilon^+(\sigma^+)$ and $\epsilon^- = \epsilon^-(\sigma^-)$.
- Write $T_{\alpha\beta}$ and the conditions on it due to energy–momentum conservation and Weyl invariance, in light–cone coordinates.

Exercise 6

Consider the global symmetries of S, i.e. the Poincaré transformations in their infinitesimal form:

$$
\begin{aligned}
X^\mu &\rightarrow X^\mu + \delta X^\mu && (6.1)\\
\delta X^\mu &= a^\mu \qquad \text{(translations)} && (6.2)\\
\delta X^\rho &= \omega_{\mu\nu}\,(M^{\mu\nu})^\rho{}_\sigma\, X^\sigma \quad \text{(Lorentz)} && (6.3)\\
(M^{\mu\nu})^\rho{}_\sigma &= \eta^{\nu\rho}\,\delta^\mu_\sigma - \eta^{\mu\rho}\,\delta^\nu_\sigma &&\\
\eta^{\mu\nu} &= \mathrm{diag}(-1,+1,\ldots,+1) && (6.4)
\end{aligned}
$$

where a^μ and $\omega_{\mu\nu}$ are σ^α–independent.

- Compute the corresponding conserved Noether currents J^μ_α and $J^{\mu\nu}_\alpha$ by computing the variation of the Polyakov action S with respect to the transformations (6.2) and (6.3) and expressing it (using the field equations) in the form:

$$\delta S = \int d\sigma d\tau\, \partial^\alpha\,(J^\mu_\alpha\, a_\mu + J^{\mu\nu}_\alpha\,\omega_{\mu\nu}) \qquad (6.5)$$

  recall that we are using here and in all the following exercises $h_{\alpha\beta} = \eta_{\alpha\beta}$

- Write the equations of motion for X^μ and solve them with the following boundary conditions:

$$
\begin{aligned}
&\underline{\text{Neumann:}} && \partial_\sigma X^\mu(\tau,\sigma=0) = \partial_\sigma X^\mu(\tau,\sigma=\pi) = 0\\
&\underline{\text{Closed string:}} && X^\mu(\tau,\sigma) \equiv X^\mu(\tau,\sigma+\pi) \quad \forall\sigma
\end{aligned}
$$

  showing that the most general solution will have the form:

$$X^\mu(\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \qquad (6.6)$$

  (express solution through Fourier mode expansion and impose the boundary conditions as constraints on the coefficients)

- Show that for the above solutions the CM momentum:

$$P^\mu = \int_0^\pi d\sigma\, J^\mu_\tau \qquad (6.7)$$

  is conserved.

- Write the expression of P^μ for the above solutions.

Exercise 7

Consider an open string satisfying the usual Neumann boundary conditions along the directions $X^0, \dots X^{D-2}$, but a different one on X^{D-1}. For the following two cases, compute the mode expansion of X^{D-1} as in the previous exercise.

- Dirichlet boundary conditions at both endpoints (DD):
$$\partial_\tau X^{D-1}(\tau, \sigma = 0) = 0\,, \qquad \partial_\tau X^{D-1}(\tau, \sigma = \pi) = 0\,. \tag{7.1}$$
This has the interpretation of an open string with both ends on a D-brane, where "D" stands for Dirichlet. Is the momentum P^{D-1} along this direction conserved?
- Dirichlet boundary condition at one endpoint and Neumann at the other endpoint (ND):
$$\partial_\tau X^{D-1}(\tau, \sigma = 0) = 0\,, \qquad \partial_\sigma X^{D-1}(\tau, \sigma = \pi) = 0\,. \tag{7.2}$$
This is an open string with one end on a D-brane and one free.
[**Hint:** *Will the frequencies of the Fourier modes still be integer?*]

Exercise 8

Show that the following functions:

$$\begin{aligned} X^0 &= A\tau \\ X^1 &= A\cos(\tau)\cos(\sigma) \\ X^2 &= A\sin(\tau)\cos(\sigma) \\ X^{i>2} &= 0 \end{aligned} \tag{8.1}$$

define a solution of the string field equations with the Neumann boundary conditions (i.e. describe a free open string). In particular show that it can be written in the form (6.6).

- Compute the energy $E = P^0$ and the angular momentum $\boldsymbol{J}$ for this solution.
- Show that the non linear constraints $T_{\alpha\beta} = 0$ are fulfilled as well, namely that:
$$(\partial_\tau X)^2 + (\partial_\sigma X)^2 = 0\,, \qquad \partial_\tau X^\mu\, \partial_\sigma X_\mu = 0 \tag{8.2}$$
- Show that:
$$\frac{E^2}{|\boldsymbol{J}|} \quad = \quad \text{const.} = \frac{1}{\alpha'} \tag{8.3}$$
and show that $T = 1/(2\pi\alpha')$
- Show that this solution describes an open string with the end points rotating at the speed of light.

Exercise 9

Consider the following parametric equations:

$$\begin{aligned} X^0 &= A\tau \\ X^1 &= A\cos(\tau)\cos(\sigma) \\ X^2 &= A\cos(\tau)\sin(\sigma) \end{aligned} \tag{9.1}$$

Show that they describe a string solution, i.e. that one can write it in the form (6.6) and that the non linear constraints in eq. (8.2) are fulfilled. Which boundary conditions does the solution fulfill in the various space directions?

- Plot the solution on the X^1, X^2 plane in time for τ varying from 0 to 2π with steps of $\pi/4$.
- Compute the conserved current J^μ_α associated with global translational invariance in space–time (see problem 6). Show that the component $P^{\mu=2}$ of the momentum defined by eq. (6.7) is not conserved.
- Compute the variation of the momentum between $\tau = 0$ and $\tau = \tau_0$ and prove the following relation:

$$P^{\mu=2}_{|\tau=\tau_0} - P^{\mu=2}_{|\tau=0} \;=\; \int_0^{\tau_0} d\tau\, \left(J^{\mu=2}_\sigma(\tau,\pi) - J^{\mu=2}_\sigma(\tau,0)\right) \tag{9.2}$$

  the above equation derives from the momentum conservation condition $\partial^\alpha J^\mu_\alpha = 0$ and the right hand side can be interpreted as the momentum flow in the direction $\mu = 2$ across the end–points of the string.

Exercise 10

Consider the following parametric equations:

$$\begin{aligned} X^0 &= 2A\tau \\ X^1 &= A\cos 2\tau\cos 2\sigma \\ X^2 &= A\sin 2\tau\cos 2\sigma \end{aligned} \tag{10.1}$$

Show that they describe a closed string solution and that one can write it in the form (6.6) and that it fulfills (8.2). Plot the solution on the X^1, X^2 plane in time for τ varying from 0 to π with steps of $\pi/8$.

Now consider the following parametric equations:

$$\begin{aligned} X^0 &= A\tau \\ X^1 &= A\cos(2\tau)\sin(2\sigma) \\ X^2 &= A\sin(4\tau)\sin(4\sigma) \end{aligned} \tag{10.2}$$

Show that although they can be written in the form (6.6) they do not fulfill the constraint (8.2).

Exercise 11

Write the components T_{++} and T_{--} of the energy momentum tensor in terms of the parameters x^μ, p^μ, α_n^μ and $\tilde{\alpha}_n^\mu$ defining the open and closed string solutions. In particular in the two cases find the corresponding expressions of the coefficients L_n and $\tilde{L}_n$ defined as follows:

$$\begin{aligned} \underline{\text{open string:}} \qquad T_{++} &= \frac{\ell^2}{2} \sum_{n=-\infty}^{\infty} L_n \, e^{-in\sigma^+} \\ \underline{\text{closed string:}} \qquad T_{++} &= 2\ell^2 \sum_{n=-\infty}^{\infty} \tilde{L}_n \, e^{-2in\sigma^+} \\ T_{--} &= 2\ell^2 \sum_{n=-\infty}^{\infty} L_n \, e^{-2in\sigma^-} \end{aligned} \tag{11.1}$$

The constraints $T_{\alpha\beta} = 0$ can be restated as $L_n = \tilde{L}_n = 0$ for all n. From the conditions $L_0 = \tilde{L}_0 = 0$ derive an expression for the mass squared $M^2 = -p^\mu p_\mu$ in terms of the coefficients α_n^μ and $\tilde{\alpha}_n^\mu$ in both the open and closed string cases.

Exercise 12

Compute in terms of the parameters x^μ, p^μ, α_n^μ and $\tilde{\alpha}_n^\mu$ defining the open and closed string solutions the Hamiltonian associated with the Polyakov action:

$$H(\tau) = \int d\sigma \left(\dot{X}^\mu \frac{dP_\mu}{d\sigma} \right) - L(\tau) \tag{12.1}$$

where $\frac{dP_\mu}{d\sigma}$ and $L(\tau)$ are defined in eqs. (3.43) and (4.1) of the lecture notes. What is the relation between H and L_0, $\tilde{L}_0$ in the open and closed string cases?

Exercise 13

Let us restrict ourselves to the quantized open string case in the light-cone gauge. Derive the commutation relations between x^i, p^i and α_n^i from equations (4.5) and (4.3) of the lecture notes. Let us define the vacuum state $|p^\mu, 0\rangle$ such that $\alpha_n^i |p^\mu, 0\rangle = 0$ for $n > 0$ and $\alpha_0^i |p^\mu, 0\rangle = p^i |p^\mu, 0\rangle$. A generic open string state is obtained by applying the creation operators α_n^i $(n < 0)$ a finite number of times to $|p^\mu, 0\rangle$. Consider the operator N defined as follows:

$$N = \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} \alpha_{-n}^i \alpha_n^i \tag{13.1}$$

Show that on the following state:

$$|p^\mu,\ N_{i,n}\rangle \;=\; \overbrace{\alpha^i_{-n}\dots\alpha^i_{-n}}^{N_{i,n}}|p^\mu,0\rangle \tag{13.2}$$

the operator N has the following value:

$$N\,|p^\mu,\ N_{i,n}\rangle \;=\; nN_{i,n}|p^\mu,\ N_{i,n}\rangle \tag{13.3}$$

Exercise 14

A solution of the string equations of motion can be written in the form (6.6). For the open string we have the further constraint that $X^\mu_L(\tau) = X^\mu_R(\tau)$ and that $X^\mu_L(\sigma^+)$ and $X^\mu_R(\sigma^-)$ are respectively periodic in σ^+ and σ^- of period 2π a part for a constant shift while for the closed string X^μ_L and X^μ_R need only be periodic in $\sigma\in(0,\pi)$ at fixed τ. The theory is also invariant under conformal transformations which can be written in the following infinitesimal form:

$$\begin{aligned}\sigma^+ &\;\to\; \sigma^+ + \epsilon^+(\sigma^+)\\ \sigma^- &\;\to\; \sigma^- + \epsilon^-(\sigma^-)\end{aligned} \tag{14.1}$$

where $\epsilon^+(\sigma^+)$ and $\epsilon^-(\sigma^-)$ are periodic in σ^+ and σ^- of period 2π. For the open string we further require that $\epsilon^+ = \epsilon^-$. In this exercise we will compute the infinitesimal generators of conformal transformations and their algebraic properties. Being periodic of period 2π in their argument, the infinitesimal functions can be expanded in Fourier series:

$$\epsilon^\pm(\sigma^\pm) \;=\; i\sum_{n=-\infty}^{+\infty}\epsilon^\pm_n e^{in\sigma^\pm} \tag{14.2}$$

where the reality condition implies that $\epsilon^\pm_{-n} = -(\epsilon^\pm_n)^\star$.
A generic scalar function $Y(\sigma^\pm)$ will transform under (14.1) as follows:

$$\begin{aligned}Y(\sigma^\pm) &\;\to\; Y(\sigma^\pm+\epsilon^\pm)\sim Y(\sigma^\pm)+\delta Y(\sigma^\pm)\\ \delta Y(\sigma^\pm) &\;=\; \epsilon^\pm\partial_\pm Y(\sigma^\pm) = i\sum_{n=-\infty}^{+\infty}\epsilon^\pm_n e^{in\sigma^\pm}\partial_\pm Y(\sigma^\pm)\end{aligned} \tag{14.3}$$

where we have used eq. (14.2) to derive the last expression. Let us define the following differential operators $\mathbf{L}^{(\pm)}_n$ on functions of $\sigma^\pm$ respectively:

$$\begin{aligned}\mathbf{L}^{(+)}_n\big(Y(\sigma^+)\big) &\;=\; ie^{in\sigma^+}\partial_+Y(\sigma^+)\\ \mathbf{L}^{(-)}_n\big(Y(\sigma^-)\big) &\;=\; ie^{in\sigma^-}\partial_-Y(\sigma^-)\\ \mathbf{L}^{(+)}_n\big(Y(\sigma^-)\big) &\;=\; 0\\ \mathbf{L}^{(-)}_n\big(Y(\sigma^+)\big) &\;=\; 0\end{aligned} \tag{14.4}$$

the product of two of these operators is defined by their consecutive action on a same function Y: $\mathbf{L}_n\,\mathbf{L}_m\,(Y) \equiv \mathbf{L}_n\,(\mathbf{L}_m\,(Y))$. The operators $\mathbf{L}_n^{(\pm)}$ are the generators of the infinitesimal conformal transformations (14.3), indeed we can write:

$$\delta Y(\sigma^{\pm}) = \sum_{n=-\infty}^{+\infty} \epsilon_n^{\pm}\,\mathbf{L}_n^{(\pm)}\,(Y) \tag{14.5}$$

- Show that taking $Y(\sigma^{\pm}) = \sigma^{\pm}$ and considering the definitions (14.4) the infinitesimal conformal transformation (14.1) follows from (14.5).
- Show that on a generic function $Y(\sigma^{\pm})$ the following relations hold:

$$\begin{aligned}
\left(\mathbf{L}_n^{(+)}\,\mathbf{L}_m^{(+)} - \mathbf{L}_m^{(+)}\,\mathbf{L}_n^{(+)}\right)\left(Y(\sigma^+)\right) &= (n-m)\mathbf{L}_{n+m}^{(+)}\left(Y(\sigma^+)\right)\\
\left(\mathbf{L}_n^{(-)}\,\mathbf{L}_m^{(-)} - \mathbf{L}_m^{(-)}\,\mathbf{L}_n^{(-)}\right)\left(Y(\sigma^-)\right) &= (n-m)\mathbf{L}_{n+m}^{(-)}\left(Y(\sigma^-)\right)\\
\left(\mathbf{L}_n^{(+)}\,\mathbf{L}_m^{(-)} - \mathbf{L}_m^{(-)}\,\mathbf{L}_n^{(+)}\right)\left(Y(\sigma^{\pm})\right) &= 0
\end{aligned} \tag{14.6}$$

- Consider the following two constant infinitesimal transformations:

$$\begin{aligned}
\delta\tau &= c\ ;\ \ \delta\sigma = 0 \qquad \text{time shift}\\
\delta\tau &= 0\ ;\ \ \delta\sigma = c \qquad \text{world sheet rotation}
\end{aligned} \tag{14.7}$$

  Show that they are generated respectively by the following differential operators:

$$\begin{aligned}
\mathcal{O}_t &= -i\,c\,(\mathbf{L}_0^{(+)} + \mathbf{L}_0^{(-)})\\
\mathcal{O}_s &= -i\,c\,(\mathbf{L}_0^{(+)} - \mathbf{L}_0^{(-)})
\end{aligned} \tag{14.8}$$

  namely that for the time shift $\delta\tau = \mathcal{O}_t(\tau) = c$ and $\delta\sigma = \mathcal{O}_s(\sigma) = 0$ while for the world sheet rotation $\delta\tau = \mathcal{O}_s(\tau) = 0$ and $\delta\sigma = \mathcal{O}_s(\sigma) = c$

Exercise 15

Let us define the following operators on the closed string states:

$$\begin{aligned}
\mathcal{L}_n &= \frac{1}{2}\sum_{m=-\infty}^{+\infty}\sum_{i=1}^{D-2} :\alpha_m^i\alpha_{n-m}^i:\\
\tilde{\mathcal{L}}_n &= \frac{1}{2}\sum_{m=-\infty}^{+\infty}\sum_{i=1}^{D-2} :\tilde{\alpha}_m^i\tilde{\alpha}_{n-m}^i:
\end{aligned} \tag{15.1}$$

where as usual α_n and $\tilde{\alpha}_n$ denote the right and left moving mode operators. Show that they fulfill the following commutation relations:

$$\begin{aligned}
[\mathcal{L}_n,\,\mathcal{L}_m] &= (n-m)\mathcal{L}_{n+m} + \frac{D-2}{12}\,n(n^2-1)\delta_{n+m}\\
\left[\tilde{\mathcal{L}}_n,\,\tilde{\mathcal{L}}_m\right] &= (n-m)\tilde{\mathcal{L}}_{n+m} + \frac{D-2}{12}\,n(n^2-1)\delta_{n+m}\\
\left[\mathcal{L}_n,\,\tilde{\mathcal{L}}_m\right] &= 0
\end{aligned} \tag{15.2}$$

these relations, apart from the term $\frac{D-2}{12}\, n(n^2-1)\delta_{n+m}$ on the left hand sides, are analogous to the relations (14.6) characterizing the generators of conformal transformations (Virasoro algebra).

Consider now the quantum version of the solution to the constraints in the light-come gauge, namely:

open string:

$$\alpha_n^- = \frac{1}{p^+\ell}\left(\mathcal{L}_n - a\,\delta_n\right) \;;\; \alpha_n^+ = 0 \;(n\neq 0)$$

closed string:

$$\begin{aligned}\alpha_n^- &= \frac{2}{p^+\ell}\left(\mathcal{L}_n - a\,\delta_n\right) \;;\; \alpha_n^+ = 0 \;(n\neq 0)\\ \tilde{\alpha}_n^- &= \frac{2}{p^+\ell}\left(\tilde{\mathcal{L}}_n - a\,\delta_n\right) \;;\; \tilde{\alpha}_n^+ = 0 \;(n\neq 0)\end{aligned} \tag{15.3}$$

Recalling that in the open string case $\alpha_0^\mu = \ell p^\mu$ while in the closed string case $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \ell p^\mu/2$, express in both cases $M^2 = -p^\mu p_\mu$ as function of α_n^i, $\tilde{\alpha}_n^i$ and a and in terms of the operator N defined for the open string case in equation (13.1) and for the closed string case by:

$$N = \sum_{n=1}^{\infty}\sum_{i=1}^{D-2}\left(\alpha_{-n}^i\alpha_n^i + \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i\right) \tag{15.4}$$

What is the lowest value of M^2 in both cases? Show that in the case $a=1$ these states are tachyons.

Derive in the open string case all the commutation relations between x^μ, p^μ and α_n^μ represented in the table in section 4.3 of the lecture notes.

Consider now the following quantum version of the expressions found in exercise 11 for the coefficients L_n and $\tilde{L}_n$ of the Fourier expansion of $T_{\pm\pm}$ in terms of the coefficients α_n and $\tilde{\alpha}_n$:

$$\begin{aligned}L_n &= \frac{1}{2}\sum_{m=-\infty}^{+\infty} :\alpha_m^\mu \alpha_{n-m\,\mu}:\\ \tilde{L}_n &= \frac{1}{2}\sum_{m=-\infty}^{+\infty} :\tilde{\alpha}_m^\mu \tilde{\alpha}_{n-m\,\mu}:\end{aligned} \tag{15.5}$$

Using equations (15.3) show that in the light-cone gauge $L_n = \tilde{L}_n = a\,\delta_n$.

Consider the open string case in which $\mathcal{L}_n = \tilde{\mathcal{L}}_n$. Show that the following relations hold:

$$\left[\alpha_n^-, \alpha_m^-\right] = (n-m)\frac{\alpha_{n+m}^-}{p^+\ell} + \frac{1}{(p^+\ell)^2}\left(\frac{D-2}{12}\, n(n^2-1) + 2na\right)\delta_{n+m} \tag{15.6}$$

Exercise 16

Find values of the mass squared M^2 for the following states using the value $a = 1$:

open string:

$$
\begin{aligned}
|\psi_1\rangle &= \alpha^i_{-1}|0\rangle \\
|\psi_2\rangle &= \alpha^i_{-1}\alpha^i_{-1}\alpha^j_{-2}|0\rangle
\end{aligned}
$$

closed string:

$$
\begin{aligned}
|\chi_1\rangle &= \alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle \\
|\chi_2\rangle &= \alpha^i_{-1}\alpha^i_{-1}\tilde{\alpha}^j_{-2}|0\rangle
\end{aligned} \tag{16.1}
$$

Exercise 17

Consider the transverse components of the open string operator $X^{tr}(\tau,\sigma) = X^i(\tau,\sigma)$, $(i = 1, \ldots, D-2)$. We wish to compute the propagator associated with the transverse modes of an open string. Let us extend for convenience the definition of normal ordering to the operators p^i, x^i in the following way:

$$
\begin{aligned}
:\alpha^i_n\alpha^j_{-n}: &= :\alpha^j_{-n}\alpha^i_n: = \alpha^j_{-n}\alpha^i_n \; ; \;\; (n > 0) \\
:p^i x^j: &= :x^j p^i: = x^j p^i
\end{aligned} \tag{17.1}
$$

Show that:

$$
\begin{aligned}
X^i(\tau,\sigma)X^j(\tau',\sigma') &= :X^i(\tau,\sigma)X^j(\tau',\sigma'): + \delta^{ij}\, G(\tau,\, \tau',\, \sigma,\, \sigma') \\
G(\tau,\, \tau',\, \sigma,\, \sigma') &= -\frac{1}{4}\log\Big[(e^{i\sigma'^{+}} - e^{i\sigma^{+}})(e^{i\sigma'^{-}} - e^{i\sigma^{-}})\Big] + \\
&\quad \mathrm{R}\,(\tau \to \tau', \sigma \to \sigma')
\end{aligned} \tag{17.2}
$$

where “$\mathrm{R}\,(\tau \to \tau', \sigma \to \sigma')$” denotes terms which are not divergent in the limit $\tau \to \tau', \sigma \to \sigma'$. Find the expressions of G and therefore of $\mathrm{R}\,(\tau \to \tau', \sigma \to \sigma')$.
[**Hint:** *Write the open string solution $X^i(\tau,\sigma)$ in the form (taking the scale $\ell = 1$):*

$$
\begin{aligned}
X^i(\tau,\sigma) &= x^i(\tau) + A^i(\tau,\sigma) + A^{i\dagger}(\tau,\sigma) \\
x^i(\tau) &= x^i + p^i\tau \\
A^i(\tau,\sigma) &= i\sum_{n=1}^{\infty}\frac{\alpha^i_n}{n}\,e^{-in\tau}\,\cos(n\sigma) \\
A^{i\dagger}(\tau,\sigma) &= -i\sum_{n=1}^{\infty}\frac{\alpha^i_{-n}}{n}\,e^{in\tau}\,\cos(n\sigma)
\end{aligned} \tag{17.3}
$$

show that the only non normal ordered terms in the product $X^i(\tau,\sigma)X^j(\tau',\sigma')$ are $p^i x^j\tau$ and $A^i(\tau,\sigma)\,A^{i\dagger}(\tau',\sigma')$. These terms will be rewritten in terms of their normal *products and of the following commutators:* $[p^i, x^j]\,\tau$ *and* $\left[A^i(\tau,\sigma),\, A^{j\dagger}(\tau',\sigma')\right]$*. Using the formula:*

$$\sum_{n=1}^{\infty}\frac{x^n}{n} \;=\; -\log\left(1-x\right) \tag{17.4}$$

from the expression of these commutators deduce $G(\tau,\,\tau',\,\sigma,\,\sigma')$*.]*
Show that for $\tau\neq\tau'$ and $\sigma\neq\sigma'$, $G(\tau,\,\tau',\,\sigma,\,\sigma')$ fulfills the equation of motion:

$$\left(\partial_\tau^2-\partial_\sigma^2\right)\,G(\tau,\,\tau',\,\sigma,\,\sigma') \;=\; 0 \tag{17.5}$$

where the partial derivation is ment *only* with respect to τ, σ and not with respect to τ', σ'. From equations (17.2) show that the function G is the *propagator* of each transverse mode from the point (τ',σ') to the point (τ,σ), namely show that for $\tau>\tau'$:

$$\langle 0|X^i(\tau,\sigma)X^j(\tau',\sigma')|0\rangle \;=\; \delta^{ij}\,G(\tau,\,\sigma,\,\tau',\,\sigma')\;\;(\tau>\tau') \tag{17.6}$$

In the following we shall always consider $\tau>\tau'$.
We wish to show that $G(\tau,\,\tau',\,\sigma,\,\sigma')$ in the whole τ, σ plane fulfills the following equation:

$$\left(\partial_\tau^2-\partial_\sigma^2\right)\,G(\tau,\,\tau',\,\sigma,\,\sigma') \;=\; i\pi\,\delta(\tau-\tau')\delta(\sigma-\sigma') \tag{17.7}$$

Only the terms in G which diverge for $\tau\to\tau'$ and $\sigma\to\sigma'$ (see equation (17.2)) contribute to the delta functions on the right hand side while the regular terms in $\mathrm{R}\left(\tau\to\tau',\sigma\to\sigma'\right)$ are fixed by other boundary conditions which we shall not consider here. Let us introduce the Green function $\tilde{G}(\tau-\tau',\,\sigma-\sigma')$ of the open string field equation:

$$\left(\partial_\tau^2-\partial_\sigma^2\right)\,\tilde{G}(\tau-\tau',\,\sigma-\sigma') \;=\; \delta(\tau-\tau')\delta(\sigma-\sigma') \tag{17.8}$$

It is useful to express a solution of the above equation in terms of Fourier transforms with respect to the variables $\tau-\tau'\in(-\infty,\,\infty)$ and $\sigma-\sigma'\in(-\pi,\,\pi)$. Let us define the following new variables: $\xi^0=(\tau-\tau')\Delta$, $\xi^1=(\sigma-\sigma')\Delta$ where Δ is a scale which we shall send to infinity in order to work with coordinates ξ^α which run from $-\infty$ to $+\infty$ (indeed ξ^1 will take values in the interval $(-\pi\Delta,\,\pi\Delta)$ which becomes $(-\infty,\,\infty)$ in the limit $\Delta\to\infty$). The momenta associated with ξ^α are denoted by k^α. In particular k^1 is quantized as n/Δ (n integer) and in the limit $\Delta\to\infty$ becomes a continuous variable. Therefore the following approximation holds:

$$\sum_{n=0}^{\infty}f(\sigma-\sigma')\,e^{in(\sigma-\sigma')} \;\sim\; \Delta\int_0^\infty dk^1\,f(\xi^1/\Delta)\,e^{ik^1\xi^1} \tag{17.9}$$

Show that the following function:

$$\tilde{G}(\tau - \tau', \sigma - \sigma') = -\int_{-\infty}^{\infty} \frac{dk^1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0\xi^0 + ik^1\xi^1}}{(k^0)^2 - (k^1)^2 + i\epsilon}$$
$$\xi^\alpha = \Delta(\sigma^\alpha - \sigma'^\alpha) \qquad (17.10)$$

fulfills equ. (17.8) by using the following useful relations:

$$\int_{-\infty}^{\infty} dx e^{ixy} = 2\pi\,\delta(y)$$
$$\delta(ay) = \frac{1}{a}\delta(y) \qquad (17.11)$$

(ϵ is an infinitesimal parameter used for regularizing the integral).
Let us show that the singular parts of G and of $\tilde{G}$ as given by equ. (17.10) are proportional. In this way we would show that G is proportional to the Green function of the open string equation of motion.
We start by computing $\tilde{G}$ from equ. (17.10) as a function of the world sheet coordinates. Show that:

$$\int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0\xi^0 + ik^1\xi^1}}{(k^0)^2 - (k^1)^2 + i\epsilon} = -\frac{i}{2(|k^1| - i\epsilon)} e^{-i|k^1|\xi^0 + ik^1\xi^1} \qquad (17.12)$$

[**Hint:** *Compute the integral in the complex k^0 plane and write the denominator as the product of two simple poles:*

$$(k^0)^2 - (k^1)^2 + i\epsilon = \left[k^0 - (|k^1| - i\epsilon)\right]\left[k^0 - (-|k^1| + i\epsilon)\right] \qquad (17.13)$$

since $\xi^0 > 0$ we should close the contour of integration in the lower plane so to include the pole $|k^1| - i\epsilon$ and then compute the integral as the residue in that pole using the formula:

$$\oint_{C_{z_0}} dz \frac{f(z)}{z - z_0} = -2\pi i f(z_0) \qquad (17.14)$$

where $f(z)$ is a regular function in z_0 and C_{z_0} is a contour around z_0 oriented clockwise.]
We are left with the integral over k^1 which is divergent in $k^1 = 0$ in the limit $\epsilon \to 0$. It is useful to express this integral in terms of its principal part by using the relation:

$$\frac{1}{z \pm i\epsilon} = \mathrm{PP}\left(\frac{1}{z}\right) \mp i\delta(z) \qquad (17.15)$$

Show that the second term on the right hand side contributes to $\tilde{G}$ with a constant term. Since we are interested only in the singular part of $\tilde{G}$ we shall ignore this term and simply substitute the integral in k^1 with its principal part which can be expressed as follows:

$$\mathrm{PP}\int_{-\infty}^{\infty} dk^1\, f(k^1) \;=\; \lim_{\Delta\to\infty}\left(\int_{1/\Delta}^{\infty} dk^1\, f(k^1) + \int_{-\infty}^{-1/\Delta} dk^1\, f(k^1)\right)$$

Show that:

$$\begin{aligned}\tilde{G}(\tau-\tau',\,\sigma-\sigma') &= \mathrm{PP}\int_{-\infty}^{\infty}\frac{dk^1}{2\pi}\,\frac{i}{2|k^1|}\,e^{-i|k^1|\xi^0+ik^1\xi^1}\\ &= \frac{i}{4\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n}\,e^{-in(\sigma^- -\sigma'^-)}+\sum_{n=1}^{\infty}\frac{1}{n}\,e^{-in(\sigma^+ -\sigma'^+)}\right) \qquad (17.16)\end{aligned}$$

[**Hint:** *Use the property that $k^1 = n/\Delta$ and therefore one has*

$$\int_{1/\Delta}^{\infty} dk^1 \;\sim\; (1/\Delta)\sum_{n=1}^{\infty}$$

for large Δ.]
From (17.16) show that the functions $\tilde{G}$ and G are proportional a part from terms which are not divergent in the limit $\tau \to \tau'$ and $\sigma \to \sigma'$ and from this deduce that equation (17.7) holds.

Exercise 18

Show that:

$$\begin{aligned} :e^{ik_1^i X^i(\tau,0)}::e^{ik_2^i X^i(0,0)}: &= \;:e^{ik_1^i X^i(\tau,0)}\,e^{ik_2^i X^i(\tau,0)}:\; e^{-(k_1\cdot k_2)\,G(\tau,0,0,0)} = \\ &\quad (1-x)^{k_1\cdot k_2}\,x^{-k_1\cdot k_2}\\ x &= e^{-i\tau} \qquad (18.1)\end{aligned}$$

by using the expression for the propagator G found in the previous exercise and the basic formulas in section 6 of the lecture notes.

Exercise 19

Anticommuting c-numbers. Consider anticommuting c-numbers θ_i, satisfying

$$\theta_i \theta_j = -\theta_j \theta_i\,.$$

The numbers θ_i are taken to be real, i.e. $\theta_i^\dagger = \theta_i$.

- Define a function $F(\theta_1, \theta_2, \theta_3)$, where the θ_i are 3 anticommuting variables, called *Grassmann* variables. Show that there are 8 terms (*monomials*) in the polynomial decomposition of the function in terms of the θ variables. How many independent terms do you expect in the decomposition of a function of n Grassmann variables? We first look at differentiation. The *left* derivative of a function is obtained by differentiating its monomials and resumming the result. To calculate the left derivative with respect to θ_i we must, in every monomial, permute θ_i to the left and then drop it. Let $\epsilon[i]$ be the sign of the permutation needed to bring θ_i to the left, and $\epsilon[i] = 0$ when θ_i does not occur in the monomial. The left derivative of a monomial can then be written in the following way,

$$\frac{\overrightarrow{\partial}}{\partial \theta_{i_k}} \left(\theta_{i_1} \cdots \theta_{i_{k-1}} \theta_{i_k} \theta_{i_{k+1}} \cdots \theta_{i_m}\right) = \epsilon[i_k] \left(\theta_{i_1} \cdots \theta_{i_{k-1}} \theta_{i_{k+1}} \cdots \theta_{i_m}\right). \tag{19.1}$$

  Analogously, one can define a *right* derivative.

- Define a function $F(\theta_1, \theta_2)$, calculate

$$\frac{\overrightarrow{\partial}}{\partial \theta_1} \quad F(\theta_1, \theta_2), \text{ and}$$
$$\frac{\overrightarrow{\partial}}{\partial \theta_2} \quad F(\theta_1, \theta_2).$$

  The definition of the left derivative corresponds to the variational statement

$$\delta_\theta F(\theta) = \delta\theta \, \frac{\overrightarrow{\partial}}{\partial \theta} \, F(\theta), \tag{19.2}$$

  where $\delta\theta$ is a Grassmann parameter. Note the order of the terms in the above expression.

- For the right derivative write down the corresponding variational statement.
- Consider a $\psi(\theta_i)$ which is a function of the θ_i ($i = 1, \ldots, n$) variables. Work out

$$\frac{\overrightarrow{\partial}}{\partial \theta_j} \, F(\psi(\theta_i)); \quad \frac{\overleftarrow{\partial}}{\partial \theta_j} \, \left(F(\psi(\theta_i))\right).$$

Again, the order in which the terms appear is important. This is an immediate consequence of working with anticommuting variables. Now look at Leibniz' rule:

- Work out
$$\frac{\vec{\partial}}{\partial\theta}\left(F(\theta)\ G(\theta)\right).$$
- Calculate
$$\frac{\vec{\partial}}{\partial\theta}\exp(\theta)$$

We now construct the analogue of the indefinite one-dimensional integral,

$$\int_{-\infty}^{\infty} dx\ f(x),$$

for the case of anticommuting c-numbers, which we denote by

$$\int d\theta F(\theta).$$

We want it to obey the following property of the integral over commuting variables,

$$\int_{-\infty}^{\infty} dx\ f(x) = \int_{-\infty}^{\infty} dx\ f(x+a) \tag{19.3}$$

with a finite.

- Consider a function $F(\theta)$ of one Grassmann variable θ. Show that requiring the property (19.3) to hold leads to the requirement

$$\int d\theta\ [\text{any element not depending on } \theta] = 0.$$

Compare with differentiating a function of commuting variables. We are then left with an integral over θ, which we normalize to unity,

$$\int d\theta\ \theta \equiv 1.$$

Exercise 20

Let us consider the change of variables from τ, σ to z, $\overline{z}$. After a Wick rotation we write $\tau = i\sigma_2$ where σ_2 is the real Euclidean time and we rename σ as σ_1. If we define $w = \sigma^+ = i\sigma_2 + \sigma_1$ so that $\overline{w} = -\sigma^- = -i\sigma_2 + \sigma_1$ then z, $\overline{z}$ are defined as follows:

$$\begin{array}{lll} \underline{\text{open string:}} & z = e^{iw} \; ; & \overline{z} = e^{-i\overline{w}} \\ \underline{\text{closed string:}} & z = e^{2iw} \; ; & \overline{z} = e^{-2i\overline{w}} \end{array} \tag{20.1}$$

- For $\sigma_2 \in (-\infty, +\infty)$ and $\sigma_1 \in (0, \pi)$ describe the space spanned by z for the open and closed string cases and draw the equal–time curves. Which points correspond to $\sigma_2 \to \pm\infty$?
- Write as functions of the complex variables the open and closed string solutions $X^\mu(z, \overline{z})$ together with their holomorphic and anti–holomorphic derivatives $\partial_z X^\mu(z, \overline{z})$, $\partial_{\overline{z}} X^\mu(z, \overline{z})$.
- Rewrite the Green function $G(\tau, \tau', \sigma, \sigma')$ computed in exercise 17 as a function $G(z, \overline{z}, z', \overline{z}')$ in the complex variables. Compute it for the closed string as well.
- Consider the variable $\hat{z} = \log(z)$. For $\sigma_2 \in (-\infty, +\infty)$ and $\sigma_1 \in (0, \pi)$ describe the space spanned by $\hat{z}$ for the open and closed string cases and draw the equal–time curves.

Exercise 21

Let us consider conformal transformations in the complex coordinate notation. In these coordinates conformal transformations have the form: $z \to z'(z)$ and $\overline{z} \to \overline{z}'(\overline{z})$. We shall denote in the sequel by $\Phi_{h,\bar{h}}(z, \overline{z})$ a tensor with h holomorphic and $\bar{h}$ anti–holomorphic lower indices. Under a conformal transformation $\Phi_{h,\bar{h}}$ transforms as follows:

$$\Phi_{h,\bar{h}}(z, \overline{z}) \quad \to \quad \Phi'_{h,\bar{h}}(z, \overline{z}) = \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \overline{z}'}{\partial \overline{z}}\right)^{\bar{h}} \Phi_{h,\bar{h}}(z'(z), \overline{z}'(\overline{z})) \tag{21.1}$$

- Write the transformation property of $\Phi_{h,\bar{h}}$ under $z \to e^{i\theta}\, z$, $\overline{z} \to e^{-i\theta}\, \overline{z}$ where θ is a constant angle.
- Write the transformation property of $\Phi_{h,\bar{h}}$ under $z \to z' = \log(z)$, $\overline{z} \to \overline{z}' = \log(\overline{z})$.

Consider an infinitesimal conformal transformation $z \to z' = z + \epsilon(z)$ and $\overline{z} \to \overline{z}' = \overline{z} + \overline{\epsilon}(\overline{z})$. Show that:

$$\begin{aligned} \Phi'_{h,\bar{h}} &= \Phi_{h,\bar{h}} + \delta\Phi_{h,\bar{h}} \\ \delta\Phi_{h,\bar{h}} &= \left(h\,(\partial\epsilon) + \bar{h}\,(\overline{\partial}\overline{\epsilon}) + \epsilon\,\partial + \overline{\epsilon}\,\overline{\partial}\right)\Phi_{h,\bar{h}} \end{aligned} \tag{21.2}$$

where we have used the notation: $\partial = \partial/\partial z$ and $\overline{\partial} = \partial/\partial \overline{z}$.

Consider now the quantized closed string and let $\Phi_{h,\bar{h}}$ be an operator on the Hilbert space of states which transforms under conformal transformations as in eq (21.1) (*primary operator*) . We wish to define generators of infinitesimal conformal transformations T_ϵ, $\overline{T}_{\overline{\epsilon}}$ on the Hilbert space such that:

$$\delta\Phi_{h,\bar{h}} \;=\; [T_\epsilon,\, \Phi_{h,\bar{h}}] + [\overline{T}_{\overline{\epsilon}},\, \Phi_{h,\bar{h}}] \qquad (21.3)$$

Let us consider the Laurent expansion of ϵ, $\overline{\epsilon}$:

$$\epsilon = \sum_{n=-\infty}^{\infty} \epsilon_n\, z^{n+1} \qquad \overline{\epsilon} = \sum_{n=-\infty}^{\infty} \overline{\epsilon}_n\, \overline{z}^{n+1}$$

and define T_ϵ, $\overline{T}_{\overline{\epsilon}}$ in terms of some operators ℓ_n and $\bar{\ell}_n$ as follows:

$$T_\epsilon \;=\; \sum_{n=-\infty}^{\infty} \epsilon_n\, \ell_n \quad \overline{T}_{\overline{\epsilon}} = \sum_{n=-\infty}^{\infty} \overline{\epsilon}_n\, \bar{\ell}_n \qquad (21.4)$$

Show that if the following commutation relations hold:

$$\begin{aligned}\left[\ell_n,\, \Phi_{h,\bar{h}}\right] &= z^n(z\partial + h(n+1))\Phi_{h,\bar{h}} \\ \left[\bar{\ell}_n,\, \Phi_{h,\bar{h}}\right] &= \overline{z}^n(\overline{z}\partial + \bar{h}(n+1))\Phi_{h,\bar{h}}\end{aligned} \qquad (21.5)$$

formula (21.3) yields the transformation rule (21.2). Compare formulas (21.5) and (21.4) with the analogous formulae of exercise 14. Observe the differences with respect to exercise 14: there conformal transformations were considered for simplicity only on scalar functions Y (which have $h = \bar{h} = 0$); the infinitesimal generator had the same form as in (21.4), but its action on the function Y is a differential operation, while in the present exercise its action on operators is expressed in terms of commutators (21.5); the light cone indices $\pm$ are substituted by holomorphic/anti–holomorphic indices in the present exercise.

Operators on free closed string states are expressed in terms of α^i_n and $\tilde{\alpha}^i_n$. An asymptotic string state emitted from a point $z, \overline{z}$ of the world sheet is described by a local *primary* operator $V(z,\overline{z})$ (i.e. transforming as in equ.(21.1) with definite values of h, $\bar{h}$) which is called *vertex operator*.

Consider the vertex operator $V(z,\overline{z}) =: e^{ik^i X^i(z,\overline{z})} :$ (from now on, for the vertex operators, we shall consider only space–like momenta $k^\mu \equiv k^i$) describing the emission of a tachyon of momentum k^i (verify indeed that $[p^i, V] = k^i\, V$). If for ℓ_n and $\bar{\ell}_n$ we take $\mathcal{L}_n$ and $\tilde{\mathcal{L}}_n$ defined in exercise 15, show that the commutator of these operators with V have the expression on the right hand side of eqs. (21.5) for suitable values of h, $\bar{h}$. Find these values. Repeat this exercise for the vertex operator $V(z,\overline{z}) =: \partial X^j\, \overline{\partial} X^i\, e^{ik\cdot X(z,\overline{z})} :$.

In general infinitesimal conformal transformations on the closed string Hilbert space are generated by T_ϵ, $\overline{T}_{\overline{\epsilon}}$ which are expressed in terms of $\mathcal{L}_n$ and $\tilde{\mathcal{L}}_n$.

There is a remark to be done about conformal transformations and the light–cone gauge. In the light–cone gauge the reparametrization invariance of the string action is totally fixed and with it also the conformal transformations which are generated by T_{++} and T_{--}. Indeed on the Hilbert space the Fourier modes of T_{++} and T_{--}, which are L_n and $\tilde{L}_n$ and are the generators of the conformal transformations are fixed to the value $\delta_{n,0}$ (see exercise 15). The generators $\mathcal{L}_n$ and $\tilde{\mathcal{L}}_n$ have the same expression as L_n and $\tilde{L}_n$ but involve only oscillators along transverse directions. They close a Virasoro algebra as well but with a different central charge given by $c = D - 2$, and thus are generators of conformal transformations though we do not expect the action to be invariant with respect to it (since all reparametrization invariance have been fixed).

In the framework of covariant quantization the constraint $T_{\alpha\beta} = 0$ is imposed on the physical states and translated into the conditions: $L_n|phys\rangle = \tilde{L}_n|phys\rangle = 0$ $(n > 0)$; $(L_0 - 1)|phys\rangle = (\tilde{L}_0 - 1)|phys\rangle = 0$. To summarize the $\{L_n,\, \tilde{L}_n\}$ and $\{\mathcal{L}_n,\, \tilde{\mathcal{L}}_n\}$ conformal algebras are different and with respect to them physical states will have different conformal weights $\{h,\, \bar{h}\}$. In particular the conformal weights with respect to the first are bound by the constraints written above to be $(1, 1)$, while the conformal weights corresponding to the action of the second algebra are not constrained.

Exercise 22

There is a one to one correspondence between vertex operators $V(z, \overline{z})$ and asymptotic in–coming or our–going states of a string (free states). The in–coming string state $|V,\, in\rangle$ associated with the operator $V(z, \overline{z})$ is defined through the asymptotic limit $\sigma_2 \to \infty$ of $V(z, \overline{z})|0\rangle$ (recall that $|0\rangle$ is the vacuum state with vanishing CM momentum), similarly the corresponding out–going free state $\langle V,\, out|$ is obtained by performing on the state $\langle 0|V(z, \overline{z})$ the opposite limit $\sigma_2 \to -\infty$. Use the complex notation and find the in–coming and out–going states corresponding to the following vertex operators (for simplicity in what follows the momentum k^μ is always space-like):

$$
\begin{array}{ll}
\underline{\text{open string:}} & :e^{ik^i X^i(z,\overline{z})}: \\
 & :\partial X^j\, e^{ik\cdot X(z,\overline{z})}: \\
 & \\
\underline{\text{closed string:}} & :e^{ik^i X^i(z,\overline{z})}: \\
 & :\partial X^j\, \overline{\partial} X^i\, e^{ik\cdot X(z,\overline{z})}:
\end{array}
\tag{22.1}
$$

Compute the mass squared M^2 of the in–coming asymptotic states computed above either through direct evaluation of M^2 on $|V,\, in\rangle$ or by performing the limit $\sigma_2 \to \infty$ on $M^2(V(z, \overline{z})|0\rangle)$.

[**Hint:** *Express* $M^2(V(z, \overline{z})|0\rangle)$ *as* $[M^2,\, V(z, \overline{z})]\,|0\rangle + V(z, \overline{z})\,M^2|0\rangle$]

Exercise 23

Representations of the Clifford algebra. The Clifford algebra of Dirac matrices:

$$\{\gamma^\mu \gamma^\nu\} = -2\eta^{\mu\nu} \tag{23.1}$$

cannot be represented in any dimensions. Show, that for representing the γ^μ matrices in $2n$ spacetime dimensions we need at least $2^n \times 2^n$ matrices. To do that, construct n linear operators out of the γ matrices, eg.

$$a_i := \frac{1}{\sqrt{2}}(\gamma_{2i-2} + i\gamma_{2i-1}) \quad i = 1..n$$

use a representation[1] for which

$$\gamma^\dagger = \gamma^0 \quad \gamma^{i\dagger} = -\gamma^i \quad i = 1..2n-1.$$

Show that

$$\{a_n, a_m^\dagger\} = \delta_{m,n},$$

define a vacuum $|0\rangle$ and determine the number of different states which the creation operators can create from the vacuum. If we want to impose further conditions (like Weyl or Majorana spinor) we will find further constraints. Check that for two dimensions the following definition

$$\gamma^0 = \sigma^2 \quad \gamma^1 = i\sigma^1$$

(where σ^i means the i-th Pauli matrix) satisfies the algebra (23.1). Now with an inductive procedure we can construct a representation of 2^n dimensions if the spacetime has $2n$ dimensions (that is we will have 2n $2^n \times 2^n$ matrices). Suppose that we have the algebra for $d = 2n-2$.($\hat{\gamma}^\mu$'s are given for $\mu = 0...2n-3$ and (23.1) is satisfied). Then we define two more matrices according to the following:

$$\gamma^{2n-2} = i\mathbf{1} \times \sigma^1 \quad \gamma^{2n-1} = i\mathbf{1} \times \sigma^2 \tag{23.2}$$

and extend the first $2n-2$ as

$$\gamma^\mu = \hat{\gamma}^\mu \times (-\sigma_3) \quad \mu = 1..2n-1$$

$\mathbf{1}$ is the unit matrix in 2^{n-1} dimensions. ($(A \times B)_{ijkl} = A_{ik}B_{jl}$ is the formula for the indices of the direct product matrix.) That is these two matrices are blockdiagonal ones, each entry of the appropriate Pauli matrix is the entry times the $2^{n-1} \times 2^{n-1}$ $\hat{\gamma}^\mu$ or unit matrix. Check that (23.1) is then satisfied in 2n dimensions. What about the case of odd dimensions ? Show that in $2n+1$ dimensions $2^n \times 2^n$ matrices still can represent the algebra. [**Hint:** *Extend the case of* $2n$ *with* $\gamma^{2n} = \gamma$ *defined as (23.5) further in the exercise.*]

[1] One needs to prove that this representation always exists.

Since the right hand side of (23.1) is real the relation for the anticommutator of the complex conjugate γ's are the same:

$$\{\gamma^{\mu *}\gamma^{\nu *}\} = -2\eta^{\mu\nu}$$

The complex conjugate representation is equivalent, that is

$$\gamma^{\mu *} = B\gamma^{\mu}B^{-1} \quad \mu = 1...2n-1 \tag{23.3}$$

where both B_1 and B_2 are unitary. An explicit construction for such a matrix is the following:

$$B_1 = \gamma^3\gamma^5...\gamma^{2n-1} \quad B_2 = \gamma^0\gamma^1\gamma^2\gamma^4...\gamma^{2n-2} \tag{23.4}$$

Show that

$$B_1\gamma^{\mu}B_1^{-1} = (-1)^n\gamma^{\mu *} \quad B_2\gamma^{\mu}B_2^{-1} = (-1)^{n-1}\gamma^{\mu *}$$

that is one of them always satisfies (23.3).
Define now

$$\gamma = c\gamma^0\gamma^1...\gamma^{2n-1} \tag{23.5}$$

Determine the value of c by the requirement that $(\gamma)^2 = \mathbf{1}$. Can we generalize the relations (23.4) to odd $(2n+1)$ dimensions ?
The spinors on which the γ^{μ} matrices act are representations of the Lorentz group with the generator

$$\sigma^{\mu\nu} = -\frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]$$

where the bracket is the commutator. With respect to the full Lorentz group these are not always irreducible representations, the familiar projections,

$$P_L = \frac{1-\gamma}{2} \quad P_R = \frac{1+\gamma}{2}$$

project out irreducible subspaces. In physical language we can find chiral (left and right handed) fermions, which are not mixed under the action of the Lorentz group. They are then called *Weyl spinors.*
When can one find real spinor representations ? In other words one wants to impose the following condition

$$\psi = B\psi^* \tag{23.6}$$

Show that ψ and $B\psi^*$ transforms according to the same representation of the Lorentz group. Show, that from the above condition requirement $B^*B = 1$ follows. Verify the following formulae for the explicit B_1 and B_2:

$$B_1B_1^* = (-1)^{\frac{n(n-1)}{2}} \quad B_2B_2^* = (-1)^{\frac{(n-1)(n-2)}{2}}$$

The spinors for which (23.6) holds are called *Majorana spinors.*

Verify that

$$B_i(\frac{1\pm\gamma}{2}\psi)^* = \frac{1\pm(-1)^{(n+1)}}{2}B_i\psi^* \quad i=1,2$$

that is the in certain spacetime dimensions

$$(\psi_L)^* = \psi_L \quad (\psi_R)^* = \psi_R$$

that is in certain spacetime dimensions $\psi_L^* = \psi_L$, $\psi_R^* = \psi_R^*$. In which spacetime dimensions are there Weyl, Majorana, Weyl-Majorana spinors ? [**Hint:** *Use the given explicit representation of B's and γ^μ's.*]

Exercise 24

Check explicitly the calculations leading to (11.22) and (11.23) of the lecture notes.

Exercise 25

Supersymmetric particle. Consider the action of a massless *super particle* propagating in D–dimensional Minkowski space, which is obtained from the particle world line action by adding to the D bosonic fields $x^\mu(\tau)$, the contribution from D Majorana fermions $\psi^\mu(\tau)$ ($\psi^{\mu\star} = \psi^\mu$) which describe the corresponding *super partner*:

$$S = -\frac{1}{2}\int d\tau\, e\left((\dot{x}^\mu)^2 - i\psi^\mu\dot{\psi}_\mu\right) \tag{25.1}$$

Compute the field equations for the fields x^μ, ψ^μ, e and show that the field equation for e amounts to a constraint on the other fields. Consider the following super symmetry transformation:

$$\delta x^\mu \ = \ i\epsilon\psi^\mu \ ; \quad \delta\psi^\mu = \epsilon\dot{x}^\mu \tag{25.2}$$

where ϵ is an infinitesimal real Grassmann parameter (recall that on Grassmann variables the complex conjugation is defined to have the property that $(\epsilon_1\epsilon_2)^* = \epsilon_2^*\epsilon_1^*$ just like Hermitian conjugation for matrices. Therefore you can show that the product of two real Grassmann variables acts as an imaginary c–number). The fields x^μ and ψ^μ are called super partners because they transform into each other by super symmetry. If we denote by δ_1 and δ_2 two infinitesimal super symmetry transformations parametrized by ϵ_1 and ϵ_2 respectively, using (25.2), show that:

$$\delta_1(\delta_2 x^\mu) - \delta_2(\delta_1 x^\mu) \ = \ \delta\tau\,\frac{dx^\mu}{d\tau} \tag{25.3}$$

and find the expression of $\delta\tau$. From the above result we see that the commutator of two super symmetry transformations amounts to a space–time coordinate translation. If the action were invariant under *local* super symmetry then (25.3) would guarantee its invariance under local reparametrization as well. Show that under the transformation (25.2) if ϵ depends on τ we can write:

$$\delta S = -2 \int d\tau \left(i\dot{\epsilon} J - \frac{ie}{2} \frac{d}{d\tau}(\epsilon \dot{x} \psi) + \text{field equations} \right) \tag{25.4}$$

and find the expression for J. Is the action S invariant under global super symmetry transformations (i.e. $\dot{\epsilon} = 0$)? Under which constraint is the action invariant under local super symmetry transformations? Since our action is invariant under local reparametrization, in view of (25.3), we would like this invariance to be a consequence of local super symmetry, then we shall impose the local super symmetry constraint computed above. This constraint, as explained in the lecture notes, allows to have finite fermionic mass spectrum, and the consequent local super symmetry will allow to gauge away non–physical longitudinal modes of ψ^μ.

Consider the canonical quantization of the super particle setting for simplicity $e = 1$. Show form the canonical commutation relations that $\{\psi^\mu(\tau), \psi^\nu(\tau)\} = \eta^{\mu\nu}$.

[**Hint:** *In order to work with a well defined canonical momentum* Π_ψ *associated with* ψ *it is advisable to follow the procedure described in section 11.4 of the lecture notes.*]

What can you conclude about the states generated by ψ^μ?

We can construct a locally super symmetric action by introducing a new auxiliary Grassmann valued field ν to be regarded as the super partner of the auxiliary field e and modifying the action as follows:

$$S = -\frac{1}{2} \int d\tau \left(e(\dot{x}^\mu)^2 - ie\psi^\mu \dot{\psi}_\mu - 2i\nu \dot{x}^\mu \psi_\mu \right) \tag{25.5}$$

Show that this action is invariant under the following local super symmetry transformations:

$$\delta x^\mu = i\epsilon\psi^\mu \,; \quad \delta\psi^\mu = \epsilon \dot{x}^\mu \,; \quad \delta e = -2i\epsilon\nu \,; \quad \delta\nu = \dot{\epsilon} e - \frac{1}{2}\epsilon\dot{e} \,. \tag{25.6}$$

Compute the field equations for the various fields and verify that the field equation for ν yields the local super symmetry constraint derived previously. If we regard the physical states of this theory as space–time fields what condition does the local super symmetry constraint imply on these fields?

Locally super symmetric theories are called super gravity theories since they include invariance under general coordinate transformation, which is the symmetry of General Relativity, as a consequence of local super symmetry. The field e is called the *graviton* and its superpartner ν the *gravitino*. In one and two dimensions both these fields are non–propagating.

Exercise 26

Consider the superstring action (eq. (11.45) of the lecture notes):

$$S = -\frac{1}{2\pi}\int d^2\sigma\,\left(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right) \tag{26.1}$$

Show that under the super symmetry trnsformations in eq. (11.44) of the lecture notes the action transforms as in (11.46). What is the constraint for *local* super symmetry?

Also in this case this constraint can be derived from a locally super symmetric action as the field equation of an auxiliary gravitino χ^α, super partner of the graviton e^a_α. The construction of this action follows the same lines as in the super symmetric particle case, i.e. with the addition of a term of the form $\overline{\chi}^\alpha J_\alpha$. In this case however the addition of a further quadratic term in the gravitino field is required by local super symmetry invariance. The derivation of this action is described in detail in section 4.3.5 of the Green–Schwarz–Witten book. It is lengthy and we shall not deal with it in this exercise. However there are some properties of Majorana spinors in two dimensions which are needed for this derivation and are useful to derive. Show that:

$$\overline{\psi}^\mu \rho^\alpha \nabla_\alpha \psi_\mu = \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\,; \qquad \psi^\mu_A \overline{\psi}_{B\mu} = -\frac{1}{2}\,\overline{\psi}_\mu \psi^\mu\,\delta_{AB}\,. \tag{26.2}$$

Exercise 27

Given local left–moving and right–moving fields $\Phi(\tau+\sigma)$, $\tilde{\Phi}(\tau-\sigma)$ on the world sheet of a string ($\Phi \equiv \tilde{\Phi}$ for the open string), which are expanded in Fourier modes as follows:

$$\begin{aligned}
\underline{\text{open string:}}\qquad \Phi(\tau\pm\sigma) &= \frac{1}{\Delta}\sum_n \Phi_n\, e^{-in(\tau\pm\sigma)}\\
\underline{\text{closed string:}}\qquad \Phi(\tau+\sigma) &= \frac{1}{\Delta}\sum_n \Phi_n\, e^{-2in(\tau+\sigma)}\\
\tilde{\Phi}(\tau-\sigma) &= \frac{1}{\Delta}\sum_n \tilde{\Phi}_n\, e^{-2in(\tau-\sigma)}
\end{aligned} \tag{27.1}$$

where Δ is a normalization factor. Show that the following inverse relations hold:

$$\begin{aligned}
\underline{\text{open string:}}\qquad \Phi_n &= \frac{\Delta}{2\pi}\int_0^\pi d\sigma\,\left[\Phi(\tau-\sigma)\, e^{in(\tau-\sigma)} + \Phi(\tau+\sigma)\, e^{in(\tau+\sigma)}\right]\\
\underline{\text{closed string:}}\qquad \Phi_n &= \frac{\Delta}{\pi}\int_0^\pi d\sigma\,\Phi(\tau+\sigma)\, e^{2in(\tau+\sigma)}\\
\tilde{\Phi}_n &= \frac{\Delta}{\pi}\int_0^\pi d\sigma\,\tilde{\Phi}(\tau-\sigma)\, e^{2in(\tau-\sigma)}
\end{aligned} \tag{27.2}$$

Let Φ and $\tilde{\Phi}$ be fermionic fields with Neveu-Schwarz boundary conditions, express them as Fourier series and find the inverse transformations yielding the coefficients. Using the above relations derive the expressions for the Fourier coefficients α_n^μ, d_n^μ, b_r^μ, L_n, F_n, G_r as functions of $\partial_+ X^\mu$, ψ_+^μ(Ramond), ψ_+^μ (Neveu–Schwarz), T_{++}, J_+ (Ramond), J_+ (Neveu–Schwarz) respectively and the corresponding left–handed quantities, for open and closed strings. Recall that:

<u>open string:</u>

$$\begin{aligned}
\partial_\pm X^\mu &= \frac{1}{2} \sum_n \alpha_n^\mu e^{-in(\tau \pm \sigma)} \\
\psi_\pm^\mu \,(\mathrm{R}) &= \frac{1}{\sqrt{2}} \sum_n d_n^\mu e^{-in(\tau \pm \sigma)} \\
\psi_\pm^\mu \,(\mathrm{NS}) &= \frac{1}{\sqrt{2}} \sum_{r \in Z+1/2} b_r^\mu e^{-ir(\tau \pm \sigma)} \\
T_{\pm\pm} &= \frac{1}{2} \sum_n L_n \, e^{-in(\tau \pm \sigma)} \\
J_\pm \,(\mathrm{R}) &= \frac{1}{2\sqrt{2}} \sum_n F_n \, e^{-in(\tau \pm \sigma)} \\
J_\pm \,(\mathrm{NS}) &= \frac{1}{2\sqrt{2}} \sum_{r \in Z+1/2} G_r \, e^{-ir(\tau \pm \sigma)}
\end{aligned}$$

<u>closed string:</u>

$$\begin{aligned}
\partial_+ X^\mu &= \sum_n \alpha_n^\mu e^{-2in(\tau + \sigma)} & \partial_- X^\mu &= \sum_n \tilde{\alpha}_n^\mu e^{-2in(\tau - \sigma)} \\
\psi_+^\mu \,(\mathrm{R}) &= \sum_n d_n^\mu e^{-2in(\tau + \sigma)} & \psi_-^\mu \,(\mathrm{R}) &= \sum_n \tilde{d}_n^\mu e^{-2in(\tau - \sigma)} \\
\psi_+^\mu \,(\mathrm{NS}) &= \sum_{r \in Z+1/2} b_r^\mu e^{-2ir(\tau + \sigma)} & \psi_-^\mu \,(\mathrm{NS}) &= \sum_{r \in Z+1/2} \tilde{b}_r^\mu e^{-2ir(\tau - \sigma)} \\
T_{++} &= 2 \sum_n L_n \, e^{-2in(\tau + \sigma)} & T_{--} &= 2 \sum_n \tilde{L}_n \, e^{-2in(\tau - \sigma)} \\
J_+ \,(\mathrm{R}) &= \sum_n F_n \, e^{-2in(\tau + \sigma)} & J_- \,(\mathrm{R}) &= \sum_n \tilde{F}_n \, e^{-2in(\tau - \sigma)} \\
J_+ \,(\mathrm{NS}) &= \sum_{r \in Z+1/2} G_r \, e^{-2ir(\tau + \sigma)} & J_- \,(\mathrm{NS}) &= \sum_{r \in Z+1/2} \tilde{G}_r \, e^{-2ir(\tau - \sigma)}
\end{aligned}$$

Use the above definitions to express the constraints L_n, F_n, G_r and the corresponding tilded quantities in terms of field coefficients for both open and closed strings.

Exercise 28

From the canonical commutation/anti–commutation relations:

$$\begin{aligned}
\left[\dot{X}^\mu(\sigma,\tau),\, X^\nu(\sigma',\tau)\right] &= -i\pi\eta^{\mu\nu}\delta(\sigma-\sigma') \\
\left[X^\mu(\sigma,\tau),\, X^\nu(\sigma',\tau)\right] &= \left[\dot{X}^\mu(\sigma,\tau),\, \dot{X}^\nu(\sigma',\tau)\right] = 0 \\
\{\psi^\mu_\pm(\sigma,\tau),\, \psi^\nu_\pm(\sigma',\tau)\} &= \pi\eta^{\mu\nu}\delta(\sigma-\sigma') \\
\{\psi^\mu_\pm(\sigma,\tau),\, \psi^\nu_\mp(\sigma',\tau)\} &= 0
\end{aligned} \tag{28.1}$$

using the results and conventions of the previous exercise, deduce the commutation/anti–commutation relations among α^μ_n, d^μ_n, b^μ_r and the corresponding left–moving quantities (see eqs. (11.39), (11.40) of the lecture notes) as operators in the *unconstrained* Hilbert space. Write the constraints L_n, F_n, G_r, $\tilde{L}_n$, $\tilde{F}_n$, $\tilde{G}_r$ as normal ordered operators (recall that the definition of normal order for fermionic operators is the same as in the bosonic case except for a minus sign required every time the order of two operators is inverted: $:d_n d_{-n} := -d_{-n}d_n = -:d_{-n}d_n:$). Deduce the commutation/anti–commutation relations among L_n, F_n, G_r, $\tilde{L}_n$, $\tilde{F}_n$, $\tilde{G}_r$.

When a constraint is expressed as a quadratic function of quantities which, as operators, do not commute (bosonic operators) or anti–commute (fermionic operators), then their implementation at the quantum level suffers from an ordering ambiguity and therefore requires the introduction of a constant. For the bosonic string we saw that the only constraint with this ambiguity was L_0 and therefore a constant a was introduced so that in the light–cone quantization the constraints had the form $L_n - a\delta_n = 0$. In the superstring case, which of the constraints requires an ordering constant in the R and NS sectors? The ordering constants in the two sectors are in general unrelated. Consider the R sector and suppose we implement the constraint F_0 in the form $F_0 - \mu = 0$, μ being a c–number, find the relation between the constant a for L_0 and μ (use the relation $\{F_0,\, F_0\} \sim L_0$). Is it consistent to set $\mu = 0$? What consequence dose this have on the value of a for the R sector?

Let us denote by $|0\rangle_R$ and $|0\rangle_{NS}$ the vacua of the R and NS sectors (for the open string or for a single left or right mover sector of the closed string) respectively, show that $|0\rangle_{NS}$ has degeneracy one while for $D = 2n$ $|0\rangle_R$ has degeneracy 2^n and therefore describes a fermionic state (show that the action of d^μ_0 on $|0\rangle_R$ does not change the energy of the state and use the anti–commutation relations among the d^μ_0.). How many sectors does the closed string have? Write the vacua for each closed string sector.

Exercise 29

Light–cone quantization. Consider the gauge fixing conditions $X^+ = x^+ + p^+\tau$ and $\psi^+ = 0$. Show that they are consistent with supersymmetry transformations (equations (11.44) of the lecture notes). Solve them in terms of the α_n^+, d_n^+, b_r^+, $\tilde{\alpha}_n^+$, $\tilde{d}_n^+$, $\tilde{b}_r^+$ in the R, NS, left and right–moving sectors. Use these conditions to solve the constraints by expressing α_n^-, d_n^-, b_r^-, $\tilde{\alpha}_n^-$, $\tilde{d}_n^-$, $\tilde{b}_r^-$ in terms of the corresponding transverse components for the relevant sectors in the open and closed string cases using $a = 0$ in the R sector (see previous exercise) and $a = 1/2$ in the NS sector (recall that $\alpha_0^\mu = p^\mu$ for the open string and $\alpha_0^\mu = \tilde{\alpha}_0^\mu = p^\mu/2$ for the closed string). Write the formula for M^2 in the various sectors of the open and closed string. Write explicitly the tachyonic and massless open and closed string states.

Exercise 30

Find the expression of the vacuum expectation value $G^{F\,\mu\nu}$ of the product of two fermionic fields:

$$\langle 0|\psi^\mu(\tau,\,\sigma)\,\psi^\nu(\tau',\,\sigma')|0\rangle \;=\; G^{F\,\mu\nu}(\tau,\,\sigma,\,\tau',\,\sigma') \tag{30.1}$$

as for the bosonic case, the function G^F is the Green function of the fermionic field equation and, when time ordered, yields the fermion propagator.
[**Hint:** *Express the product of the two fields in terms of its normal ordered expression:*

$$\psi^\mu(\tau,\,\sigma)\,\psi^\nu(\tau',\,\sigma') =:\psi^\mu(\tau,\,\sigma)\,\psi^\nu(\tau',\,\sigma'):+\,G^{F\,\mu\nu}(\tau,\,\sigma,\,\tau',\,\sigma')$$

]

Exercise 31

On $SO(1,9)$ and $SO(8)$. The groups $SO(1,9)$ and $SO(8)$ are defined as those groups of transformations whose action on the corresponding vector representations V^μ, v^i ($\mu = 0,\dots,9$, $i = 1,\dots,8$) leaves the metrics $\eta_{\mu\nu}$ and δ_{ij} invariant:

$$\begin{aligned} U^\mu{}_\nu &\in SO(1,9) \;\Leftrightarrow\; U^\rho{}_\mu\,\eta_{\rho\sigma}\,U^\sigma{}_\nu \;=\; \eta_{\mu\nu} && (31.1)\\ \tilde{U}^i{}_j &\in SO(8) \;\Leftrightarrow\; \tilde{U}^k{}_i\,\delta_{kn}\,\tilde{U}^n{}_j \;=\; \delta_{ij} && (31.2)\\ \eta^{\mu\nu} &= \mathrm{diag}(-,\,+,\,\dots,\,+) \end{aligned}$$

The vector representation of $SO(8)$, described by v^i, is usually denoted by $\mathbf{8}_v$. Let us define the following matrices:

$$\begin{aligned} \mathbb{M}^\mu{}_\nu &= w_{\rho\sigma}\,(M^{\rho\sigma})^\mu{}_\nu \;;\; (M^{\rho\sigma})^\mu{}_\nu \;=\; \eta^{\sigma\mu}\,\delta^\rho_\nu - \eta^{\sigma\nu}\,\delta^\rho_\mu \\ \tilde{\mathbb{M}}^i{}_j &= w_{kn}\,(\tilde{M}^{kn})^i{}_j \;;\; (\tilde{M}^{kn})^i{}_j \;=\; \delta^{ni}\,\delta^k_j - \delta^{ki}\,\delta^n_j && (31.3) \end{aligned}$$

where $w_{\mu\nu}$ and its restriction w_{ij} to the i, j indices are antisymmetric matrices of parameters. Show that $\mathbb{M}^\mu{}_\nu$ and $\tilde{\mathbb{M}}^i{}_j$ are infinitesimal generators of $SO(1,9)$ and $SO(8)$ respectively in their vector representations, i.e. that $U^\mu{}_\nu = \delta^\mu{}_\nu + \mathbb{M}^\mu{}_\nu$ and $\tilde{U}^i{}_j = \delta^i{}_j + \tilde{\mathbb{M}}^i{}_j$ fulfill equations (31.1) and (31.2) respectively to first order in w. How many independent generators do $SO(1,9)$ and $SO(8)$ have? Using the definitions (31.3) show that the $SO(1,9)$ infinitesimal generators $M^{\mu\nu}$ fulfill the following commutation relations:

$$[M^{\mu\nu},\, M^{\rho\sigma}] \;=\; \eta^{\mu\rho}\, M^{\nu\sigma} + \eta^{\nu\sigma}\, M^{\mu\rho} - \eta^{\nu\rho}\, M^{\mu\sigma} - \eta^{\mu\sigma}\, M^{\nu\rho} \tag{31.4}$$

and that the $SO(8)$ generators $\tilde{M}^{ij}$ fulfill relations obtained by restricting eq. (31.4) to the i, j indices.

Consider the $SO(1,9)$ and $SO(8)$ spinorial representations. What are the dimensions of an $SO(1,9)$ and an $SO(8)$ spinors? Let us introduce the $SO(1,9)$ and $SO(8)$ Clifford algebras Γ^μ, γ^i fulfilling $\{\Gamma^\mu,\, \Gamma^\nu\} = -2\,\eta^{\mu\nu}$ and $\{\gamma^i,\, \gamma^j\} = -2\,\delta^{ij}$ (notice here a sign difference with respect to the definition of the $SO(8)$ gamma matrices given in the lecture notes). Show that the matrices $M^{\mu\nu} = \Gamma^{\mu\nu}/2 = (\Gamma^\mu\,\Gamma^\nu - \Gamma^\nu\,\Gamma^\mu)/4$ and $\tilde{M}^{ij} = \gamma^{ij} = (\gamma^i\,\gamma^j - \gamma^j\,\gamma^i)/4$ are the generators of $SO(1,9)$ and of $SO(8)$ in the spinorial representation respectively, namely that they fulfill the corresponding commutation relations. To give an explicit representation to these matrices it is useful to introduce the tensor product notation: given two matrices $A_{n\times n} = (a_{ij})$ and $B_{m\times m} = (a_{ij})$, the matrix $C_{nm\times nm} = A_{n\times n} \otimes B_{m\times m}$ is an $nm \times nm$ which is obtained by substituting each entry b_{ij} of $B_{m\times m}$ with the $n\times n$ block $b_{ij}\, A_{n\times n}$. For example consider:

$$\tau^1 \otimes \tau^2 \;=\; \begin{pmatrix} 0_{2\times 2} & -i\tau^1 \\ i\tau^1 & 0_{2\times 2} \end{pmatrix} \;=\; \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{31.5}$$

where τ^i are the Pauli matrices. This product can be iterated by defining $A\otimes B\otimes C = A\otimes(B\otimes C)$. Show that:

$$(A_1 \otimes B_1)\cdot(A_2\otimes B_2) \;=\; (A_1\cdot A_2)\otimes(B_1\cdot B_2) \tag{31.6}$$

Show that the γ^i can be expressed as the tensor product of two by two matrices in the following way:

$$\begin{aligned}
&\gamma^1 = i\tau^1\otimes \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} && \gamma^2 = i\tau^2\otimes \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\\
&\gamma^3 = i\tau^3\otimes \tau^1\otimes\mathbb{1}\otimes\mathbb{1} && \gamma^4 = i\tau^3\otimes \tau^2\otimes\mathbb{1}\otimes\mathbb{1}\\
&\gamma^5 = i\tau^3\otimes \tau^3\otimes\tau^1\otimes\mathbb{1} && \gamma^6 = i\tau^3\otimes \tau^3\otimes\tau^2\otimes\mathbb{1}\\
&\gamma^7 = i\tau^3\otimes \tau^3\otimes\tau^3\otimes\tau^1 && \gamma^8 = i\tau^3\otimes \tau^3\otimes\tau^3\otimes\tau^2
\end{aligned} \tag{31.7}$$

The above representation is different from the one given in the lecture notes in which all the γ^i are imaginary and the $SO(8)$ generators already appear in a block diagonal form according to the decomposition into chiral representations. In the representation (31.7) however the generators which we will need to diagonalize in the sequel are diagonal to start with.

Compute the matrix $\gamma = \gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^5 \gamma^6 \gamma^7 \gamma^8$ and show that it is the chirality matrix for the $SO(8)$ spinor representation, namely that $(\gamma)^2 = \mathbb{1}_{16\times16}$ and that $\{\gamma, \gamma^i\} = 0$. Is γ diagonal? What are its eigenvalues? Show that the matrix $B = \tau^2 \otimes \tau^1 \otimes \tau^2 \otimes \tau^1$ fulfills $(B)^2 = \mathbb{1}_{16\times16}$ and $B^{-1}\gamma^i B = \gamma^{i*}$. Using the results of exercise 23 and the matrices B and γ express the Majorana and the Weyl conditions on an $SO(8)$ spinor λ and show that they are compatible, so that a Majorana–Weyl spinor can be defined.

Now show that the following matrices realize the $SO(1,9)$ Clifford algebra:

$$\begin{aligned} \Gamma^0 &= \gamma \otimes \tau^2 \\ \Gamma^i &= \gamma^i \otimes \mathbb{1}_{2\times2} \\ \Gamma^9 &= i\gamma \otimes \tau^1 \end{aligned} \tag{31.8}$$

Compute the chirality matrix Γ. Is Γ diagonal? What are its eigenvalues? Show that the matrix $B' = \tau^2 \otimes \tau^1 \otimes \tau^2 \otimes \tau^1 \otimes \tau^3$ fulfills $(B')^2 = \mathbb{1}_{32\times32}$ and $B'^{-1}\Gamma^\mu B' = \Gamma^{\mu*}$. In terms of B' and Γ write the Majorana and the Weyl conditions on an $SO(1,9)$ spinor ξ and show that they are compatible, so that a Majorana–Weyl spinor can be defined. The group $SO(8)$ is contained inside $SO(1,9)$. The subset of the $SO(1,9)$ generators which are also $SO(8)$ generators are represented by the 32×32 matrices $M^{ij} = \Gamma^{ij}/2$. Write these generators in terms of the $SO(8)$ generators given by the 16×16 matrices $\tilde{M}^{ij} = \gamma^{ij}/2$ using the tensor product notation.

Just as for states in quantum mechanics, in order to describe a basis of the $SO(1,9)$ spinorial representation we need to define a *complete set* of $SO(1,9)$ generators, i.e. a maximal set of commuting generators, so that each element of the basis of can be labeled by the simultaneous eigenvalues of these operators. A complete set of $SO(1,9)$ generators consists of five elements $\{H_a\}$ $a = 0,\dots,4$. Show that the following $SO(1,9)$ generators commute:

$$\begin{aligned} H_0 &= \frac{1}{2}\Gamma^0\Gamma^9 \;;\; H_1 = \frac{1}{2}\Gamma^1\Gamma^2 \;;\; H_2 = \frac{1}{2}\Gamma^3\Gamma^4 \;;\; H_3 = \frac{1}{2}\Gamma^5\Gamma^6 \\ H_4 &= \frac{1}{2}\Gamma^7\Gamma^8 \end{aligned} \tag{31.9}$$

Compute their expression as tensor product of two by two matrices and show that as 32×32 matrices they are diagonal. Show that H_0 has real eigenvalues, to be denoted by s_0, as opposite to $H_{k=1,2,3,4}$ which have imaginary eigenvalues, to be denoted by is_k, s_k being real. You may also convince yourself that this abelian set cannot be enlarged, i.e. that there is no other $SO(1,9)$ generator commuting with all of them. A basis for the $SO(1,9)$ spinorial representation can therefore be written in the form $\{|s_0, s_1, s_2, s_3, s_4\rangle\}$, each element being a Majorana spinor, simultaneous eigenvector of the H_a:

$$H_a\,|s_0, s_1, s_2, s_3, s_4\rangle = s_a\,|s_0, s_1, s_2, s_3, s_4\rangle \tag{31.10}$$

Write the elements of this basis explicitly according to the various combinations of values s_a. Show that all these spinors have definite chirality and that

$$\Gamma\,|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle \;=\; 32\, s_0 s_1 s_2 s_3 s_4\, |s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle \qquad (31.11)$$

Let us apply the same procedure for constructing a basis of the $SO(8)$ spinorial representation. Show that a complete set of generators in $SO(8)$ is provided by the following four:

$$h_1 \;=\; \frac{1}{2}\gamma^1\gamma^2\;;\; h_2 = \frac{1}{2}\gamma^3\gamma^4\;;\; h_3 = \frac{1}{2}\gamma^5\gamma^6\;;\; h_4 = \frac{1}{2}\gamma^7\gamma^8 \qquad (31.12)$$

compute the expression of h_k as tensor product of two by two matrices and show that as 16×16 matrices they are diagonal with imaginary eigenvalues $i s_k$. A basis for the $SO(8)$ spinorial representation can therefore be written in the form of Majorana spinors $\{|s_1,\, s_2,\, s_3,\, s_4\rangle\}$. Write these basis explicitly with the various labels s_k. Show that all these spinors have definite chirality and that

$$\gamma\,|s_1,\, s_2,\, s_3,\, s_4\rangle \;=\; 16\, s_1 s_2 s_3 s_4\, |s_1,\, s_2,\, s_3,\, s_4\rangle \qquad (31.13)$$

group the elements $|s_1,\, s_2,\, s_3,\, s_4\rangle$ according to their chirality and show that each chiral representation is generated by 8 elements. The representation with positive chirality is conventionally denoted by $\mathbf{8}_s$, the one with negative chirality by $\mathbf{8}_c$. Show that $H_k = h_k \otimes \mathbb{1}_{2\times 2}$ and therefore they belong to the $SO(8)$ subgroup of $SO(1,9)$. Using this property show that the spinors $|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle$ can be grouped in $SO(8)$ representations $\mathbf{8}_s$ and $\mathbf{8}_c$. For each of these representations write the eigenvalues of Γ and of H_0 (s_0).

[**Hint:** *Prove that the $SO(1,9)$ generator H_0 commutes with the $SO(8)$ subgroup of $SO(1,9)$. Therefore H_0 has a fixed eigenvalue s_0 on each irreducible representation of $SO(8)$ contained in the $SO(1,9)$ spinorial representation, by Shur's lemma.*]

Consider the vacuum of the Ramond sector $|k,\, 0\rangle_R$ with momentum k^μ. It was shown to be an $SO(1,9)$ spinor, and therefore is can be expanded in the basis $|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle$. Let us implement on this state the constraint $p^\mu\,\Gamma_\mu\,|k,\, 0\rangle_R = 0$. Since this state was shown to be massless ($k^\mu k_\mu = 0$) we can consider a frame in which $k^0 = k$, $k^9 = k$ and $k^i = 0$. Find the elements of the basis $|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle$ fulfilling in this frame the constraint $p^\mu\,\Gamma_\mu\,|k,\, 0\rangle_R = 0$. Which $SO(8)$ representations do they belong to? Label them by the s_0 eigenvalue and the $SO(1,9)$ chirality.

[**Hint:** *Write the constraint in the chosen frame as a projector on H_0 eigenspaces:*

$$p^\mu\,\Gamma_\mu \;=\; -2\,k\,\Gamma^0\,(\frac{1}{2} - H_0)$$

]

After imposing the GSO condition on $|k,\, 0\rangle_R$, namely the condition on the vacuum to have a definite $SO(1,9)$ chirality, write the spinors $|s_0,\, s_1,\, s_2,\, s_3,\, s_4\rangle$ contributing to this state for the two different eigenvalues of Γ. Which $SO(8)$ representations do they belong to?

Exercise 32

In the present exercise we shall show that the world sheet of a propagating closed string is conformally equivalent to a sphere with the asymptotic states coinciding with the two polar points.

Consider a closed string propagating in space–time. Represent its world–sheet as a cylinder parametrized by the complex coordinate $w = i\sigma_2 + \sigma_1$, where as usual σ_2 is the Euclidean time coordinate which runs from $-\infty$ (incoming string) to $+\infty$ (outgoing string) and σ_1 the angular coordinate in the interval $(0, \pi)$. Since we are dealing with a closed string the points with $\sigma_1 = 0$ and $\sigma_1 = \pi$ are identified. In the conformal gauge the metric in the coordinates w, $\bar{w}$ can be written as:

$$ds^2 = e^{\phi}\, dw\, d\bar{w} \tag{32.1}$$

write the metric in the coordinates $z = e^{2iw}$, $\bar{z}$. Represent in these coordinates the points corresponding to the asymptotic incoming and outgoing string. Now let us make a suitable choice for the conformal factor, namely:

$$e^{\phi} = 16\,\frac{|z|^2}{(1+|z|^2)^2} \tag{32.2}$$

show that the corresponding ds^2 is the metric on a sphere described by stereographic coordinates and that the asymptotic states coincide with the two polar points. [**Hint:** *Perform the change of variables* $z, \bar{z} \to \theta, \varphi$ *where* $z = \cot(\frac{\theta}{2})\, e^{i\varphi}$]

In general, for a more complicated process involving several incoming and outgoing strings, it can be shown that, at tree level (that is if the surface described by the interacting strings has no holes), the conformal factor of the metric can be chosen so as to map the surface of the initial diagram into a sphere in which the asymptotic states are represented by points or *punctures*, where the corresponding vertex operators are inserted. Analogously higher loop diagrams will be mapped into two dimensional surfaces with a certain number of holes (counted by the genus g of the surface) and punctures corresponding to the asymptotic states.

Exercise 33

Consider type IIA and IIB superstring theories. Write the explicit state realization of the NS–NS massless modes, namely of the scalar field ϕ, the ten dimensional metric G_{ij} and the two form B_{ij} (also called Kalb–Ramond field). As far as the R–R sector is concerned, the zero mode states are expressed as tensor product of the left and right mover sector ground states (we shall use the notation of the lecture notes and denote by $|\psi_L\rangle$, $|\psi_R\rangle$ the R ground states with positive and negative $SO(8)$ chirality respectively (which we have called $\mathbf{8}_s$, $\mathbf{8}_c$ in exercise 31). The states corresponding to the left or right moving sector are distinguished only by their positions (to the left or to the right) in the tensor products). For type IIA and IIB the zero modes of the R–R sector are $|\psi_L\rangle\langle\psi_R|$ and $|\psi_L\rangle\langle\psi_L|$. These states have to be considered as 8×8 matrices in the spinor space and therefore can be decomposed in a complete basis of generators of this space, namely $\Gamma^{i_1 i_2 \ldots i_k}$ for $k = 1, \ldots, 8$. Show that only the matrices $\Gamma^{i_1 i_2 \ldots i_k}$ for $k = 1, \ldots, 4$ contribute to the expansions of $|\psi_L\rangle\langle\psi_R|$ and $|\psi_L\rangle\langle\psi_L|$.

As a first exercise let us consider the simpler case of bi–spinors ξ^α, χ^β. Show that, if we denote by $M^{\alpha\beta} = \xi^\alpha \chi^\beta$, the following expansion is true:

$$\begin{aligned} M &= \sum_i c_i \,\mathrm{Tr}(M^T \tau^i)\, \tau^i \\ c_i &= \frac{1}{\mathrm{Tr}(\tau^{iT} \tau^i)} \end{aligned} \tag{33.1}$$

where τ^i are the Pauli matrices and $\mathrm{Tr}(M^T \tau^i) = \xi^T \tau^i \chi$.

Similarly we can write:

$$\begin{aligned} |\psi_L\rangle\langle\psi_R| &= \sum_k c_k\, C^A_{i_1 \ldots i_k}\, \Gamma^{i_1 i_2 \ldots i_k} \\ |\psi_L\rangle\langle\psi_L| &= \sum_k c_k\, C^B_{i_1 \ldots i_k}\, \Gamma^{i_1 i_2 \ldots i_k} \\ C^A_{i_1 \ldots i_k} &= \langle\psi_L|\Gamma^{i_1 i_2 \ldots i_k}|\psi_R\rangle \\ C^B_{i_1 \ldots i_k} &= \langle\psi_L|\Gamma^{i_1 i_2 \ldots i_k}|\psi_L\rangle \end{aligned} \tag{33.2}$$

where c_k are numerical constants. Show that the fields $C^A_{i_1 \ldots i_k}$ are non vanishing only for $k = 1, 3$ and $C^B_{i_1 \ldots i_k}$ are non vanishing only for $k = 0, 2, 4$. Which $SO(8)$ representations do the C^A and C^B tensors belong to?

Write the explicit state realization of the NS–R and R–NS massless modes. Perform the counting of fermionic and bosonic degrees of freedom.

Exercise 34

In the present exercise we shall study the unoriented closed superstring theory. Unoriented theories are invariant under the transformation $\Omega : \sigma \to \pi - \sigma$. What is the effect of this transformation on the left and right mode operators? Show that only type IIB theory can be made invariant under Ω. Show that the NS–NS fields ϕ, G_{ij} are even under Ω and that B_{ij} is odd [**Hint:** *Write the corresponding states as components of* $b^i_{-1/2}\, \tilde{b}^j_{-1/2}\, |0, 0\rangle$ *and show that the action of* Ω *amounts to* $i \leftrightarrow j$.]

As far as the R–R sector is concerned, show that under Ω C_{ij} is even and C_{ijkl} is odd. As far as the fermion fields are concerned show that the effect of Ω is to switch NS-R $\leftrightarrow$ R–NS and therefore only a symmetric combination of the two survives (is even). Perform a counting of the fermionic and bosonic degrees of freedom and motivate the fact that the amount of supersymmetry is $\mathcal{N} = 1$.