A COLLECTION OF EXERCISES IN TWO-DIMENSIONAL PHYSICS PART I

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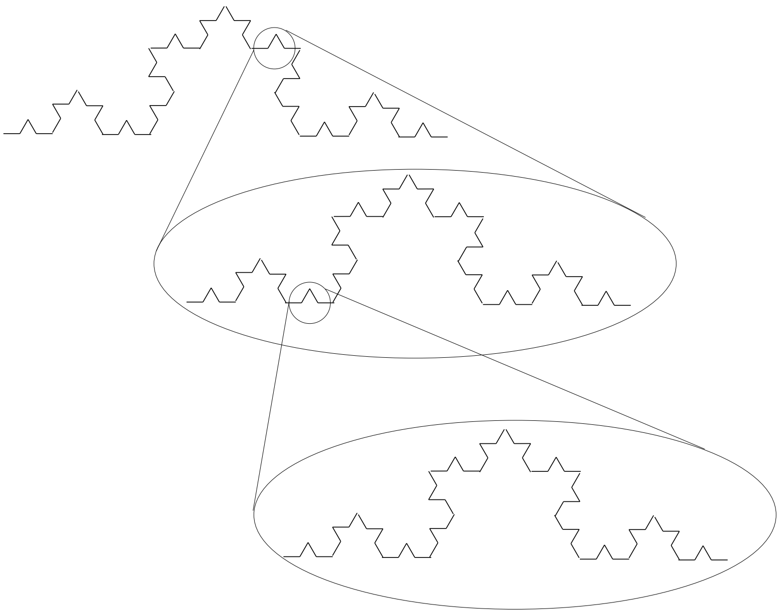

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PROLEGOMENA

A DEFENSE FOR TWO DIMENSIONS
or
AN EXTRAORDINARY PREFACE

In his classic book Flatland [1], E. A. Abbott described an imaginary two-dimensional world, embedded in three dimensions, and populated by two-dimensional figures that think, speak, and have all the human emotions. The author, a schoolmaster with a classics background who was interested in literature and theology, wrote his science fiction fantasy in pure mathematics for entertainment. He published it under a pseudonym to avoid any negative criticism resulting from it reflecting poorly on his more formal works. Apparently, he never imagined that this work not only would entertain many generations of physicists and mathematicians, but also would contain concepts that future scientists would work on.

Almost 120 years after Flatland was written[1], much has happened to our ideas and concepts of spaces with various dimensions. Relativity placed humans in a four-dimensional space-time; more recently, String Theory claims that the dimensionality of our universe has to be updated to 10 or 11 dimensions. We have thus learnt to feel comfortable in a space of many dimensions.

We have also become more comfortable in spaces of few dimensions. Two-dimensional science has grown to one of the most important, well-controlled, and well-developed areas of today's physics and mathematics. Abstract mathematics and theoretical physics have established many breathtaking results, while condensed matter physics has devised many systems that behave as two-dimensional, some of which have great impact on our society.

Of course, despite these two-dimensional applications, many physicists have argued that 'our universe' could not have been two-dimensional, that life could not have existed in two dimensions, and that three dimensions is the minimal feasible dimensionality for our world. In particular, S. Hawking in talks and M. Kaku in his best-seller Hyperspace have argued that the digestive system of living beings would separate them in two disjoint sets in two dimensions and, therefore, even this simple argument can rule out a universe in two

[1]The first edition was published in 1880, while a second corrected edition was printed in 1884. Beyond the mathematical ideas explored, the book reflects many attitudes that were normative in Western society in the late nineteenth century, but that today are rejected as offensive and ill-informed.

dimensions. However, this argument is too quick and facile (although it does point out that evolution on Earth has selected a digestive system suitable to three dimensions, and inappropriate for Flatland). Indeed, if one puts one's mind to it, a model for life in two spatial dimensions can be developed, as is done quite thorougly in A. K. Dewdney's book Planiverse [182].

So, after all, life in two dimensions might be possible. And there may even be other universes out there that realize such a possibility. We may never discover such places. But we can certainly study them. And we will be glad that we did so!

CJE'S ORDINARY PREFACE

In contrast to Warren Siegel [596], when I was a graduate student I used to love books that contained in the title "Introduction to ...". This is probably because, by the time I was a graduate student, many good books delivering on this promise[2] had been written.

Even as an undergraduate student, I recall (with some kind of nostalgia for those years) that I was lucky enough to read some great introductory books. Written on the standard core of physics, those books usually would not include in the title the previous phrase. Among these books, I especially liked Problems in Quantum Mechanics by Constantinescu and Magyari. The book is nicely written and printed. It contains a number of problems on Quantum Mechanics (QM), divided thematically into chapters. Each chapter contains a brief summary of the theory, a collection of problems, and finally their solutions. I have been influenced so greatly by this book, that the architecture of the present document is modeled after it — although I can only hope that we approach the quality of that book's presentation.

Although the subject of the present document is distinct from the subject of its stylistic sibling, the two are not completely unrelated. What one learns in [162], can be used here. After all, QM is 1-dimensional Quantum Field Theory (QFT); the present document explores the developments in 2-dimensional QFT.

During the last two decades, we have witnessed an amazing explosion in the progress of mathematical physics. String Theory[3] has emerged as the leading candidate for the Theory of Everything (TOE), and has led to revolutions in our understanding of the principles underlying fundamental physics. In conjunction with the rise of string theory, we have witnessed the discovery of and advancements in many other areas of mathematical physics: Conformal Field Theory (CFT), Integrable Models (IMs), 2-Dimensional Gravity, Quantum Groups (QGs), and Dualities in QFT, to name just a few areas. The developments in each field have influenced the developments in the other fields, at times in profound and radical ways, at other times in subtler and more controlled fashion.

Thus, I embarked on the preparation of this collection of problems, for those who wish to study mathematical physics and want to see some solved problems to whet their appetite. I hope that people who learn the subjects treated in this collection of problems will find it useful. In this sense, it might also prove useful to people who teach material related to the subjects explored herein.

In his book, Siegel also writes, "It is therefore simultaneously the best time for someone to read a book and the worst time for someone to write one." This sentence remains true

[2] This has already become better with the creation of the LANL archives by P. Ginsparg. Many beautifully written articles which include "Introduction to ..." in their titles are posted there frequently in a variety of fields. Papers of all flavors are available there for virtually any taste.

[3] According to the standard lore of naming theories in Field Theory, the name *Quantum Nematodynamics* (QND) is more appropriate. Perhaps the name *String Theory* should be used only at the classical level. Another possible name, given the recent developments regarding branes, is *Quantum Branodynamics* (QBD); however, I personally would vote in favor of QND as I find *Quantum Nematodynamics* the most euphonic choice.

for this collection as well. Recent times have proven so fertile for mathematical physics that many results are considered 'common knowledge' just a few days after they are posted on the LANL archives. As a result, no review or book can cover completely such a vast terrain. The present document is certainly no exception. It covers only a fraction of the relevant subjects, and even for those covered, only a limited number of possible themes have been touched. However, depending on the interest, we hope that in a later edition, more topics will be added, and more exercises within each area.

We have partitioned the material in 15 chapters. Although there are not always sharp boundaries among these chapters, this approach was taken to enhance the pedagogical value of this work.

Finally, we would like to say that for such a subject, it is impossible to give an exhaustive list of references. We primarily cite review papers and books, as those may be of great help for the reader who would like to study the material. However, we have also included many significant original papers in the bibliography that might not be cited in the text. In this spirit, we would like apologize to all people whose works are not cited; there is no way to be exhaustive, nor have we attempted to be, and in some cases such omissions are doubtless a result of our ignorance. If this document serves as an introduction to the field of two-dimensional physics and its literature, it will have served its purpose.

DAS'S ORDINARY PREFACE

The very first paper I published offered new demonstrations of the integrability of some non-linear sigma models, with and without supersymmetry, in two spacetime dimensions. While that paper predates (by just a little bit) the revolution in 2-dimensional physics that this manuscript addresses, I am struck by how much the topics of that paper are echoed here.

If I trace my own personal history of interest in the topics covered here, there are two pivotal moments, beyond that first paper of mine. The first moment is the ICGTMP (International Colloquium on Group Theoretical Methods in Physics) held in Montreal in 1988. Those were heady times. The conference itself was a cornucopia of string theory, conformal field theory, and quantum groups, held in a political context that allowed attendance by an extensive collection of physicists and mathematicians from around the globe. How could one not be hooked?

The second moment was my decision to pursue a career in a liberal arts institution. These are places not well-understood outside the USA; they are colleges with no graduate students or postdocs, but that does not mean that they are void of research. On the contrary, my colleagues are some of the most vibrant minds I know. But it does mean that many of us do not have the benefit of spending time each day, bumping into colleagues in the hallway or seminar room, and learning things almost by osmosis. The value of such a pedagogical document as the present one of course transcends my own personal context; but this context has made clearer to me what the value of such a manuscript is.

My co-author Costas Efthimiou has been the driving force behind this project, and the rationale he presents above is indeed the same rationale that drew me into this project, and I am glad to have been drawn in. There is no need for me to repeat the ideas Costas has expressed above. But I will express my hope that this document will find multiple uses, from students beginning their explorations of theoretical physics in graduate school, to established scientists trying to move into new areas of research, to faculty seeking inspiration for their courses.

ACKNOWLEDGEMENTS

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Comments and criticism are welcomed and greatly encouraged.

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Other mistakes may perchance...await the penetrating glance of some critical reader, to whom the joy of discovery, and the intellectual superiority which he will thus discern, in himself, to the author of this little book, will, I hope, repay to some extent the time and trouble its perusal may have cost him!
LEWIS CARROLL

COMMENTS

1. When you read this document, please keep in mind that the document is still in its infancy. Most sections are brief, and the presentation at times somewhat abbreviated. The reader is also warned that, despite our best efforts, no doubt many typos remain. We apologize, too, that different conventions may still be used in different parts of the document! This is due to the fact that several sections have their origins in projects undertaken before conceiving the plan to prepare one comprehensive pedagogical collection. (Some would say this might even be valuable, preparing the reader for the array of conventions in the published literature!) We hope that the reader nonetheless finds this work valuable and beneficial. If all proceeds according to our expectations, when the document reaches its adult stage, it will have been cured of all these childhood diseases!

2. On the cover page, a release number

   RELEASE $N.n$

   is given (see on the cover). It should be interpreted as follows. A higher release number of course signals a newer version. A larger n means that *simple* typos have been corrected, wording may have been improved, conventions and notation may have been uniformized, additional references may have been added, but *no essential* changes have been made. Reprinting the document is in this case *strongly discouraged* — save the forests! A larger N number means that new material has been added (e.g. new problems in previously existing sections, new sections in previously existing chapters, or even new chapters) or *conceptual or other important* mistakes have been corrected.

3. Of course, there are many topics that could be added to the present document. A list of topics that would appear as natural extensions to our work would include (see the list of abbreviations on page xi):

   - Background Material in QFT and in Mathematics

- Supersymmetric CFT
- Higher Genus CFT
- CFT in $D > 2$ Dimensions
- IMs in QM
- Classical IMs
- Quantization of Classical IMs
- Bethe Ansatz
- Form Factors for IMs
- Boundary IMs
- Vertex Models
- Applications to Condensed Matter
- Knot Theory
- Matrix Models
- Topological Field Theories
- String Theory
- Seiberg-Witten (SW) Theory and IMs

As the release number increases, this list of missing items should shorten until it *dissappears* (what optimism!!!). However, even if this occurs, it would by no means imply that we had achieved a complete coverage of these topics; it should be only interpreted as a completion of *our* target, which is to allow the reader a foundation for indulging in the exploration of mathematical physics and physical mathematics.

4. Abbreviations are in general defined at the place of their first occurrence. However, especially if you do not read this document sequentially, relying on this for definitions may be somewhat cumbersome. Certain well-known abbreviations may even not be expanded in any place in the document. Therefore, we have included a table of abbreviations (Table 1), which appears on the following page.

Abbreviation	Explanation
BCFT	Boundary Conformal Field Theory
CFT	Conformal Field Theory
CBC	Conformal Boundary Condition
CGC	Coulomb Gas Construction
CGF	Coulomb Gas Formulation
DSZ	Dirac-Schwinger-Zwanziger
GUT	Grand Unified Theory
IM	Integrable Model
LANL	Los Alamos National Laboratory
l.h.s.	left hand side
MM	Minimal Model
OPE	Operator Product Expansion
QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
QFT	Quantum Field Theory
QG	Quantum Group
QM	Quantum Mechanics
RCFT	Rational Conformal Field Theory
reg	Non-Singular Terms in Operator Product Expansion
r.h.s.	right hand side
RSOS	Restricted Solid-on-Solid
SCFT	Supersymmetric Conformal Field Theory
SG	Sine-Gordon
S-matrix	Scattering Matrix
SOS	Solid-on-Solid
SUSY	Supersymmetry
SUGRA	Supergravity
SW	Seiberg-Witten
TOE	Theory of Everything
UMM	Unitary Minimal Model
W-matrix	Wall (or Reflection) Matrix
w.r.t.	with respect to
WZWN	Wess-Zumino-Witten-Novikov
YBE	Yang-Baxter Equation

Table 1: Table of abbreviations and acronyms used in this document.

Contents

PART I

PART II

Warning! Part II has not yet been released. The contents and page numbers for Part II are therefore preliminary, and subject to change.

Chapter 1

CURIOSITIES IN TWO DIMENSIONS

> But to me, proficient though I was in Flatland Mathematics, it was by no means a simple matter...although I saw the facts before me, the causes were as dark as ever.
>
> A. Square

References: Good introductions to spin and statistics in two dimensions are found in [275, 478]. A recent review on generalized statistics in 1+1 dimensions is [553]. Bosonization is discussed in [607]. The conformal transformations are discussed in many books on complex analysis, e.g. [519].

1.1 BRIEF THEORY

2-dimensional mathematics and physics are quite special. The lack of 'extra space' imposes such great constraints that the particles and their interactions adhere to a highly restrictive set of properties. This chapter examines some of the basic implications of a 2-dimensional world. This lays the groundwork for what follows in the subsequent chapters.

1.1.1 Statistics in Two Dimensions

It is well-known that the path integral on a multipy-connected space $\mathcal{C}$ decomposes into a sum of path integrals (one for each class of the first homotopy group), with the contributions from each sector weighted by a factor dependent on the equivalence class only, so that

$$Z(x,y) \;=\; \sum_{\alpha\in\pi_1(\mathcal{C})} \chi(\alpha)\, Z_\alpha(x,y) \ .$$

The weights $\chi(\alpha)$ are restricted by two physical requirements:
(a) Physical observables cannot depend on the mesh used to calculate the homotopy classes.
(b) The weighted sum must satisfy the standard convolutive property

$$Z(x,y) \;=\; \int dz\, Z(x,z) Z(z,y) \ .$$

As a result, the weights satisfy the constraints

$$\left.\begin{array}{r}|\chi(\alpha)| = 1\ , \quad \text{and} \\ \chi(\alpha)\chi(\beta) = \chi(\alpha\beta)\ .\end{array}\right\} \tag{1.1}$$

Therefore the weights χ provide a 1-dimensional unitary representation of the fundamental group $\pi_1(\mathcal{C})$.

Given n particles in d dimensions, the space of all allowed configurations would be $\overline{\mathcal{C}}_n = \mathbb{R}^{dn} \setminus D$, where D is the subset of $\mathbb{R}^{dn}$ where at least two particles have identical position vectors. When the particles are of identical kind, configurations that differ by a permutation are not distinct. To avoid multiple enumeration of the configurations, one must divide by the permutation group for n objects S_n. In this way, the configuration space $\mathcal{C}_n$ of n identical particles reads

$$\mathcal{C}_n \ = \ \frac{\overline{\mathcal{C}}_n}{S_n} \ = \ \frac{\mathbb{R}^{dn} \setminus D}{S_n}\ .$$

In $d > 2$ dimensions, $\mathcal{C}_n$ is simply-connected and the fundamental group is $\pi_1(\mathcal{C}_n) = S_n$. The permutation group has only two unitary abelian representations; either

$$\chi(P) \ = \ 1\ , \qquad \forall P \in S_n\ ,$$

or

$$\chi(P) \ = \ \begin{cases} +1\ , & \text{if } P \text{ is even}\ , \\ -1\ , & \text{if } P \text{ is odd}\ . \end{cases}$$

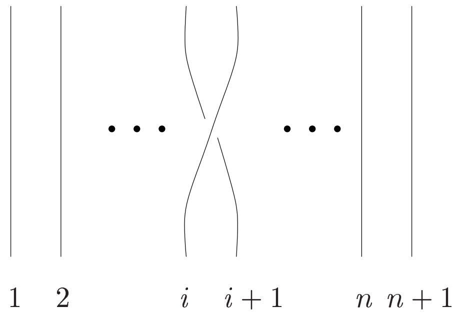
These representations correspond to bosons and fermions, respectively.

In $d = 2$ dimensions, $\mathcal{C}_n$ is multiply-connected and $\pi_1(\mathcal{C}_n)$ is more complicated. In fact, $\pi_1(\mathcal{C}_n) = B_n$, the **braid group** of n strings. Combinatorially, B_n is generated by a set of generators σ_i, $i = 1, 2, \ldots, n$, obeying the relations

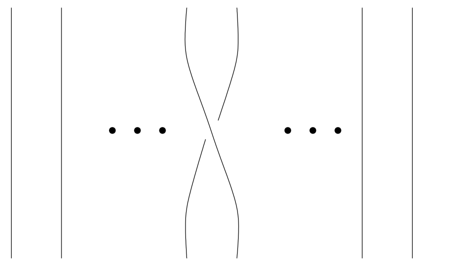
$$\sigma_i\sigma_{i+1}\sigma_i \ = \ \sigma_{i+1}\sigma_i\sigma_{i+1}\ , \tag{1.2}$$

$$\sigma_i\sigma_j \ = \ \sigma_j\sigma_i\ , \quad |i-j| > 1\ . \tag{1.3}$$

Intuitevely, we can think of the generator σ_i as the operation that twists the i-th and $(i+1)$-th strings, as depicted here:



The inverse σ_i^{-1} of the operator σ_i interweaves the same strings in the opposite way:



Any element $\sigma \in B_n$ can be represented as set of n interwoven strings, that is, as an ordered product of the generators σ_i and their inverses σ_i^{-1}. Relation (1.3) says that the twistings of two well-separated pairs of strings are independent. Relation (1.2) identifies two ways of interweaving three successive strings that are equivalent.

Regarding the weights χ, relations (1.2) and (1.3) require that the phase be universal

$$\chi(\sigma_i) \;=\; \chi(\sigma_j) \ , \qquad \forall i, j \ .$$

We write

$$\chi(\sigma_i) \;=\; e^{i\theta} \ .$$

A path (braid) $\sigma \in \pi_1(\mathcal{C}_n)$ represented by the product

$$\sigma \;=\; \prod_{k=1}^{n} \sigma_{i_k} \ ,$$

is thus weighted by the factor

$$\chi(\sigma) \;=\; \prod_{k=1}^{n} \chi(\sigma_{i_k}) \;=\; e^{i\theta \sum_{k=1}^{n} \mathrm{sgn}\sigma_{i_k}} \ ,$$

where

$$\mathrm{sgn}\sigma_{i_k} \;=\; \begin{cases} +1 \ , & \text{if } \sigma_{i_k} \text{ is a generator} \ , \\ -1 \ , & \text{if } \sigma_{i_k} \text{ is an inverse generator} \ . \end{cases}$$

1.1.2 Bosonization/Fermionization

In two dimensions, we have seen that particles may obey a variety of statistics, in contrast with higher dimensions, in which particles must obey either fermionic or bosonic statistics. However, one might still expect that each of the choices would be related to a particular choice of the intrinsic spin of the particles:

$$\theta\text{-statistics} \leftrightarrow \text{particles with spin } s_\theta \ .$$

Surprisingly, spin is not an intrinsic property in two dimensions, and statistics are a matter of convention. For example, given any bosonic operators $b(p)$ and $b^\dagger(p)$ that satisfy the commutation relations

$$\begin{aligned} \left[b(p), b^\dagger(p')\right] &= p^0 \delta(p-p') \ , \\ \left[b^\dagger(p), b^\dagger(p')\right] = \left[b(p), b(p')\right] &= 0 \ , \end{aligned}$$

one can construct operators $f(p)$ and $f^\dagger(p)$, defined by

$$\left.\begin{aligned} f^\dagger(p) &= b^\dagger(p)\, e^{-i\pi \int_{p_1}^{+\infty} \frac{dk_1}{k_0} b^\dagger(k) b(k)} \ , \\ f(p) &= e^{i\pi \int_{p_1}^{+\infty} \frac{dk_1}{k_0} b^\dagger(k) b(k)}\, b(p) \ , \end{aligned}\right\} \tag{1.4}$$

which are fermionic operators that satisfy the anticommutation relations

$$\begin{aligned} \left\{f(p), f^\dagger(p')\right\} &= p^0 \delta(p-p') \ , \\ \left\{f^\dagger(p), f^\dagger(p')\right] = \left\{f(p), f(p')\right\} &= 0 \ . \end{aligned}$$

Thus there is a 1-to-1 mapping between the states that are symmetric and those that are antisymmetric under permutations!

Having no intrinsic spin in two dimensions, we use the term "spin" to refer to the Lorentz spin, i.e., to the eigenvalue of a state under the boost operator. Using Wigner's classic analysis, the scalar states can be chosen to transform under boosts as

$$U(\alpha)\, |\, p_0, p_1 \,\rangle = \left|\, p_0', p_1' \,\right\rangle \ ,$$

where

$$\begin{bmatrix} p_0' \\ p_1' \end{bmatrix} = \begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \ .$$

One can define the states

$$|\, p_0, p_1; s \,\rangle \equiv \left(\frac{p_0 - p_1}{m}\right)^s |\, p_0, p_1 \,\rangle \ ,$$

which then transform by

$$U(\alpha)\, |\, p_0, p_1; s \,\rangle = e^{-s\alpha} \left|\, p_0', p_1'; s \,\right\rangle \ ,$$

i.e., they have Lorentz spin s. Zero mass states have a larger symmetry group — the conformal group.

The issues we have enountered in this section will arise many times throughout this document.

1.1.3 The Conformal Group in Two Dimensions

Let $f : M_1 \to M_2$ be a map from the n-manifold M_1 to the n-manifold M_2. Then, f is said to be a **conformal transformation** if $\forall p \in M_1$, given any $\boldsymbol{v}, \boldsymbol{u} \in T_pM_1$,

$$\boldsymbol{g}'_{f(p)}(f_*(\boldsymbol{v}), f_*(\boldsymbol{u})) \;=\; \Omega^2\, \boldsymbol{g}_p(\boldsymbol{v}, \boldsymbol{u}) \;.$$

In the above equation, $\boldsymbol{g}$ and $\boldsymbol{g}'$ are the metrics on M_1 and M_2 respectively and f_* is the induced map $f_* : T_pM_1 \to T_{f(p)}M_2$. When $M_1 = M_2$, f is called a **conformal isometry** of M_1. It is very instructive to write the transformation rule explicitly:

$$\Omega^2(p)\, g_{\mu\nu}(p) \;=\; \frac{\partial f^\rho}{\partial x^\mu}\,\frac{\partial f^\sigma}{\partial x^\nu}\, g'_{\rho\sigma}(f(p)) \;. \tag{1.5}$$

A conformal transformation leaves invariant the angle θ between two vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ in the tangent space, since this angle obeys

$$\cos\theta \;\equiv\; \frac{\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{u})}{\sqrt{\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v})\, \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{u})}} \;.$$

We may say that a conformal transformation changes only the size or scale of an object while keeping its shape fixed; this intuitive visualization of the transformation is more faithful at small distances.

If for a (pseudo-)Riemannian manifold M, there is an atlas $\{U_j, \phi_j\}$ such that the metric tensor $\boldsymbol{g}_j$ on each patch is conformally related to the flat metric, i.e., $\boldsymbol{g}_j = \Omega_j^2\, \boldsymbol{\eta}$, then the manifold M is said to be **conformally flat**.

Let $f^\mu(\boldsymbol{x})$ be a conformal isometry of the metric manifold M, i.e., suppose that

$$g_{\mu\nu}(\boldsymbol{x}) \;=\; \Omega^2\, g_{\alpha\beta}(\boldsymbol{f}(\boldsymbol{x}))\, \frac{\partial f^\alpha}{\partial x^\mu}\,\frac{\partial f^\beta}{\partial x^\nu} \;. \tag{1.6}$$

Without loss of generality, we can work with infinitesimal displacements

$$f^\mu(\boldsymbol{x}) \;=\; x^\mu + \epsilon\, \xi^\mu(\boldsymbol{x}) \;, \qquad |\epsilon| \ll 1 \;, \tag{1.7}$$

since any finite displacement can be made from an infinite sum of infinitesimal transformations. For infinitesimal conformal transformations, we must have

$$\Omega^2 \;=\; 1 + \epsilon\, \psi + \mathcal{O}(\epsilon^2) \;. \tag{1.8}$$

Using (1.7) and (1.8), equation (1.6) becomes

$$(1 + \epsilon\, \psi)\, g_{\mu\nu} \;=\; g_{\mu\nu} + \epsilon\, g_{\mu\beta}\, \partial_\nu \xi^\beta + \epsilon\, g_{\alpha\nu}\, \partial_\mu \xi^\alpha + \epsilon\, \partial_\rho g_{\alpha\nu}\, \xi^\rho + \mathcal{O}(\epsilon^2) \;,$$

or

$$\xi^\rho\, \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho\, g_{\rho\nu} + \partial_\nu \xi^\rho\, g_{\mu\rho} \;=\; \psi\, g_{\mu\nu} \;. \tag{1.9}$$

The scalar ψ is determined by self-consistency. Multiplying the last equation by $g^{\nu\mu}$ and summing over the indicated repeated indices, we find

$$\xi^\rho\, g^{\nu\mu}\, \partial_\rho g_{\mu\nu} + 2\,\partial_\rho\xi^\rho \;=\; n\,\psi\ .$$

Substituting for ψ in equation (1.9), we finally obtain

$$\xi^\rho\,\partial_\rho g_{\mu\nu} + \partial_\mu\xi^\rho\, g_{\rho\nu} + \partial_\nu\xi^\rho\, g_{\mu\rho} \;=\; \frac{1}{n}\,(\xi^\rho\, g^{\kappa\sigma}\,\partial_\rho g_{\sigma\kappa} + 2\,\partial_\rho\xi^\rho)\, g_{\mu\nu}\ . \tag{1.10}$$

This is the **conformal Killing equation**. The solutions $\boldsymbol{\xi}$ of this equation are called the **conformal Killing vector fields** associated with the given metric. The above discussion makes obvious that if $\psi = 0$, i.e., if we are considering isometries of M, then the conformal Killing equation reduces to the simpler form

$$\xi^\rho\,\partial_\rho g_{\mu\nu} + \partial_\mu\xi^\rho\, g_{\rho\nu} + \partial_\nu\xi^\rho\, g_{\mu\rho} \;=\; 0\ . \tag{1.11}$$

This equation is called simply the **Killing equation**; the corresponding solutions $\boldsymbol{\xi}$ are called the **Killing vector fields**.

For a conformally flat n-dimensional manifold, the conformal Killing equation gives

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu \;=\; \frac{2}{n}\,\eta_{\mu\nu}\,\partial_\rho\xi^\rho\ . \tag{1.12}$$

The solutions of this equation generate the **conformal group**. For $d > 2$, the conformal group is finite-dimensional. This group in two dimensions is infinite-dimensional, imposing great restrictions on any conformally invariant field theory. In particular, the above equation (1.12) reduces to the Cauchy-Riemann conditions

$$\frac{\partial\xi_2}{\partial x^1} \;=\; -\,\frac{\partial\xi_1}{\partial x^2}\ , \qquad \frac{\partial\xi_1}{\partial x^1} \;=\; \frac{\partial\xi_2}{\partial x^2}\ .$$

Using complex coordinates

$$z \;=\; x^1 + ix^2\ , \qquad \overline{z} \;=\; x^1 - ix^2\ ,$$

and the function $\xi = \xi_1 + i\,\xi_2$, the Cauchy-Riemann conditions guarantee that ξ is a holomorphic function, i.e., that

$$\frac{\partial\xi}{\partial\overline{z}} \;=\; 0\ .$$

The conformal transformations (1.5) are thus realized in two dimensions by

$$z \;\to\; z' = f(z)\ , \qquad \overline{z} \;\to\; \overline{z}' = \overline{f}(\overline{z})\ . \tag{1.13}$$

This decoupling of variables allows one to handle the coordinates z and $\overline{z}$ as independent; the reality condition $\overline{z} = z^*$ can be imposed at the end of the day.

1.1.4 Commonly Used Conformal Transformations

In this document, conformal transformations will underlie nearly all that we do. Therefore, it is worthwhile to study some of them and become familiar with their properties.

A fundamental map is

$$w = e^z \tag{1.14}$$

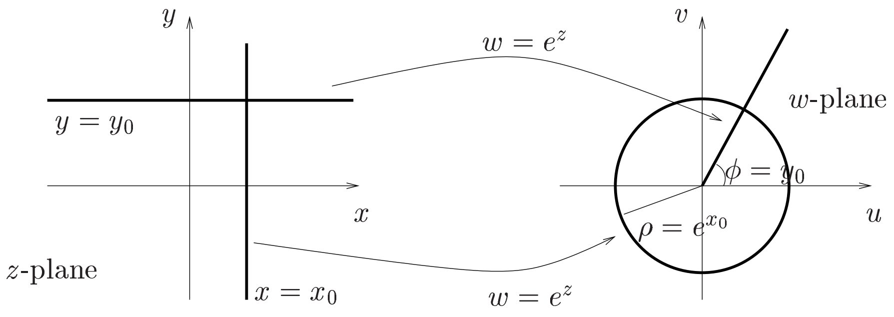
from the z-plane to the w-plane. Decomposing z and w into real and imaginary parts via $z = x + iy$ and $w = u + iv$, this map can be expressed in real coordinates as

$$\begin{aligned} u &= e^x \cos y \ , \\ v &= e^x \sin y \ . \end{aligned}$$

Therefore

$$\begin{aligned} u^2 + v^2 &= e^{2x} \ , \\ v &= u \tan y \ , \end{aligned}$$

and the line $x = x_0$ is mapped to a circle with radius $\rho = e^{x_0}$. The mapping (1.14) transforms lines parallel to the imaginary axis into circles around the origin. If x is the temporal coordinate, then time ordering on the z-plane becomes radial ordering on the w-plane. A line $y = y_0$ is mapped to the half-line $v = (\tan y_0)u$. Thus the mapping (1.14) also transforms lines parallel to the real axis into rays directed outward from the origin.



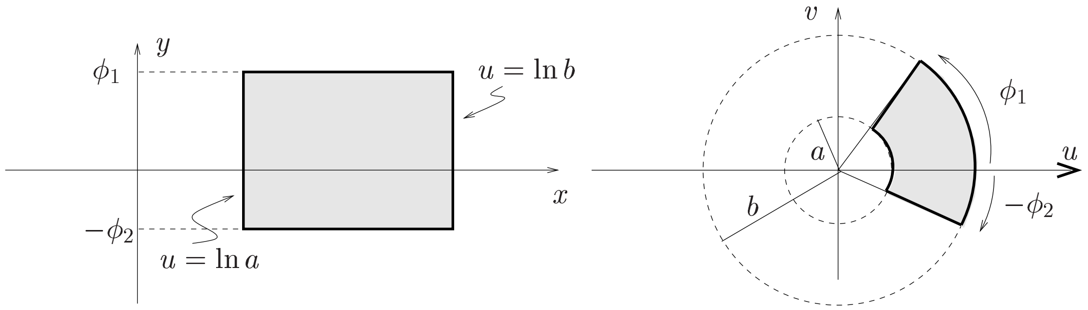
Additional properties may be derived considering the inverse transformation of (1.14),

$$z = \ln w \ . \tag{1.15}$$

Using the polar coordinates ρ and ϕ on the w-plane, with $w = \rho e^{i\phi}$, (1.15) becomes

$$\left.\begin{aligned} x &= \ln \rho \ , \\ y &= \phi \ . \end{aligned}\right\} \tag{1.16}$$

Equation (1.16) says that an arc $\phi_1 \leq \phi \leq \phi_2$ of the circle $\rho = a$ on the w-plane is mapped to a segment on the line $x = \ln a$. The segment starts at $y = \phi_1$ and ends at $y = \phi_2$. Also, a segment $a \leq \rho \leq b$ lying on the ray $\phi = \phi_1$ is mapped to a segment on the line $y = \phi_1$. The segment starts at $x = \ln a$ and ends at $x = \ln b$. All this together means that any rectangle in the z-plane becomes an angular sector in the w-plane.



An important theorem in conformal mapping is the **Riemann mapping theorem**:

Theorem [Riemann]
Given two arbitrary simply-connected domains D and D' whose boundaries contain more than one point, there always exists an analytic function $w = f(z)$ that maps D onto D'. The function $f(z)$ depends on three real parameters.

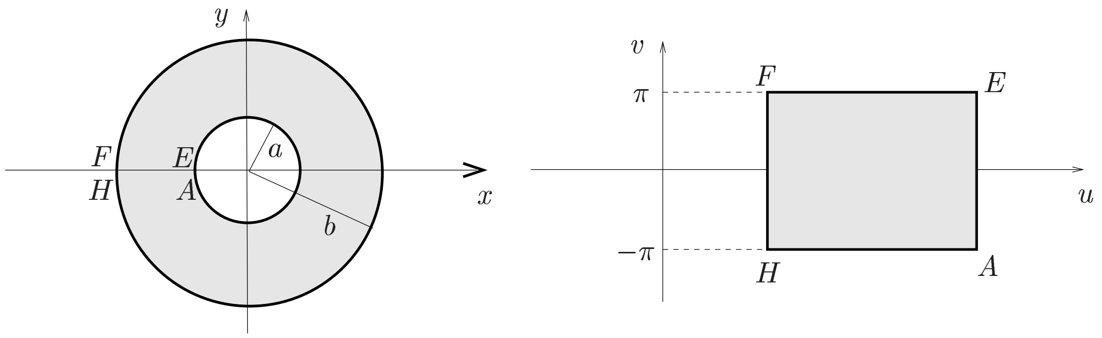
Usually D' is taken to be the interior of the unit disc. We can restate the Riemann mapping theorem in this case as:

Theorem
Any simply-connected domain D whose boundary contains more than one point can be mapped conformally onto the interior of the unit disc. Moreover, it is possible to choose any one point of D and a direction through this point to be mapped to the origin and a direction through the origin; these choices render the map unique.

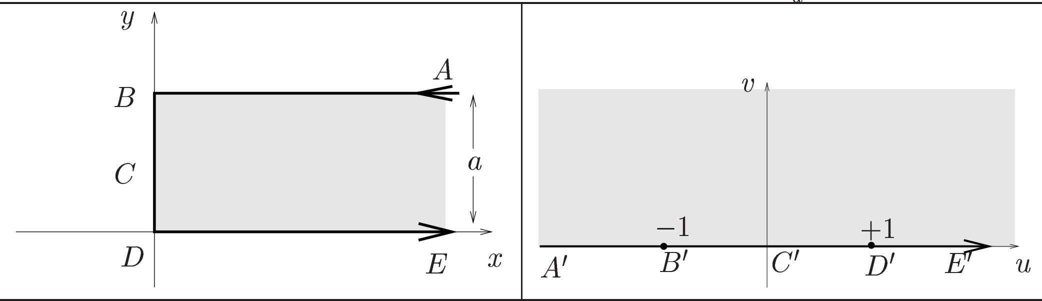
The Riemann mapping theorem, upon careful examination, can be extended to the case that D is bounded between two closed simple curves, one inside the other. In this case, the map is to an annulus.

We can employ similar considerations for various conformal maps. We tabulate some results to be used in later sections in table 1.1.

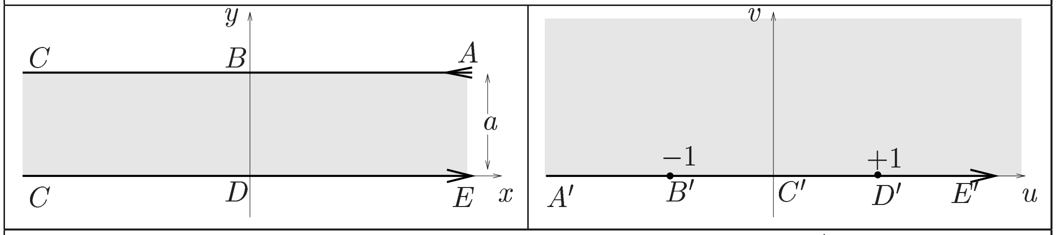
In the last entry of the table 1.1, imagine that we take $\phi_1 = \pi$ and $\phi_2 = -\pi$. In this case, the angular sector becomes a ring, i.e., a domain that is not simply-connected. Its image on the w-plane appears to be a simply-connected rectangle.



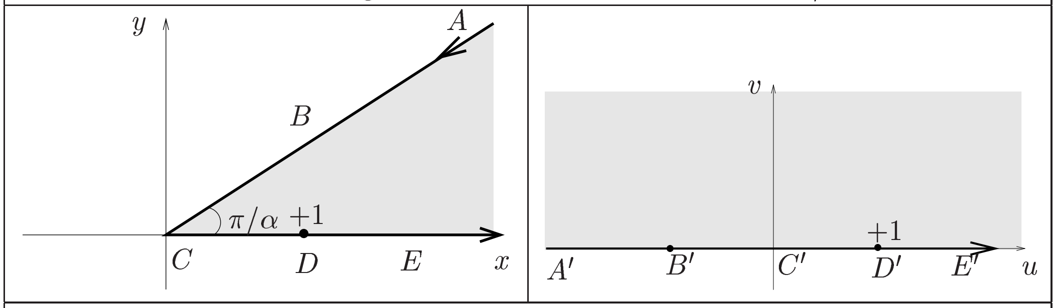
semi-infinite strip to half-plane $w = \cosh \frac{\pi z}{a}$



infinite strip to half-plane $w = e^{\frac{\pi z}{a}}$



infinite wedge to half-plane $w = z^{\alpha}$, $\alpha \geq 1/2$



angular sector to rectangle $w = \ln z$

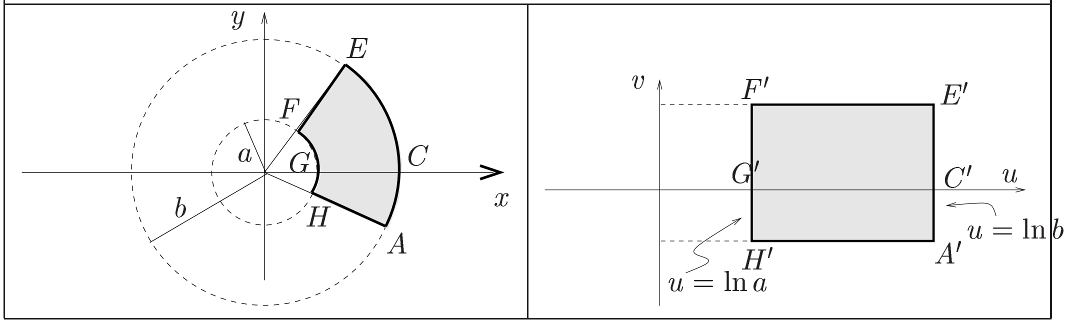
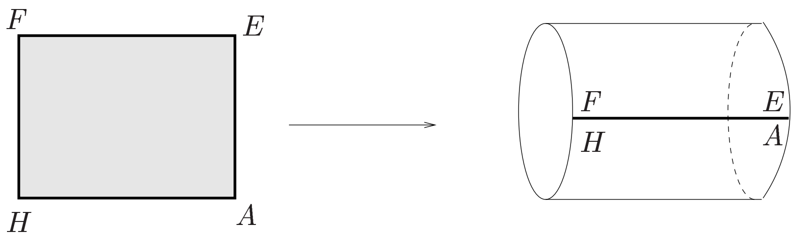


Table 1.1: Some commonly used conformal transformations.

plane to cylinder	$w = \frac{a}{2\pi} \ln z$
upper half-plane to infinite strip without periodic boundary conditions on the edges	$w = \frac{a}{\pi} \ln z$

Table 1.2: Two different conformal transformations.

This seems to contradict the Riemann mapping theorem. However, one must notice that the edges FE and HA coincide now on the z-plane and therefore, under the mapping, they should be identified on the w-plane too. This means that the rectangle is really a cylinder which is also not a simply-connected domain.



A final comment is in order here. In applications of the conformal transformations, special care must be given to all quantities and constants in the transformation. Sometimes, 'innocent' factors are manifestations of a very subtle situation! For example, in Table 1.2 we give two transformations that differ only by an 'innocent' factor of 2; nonetheless (see the second and fourth entries in Table 1.1), they are quite different!

1.1.5 Symmetries of the S-Matrix

One of the goals of the machinery of QFT is to devise methods for determining the S-matrix — i.e., the array of probabilities for all possible events — and to carry out the program and calculate the S-matrix in particular theories, especially those that are applicable to the real world. Clearly, the symmetries of the S-matrix constrain its possible form. (In the Lagrangian framework, all symmetries of the Lagrangian that are not spontaneously broken appear as symmetries of the S-matrix.) Hence, knowing the possible symmetries of the S-matrix is both useful and informative.

For a QFT theory defined on a flat manifold, one always assumes that the Poincaré group $\mathcal{P}$ is a symmetry of the S-matrix. This is a 'must' in the construction of any physical model. The full symmetry group of the S-matrix must thus include $\mathcal{P}$ as a subgroup. Attempts to find physical symmetry groups that included $\mathcal{P}$ in a non-trivial way led ultimately to a no-go theorem known as the Coleman-Mandula theorem. Under very mild conditions, Coleman and Mandula proved that that there can be no exotic space-time symmetries of the S-matrix. More exactly, the statement of the theorem is as follows [162]:

Theorem [Coleman and Mandula]

Let G be a connected symmetry (Lie) group of the S-matrix, and let the following five conditions hold:

1. (Lorentz invariance) G contains a subgroup locally isomorphic to $\mathcal{P}$.
2. (Particle finiteness) All particle types correspond to positive energy representations of $\mathcal{P}$. For any finite M, there is only a finite number of particle types with mass less than M.
3. (Weak elastic analyticity) Elastic scattering amplitudes are analytic functions of the center-of-mass energy s and invariant momentum transfer t, in some neighborhood of the physical region, except at normal thresholds.
4. (Occurrence of scattering) Let $|\,p\,\rangle$ and $|\,p'\,\rangle$ be any two 1-particle momentum eigenstates, and let $|\,p,p'\,\rangle$ be the 2-particle state made from these. Then
$$T\,|\,p,p'\,\rangle \neq 0\ , \qquad S = 1 - i(2\pi)^4\delta^4(P-P')\,T\ ,$$
except perhaps for certain isolated values of s. Phrased physically, this assumption is that at almost all energies, any two plane waves scatter.
5. (An ugly technical assumption) The kernels of the generators of G, considered as integral operators in momentum space, are distributions. More precisely: There is a neighborhood of the identity in G such that every element in G in this neighborhood lies on some 1-parameter group $g(t)$. Further, if x and y are any two states in the the set $\mathcal{D}$ of all 1-particle states whose momentum space wave functions are test functions, then
$$-i\frac{d}{dt}\,\langle\, x|g(t)|y\,\rangle \;=\; \langle\, x|A|y\,\rangle$$
exists at $t=0$, and defines a continuous function of x and y, linear in y and antilinear in x.

Then as long as the S-matrix is not trivial (i.e., as long as the S-matrix is not that of a set of non-interacting objects), G is locally isomorphic to the direct product of an internal symmetry group $\mathcal{I}$ and the the Poincaré group:
$$\mathrm{G} \;=\; \mathcal{I}\otimes\mathcal{P}\ .$$

A corollary of the Coleman-Mandula theorem is that the generators I_a of $\mathcal{I}$ must commute with the momentum generators P_μ and the angular momentum generators $J_{\mu\nu}$.
$$[I_a, J_{\mu\nu}] = [I_a, P_\mu] = 0\ .$$
Therefore, if we classify the particles in families according to representations of $\mathcal{I}$, then all particles in the same representation must have the *same mass* and the *same spin*, since
$$\left[I_a, P^2\right] = \left[I_a, W^2\right] = 0\ ,$$
where W^μ is the Pauli-Lubarski vector
$$W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}\frac{P^\sigma}{\sqrt{P^2}}\ .$$

There are three important loopholes in this theorem, two of which were known from the time the Coleman-Mandula theorem was proven.

- If assumption 2 is relaxed, the space-time symmetry group can be extended to the conformal group, which includes $\mathcal{P}$ as a subgroup. However, if the other assumptions are preserved, the final conclusion is almost the same, with the Poincaré group replaced by the conformal group, as Coleman and Mandula discussed in their original article.

- The theorem makes use of Lie groups, i.e., groups whose algebra is based on commutation relations of the form
$$[T_i, T_j] \;=\; c_{ij}^k\, T_k \ .$$
Once one assumes that generalized commutation relations are possible for the generators, the theorem does not apply. In particular, supersymmetry (SUSY) includes commutation and anticommutation relations in its algebra,
$$\begin{aligned} [E_a, E_b] &= f_{ab}^c\, E_c \ , \\ [E_a, O_i] &= m_{ai}^j\, O_j \ , \\ \{O_i, O_j\} &= c_{ij}^k\, E_k \ . \end{aligned}$$
One can re-work the analysis of the Coleman-Mandula theorem in the case that Lie groups and supergroups are allowed [359]. The results are exactly analogous to those of the original theorem, with the conclusion being that the most general possible supersymmetry of the S-matrix is as before, except with the Poincaré (conformal) group replaced by the super-Poincaré (superconformal) group.

- The theorem assumes that the S-matrix is an analytic function of at least two Mandelstam variables s and t. However, this can be true only in $d+1$ space-time dimensions, with $d \geq 2$. In $1+1$ dimensions, there is only one Mandelstam variable, and therefore this assumption is not valid. Stated physically, in $1 + 1$ dimensions, there can only be forward or backward scattering, and hence there is no prospect of analyticity in the scattering angle. Indeed, this is not just a technical observation. In $1 + 1$ dimensions, theories exist which have a non-trivial S-matrix *and* which also have extended space-time Lie symmetries that contain the Poincaré group in a non-trivial way.

1.2 EXERCISES

1. Consider Maxwell's equations in $d+1$ dimensions. What is the electrostatic energy between two charge distributions $\rho_1(\vec{r})$ and $\rho_2(\vec{r})$?

2. Show that the configuration space $\mathcal{C}_2$ of two identical particles in d spatial dimensions is the manifold
$$\mathcal{C}_2 \;=\; \mathbb{R}^{d+1} \times \mathbb{R}P^{d-1} \;.$$
Then prove that
$$\pi_1(\mathcal{C}_2) \;=\; \begin{cases} \mathbb{Z}\,, & \text{if } d=2 \\ \mathbb{Z}_2\,, & \text{if } d\geq 3\,, \end{cases}$$
and from this deduce that in $d \geq 3$ only fermions and bosons can exist, while in $d=2$ particles with different statistics are allowed.

3. (a) Is the posssibility of assigning many spins to the same state in two dimensions a new phenomenon compared to four dimensions?

   (b) The map (1.4) can be be defined (in an analogous way) in any number of dimensions. Why do we then claim that statistics is a matter of convention only in two dimensions?

4. (a) **Green's theorem** on the plane states: If P and Q are continuous functions with continuous partial derivatives on the domain D, then
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy \;=\; \oint_{\partial D} (P dx + Q dy) \;.$$
Rewrite this theorem using complex variables.

   (b) Use the result of part (a) to find a representation of the δ-function in complex coordinates.

5. Trying to extend the Riemann mapping theorem in the case of multiply-connected domains, we face the problem that domains of the same order of connectivity are not necessarily conformally mapped onto each other [519]. This exercise studies this issue in one simple case.

   Consider the circular rings $D = \{1 < |z| < r\}$ and $D' = \{1 < |w| < R\}$, where $r \neq R$. Assume that there is a function $w = f(z)$ which maps D onto D'. Show that this would require $r = R$, which is contrary to the hypothesis $r \neq R$.

6. Show that all the 2-dimensional Riemannian manifolds are conformally flat.

7. Find all Killing vectors of an n-dimensional flat manifold. Compute the algebra they generate (which is the **Poincaré algebra**).

8. Write down the classical Poincaré algebra (i.e., using Poisson brackets) in $1+1$ dimensions, and find a representation for a system of N particles interacting via a nearest-neighbor potential [573].

9. Equation (1.11) has been derived for a general metric on a manifold M. Show that if the metric is a Levi-Civita metric, then the Killing equation can be brought to the more symmetric form
$$\xi_{\mu,\nu} + \xi_{\nu,\mu} \ = \ 2\,\Gamma^{\rho}_{\mu\nu}\,\xi_{\rho} \ . \tag{1.17}$$

10. Using equation (1.17), find all Killing vectors of the 2-sphere S^2 with the standard metric
$$\boldsymbol{g} \ = \ \sin^2\theta\, d\phi \otimes d\phi + d\theta \otimes d\theta \ .$$
Compute the algebra they generate.

11. Using equation (1.17), find all Killing vectors of the Poincaré half-plane H^2, for which the metric is
$$\boldsymbol{g} \ = \ \frac{dx \otimes dx + dy \otimes dy}{y^2} \ .$$
Compute the algebra they generate, and integrate it to find the corresponding group.

12. Find all conformal Killing vectors of an n-dimensional conformally flat manifold. Compute the algebra they generate.

13. Recall from Exercise 6 that a 2-dimensional manifold is conformally flat. Thus the result of the last problem applies locally in any patch of a 2-dimensional manifold. From that result, find the global conformal algebra on the 2-sphere S^2.

14. **Mandelstam variables**: Show that in D space-time dimensions, for the scattering of N particles (incoming and outgoing all counted), we can define $(D-1)\,N - \frac{D(D+1)}{2}$ independent Mandelstam variables. From this result, confirm that in two space-time dimensions, the 2-particle scattering is determined by one Mandelstam variable.

15. **A flavor of the Coleman-Mandula theorem**: For a hypothetical model, assume the existence of a second order tensorial conserved charge $Q_{\mu\nu}$ different from the angular momentum $J_{\mu\nu}$. Study the consequences of such an additional conservation law for the n-to-n elastic scattering process
$$a_1(p_1^i) + a_2(p_2^i) + \cdots a_n(p_n^i) \longrightarrow a_1(p_1^f) + a_2(p_2^f) + \cdots a_n(p_n^f),$$
and show that only in 1+1 dimensions is there no conflict between this conservation law and the well-known analyticity properties of the S-matrix .

1.3 SOLUTIONS

1. Consier a point charge q in d spatial dimensions. We can imagine surrounding the charge by a $(d-1)$-sphere S^{d-1}. Then Gauss's Law in integral form for electric field $\boldsymbol{E}$ of the charge q reads

$$\oint_{S^{d-1}} \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{q}{\varepsilon_0} .$$

The rotational symmetry of the configuration requires that $\boldsymbol{E}$ be a function of the radial distance r only, and that it point along the radial direction. Therefore

$$\begin{aligned} \frac{q}{\varepsilon_0} &= E(r) \oint_{S^{d-1}} dS \\ &= E(r)\, \Omega_{d-1}\, r^{d-1} , \end{aligned}$$

where E is the magnitude of the electric field, and Ω_{d-1} is the $(d-1)$-dimensional total solid angle, which is easily computed to be

$$\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} .$$

Therefore

$$E(r) = \frac{\Gamma(d/2)}{2\pi^{d/2-1}\varepsilon_0} \frac{q}{r^{d-1}} .$$

The potential is then given by

$$\phi(r) = \begin{cases} -\frac{1}{2\varepsilon_0}\, q\, r , & \text{if } d = 1 , \\ -\frac{1}{2\pi\varepsilon_0}\, q \,\ln r , & \text{if } d = 2 , \\ -\frac{\Gamma(d/2)}{4\pi^{d/2}\varepsilon_0} \frac{q}{r^{d-2}} , & \text{if } d > 2 . \end{cases}$$

Consequently, the interaction between the two charge distributions $\rho_1(\vec{r})$ and $\rho_2(\vec{r})$ is

$$U = \begin{cases} -\frac{1}{2\varepsilon_0} \int dr_1 dr_2\, \rho_1(r_1)|r_1 - r_2|\rho_2(r_2) , & \text{if } d = 1 , \\ -\frac{1}{2\pi\varepsilon_0} \int d^2r_1 d^2r_2\, \rho_1(r_1) \ln|r_1 - r_2|\rho_2(r_2) , & \text{if } d = 2 , \\ -\frac{\Gamma(d/2)}{4\pi^{d/2}\varepsilon_0} \int d^dr_1 d^dr_2\, \rho_1(r_1) \frac{1}{r^{d-2}} \rho_2(r_2) , & \text{if } d > 2 . \end{cases}$$

---

2. In d dimensions, the locus of particles is described by their position vectors $(\boldsymbol{r}_1, \boldsymbol{r}_2)$, with the restriction $\boldsymbol{r}_1 \neq \boldsymbol{r}_2$. As usual, we define the center of mass position vector $\boldsymbol{R}$ and the relative position vector $\boldsymbol{r}$ by

$$\boldsymbol{R} = \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2} , \qquad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 .$$

In fact, the vector $\boldsymbol{r}$ can furthermore be decomposed into its magnitude r and a unit vector $\boldsymbol{u}_r$ parallel $\boldsymbol{r}$. If the particles were distinguishable, the configuration space would be

$$\begin{aligned}
\overline{\mathcal{C}}_2 &= \{(\boldsymbol{R},\boldsymbol{r}) \mid \boldsymbol{R},\boldsymbol{r} \in \mathbb{R}^d,\ \boldsymbol{r} \neq 0\} \\
&= \{(\boldsymbol{R},\boldsymbol{u},r) \mid \boldsymbol{R} \in \mathbb{R}^d,\ \boldsymbol{u} \in S^{d-1}, r \in \mathbb{R}^*_+\} \\
&= \mathbb{R}^d \times S^{d-1} \times \mathbb{R}^*_+ \\
&\sim \mathbb{R}^d \times S^{d-1} \times \mathbb{R} \\
&= \mathbb{R}^{d+1} \times S^{d-1} .
\end{aligned}$$

For indistinguishable particles, we must divide by their permutation group $S_2 = \mathbb{Z}_2$ to avoid double counting. Thus the configuration space for two identical particles is

$$\mathcal{C}_2 = \frac{\overline{\mathcal{C}}_2}{\mathbb{Z}_2} = \mathbb{R}^{d+1} \times \frac{S^{d-1}}{\mathbb{Z}_2} = \mathbb{R}^{d+1} \times \mathbb{R}P^{d-1} ,$$

since the manifold $S^{d-1}/\mathbb{Z}_2$ is $\mathbb{R}P^{d-1}$, the $(d-1)$-dimensional real projective space. The first two such spaces are $\mathbb{R}P^1 = S^1$ and $\mathbb{R}P^2 = \mathrm{SO}(3)$.

Since

$$\pi_1(M_1 \times M_2) = \pi_1(M_1) \times \pi_1(M_2) ,$$

to find the first homotopy group of $\mathcal{C}_2$, we notice that any non-trivial contribution will come from $\mathbb{R}P^{d-1}$, since $\mathbb{R}^n$ is homotopically trivial. For $d = 2$, we thus immediately find that $\pi_1(\mathcal{C}_2) = \pi_1(\mathbb{Z}) = \mathbb{Z}$. For $d \geq 3$, we must find $\pi_1(S^{d-1}/\mathbb{Z}_2)$. According to a theorem of algebraic topology, if a manifold M is homotopically trivial, and a discrete group Γ acts on M effectively, then $\pi_1(M/\Gamma) = \Gamma$. In our case, $\pi_1(S^{d-1}) = 0$ (you cannot lasso the sphere) and $\mathbb{Z}_2$ acts effectively on it. Therefore $\pi_1(\mathbb{R}P^{d-1}) = \mathbb{Z}_2$.

From (1.1), we see that the weights χ are unitary 1-dimensional representations of $\mathbb{Z}$ for $d = 2$ and of $\mathbb{Z}_2$ for $d \geq 3$. Now, $\mathbb{Z}_2$ has only two such representations, namely 1 and -1. The first corresponds to bosons and the second to fermions. On the other hand, $\mathbb{Z}$ allows for a continuum of choices parametrized by a variable θ, with $\chi(n) \mapsto e^{in\theta}$. When $\theta = 0$ the statistics are bosonic, and when $\theta = \pi$ the statistics are fermionic, while other values of θ lead to exotic statistics.

---

3. (a) The possibility of assigning many spins to the same state is known in four dimensions. A free particle of spin s can be described equivalently by many relativistic wave equations, with fields which transform differently under the Lorentz group.

   (b) In d spatial dimensions, one can certainly define the map

$$\begin{aligned}
f^\dagger(p) &= b^\dagger(p)\, e^{-i\pi \int \frac{d^d k_1}{k_0} b^\dagger(k) b(k)} , \\
f(p) &= e^{i\pi \int \frac{d^d k}{k_0} b^\dagger(k) b(k)}\, b(p) .
\end{aligned}$$

However, this map is Lorentz invariant only in two dimensions.

---

4. Let $A(z,\overline{z})$ be a complex function

$$A(z,\overline{z}) \;=\; P(x,y) + iQ(x,y)$$

of $z = x + iy$. Then

$$\begin{aligned}
\oint_{\partial D} A(z,\overline{z}) dz &= \oint_{\partial D} (P + iQ)\,(dx + idy) \\
&= \oint_{\partial D} (Pdx - Qdy) + i \oint_{\partial D} (Qdx + Pdy) \\
&= -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy + i \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dxdy \\
&= -\iint_D \left(\frac{\partial Q}{\partial x} + i\frac{\partial Q}{\partial y}\right) dxdy + i \iint_D \left(\frac{\partial P}{\partial x} + i\frac{\partial P}{\partial y}\right) dxdy \\
&= -\iint_D 2\frac{\partial Q}{\partial \overline{z}}\, dxdy + i \iint_D 2\frac{\partial P}{\partial \overline{z}}\, dxdy \\
&= 2i \iint_D \frac{\partial}{\partial \overline{z}}(P + iQ)\, dxdy \\
&= 2i \iint_D \frac{\partial A}{\partial \overline{z}}\, dxdy \;.
\end{aligned}$$

Finally, since

$$dxdy \;=\; \left|\frac{\partial(x,y)}{\partial(z,\overline{z})}\right| dzd\overline{z} \;=\; \frac{dzd\overline{z}}{|-2i|} \;=\; \frac{dzd\overline{z}}{2} \;,$$

we write

$$\oint_{\partial D} A(z,\overline{z})\frac{dz}{i} \;=\; \iint_D \frac{\partial A}{\partial \overline{z}}\, dzd\overline{z} \;. \tag{1.18}$$

In the same way, we can prove

$$\oint_{\partial D} B(z,\overline{z})\frac{d\overline{z}}{i} \;=\; -\iint_D \frac{\partial B}{\partial z}\, dzd\overline{z} \;.$$

Putting the two results together,

$$\oint_{\partial D} A(z,\overline{z})\frac{dz}{i} + B(z,\overline{z})\frac{d\overline{z}}{i} \;=\; \iint_D \left(\frac{\partial B}{\partial z} + \frac{\partial A}{\partial \overline{z}}\right) dzd\overline{z} \;.$$

(b) Let $A(z)$ be a holomorphic function. Any representation of the δ-function must satisfy the equation

$$A(x,y) \;=\; \iint_D \delta(x - x')\delta(y - y')\, A(x',y')\, dxdy \;,$$

or

$$A(w) \;=\; \iint_D \delta^{(2)}(z-w)\, A(z)\, \frac{dz d\overline{z}}{2}\;.$$

We will show that this is the case for the representation

$$\delta^{(2)}(z-w) \;=\; \frac{1}{2\pi}\,\overline{\partial}\frac{1}{z-w}\;. \tag{1.19}$$

Indeed

$$\begin{aligned}\iint_D \delta^{(2)}(z-w)\, A(z)\, dz d\overline{z} \;&=\; \frac{1}{2\pi}\iint_D \overline{\partial}\frac{A(z)}{z-w}\, dz d\overline{z}\\ &=\; \oint_{\partial D}\frac{dz}{2\pi i}\,\frac{A(z)}{z-w}\\ &=\; A(w)\;.\end{aligned}$$

In the same way

$$\delta^{(2)}(z-w) \;=\; \frac{1}{2\pi}\,\partial\frac{1}{\overline{z}-\overline{w}}\;.$$

5. We consider the analytic function

$$\begin{aligned} g(z) \;&=\; \ln r \ln f(z) - \ln R \ln z\\ &=\; \ln r \ln\Big(|f(z)|\, e^{i\arg f(z)}\Big) - \ln R \ln\left(|z|\, e^{i\arg z}\right)\\ &=\; [\ln r \ln|f(z)| - \ln R \ln|z|] + i\,[\ln r \arg f(z) - \ln R \arg z]\\ &\equiv\; h(z) + i\,\alpha(z)\;.\end{aligned}$$

The real part of $g(z)$,

$$h(z) \;=\; \ln r \ln|f(z)| - \ln R \ln|z|\;,$$

is obviously a harmonic function on D. On the boundaries $C_1 = \{|z| = 1\}$ and $C_2 = \{|z| = r\}$, $h(z)$ vanishes:

$$h(z) \;=\; 0 \qquad \text{on } C_1 \text{ and } C_2\;.$$

From Liouville's theorem, then,

$$h(z) \;=\; 0 \qquad \text{on } D\;.$$

From this result and the Cauchy-Riemann conditions

$$\frac{\partial h}{\partial x} = \frac{\partial \alpha}{\partial y} \quad \text{and} \quad \frac{\partial h}{\partial y} = -\frac{\partial \alpha}{\partial x} ,$$

we conclude that

$$\alpha(z) = \text{const} \equiv a .$$

However,

$$\alpha(z) = \ln r \arg f(z) - \ln R \arg z = w \ln r - \arg z \ln R .$$

When we tranverse the circle $|z| = r$ counterclockwise, we also tranverse the circle $|w| = R$ counterclockwise. If the map is 1-to-1, after a full rotation

$$z \mapsto z + 2\pi i , \qquad w \mapsto w + 2\pi i ,$$

which implies

$$\alpha(z) \mapsto \alpha(z) + 2\pi (\ln r - \ln R) .$$

But if $\alpha(z)$ is going to be a constant, the extra term must vanish, thus requiring

$$\ln r = \ln R \Rightarrow r = R .$$

This exercise clearly shows that the "conformal type" of a ring is described by the ratio $\tau = r_1/r_2$ ($r_2 = 1$ above). This ratio is know as the **modulus** of the ring. As long as two rings have different moduli, they are of different "conformal type" although they have the same connectivity.

6. The most general metric in the patch[1] U_j of the 2-manifold M has the form

$$\boldsymbol{g} = g_{11}\, dx \otimes dx + g_{12}\, dx \otimes dy + g_{21}\, dy \otimes dx + g_{22}\, dy \otimes dy .$$

We rewrite the above expression in the form

$$\boldsymbol{g} = \left(\sqrt{g_{11}}\, dx + \frac{g_{12} + i\sqrt{g}}{\sqrt{g_{11}}}\, dy \right) \otimes \left(\sqrt{g_{11}}\, dx + \frac{g_{12} - i\sqrt{g}}{\sqrt{g_{11}}}\, dy \right) .$$

Let us concentrate on the 1-form

$$\sqrt{g_{11}}\, dx + \frac{g_{12} + i\sqrt{g}}{\sqrt{g_{11}}}\, dy .$$

[1] All the quantities in this problem depend on the patch, and so should have a label j; for simplicity, we omit these labels.

Can it be seen as the differential of a function $z = z(x, y)$? In other words, is the above expression equal to

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

for some $z(x, y)$? If it were, this would imply

$$\begin{aligned} \frac{\partial z}{\partial x} &= \sqrt{g_{11}}\ , \\ \frac{\partial z}{\partial y} &= \frac{g_{12} + i\sqrt{g}}{\sqrt{g_{11}}}\, dy\ , \end{aligned}$$

and since for a continuous function z with continuous derivatives,

$$\frac{\partial}{\partial y}\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y}\ ,$$

which leads to the constraint

$$\frac{\partial}{\partial y}\sqrt{g_{11}} = \frac{\partial}{\partial x}\frac{g_{12} + i\sqrt{g}}{\sqrt{g_{11}}}\ .$$

Obviously this cannot be true in general; the l.h.s. is real, while the r.h.s. is complex. This difficulty is overcome by introducing a complex function $\lambda(x, y)$ such that

$$\frac{\partial}{\partial y}(\lambda\sqrt{g_{11}}) = \frac{\partial}{\partial x}\left(\lambda\,\frac{g_{12} + i\sqrt{g}}{\sqrt{g_{11}}}\right)\ .$$

This equation in fact determines the function λ. Then

$$dz = \lambda\left(\sqrt{g_{11}}\, dx + \frac{g_{12} + i\sqrt{g}}{\sqrt{g_{11}}}\, dy\right) \equiv du + i dv\ .$$

The new coordinates $u = u(x, y)$ and $v = v(x, y)$ introduced above are known as the **isothermal coordinates**. In terms of these coordinates

$$\boldsymbol{g} = \lambda^{-1}\, dz \otimes (\overline{\lambda})^{-1}\, d\overline{z} = \frac{1}{|\lambda|^2}\,(du \otimes du + dv \otimes dv)\ .$$

7. In this case $g_{\mu\nu} = \eta_{\mu\nu}$, and so the Killing equation (1.42) simplifies to

$$\partial_\mu \xi_\nu = -\,\partial_\nu \xi_\mu\ . \tag{1.20}$$

This equation implies also

$$\partial_\rho \partial_\mu \xi_\nu = -\,\partial_\rho \partial_\nu \xi_\mu\ . \tag{1.21}$$

In the same way

$$\partial_\mu \partial_\rho \xi_\nu \;=\; -\,\partial_\mu \partial_\nu \xi_\rho \;. \tag{1.22}$$

Notice now that the l.h.s of equations (1.21) and (1.22) are equal and therefore

$$\partial_\rho \partial_\nu \xi_\mu \;=\; \partial_\mu \partial_\nu \xi_\rho \;.$$

Applying equation (1.20) in the r.h.s. of the last equation, we find

$$\partial_\rho \partial_\nu \xi_\mu \;=\; -\,\partial_\rho \partial_\nu \xi_\mu \;,$$

which necessarily implies that $\boldsymbol{\xi}$ is first order in $\boldsymbol{x}$:

$$\partial_\rho \partial_\nu \xi_\mu \;=\; 0 \;.$$

Therefore, the allowed Killing vector fields for a flat manifold are of two kinds:

(i) translations:

$$\xi^\mu_{(i)} \;=\; \delta^\mu_i \;.$$

The index i labels the n independent vectors.

(ii) rotations:

$$\xi^\mu \;=\; \omega_{\mu\nu}\, x^\nu \;.$$

Notice that equation (1.20) requires

$$\omega_{\mu\nu} \;=\; -\,\omega_{\nu\mu} \;.$$

Thus, the general Killing vector $\boldsymbol{\xi}$ on the Euclidean plane has the form

$$\boldsymbol{\xi} = a_\mu\, \partial^\mu + \frac{1}{2}\,\omega_{\mu\nu}\,(x^\mu \partial^\nu - x^\nu \partial^\mu) \;.$$

We usually define the vector fields

$$\boldsymbol{P}_\mu \;\equiv\; -\,i\partial_\mu \;, \qquad \boldsymbol{L}_{\mu\nu} \;\equiv\; i(x_\mu \partial_\nu - x_\nu \partial_\mu) \;,$$

which are the well-known generators of translations (linear momentum) and rotations (angular momentum), respectively. The algebra generated by them is denoted $\mathfrak{iso}(n)$, and the corresponding group is denoted ISO(n). Physicists usually call these the Poincaré algebra and the **Poincaré group**, respectively, and use the symbol $\mathcal{P}$ for the group. One can easily check that the Poincaré algebra has the commutators

$$\begin{aligned}
[\boldsymbol{L}_{\mu\nu}, \boldsymbol{P}_\rho] \;&=\; i\,(\eta_{\nu\rho}\boldsymbol{P}_\mu - \eta_{\mu\rho}\boldsymbol{P}_\nu) \;, \\
[\boldsymbol{L}_{\mu\nu}, \boldsymbol{L}_{\rho\tau}] \;&=\; i\,(\eta_{\mu\tau}\boldsymbol{L}_{\nu\rho} + \eta_{\nu\rho}\boldsymbol{L}_{\mu\tau} - \eta_{\mu\rho}\boldsymbol{L}_{\nu\tau} - \eta_{\nu\tau}\boldsymbol{L}_{\mu\rho}) \;.
\end{aligned}$$

8. In the (1+1)-dimensional case, we have only 2 components of the linear momentum, P_1 and P_2, and one component of the angular momentum $L \equiv L_{01}$. From the result of the previous problem (or by a quick computation) we find that the classical corresponding Poincaré algebra reads:

$$\begin{aligned}
\{L, P_0\}_{PB} &= P_1 \ , && (1.23)\\
\{L, P_1\}_{PB} &= P_0 \ , && (1.24)\\
\{P_0, P_1\}_{PB} &= 0 \ . && (1.25)
\end{aligned}$$

Introducing the operators

$$P_\pm \equiv P_0 \pm P_1 \ ,$$

we can rewrite (1.23)-(1.25) in the form

$$\begin{aligned}
\{L, P_\pm\}_{PB} &= \pm P_\pm \ , && (1.26)\\
\{P_+, P_-\}_{PB} &= 0 \ . && (1.27)
\end{aligned}$$

Now for a system of N particles with coordinates x_i and momentum p_i, we write

$$\begin{aligned}
\mathcal{P}_\pm &= \sum_{i=1}^N e^{\pm p_i} V_i(x_1, \ldots, x_N) \ ,\\
\mathcal{L}_\pm &= \sum_{i=1}^N x_i \ .
\end{aligned}$$

We will now find the conditions on the functions V_i under which $\mathcal{P}_\pm$ and $\mathcal{L}$ give a representation of the Poincaré algebra. The boost operator $\mathcal{L}$ has been chosen such that the Poisson bracket (1.26) is satisfied immediately:

$$\begin{aligned}
\{\mathcal{L}, \mathcal{P}_\pm\}_{PB} &= \sum_{k=1}^N \left(\frac{\partial \mathcal{L}}{\partial x_k} \frac{\partial \mathcal{P}_\pm}{\partial p_k} - \frac{\partial \mathcal{P}_\pm}{\partial x_k} \frac{\partial \mathcal{L}}{\partial p_k} \right)\\
&= \sum_{k=1}^N \left(\sum_{i=1}^N \delta_{ik} \right) \sum_{j=1}^N e^{\pm p_j} (\pm \delta_{jk}) V_j - 0\\
&= \pm \sum_{i=1}^N e^{\pm p_i} V_i\\
&= \pm \mathcal{P}_\pm \ .
\end{aligned}$$

We now impose condition (1.27):

$$\{\mathcal{P}_+, \mathcal{P}_-\}_{PB} = \sum_{k=1}^N \left(\frac{\partial \mathcal{P}_+}{\partial x_k} \frac{\partial \mathcal{P}_-}{\partial p_k} - \frac{\partial \mathcal{P}_-}{\partial x_k} \frac{\partial \mathcal{P}_+}{\partial p_k} \right)$$

$$
\begin{aligned}
&= \sum_{k=1}^{N}\left(\sum_{i=1}^{N} e^{p_i}\partial_k V_i \sum_{j=1}^{N} e^{-p_j} V_j(-\delta_{jk}) - \sum_{i=1}^{N} e^{p_i} V_i \delta_{ik} \sum_{j=1}^{N} e^{-p_j}\partial_k V_j\right) \\
&= -\sum_{i=1}^{N}\sum_{j=1}^{N} e^{p_i-p_j}\left(V_i\partial_i V_j + V_j\partial_j V_i\right) \\
&= -\sum_{i\neq j} e^{p_i-p_j}\left(V_i\partial_i V_j + V_j\partial_j V_i\right) - \sum_i \left(V_i\partial_i V_i + V_i\partial_i V_i\right) \\
&= -\sum_{i\neq j} e^{p_i-p_j}\left(V_i\partial_i V_j + V_j\partial_j V_i\right) - \sum_i \partial_i V_i^2 \ .
\end{aligned}
$$

The r.h.s. will vanish for all values of the momenta if and only if

$$
\begin{aligned}
V_i\partial_i V_j + V_j\partial_j V_i &= 0 \ , \qquad i \neq j \ , \\
\sum_i \partial_i V_i^2 &= 0 \ .
\end{aligned}
$$

The first of these equations is satisfied if the functions V_i have the functional form:

$$
V_i \ = \ f(x_{i-1}-x_i)\, f(x_i - x_{i-1}) \ .
$$

Indeed, if i and j are well-separated, $|i-j| \geq 2$, then none of the points x_{i-1}, x_i and x_{i+1} coincide with x_j, and $\partial_j V_i = 0$. If $|i-j| = 1$, we may assume without loss of generality that $i = j+1$. Then

$$
\begin{aligned}
\frac{\partial_{j+1} V_j}{V_j} + \frac{\partial_j V_{j+1}}{V_{j+1}} &= \frac{\partial_{j+1} f(x_{j-1}-x_j)}{f(x_{j-1}-x_j)} + \frac{\partial_{j+1} f(x_j - x_{j+1})}{f(x_j - x_{j+1})} \\
&\quad + \frac{\partial_j f(x_j - x_{j+1})}{f(x_j - x_{j+1})} + \frac{\partial_j f(x_{j+1} - x_{j+2})}{f(x_{j+1} - x_{j+2})} \\
&= 0 - \frac{f'(x_j - x_{j+1})}{f(x_j - x_{j+1})} + \frac{f'(x_j - x_{j+1})}{f(x_j - x_{j+1})} + 0 \ = \ 0 \ .
\end{aligned}
$$

Finally, we must specify f using the last condition $S \equiv \sum_i \partial_i V_i^2 = 0$. Notice that if we define periodic boundary conditions

$$
x_0 \ = \ x_N \ , \qquad x_{N+1} \ = \ x_1 \ ,
$$

or if we set

$$
x_0 \to -\infty \ , \qquad x_{N+1} \to +\infty \ ,
$$

then

$$
\sum_{i=1}^{N} e^{x_i - x_{i+1}} \ = \ \sum_{i=1}^{N} e^{x_{i-1}-x_i} \ .
$$

This is a trivial identity that we can rearrange to obtain the function f! Indeed:

$$\begin{aligned} 0 &= ab\sum_{i=1}^{N}\left(e^{x_i-x_{i+1}}-e^{x_{i-1}-x_i}\right) \\ &= \sum_{i=1}^{N}\partial_i\left(a^2+ab\,e^{x_i-x_{i+1}}+ba+ab\,e^{x_{i-1}-x_i}+b^2\,e^{x_{i-1}-x_{i+1}}\right) \\ &= \sum_{i=1}^{N}\partial_i\left[\left(a+be^{x_{i-1}-x_i}\right)\left(a+be^{x_i-x_{i+1}}\right)\right]\ , \end{aligned}$$

which shows that the function

$$f^2(y) = a+b\,e^y$$

is a solution to our problem.

The system thus obtained has the Hamiltonian

$$\mathcal{P}_0 = \sum_{i=1}^{N}\cosh p_i\,\sqrt{a+be^{x_{i-1}-x_i}}\,\sqrt{a+be^{x_i-x_{i+1}}}$$

and is known as the **relativistic Toda system** [573].

9. For a Levi-Civita metric

$$\Gamma^{\kappa}_{\rho\nu} = \frac{1}{2}\,g^{\kappa\lambda}\left(\partial_\rho g_{\nu\lambda}+\partial_\nu g_{\lambda\rho}-\partial_\lambda g_{\rho\nu}\right)\ .$$

Thus

$$\begin{aligned} \Gamma^{\kappa}_{\rho\nu}g_{\kappa\mu} &= \frac{1}{2}\,g^{\kappa\lambda}\,g_{\kappa\mu}\left(\partial_\rho g_{\nu\lambda}+\partial_\nu g_{\lambda\rho}-\partial_\lambda g_{\rho\nu}\right) \\ &= \frac{1}{2}\,\delta^{\lambda}_{\mu}\left(\partial_\rho g_{\nu\lambda}+\partial_\nu g_{\lambda\rho}-\partial_\lambda g_{\rho\nu}\right) \\ &= \frac{1}{2}\left(\partial_\rho g_{\nu\mu}+\partial_\nu g_{\mu\rho}-\partial_\mu g_{\rho\nu}\right)\ . \end{aligned}$$

Interchanging μ and ν, we obtain

$$\Gamma^{\kappa}_{\rho\mu}g_{\kappa\nu} = \frac{1}{2}\left(\partial_\rho g_{\mu\nu}+\partial_\nu g_{\nu\rho}-\partial_\nu g_{\rho\mu}\right)\ .$$

Summing the two preceding results yields

$$\Gamma^{\kappa}_{\rho\nu}g_{\kappa\mu}+\Gamma^{\kappa}_{\rho\mu}g_{\kappa\nu} = \partial_\rho g_{\mu\nu}\ ,$$

where the symmetry of the metric tensor was used. We finally substitute this result in (1.42), and find

$$\begin{aligned}
0 &= \xi^\rho \left(\Gamma^\kappa_{\rho\nu} g_{\kappa\mu} + \Gamma^\kappa_{\rho\mu} g_{\kappa\nu}\right) + \partial_\mu \xi^\kappa\, g_{\kappa\nu} + \partial_\nu \xi^\kappa\, g_{\kappa\mu} \\
&= g_{\kappa\mu} \left(\Gamma^\kappa_{\rho\nu}\xi^\rho + \partial_\nu \xi^\kappa\right) + g_{\kappa\nu} \left(\Gamma^\kappa_{\rho\mu}\xi^\rho + \partial_\mu \xi^\kappa\right) \\
&= g_{\kappa\mu}\, \xi^\kappa{}_{;\nu} + g_{\kappa\nu}\, \xi^\kappa{}_{;\mu} \\
&= (g_{\kappa\mu}\, \xi^\kappa)_{;\nu} + (g_{\kappa\nu}\, \xi^\kappa)_{;\mu} \\
&= \xi_{\mu;\nu} + \xi_{\nu;\mu} \\
&= (\partial_\nu \xi_\mu - \Gamma^\rho_{\mu\nu}\, \xi_\rho) + (\partial_\mu \xi_\nu - \Gamma^\rho_{\mu\nu}\, \xi_\rho) \\
&= \xi_{\mu,\nu} + \xi_{\nu,\mu} - 2\,\Gamma^\rho_{\mu\nu}\, \xi_\rho \ ,
\end{aligned}$$

where we have used the properties

$$\begin{aligned}
g_{\kappa\mu;\rho} &= 0 \ , \\
\xi_{\mu;\nu} &= \partial_\nu \xi_\mu - \Gamma^\rho_{\mu\nu}\, \xi_\rho \ .
\end{aligned}$$

---

10. For the standard metric of the sphere, the Christoffel symbols are

$$\begin{aligned}
\Gamma^\phi_{\phi\,\phi} = \Gamma^\phi_{\theta\,\theta} = 0 \ , &\qquad \Gamma^\phi_{\theta\,\phi} = \Gamma^\phi_{\phi\,\theta} = \cot\theta \ , \\
\Gamma^\theta_{\theta\,\theta} = \Gamma^\theta_{\theta\,\phi} = \Gamma^\theta_{\phi\,\theta} = 0 \ , &\qquad \Gamma^\theta_{\phi\,\phi} = -\sin\theta\cos\theta \ .
\end{aligned}$$

Using these expressions in (1.17), we find

$$\partial_\theta \xi_\theta = 0 \ , \tag{1.28}$$

$$\partial_\phi \xi_\phi + \sin\theta\, \cos\theta\, \xi_\theta = 0 \ , \tag{1.29}$$

$$\partial_\phi \xi_\theta + \partial_\theta \xi_\phi - \sin\theta\, \cos\theta\, \xi_\theta = 0 \ . \tag{1.30}$$

This system can be solved in straightforward fashion. Equation (1.28) requires that ξ_θ be a function of ϕ only, and so

$$\xi_\theta = \Phi(\phi) \ . \tag{1.31}$$

Substituting this result in equation (1.29), we can also solve for ξ_ϕ, and find

$$\xi_\phi = -\sin\theta\, \cos\theta\, \tilde{\Phi}(\phi) + \Theta(\theta) \ , \tag{1.32}$$

where $\Theta(\theta)$ is an arbitrary function of θ and

$$\tilde{\Phi}(\phi) \equiv \int^\phi d\phi'\, \Phi(\phi') \ .$$

Finally, equation (1.30) gives

$$\partial_\phi \Phi + \tilde{\Phi} \;=\; -\,\partial_\theta \Theta + 2\,\cot\theta\,\Theta\ .$$

The ϕ-dependent terms are now separated from the θ-dependent terms and therefore each term must be constant:

$$\begin{aligned} \frac{d\Phi}{d\phi} + \tilde{\Phi} \;&=\; c_1\ , \\ -\frac{d\Theta}{d\theta} + 2\,\cot\theta\,\Theta \;&=\; c_1\ . \end{aligned} \tag{1.33}$$

The second equation is a first-order differential equation and can be solved with well-known techniques. In particular, multiplying it by $1/\sin^2\theta$, we find

$$\frac{d}{d\theta}\left(\frac{\Theta}{\sin^2\theta}\right) \;=\; -\,\frac{c_1}{\sin^2\theta} \;\Rightarrow\; \Theta(\theta) \;=\; (c_1\,\cot\theta + c_2)\,\sin^2\theta\ . \tag{1.34}$$

Now, differentiating equation (1.33), we arrive at a simple equation,

$$\frac{d^2\Phi}{d\phi^2} + \Phi \;=\; 0 \;\Rightarrow\; \Phi(\phi) \;=\; a_1\,\sin\phi - a_2\,\cos\phi\ .$$

From this

$$\tilde{\Phi}(\phi) \;=\; -\,\frac{d\Phi}{d\phi} + c_1 \;=\; a_1\,\cos\phi + a_2\,\sin\phi + c_1\ .$$

The components of the Killing vector are thus

$$\begin{aligned} \xi^\theta \;&=\; g^{\theta\theta}\,\xi_\theta \;=\; a_1\,\cos\phi + a_2\,\sin\phi\ , \\ \xi^\phi \;&=\; g^{\phi\phi}\,\xi_\phi \;=\; -\,a_1\,\cot\phi\,\sin\phi + a_2\,\cot\theta\,\cos\phi + c_2\ . \end{aligned}$$

The Killing vector $\boldsymbol{\xi} = \xi^\theta\,\partial_\theta + \xi^\phi\,\partial_\phi$ is usually written in the form

$$\boldsymbol{\xi} \;=\; -\,a_1\,\boldsymbol{L}_1 + a_2\,\boldsymbol{L}_2 + c_2\,\boldsymbol{L}_3\ ,$$

where

$$\begin{aligned} \boldsymbol{L}_1 \;&=\; -\cos\phi\,\partial_\theta + \cot\theta\,\sin\phi\,\partial_\phi\ , \\ \boldsymbol{L}_2 \;&=\; \sin\phi\,\partial_\theta + \cot\theta\,\cos\phi\,\partial_\phi\ , \\ \boldsymbol{L}_3 \;&=\; \partial_\phi\ . \end{aligned}$$

These vectors generate the $\mathfrak{su}(2)$ algebra:

$$\begin{aligned} [\boldsymbol{L}_1, \boldsymbol{L}_2] \;&=\; \boldsymbol{L}_3\ , \\ [\boldsymbol{L}_2, \boldsymbol{L}_3] \;&=\; \boldsymbol{L}_1\ , \\ [\boldsymbol{L}_3, \boldsymbol{L}_1] \;&=\; \boldsymbol{L}_2\ . \end{aligned}$$

11. From the given metric, one finds that the Christoffel symbols are

$$\Gamma^x_{xx} = \Gamma^x_{yy} = 0 \; , \qquad \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y} \; ,$$
$$\Gamma^y_{xy} = \Gamma^y_{yx} = 0 \; , \qquad \Gamma^y_{xx} = -\Gamma^y_{yy} = \frac{1}{y} \; .$$

Then, the Killing equations (1.17) for the Poincaré half-plane are

$$\partial_x \xi_x - \frac{1}{y}\,\xi_y = 0 \; , \tag{1.35}$$
$$\partial_y \xi_y + \frac{1}{y}\,\xi_y = 0 \; , \tag{1.36}$$
$$\partial_x \xi_y + \partial_y \xi_x + \frac{2}{y}\,\xi_x = 0 \; . \tag{1.37}$$

Equation (1.36) is solved easily, to wit,

$$\partial_y (y\,\xi_y) \;=\; 0 \;\Rightarrow\; \xi_y \;=\; \frac{X(x)}{y} \; ,$$

where $X(x)$ is a function of x only. Substituting this in equation (1.35), we find

$$\xi_x \;=\; \frac{\tilde{X}(x)}{y^2} + Y(y) \; ,$$

where $Y(y)$ is a function of y and

$$\tilde{X}(x) \;\equiv\; \int^x dx'\, X(x') \; .$$

Equation (1.37) also gives

$$\frac{dX(x)}{dx} \;=\; -\,y\,\frac{dY(y)}{dy} - 2\,Y(y) \; .$$

In this formula, the variables x and y are separated, and therefore each term has to be a constant:

$$\frac{dX(x)}{dx} \;=\; 2a \; ,$$
$$-y\,\frac{dY(y)}{dy} - 2\,Y(y) \;=\; 2a \; .$$

The solutions are

$$\begin{aligned} X(x) &= 2a\,x + b\ , \\ Y(y) &= -a + \frac{c}{y^2}\ . \end{aligned}$$

The components of the Killing vector are

$$\begin{aligned} \xi^x &= g^{xx}\,\xi_x = a\,(x^2 - y^2) + b\,x + c\ , \\ \xi^y &= g^{yy}\,\xi_y = 2a\,xy + b\,y\ . \end{aligned}$$

Thus, in general here we have

$$\boldsymbol{\xi} = c\,\boldsymbol{L}_1 + b\,\boldsymbol{L}_2 + a\,\boldsymbol{L}_3\ ,$$

where

$$\begin{aligned} \boldsymbol{L}_1 &= \partial_x\ , \\ \boldsymbol{L}_2 &= x\,\partial_x + y\,\partial_y\ , \\ \boldsymbol{L}_3 &= (x^2 - y^2)\,\partial_x + 2xy\,\partial_y\ . \end{aligned}$$

The above vector fields generate the algebra

$$\begin{aligned} [\boldsymbol{L}_1, \boldsymbol{L}_2] &= -\boldsymbol{L}_1\ , \\ [\boldsymbol{L}_2, \boldsymbol{L}_3] &= -\boldsymbol{L}_3\ , \\ [\boldsymbol{L}_3, \boldsymbol{L}_1] &= 2\,\boldsymbol{L}_2\ . \end{aligned}$$

Now we would like to integrate the algebra to the corresponding group. To make contact with customary notation, let us use complex variables. In these variables, the vector vields take a nice symmetric form, namely

$$\begin{aligned} \boldsymbol{L}_1 &= \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}} \equiv \boldsymbol{l}_1 + \overline{\boldsymbol{l}}_1\ , \\ \boldsymbol{L}_2 &= z\,\frac{\partial}{\partial z} + \overline{z}\,\frac{\partial}{\partial \overline{z}} \equiv \boldsymbol{l}_2 + \overline{\boldsymbol{l}}_2\ , \\ \boldsymbol{L}_3 &= z^2\,\frac{\partial}{\partial z} + \overline{z}^2\,\frac{\partial}{\partial \overline{z}} \equiv \boldsymbol{l}_3 + \overline{\boldsymbol{l}}_3\ . \end{aligned}$$

To integrate the algebra, let us work with the holomorphic part $\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3$ only. The flow generated by $\boldsymbol{l}_1$ is

$$\frac{dz}{dt} = a\,\boldsymbol{l}_1(z) = a \ \Rightarrow\ z = z_0 + a\ ,$$

where z_0 and a are two complex constants. In the same way, the flow generated by $\boldsymbol{l}_2$ is

$$\frac{dz}{dt} = b\,\boldsymbol{l}_2(z) = b\,z \ \Rightarrow\ z = z_0\,e^{bz}\ .$$

Again, z_0 and b are two complex constants. Finally, the flow generated by $\boldsymbol{l}_3$ is

$$\frac{dz}{dt} \;=\; c\,\boldsymbol{l}_3(z) \;=\; c\,z^2 \;\Rightarrow\; z \;=\; \frac{z_0}{z_0 t - 1}\;,$$

where z_0 and c are two complex constants. The generic combination of the above transformations is represented by the overall transformation

$$z \;\rightarrow\; w \;=\; \frac{\alpha z + \beta}{\gamma z + \delta}\;, \qquad \alpha\delta - \beta\gamma \neq 0\;. \tag{1.38}$$

The set of tranformations (1.38) constitute the **Möbius group**.

12. We are interested in determining all possible conformal Killing vector fields of an n-manifold that it is conformally flat:

$$g_{\mu\nu} = \eta_{\mu\nu}\;.$$

In this case, equation (1.10) simplifies to

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu \;=\; \frac{2}{n}\,\partial^\rho \xi_\rho\, \eta_{\mu\nu}\;. \tag{1.39}$$

Acting with ∂^ν, we obtain

$$\left(1 - \frac{2}{n}\right)\,\partial_\mu(\partial \cdot \xi) + \Box \xi_\mu \;=\; 0\;. \tag{1.40}$$

We notice that the behavior of this equation is different for $n \geq 3$ and $n = 2$, since in the latter case the first term drops out of the equation. Therefore, we will examine the two cases separately.

- Case I: $n > 2$

Acting with ∂_ν in (1.40), one obtains

$$\left(1 - \frac{2}{n}\right)\,\partial_\nu \partial_\mu(\partial \cdot \xi) + \Box \partial_\nu \xi_\mu \;=\; 0\;. \tag{1.41}$$

Contracting the indices ν and μ in the last equation, we arrive at the result

$$\Box \partial \cdot \xi \;=\; 0\;. \tag{1.42}$$

Now, we rewrite equation (1.41) with the indices ν and μ interchanged, and we add the resulting equation to the original one, yielding

$$2\left(1 - \frac{2}{n}\right)\,\partial_\nu \partial_\mu(\partial \cdot \xi) + \Box(\partial_\nu \xi_\mu + \partial_\mu \xi_\nu) \;=\; 0\;.$$

However, using equations (1.39) and (1.42) successively, this is transformed to

$$\partial_\nu \partial_\mu \partial \cdot \xi \;=\; 0 \; . \tag{1.43}$$

Using this in conjunction with (1.39) yields

$$\partial_\rho \partial_\sigma \partial_\mu \xi_\nu + \partial_\rho \partial_\sigma \partial_\nu \xi_\mu \;=\; 0 \; .$$

Playing with the indices, we find that

$$\partial_\rho \partial_\sigma \partial_\mu \xi_\nu \;=\; 0 \; .$$

This is a very strong result. For $n \geq 3$, the allowed conformal Killing vector fields are at most quadratic functions of the coordinates. The four possibilities are:

(i) <u>translations</u>:

$$\xi^\mu_{(i)} \;=\; \delta^\mu_i \; .$$

The index i labels the n independent vectors.

(ii) <u>rotations</u>:

$$\xi^\mu \;=\; \omega^\mu{}_\nu \, x^\nu \; .$$

Notice that equation (1.39) requires

$$\omega_{\mu\nu} \;=\; -\,\omega_{\nu\mu} \; .$$

(iii) <u>dilatations</u>:

$$\xi^\mu \;=\; \lambda \, x^\nu \; .$$

(iv) <u>special conformal transformations</u>:

$$\xi^\mu \;=\; b^\mu \, x^2 - 2\, b^\nu \, x_\nu x^\mu \; .$$

For the above transformations, we define the respective generators as follows:

$$\begin{aligned}
\boldsymbol{P}_\mu \;&\equiv\; -i\partial_\mu \; , \\
\boldsymbol{L}_{\mu\nu} \;&\equiv\; i(x_\mu \partial_\nu - x_\nu \partial_\mu) \; , \\
\boldsymbol{D} \;&\equiv\; -i x^\mu \partial_\mu \; , \\
\boldsymbol{K}_\mu \;&\equiv\; i(2x_\mu x^\rho \partial_\rho - x^\rho x_\rho \partial_\mu) \;=\; -\,2x_\mu \boldsymbol{D} + x^\rho x_\rho \boldsymbol{P}_\mu \; .
\end{aligned}$$

Then the algebra they generate, the **conformal Poincaré algebra** in $d > 2$ dimensions, is given by:

$$\begin{aligned}
[\boldsymbol{L}_{\mu\nu}, \boldsymbol{P}_\rho] \;&=\; i\,(\eta_{\nu\rho} \boldsymbol{P}_\mu - \eta_{\mu\rho} \boldsymbol{P}_\nu) \; , \\
[\boldsymbol{L}_{\mu\nu}, \boldsymbol{L}_{\rho\tau}] \;&=\; i\,(\eta_{\mu\tau} \boldsymbol{L}_{\nu\rho} + \eta_{\nu\rho} \boldsymbol{L}_{\mu\tau} - \eta_{\mu\rho} \boldsymbol{L}_{\nu\tau} - \eta_{\nu\tau} \boldsymbol{L}_{\mu\rho}) \; , \\
[\boldsymbol{L}_{\mu\nu}, \boldsymbol{K}_\rho] \;&=\; i\,(\eta_{\nu\rho} \boldsymbol{K}_\mu - \eta_{\mu\rho} \boldsymbol{K}_\nu) \; , \\
[\boldsymbol{D}, \boldsymbol{P}_\rho] \;&=\; +i\, \boldsymbol{P}_\mu \; , \\
[\boldsymbol{D}, \boldsymbol{K}_\rho] \;&=\; +i\, \boldsymbol{K}_\mu \; , \\
[\boldsymbol{P}_\mu, \boldsymbol{K}_\nu] \;&=\; 2i\,(\eta_{\mu\nu} \boldsymbol{D} + \boldsymbol{L}_{\mu\nu}) \; .
\end{aligned}$$

- Case II: $n = 2$

In two dimensions, equation (1.39) becomes:

$$\begin{aligned} \partial_1 \xi_1 &= \partial_2 \xi_2 \ , \\ \partial_1 \xi_2 &= \partial_2 \xi_1 \ . \end{aligned}$$

But these are exactly the Cauchy-Riemann conditions! In other words, in 2-dimensions, any holomorphic function $\xi = \xi_1 + i\xi_2$ of $z = x^1 + ix^2$ is a conformal transformation. Recalling that any holomorphic function can be expanded in a Laurent series

$$\xi(z) \ = \ -\sum_n \xi_{(n+1)}\, z^{n+1} \ ,$$

we recognize that any transformation

$$z \to z' \ = \ z - \epsilon\, \xi_{(n+1)}\, z^{n+1}$$

is a conformal transformation, and thus we have an infinite dimensional algebra $\mathcal{L}$. In particular, the gererators $\boldsymbol{l}_n$, with

$$\boldsymbol{l}_n = -z^{n+1}\, \partial_z \ ,$$

satisfy the algebra

$$[l_n, l_m] \ = \ (n-m)\, l_{n+m} \ . \tag{1.44}$$

The algebra $\mathcal{L}$ given by equation (1.44) is called the **classical Virasoro algebra**.

Of course, similar considerations are valid for $\overline{\xi}(\overline{z}) = \xi_1 - i\xi_2$ and the corresponding gererators $\overline{\boldsymbol{l}}_n$. It is customary to study only half of the symmetry, keeping in the back of one's head that the full symmetry is $\mathcal{L} \times \overline{\mathcal{L}}$. Moreover, although the anti-holomorphic part of the theory is the complex conjugate of the holomorphic part, it is usually treated as an independent piece of the theory. This piece is related to the holomorphic piece only if the transformation $(z, \overline{z}) \mapsto (\overline{z}, z)$ is a symmetry of the theory.

---

13. The 2-sphere S^2 is equivalent to the compactified plane

$$S^2 \ = \ \mathbb{C} \cup \{\infty\},$$

One can put coordinates on this manifold using two patches, U_1 which excludes the point at infinity, and U_2 which excludes the origin. If z and w are the (complex) coordinates on these patches, then the equation

$$w \ = \ \frac{1}{z}$$

is the transition function from one patch to the other.

The maximal subalgebra that is globally defined on S^2 must be spanned by vector fields that are smooth on the whole of S^2. To find this maximal subalgebra, we consider the local algebra $\mathcal{L}$ on one of the patches, and then attempt to extend this to the entire other patch. Obviously, the only points at which subtleties may arise are at 0 and ∞.

From their very definition,

$$\boldsymbol{l}_n = -z^{n+1}\,\partial_z \ ,$$

it is clear that not all vector fields are well defined at the point $z = 0$. In particular, when $n < -1$, the corresponding vector fields $\boldsymbol{l}_n$ are singular at $z = 0$. Thus, only the vector fields $\boldsymbol{l}_n$, $n \geq -1$ have a chance of being globally defined.

To examine now what happens at the point $z = \infty$, we use the substitution $w = 1/z$, and examine what happens at $w = 0$. Since under this transformation

$$\boldsymbol{l}_n \mapsto \boldsymbol{l}_n \ = \ w^{-n+1}\,\partial_w \ ,$$

we see that the vector fields $\boldsymbol{l}_n$ for $n > 1$ have a singularity at $w = 0$.

Therefore, the only the vector fields $\boldsymbol{l}_n$ that are globally defined are those with $-1 \leq n \leq 1$, i.e.,

$$\begin{aligned} \boldsymbol{l}_{-1} &= \partial_z \ , \\ \boldsymbol{l}_0 &= z\,\partial_z \ , \\ \boldsymbol{l}_1 &= z^2\,\partial_z \ . \end{aligned}$$

These fields generate an $\mathfrak{sl}\,(2)$ algebra, with commutation relations

$$\begin{aligned} [\boldsymbol{l}_{-1}, \boldsymbol{l}_0] &= -\boldsymbol{l}_{-1} \ , \\ [\boldsymbol{l}_0, \boldsymbol{l}_1] &= -\boldsymbol{l}_1 \ , \\ [\boldsymbol{l}_1, \boldsymbol{l}_{-1}] &= 2\boldsymbol{l}_0 \ , \end{aligned}$$

which can be integrated to the Möbius group.

14. For a collision in which N particles (the total number of incoming particles plus the total number of outgoing particles) are involved, we have ND momentum components, but not all of them are independent. Using Lorentz invariance we can fix $D(D-1)/2$ of them, and using translational invariance we can eliminate another set of D of them. In addition, the on–shell conditions $p_i^2 \ = \ m_i^2$ reduce the independent components further by an amount N. So, finally, we have

$$ND \ - \ \frac{D(D-1)}{2}\,D \ - N \ = \ (D-1)\,N \ - \ \frac{D(D+1)}{2}$$

independent momentum components in this collision. For the particular case of two particle scattering in two dimensions, that is for $D = 2$ and $N = 4$ (two particles in plus two particles out), we thus have only one independent Mandelstam variable, which is given by

$$s \;\equiv\; (p_1 + p_2)^2 \;=\; m_1^2 + m_2^2 + 2(\epsilon_1 \epsilon_2 - k_1 k_2) \ .$$

15. Without loss of generality, we will assume that $Q_{\mu\nu}$ is traceless, since if it is not, we can define

$$Q'_{\mu\nu} \;\equiv\; Q_{\mu\nu} - \delta_{\mu\nu}\, Q^{\rho}_{\rho}$$

and work with this new tensor.

For a 1-particle state $|\, p\,\rangle$, the expectation value of the charge $Q_{\mu\nu}$ must be of the form

$$\langle p | Q_{\mu\nu} | p \rangle = A(p^2)\, p_\mu\, p_\nu + B(p^2)\, \eta_{\mu\nu} \ .$$

Taking into account the vanishing trace of $Q_{\mu\nu}$, we find

$$\langle p | Q_{\mu\nu} | p \rangle = A(p^2) \left(p_\mu\, p_\nu - \frac{1}{D}\, p^2\, \eta_{\mu\nu} \right) \ .$$

For an asymptotic state of many particles, $|\, p_1, p_2, \ldots, p_n; \mathrm{asym}\,\rangle = |\, p_1\,\rangle\ |\, p_2\,\rangle \ldots |\, p_n\,\rangle$, factorization means that $Q_{\mu\nu}$ acts additively, which means

$$\langle\, p_1, p_2, \ldots, p_n; \mathrm{asym} \,|\, Q_{\mu\nu} \,|\, p_1, p_2, \ldots, p_n; \mathrm{asym}\,\rangle \;=\; \sum_{k=1}^{n} A(p_k^2) \left(p_{k,\mu}\, p_{k,\nu} - \frac{1}{D}\, p_k^2\, \eta_{\mu\nu} \right) \ .$$

The conservation of $Q_{\mu\nu}$ ensures that its values before and after the collision will be identical, given by the condition

$$\sum_{k=1}^{n} A(m_k) \left(p^i_{k,\mu}\, p^i_{k,\nu} - \frac{1}{D}\, m_k^2\, \eta_{\mu\nu} \right) \;=\; \sum_{k=1}^{n} A(m_k) \left(p^f_{k,\mu}\, p^f_{k,\nu} - \frac{1}{D}\, m_k^2\, \eta_{\mu\nu} \right) \ ,$$

or, equivalently,

$$\sum_{k=1}^{n} A(m_k)\, p^i_{k,\mu}\, p^i_{k,\nu} \;=\; \sum_{k=1}^{n} A(m_k)\, p^f_{k,\mu}\, p^f_{k,\nu} \ .$$

The above equation is true for all n-to-n elastic reactions. In particular, it is true for the reaction in which all particles are identical (and therefore have the same rest mass). Consequently, either $A = 0$ (an utterly trivial case), or

$$\{p^i_1, p^i_2, \ldots, p^i_n\} \;=\; \{p^f_1, p^f_2, \ldots, p^f_n\} \ .$$

This says that for an elastic reaction, if there is this additional conserved quantity $Q_{\mu\nu}$, there can be only a reshuffling of the momenta of the particles; momenta cannot change values in any other way. For the 2-to-2 scattering this means that there can be only forward or backward scattering as seen in figure 1.1.

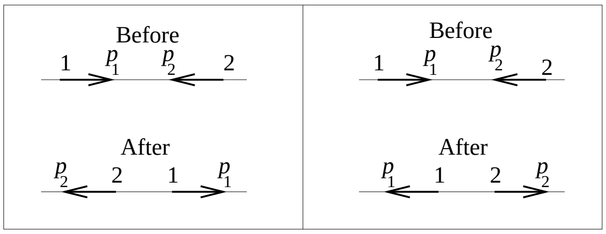


Figure 1.1: Forward and backward scattering. These are the analogues of transmission and reflection of a wavepacket in Quantum Mechanics.

Any S-matrix has to be analytic in the Mandelstam variables s and t (i.e. analytic on the complex s-t plane). However, this is impossible if only backward and forward scattering are allowed. This rules out the presence of the non-trivial conserved charge $Q_{\mu\nu}$ described here when there are at least two Mandelstam variables (except in the case of a non-interacting theory). However, as we saw in an earlier problem, in 1+1 dimensions, there is only one Mandelstam variable, and consequently no analyticity constraint. (Indeed, in 1+1-dimensions, even without an exotic conserved charge, only backward and forward scattering are possible.) Therefore, in 1+1 dimensions it is feasible to have non-trivial models in which there is an extra spacetime symmetry with an associated conserved tensorial charge.

Incidentally, let us point out the this analysis can be used to show another loophole in the Coleman-Mandula Theorem. Our discussion made use of the fact that the conserved charge $Q_{\mu\nu}$ is additive when acting on asymptotic multi-particle states. To formulate this a little more mathematically, let

$$|\, a(p_a)\, \rangle \otimes |\, b(p_b)\, \rangle$$

be a 2-particle state in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$. Additivity of the charge means that the action of $Q^{\mu\nu}_{total}$ on $\mathcal{H} \otimes \mathcal{H}$ is given by

$$Q^{\mu\nu}_{total} \;=\; Q_{\mu\nu} \otimes I + I \otimes Q_{\mu\nu}\ .$$

There are symmetries (but not Lie group symmetries) that do not satisfy this condition, and hence are possible in non-trivial physical theories. Supersymmetry and quantum groups (see chapter 9) are two such possibilities. The analogous analysis determines what range of possible symmetries can arise in these generalized cases.

Chapter 2

GENERAL PRINCIPLES OF CFT

Big fleas have little fleas upon their backs to bite them, and little fleas have lesser fleas, and so ad infinitum.
SWIFT

References: It all started with [92]. A standard reference for CFT is [333]. However, now there are a few books on the subject [186, 420], as well as a few shorter reviews for the impatient reader [68, 134, 203]. A recent review on CFT from an algebraic perspective is [308]. One can also find a review of irrational CFTs, which are not examined in the present document, in [364].

2.1 BRIEF THEORY

2.1.1 Basic Notions of CFT

The basic object of study of QFT is the set of Green's functions of the local fields of the theory. These Green's functions or correlation functions are denoted

$$\langle \mathcal{T}(A_1(x_1)A_2(x_2)...A_n(x_n))\rangle \ .$$

Usually, the time ordering operator T is not written explicitly. In the Langrangian approach, the above Green's functions can be expressed as functional integrals

$$\langle\, A_1(x_1)A_2(x_2)...A_n(x_n)\,\rangle \;\; = \;\; \frac{1}{Z}\int d\mu\,[\phi]\;\; A_1(x_1)A_2(x_2)...A_n(x_n)\, e^{-\frac{i}{\hbar}S[\phi]} \ ,$$

where $S[\phi]$ is the action for the theory, expressed in terms of a set of fundamental fields ϕ.

Any physical QFT must be translationally and rotationally invariant. Translational invariance results, via Noether's Theorem, in the energy-momentum conservation law

$$\partial^\mu T_{\mu\nu} = 0 \ , \tag{2.1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor defined by

$$T_{\mu\nu} \equiv -\frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} \ . \tag{2.2}$$

It can always be defined such that it is symmetric, with $T_{\mu\nu} = T_{\nu\mu}$. In general, QFTs are not scale invariant. Scale invariance is restored at the points where the beta function $\beta(\lambda)$ of the coupling constant λ vanishes. At these points, the dilatation current $D_\mu = T_{\mu\nu}\, x^\nu$ is conseved due to scale invariance, i.e.,

$$\partial^\mu D_\mu = \partial^\mu (T_{\mu\nu}\, x^\nu) = 0 \ .$$

Since the energy-momentum tensor is itself conserved, this conservation law for D_μ in turn implies that the energy momentum tensor of the theory is traceless

$$\Theta \ \equiv \ T^\mu_\mu \ = \ 0 \ . \tag{2.3}$$

Applying condition (2.3) in conjunction with the locality of the energy-momentum tensor implies invariance under conformal transformations, too, that is, invariance under coordinate transformations $y^\mu = y^\mu(x)$ for which the metric is invariant up to an overall factor Ω:

$$ds^2 \ = \ g_{\mu\nu}(x)\, dx^\mu dx^\nu \ = \ \Omega(y)\, g_{\mu\nu}(y)\, dy^\mu dy^\nu \ .$$

To see this, we define the conformal currents

$$J_\mu \ \equiv \ T_{\mu\nu}\, f^\nu \ ,$$

which obviously generalize the translation current $T_{\mu\nu}a^\nu$ and the dilatation current $T_{\mu\nu}x^\nu$. Clearly

$$\partial_\mu J^\mu \ = \ \frac{1}{2} T^{\mu\nu} \left(\partial_\mu f_\nu + \partial_\nu f_\mu\right) \ .$$

In d space-time dimensions, these currents will be conserved if

$$\partial_\mu f_\nu + \partial_\nu f_\mu \ = \ \frac{2}{d}\, \eta_{\mu\nu}\, \partial_\rho f^\rho \ .$$

This is simply the conformal Killing equation, which in two dimensions is solved by

$$z \ \to \ z' = f(z) \ , \qquad \overline{z} \ \to \ \overline{z}' = \overline{f}(\overline{z}) \ .$$

This decoupling of variables allows one to handle the coordinates z and $\overline{z}$ as independent; the reality condition $\overline{z} = z^*$ can be imposed at the end of the day.

In complex coordinates, the energy-momentum conservation law (2.1) becomes

$$\overline{\partial} T_{zz} = 0 \ , \qquad \partial T_{\overline{z}\overline{z}} = 0 \ , \tag{2.4}$$

where $\partial = \partial_z$, $\overline{\partial} = \partial_{\overline{z}}$. Equation (2.4) implies that

$$T_{zz} = T(z) \ , \qquad T_{\overline{z}\overline{z}} = \overline{T}(\overline{z}) \ , \tag{2.5}$$

and thus one can expand the components of the energy-momentum tensor in Laurent series

$$\begin{aligned} T(z) &= \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}} \,, \\ \overline{T}(\overline{z}) &= \sum_{n=-\infty}^{+\infty} \frac{\overline{L}_n}{\overline{z}^{n+2}} \,. \end{aligned}$$

The coefficients L_n and $\overline{L}_n$ satisfy two independent **Virasoro algebras**:

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12} n(n^2-1)\, \delta_{n+m,0} \,, \tag{2.6}$$

$$[\overline{L}_n, \overline{L}_m] = (n-m)\, \overline{L}_{n+m} + \frac{\overline{c}}{12} n(n^2-1)\, \delta_{n+m,0} \,, \tag{2.7}$$

$$[L_n, \overline{L}_m] = 0 \,. \tag{2.8}$$

The numbers c and $\overline{c}$ are called **central charges**.

The Operator Product Expansion (OPE) plays a central role in the study of CFT. According to the OPE assumption, in any QFT, the product of local operators acting at points that are sufficiently close to each other may be expanded in terms of the local fields of theory, as in

$$A_i(x)A_j(y) \sim \sum_k C_{ij}^k(x-y)\, A_k(y) \,.$$

This statement is usually an asymptotic statement in QFT. In CFT though, it becomes an exact statement,

$$A_i(x)A_j(y) = \sum_k C_{ij}^k(x-y)\, A_k(y) \,. \tag{2.9}$$

In qualitative terms, the reason this statement becomes exact is that scale invariance prevents the appearance of any length parameter l in the theory. Consequently, there is no parameter to control the expansion, and thus terms like $e^{-l/|x-y|}$ which would signal the breakdown of the exactness of the expansion (2.9) cannot arise. Note that in two dimensions the l.h.s. of equation (2.9) is time-ordered:

$$\mathcal{T}(A_i(x)A_j(y)) = \sum_k C_{ij}^k(x-y)\, A_k(y) \,.$$

When using complex variables, the time-ordering becomes radial-ordering as has been pointed out in section 1.1.4:

$$\mathcal{R}(A_i(z,\overline{z})A_j(w,\overline{w})) = \sum_k C_{ij}^k(z-w, \overline{z}-\overline{w})\, A_k(w,\overline{w}) \,.$$

By abuse of the notation, it is customary to omit[1] the symbol $\mathcal{R}$.

[1]We shall often use this convention, although not always.

One can translate the Virasoro algebra (2.6)-(2.8) into OPEs:

$$\begin{aligned}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(z_2)}{z-w} + \text{reg} \ , \\
\overline{T}(\overline{z})\overline{T}(\overline{w}) &= \frac{\overline{c}/2}{(\overline{z}-\overline{w})^4} + \frac{2\overline{T}(\overline{w})}{(\overline{z}-\overline{w})^2} + \frac{\overline{\partial T}(\overline{w})}{\overline{z}-\overline{w}} + \text{reg} \ , \\
T(z)\overline{T}(\overline{w}) &= \text{reg} \ ,
\end{aligned}$$

where "reg" refers to non-singular terms that do not blow up as $z \to w$.

One of the most exciting results of two dimensional CFT is that one can classify all possible theories. To this end, we will introduce the notion of a primary field. First, recall that a classical tensor $T_{z \ldots z \overline{z} \ldots \overline{z}}$ in two dimensions would transform under a change of variables as

$$T_{\underbrace{z \ldots z}_{n} \underbrace{\overline{z} \ldots \overline{z}}_{\overline{n}}}(z, \overline{z}) \ \mapsto \ T'_{z \ldots z \overline{z} \ldots \overline{z}}(z', \overline{z}') \ = \ \left(\frac{\partial z}{\partial z'}\right)^n \left(\frac{\partial \overline{z}}{\partial \overline{z}'}\right)^{\overline{n}} T_{z' \ldots z' \overline{z}' \ldots \overline{z}'}(z, \overline{z}) \ .$$

Taking into account that in quantum field theory, fields may acquire anomalous dimensions, we define a primary field to be a field that under a change of variables obeys the transformation law

$$\phi(z, \overline{z}) \ \mapsto \ \phi'(z', \overline{z}') \ = \ \left(\frac{\partial z}{\partial z'}\right)^\Delta \left(\frac{\partial \overline{z}}{\partial \overline{z}'}\right)^{\overline{\Delta}} \phi(z, \overline{z}) \ . \tag{2.10}$$

The real numbers Δ and $\overline{\Delta}$ are known, respectively, as the **left** and **right conformal weights** of the field ϕ. In particular, $D = \Delta + \overline{\Delta}$ is the usual anomalous scale dimension of ϕ, and $s = \Delta - \overline{\Delta}$ is the spin of ϕ.

The decoupling of the holomorphic and anti-holomorphic degrees of freedom will allow us frequently to concentrate on the holomorphic part of the theory; however, we must keep in mind that the full theory also has an anti-holomorphic part, with analogous properties.

The OPE of a primary field (2.10) with the energy-momentum tensor is

$$T(z)\phi(w, \overline{w}) = \frac{\Delta}{(z-w)^2}\, \phi(w, \overline{w}) + \frac{\partial_w \phi(w, \overline{w})}{z-w} + \text{reg} \ .$$

By a simple inspection of the $T(z)T(w)$ OPE, we can immediately conclude that the energy-momentum tensor is not a primary field, due to the extra term with the central charge. Only when $c = 0$ is $T(z)$ is a primary field, with weight 2. The extra term in the OPE of the energy-momentum tensor with itself induces a corresponding term in the 'tensorial' transformation law of $T(z)$:

$$T(z) \ \mapsto \ T'(w) \ = \ \left(\frac{\partial z}{\partial w}\right)^2 T(z) + \frac{c}{12}\, S(w, z) \ .$$

This equation may be considered as the definition of the anomalous term $S(w,z)$ which can be found to be

$$S(w,z) \;=\; \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2 \,,$$

where $z' = dz/dw$. This expression is known in the mathematics literature as the **Schwarzian derivative**.

The vacuum $|\,\emptyset\,\rangle$ of any CFT satisfies

$$L_n\,|\,\emptyset\,\rangle \;=\; 0\,, \quad n \geq -1\,.$$

For a field ϕ with conformal weight Δ, we also define the state

$$|\,\Delta\,\rangle \;\equiv\; \phi(0)\,|\,\emptyset\,\rangle\,,$$

which has the properties

$$L_0\,|\,\Delta\,\rangle \;=\; \Delta\,|\,\Delta\,\rangle\,, \qquad L_n\,|\,\Delta\,\rangle \;=\; 0\,, \quad n > 0\,.$$

The space of states

$$[\phi] \;\equiv\; \{\; L_{-n_1}L_{-n_2}...L_{-n_l}\,|\,\Delta\,\rangle\;,\; n_1 \geq n_2 \geq ... \geq n_l > 0\;\}$$

is known as the **Verma module** built over the field ϕ. The state $L_{-n_1}L_{-n_2}...L_{-n_l}\,|\,\Delta\,\rangle$ is called a **descendant** of $|\,\Delta\,\rangle$, and the quantity $\sum_{j=1}^{l} n_j$ is the **level** of such a descendant. Obviously, the Virasoro algebra maps the Verma module $[\phi]$ into itself; this implies that this space is a representation space of the Virasoro algebra.

2.1.2 Massless Free Boson

The simplest and one of the most important examples of CFT is the massless free boson, for which the action is

$$S = \frac{g}{4\pi}\iint d^2z\,\partial\Phi\overline{\partial}\Phi\,, \tag{2.11}$$

where g is a normalization parameter. (Often we will take $g = 1/2$.) The propagator is the solution of the equation

$$\overline{\partial}\partial\,\langle\,\Phi(z,\overline{z})\Phi(w,\overline{w})\,\rangle \;=\; -\frac{2\pi}{g}\,\delta^{(2)}(z-w)\,.$$

Therefore,

$$\langle\,\Phi(z,\overline{z})\Phi(w,\overline{w})\,\rangle \;=\; -\frac{1}{2g}\,\ln|z-w|^2\,. \tag{2.12}$$

The free boson can be decomposed into a holomorphic part and an anti-holomorphic part, producing the decomposition

$$\Phi(z,\overline{z}) \;=\; \phi(z) \;+\; \overline{\phi}(\overline{z})\,.$$

The result (2.12) for the propagator immediately implies

$$\langle\phi(z)\phi(w)\rangle = -\frac{1}{2g}\ln(z-w) \ , \tag{2.13}$$

$$\left\langle\,\overline{\phi}(\overline{z})\overline{\phi}(\overline{w})\,\right\rangle = -\frac{1}{2g}\ln(\overline{z}-\overline{w}) \ , \tag{2.14}$$

$$\begin{aligned}
\langle\,\partial_z\phi(z)\partial_w\phi(w)\,\rangle &= -\frac{1/2g}{(z-w)^2} \ , \\
\left\langle\,\partial_{\overline{z}}\overline{\phi}(\overline{z})\partial_{\overline{w}}\overline{\phi}(\overline{w})\,\right\rangle &= -\frac{1/2g}{(\overline{z}-\overline{w})^2} \ , \\
\left\langle\,\partial_z\phi(z)\partial_{\overline{w}}\overline{\phi}(\overline{w})\,\right\rangle &= \frac{\pi}{2g}\,\delta^{(2)}(z-w) \ .
\end{aligned}$$

The energy-momentum tensor for the massless free boson is

$$T(z) = -\,g\ :\partial\phi\partial\phi:\ \ . \tag{2.15}$$

From this, it follows that

$$T(z)T(w) = \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \mathrm{reg}\ . \tag{2.16}$$

The central charge for the massless free boson is $c = 1$. One can also easily calculate that

$$T(z)\partial_w\phi(w) = \frac{1}{(z-w)^2}\,\partial_w\phi + \frac{1}{z-w}\,\partial_w(\partial_w\phi) + \mathrm{reg}\ ,$$

i.e., the free boson field acquires no anomalous scale dimension.

Other interesting primary operators are the normal ordered exponentials

$$V_\alpha(z) = \ :e^{i\alpha\phi(z)}:\ \ , \tag{2.17}$$

usually referred to as **vertex operators**. One can easily derive the OPE

$$T(z)V_\alpha(w) = \frac{\alpha^2}{(z-w)^2}\,V_\alpha(w) + \frac{1}{z-w}\,\partial_w V(w) + \mathrm{reg}\ . \tag{2.18}$$

In other words, the conformal weights of the vertex operator (2.17) are $(\alpha^2/4g, 0)$.

2.1.3 Massless Free Fermion

Another important example of CFT is the massless free Majorana-Weyl fermion, for which the action is

$$S = \frac{\lambda}{2\pi}\iint d^2z\,\Psi\partial\!\!\!/\Psi \ , \tag{2.19}$$

where λ is a normalization parameter (which we often we will set equal to 1). The 2-dimensional spinor Ψ has two components:

$$\Psi = \begin{bmatrix} \psi \\ \overline{\psi} \end{bmatrix} .$$

The action can then be written in the equivalent form

$$S = \frac{\lambda}{2\pi} \iint d^2 z \, \left(\psi \overline{\partial} \psi + \overline{\psi} \partial \overline{\psi} \right) \ . \tag{2.20}$$

The propagator is the solution of the equation

$$\begin{aligned} \overline{\partial} \langle \, \psi(z) \psi(w) \, \rangle &= \frac{2\pi}{\lambda} \, \delta(z - w) \ , \\ \partial \left\langle \, \overline{\psi}(\overline{z}) \psi(\overline{w}) \, \right\rangle &= \frac{2\pi}{\lambda} \, \delta(\overline{z} - \overline{w}) \ . \end{aligned}$$

The result for the propagator is:

$$\begin{aligned} \langle \psi(z) \psi(w) \rangle &= \frac{1/\lambda}{z - w} \ , \\ \left\langle \, \overline{\psi}(\overline{z}) \overline{\psi}(\overline{w}) \, \right\rangle &= \frac{1/\lambda}{\overline{z} - \overline{w}} \ . \end{aligned}$$

The energy-momentum tensor for the massless free fermion is

$$T(z) = \frac{\lambda}{2} : \psi \partial \psi : \ .$$

From this it follows that

$$T(z)T(w) = \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \mathrm{reg} \ .$$

The central charge for the massless free fermion is $c = 1/2$. One can also easily verify the OPE result

$$T(z)\psi(w) = \frac{1/2}{(z-w)^2} \, \psi(w) + \frac{1}{z-w} \, \partial_w \psi(w) + \mathrm{reg} \ .$$

Like the free boson, the free fermion field acquires no anomalous scale dimension.

2.1.4 The bc-System

A generalization of the free fermion system is the so-called **bc-system** which includes two anticommuting fields $b(z, \overline{z})$ and $c(z, \overline{z})$ of weights j and $1 - j$, respectively, so that

$$\begin{aligned} T(z)b(w) &= \frac{j \, b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{z-w} + \mathrm{reg} \ , \\ T(z)c(w) &= \frac{(1-j) \, c(w)}{(z-w)^2} + \frac{\partial_w c(w)}{z-w} + \mathrm{reg} \ . \end{aligned}$$

The action for the bc-system is

$$S = \frac{1}{2\pi} \iint d^2z \, \left(b\overline{\partial} c + \overline{b} \partial \overline{c} \right) \ . \tag{2.21}$$

The holomorphic propagators of the theory are then

$$\begin{aligned} \langle \, b(z) c(w) \, \rangle &= \frac{1}{z-w} \ , \\ \langle \, c(z) b(w) \, \rangle &= \frac{1}{z-w} \ . \end{aligned}$$

The energy-momentum tensor is given by

$$T(z) \ = \ -j \ :b(z)\, \partial c(w): + (1-j) \ :\partial b(z)\, c(z): \ ,$$

from which one finds

$$T(z)T(w) \ = \ \frac{-2(6j^2-6j+1)/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg} \ .$$

The central charge for the bc-system is

$$c \ = \ -2\,(6j^2 - 6j + 1) \ .$$

A variation of the bc-system is the $\beta\gamma$**-system**, which includes two commuting fields $\beta(z,\overline{z})$ and $\gamma(z,\overline{z})$ of weights j and $1-j$, respectively, that obey the action:

$$S = \frac{1}{2\pi} \iint d^2z \, \left(\beta\overline{\partial}\gamma + \overline{\beta}\partial\overline{\gamma} \right) \ . \tag{2.22}$$

Since the 'statistics' of the theory has changed, some of the signs in various formulæ will be reversed. In particular, the holomorphic propagators of the theory are

$$\begin{aligned} \langle \, \beta(z)\gamma(w) \, \rangle &= \frac{1}{z-w} \ , \\ \langle \, \gamma(z)\beta(w) \, \rangle &= \frac{-1}{z-w} \ . \end{aligned}$$

As a result, the central charge for the $\beta\gamma$-system is

$$c \ = \ 2\,(6j^2 - 6j + 1) \ .$$

2.1.5 Boson with Background Charge

An interesting modification of the free boson theory is a model that includes a coupling between the boson and the scalar curvature[2]:

$$S \ = \ \frac{g}{4\pi} \iint d^2z \, \partial\Phi\overline{\partial}\Phi + \frac{ie_0}{4\pi} \iint d^2z \, \sqrt{\tilde{g}} R \, \Phi \ .$$

[2]Notice that the normalization factor has nothing to do with the determinant of the metric tensor which is also typically denoted by g. Here, we are momentarily denoting this determinant as $\tilde{g}$ to avoid confuction.

Incidentally, notice that this coupling is non-minimal. The constant e_0 is known as the **background charge**.

The (holomorphic) energy-momentum tensor for this theory reads:

$$T(z) \;=\; -\,g\,:\partial\phi\partial\phi:\;+\;ie_0\partial^2\phi\;, \tag{2.23}$$

Such a total derivative will not affect the status of $T(z)$ as a generator of conformal transformations, but it does modify the value of the central charge, which is now

$$c \;=\; 1-\frac{6}{g}\,e_0^2\;. \tag{2.24}$$

It also modifies the conformal weights of the vertex operators (2.17); the new weights are

$$\Delta_\alpha \;=\; \frac{\alpha(\alpha-2e_0)}{4g}\;. \tag{2.25}$$

Consistency requires

$$V_\alpha^\dagger \;=\; V_{2e_0-\alpha}\;,$$

such that

$$\left\langle V_\alpha^\dagger(z)V_\alpha(w)\right\rangle \;=\; \frac{1}{(z-w)^{2\Delta_\alpha}}\;.$$

The introduction of the background charge modifies the transformation properties of the boson such that it is consistent with (2.25). In particular,

$$\phi(z)\;\overset{z\mapsto w}{\mapsto}\;\phi(w)+\frac{ie_0}{2g}\,\ln\frac{dw}{dz}\;,$$

and so

$$e^{i\alpha\phi(z)}\;\overset{z\mapsto w}{\mapsto}\;\left(\frac{dw}{dz}\right)^{\alpha^2/4g}e^{i\alpha\phi(w)}\,e^{-\frac{\alpha e_0}{2g}\ln\frac{dw}{dz}}\;=\;\left(\frac{dw}{dz}\right)^{\frac{\alpha(\alpha-2e_0)}{4g}}e^{i\alpha\phi(w)}\;. \tag{2.26}$$

2.1.6 Minimal Models of CFT

An important subclass of CFTs is that of the so-called **Minimal Models** (MMs). These models are characterized by two relatively prime positive integers r and s, and the model corresponding to a particular pair of such integers is denoted M_{rs}. Each such model has only a finite number of primary fields ϕ_{mn} , $1\le m\le r-1,\; 1\le n\le s-1$. The corresponding conformal weights are given by

$$\Delta_{mn} \;=\; \frac{(mr-ns)^2-(r-s)^2}{4rs}\;. \tag{2.27}$$

The central charge of such a model is

$$c \;=\; 1-\frac{6(r-s)^2}{rs}\;. \tag{2.28}$$

Not all the MMs are unitary; some contain negative norm states. Calculating the norm of the the state $L_{-1}\,|\,\phi\,\rangle$, we see that

$$\begin{aligned} ||L_{-1}\,|\,\phi\,\rangle\,||^2 &= \langle\,\phi\,|\,L_{-1}^{\dagger}\,L_{-1}\,|\,\phi\,\rangle \;=\; \langle\,\phi\,|\,L_1\,L_{-1}\,|\,\phi\,\rangle \\ &= \langle\,\phi\,|\,[L_1\,,\,L_{-1}]\,|\,\phi\,\rangle \;=\; \langle\,\phi\,|\,2L_0\,|\,\phi\,\rangle \;=\; 2\Delta||\,|\,\phi\,\rangle\,||^2\;, \end{aligned}$$

where we used the fact that $L_{-n}^{\dagger} = L_n$. Therefore, for any CFT (not just a minimal model) to be unitary, it must only have primary fields with positive conformal weights. Among the MMs, then, the only unitary models are those with

$$s = r + 1\;.$$

The central charge of such models is thus

$$c \;=\; 1 - \frac{6}{r(r+1)}\;, \qquad r = 3, 4, 5, \ldots\;.$$

We call these models the Unitary Minimal Models (UMMs).

2.1.7 A Prelude to Chapters 10 and 15

We have pointed out that the left and right sectors of a CFT are studied independently. There are several reasons[3] to impose the condition $c = \overline{c}$; here, though, we will simply use the condition, rather than justifying it.

The first model of the UMMs has $(r, \overline{r}) = (3, 3)$, or $(c, \overline{c}) = (1/2, 1/2)$. This model has three primary operators Φ_{11}, Φ_{21}, and Φ_{12} with conformal weights (0, 0), (1/2, 1/2), and (1/16, 1/16), respectively.

CFT provides a classification of models according to critical behavior in 2-dimensions, and so one can identify the particular CFT model that matches the critical behavior of a known statistical mechanical system.

Consider a mechanical system with an order parameter σ. For high temperature, the 2-point correlation function of this field falls off exponentially

$$\langle\,\sigma(z,\overline{z})\sigma(0,0)\,\rangle \propto r^{-r/\xi}\;, \tag{2.29}$$

where $z = re^{i\phi}$, and ξ is called the correlation length, which is a function of the temperature T. The quantity $m = \xi^{-1}$ has the dimensions of mass, and it is a measure of how far the system is from criticality (where $m = 0$). When T equals a critical temperature T_c, the correlation function (2.29) takes the form

$$\langle\,\sigma(z,\overline{z})\sigma(0,0)\,\rangle \propto r^{-\eta}\;,$$

where η is called the **critical exponent of the order parameter**. Similarly, there are critical exponents for other quantities of the system.

[3]For example, requiring cancellation of local gravitational anomalies in order to allow a system to consistently couple to 2-dimensional gravity translates to the condition $c = \overline{c}$ [31].

The Ising system is defined on a lattice such that at each site i of the lattice there is a spin variable σ_i taking values in $\{1, -1\}$. The Hamiltonian of the system is

$$H = h \sum_i \sigma_i - J \sum_i \sigma_i \sigma_{i+1} .$$

In the continuum limit, it takes the form of a free Dirac fermion

$$H = \int dx \left[\Psi i \gamma_5 \frac{d\Psi}{dx} + m \,\overline{\Psi} \Psi + \text{h.c.} \right] .$$

The critical exponent η for the 2-dimensional Ising model is $\eta = 1/4$, i.e.

$$\langle \sigma_i \sigma_j \rangle \propto |i - j|^{-1/4} .$$

We notice the similarity with the 2-point correlation function $\langle \Phi_{12} \Phi_{12} \rangle$ of the UMM(3). This is an indication that the UMM(3) might describe the critical behavior of the Ising model. To draw this conclusion, one must find a complete map for a rest quantities of the two models. This can, in fact, be done, establishing the desired identification. In particular, the energy operator of the Ising model ε_i has a 2-point correlation function

$$\langle \varepsilon_i \varepsilon_j \rangle \propto |i - j|^{-2} ,$$

and therefore it can be identified with the primary operator Φ_{21} in the UMM(3).

UMM(3)		Ising Model	
Φ_{11}	(11)-primary operator	1	identity
Φ_{12}	(12)-primary operator	σ	spin variable
Φ_{21}	(21)-primary operator	ε	energy variable

Table 2.1: The identification of the critical behavior of the Ising model with the UMM(3).

A final comment is in order here. At the critical point the free fermion becomes massless with energy operator $\varepsilon \sim \psi(z)\overline{\psi}(\overline{z})$. Moving away from criticality is equivalent to adding a perturbation $\Phi_{pert} = \delta m \, \psi(z)\overline{\psi}(\overline{z}) = \delta m \, \Phi_{21}$ to the kinetic term (conformal action).

The Ising model is, of course, only a single case among many models for which the critical behavior has been succesfully identified with a particular CFT; some examples are listed in the following table.

model	c
Ising	1/2
tricritical Ising	7/10
3-state Potts	4/5
tricritical 3-state Potts	6/7

Table 2.2: Some statistical models for which the critical behavior has been identified with particular CFTs given by the value of the central charge.

2.2 EXERCISES

1. Consider a classical CFT on a Riemann surface M that has genus h and no boundary. Usually, we study tensorial objects transforming as

$$T(z) \mapsto \left(\frac{dz}{dw}\right)^m T(z) \ , \qquad m \in \mathbb{Z} \ .$$

Is it possible to define objects that transform in a similar way, but with fractional values of m, i.e. $m \in \mathbb{Q}$?

2. Show that if $\phi(z)$ is a primary field, then in general $\partial_z \phi(z)$ is not primary. Is there any exception?

3. Consider a possible central extension of the classical Virasoro algebra (1.44)

$$[L_n, L_m] \ = \ (n-m)\, L_{n+m} + c_{nm} \quad , \tag{2.30}$$

where

$$[c_{nm}\,,\, L_r] \ = \ 0 \ , \quad \forall n, m, r \ .$$

(a) From the Jacobi relations deduce the constraints on c_{nm}.

(b) Use the freedom to shift the L_n's by a constant (i.e., to make the change $L_n \to L_n + a_n$) to set $c_{n,0} = c_{0,n} = c_{1,-1} = 0$.

(c) Find the most general solution for c_{nm} satisfying the conditions in part (b).

4. The Kac formula predicts a null state of dimension $\Delta_{1,3} + 3$ in the module of highest weight $\Delta_{1,3}$. Find the explicit form of this null state as a descendant, i.e., in the form

$$|\,\chi\,\rangle \ = \ (L_{-3} + \cdots)\, |\,\Delta_{1,3}\,\rangle \ .$$

5. For the free boson theory described by the action (2.11), show that the Euclidean propagator is

$$\langle\, \Phi(z, \overline{z})\, \Phi(0)\,\rangle \ = \ -\,\frac{1}{2g}\, \ln |z|^2 \ .$$

6. For the free boson theory, show that:

$$\mathcal{R}\left(\partial_z \phi(z)\ :e^{i\alpha\phi(w)}:\right) \ = \ :\partial_z \phi(z) e^{i\alpha\phi(w)}: \ - \ \frac{i\alpha}{2g}\,\frac{1}{z-w}\ :e^{i\alpha\phi(w)}: \ .$$

Generalize the above identity to the case

$$\mathcal{R}\left((\partial_z \phi)^k\ :e^{i\alpha\phi(w)}:\right) \ .$$

7. Prove the following identity for a free boson:

$$\mathcal{R}\Big(:e^{i\alpha\phi(z)}: \quad :e^{i\beta\phi(w)}:\Big) \;=\; (z-w)^{\alpha\beta/2g} \quad :e^{i\alpha\phi(z)}e^{i\beta\phi(w)}: \; . \tag{2.31}$$

8. For the free boson theory, the scalar field can be expanded as

$$i\partial_z\phi \;=\; \sum_n \alpha_n\, z^{-n-1} \quad , \quad n \,\in\, \mathbb{Z}\; .$$

   Compute the commutation relations of the modes α_n.

9. Construct the Virasoro generators L_n's in terms of the modes ϕ_n for the bosonic action (2.23).

10. It is often relevant to consider the CFT of a free boson **compactified on a circle** of radius r, so that

$$\phi(\sigma+\beta,t) \sim \phi(x,t) + 2\pi r\, w \;, \quad w \in \mathbb{Z}\; .$$

    This relation is to be interpreted as follows. If the spatial coordinate is compact (i.e. a circle of length β), then the boson field is a multi-valued function with its values at any one point differing by an amount $2\pi r$.

    Discuss the CFT of the compactified boson.

11. For the free fermion theory

$$S \;=\; \frac{\lambda}{2\pi}\iint d^2z\,(\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi})\;,$$

    show that the Euclidean propagator is

$$\langle\, \psi(z)\psi(w)\,\rangle \;=\; \frac{1}{\lambda}\,\frac{1}{z-w}\;.$$

12. In the free fermion theory, consider expanding the fermion field as

$$\psi(z) \;=\; \sum_r \psi_r\, z^{-r-1/2} \quad , \quad r \,\in\, \mathbb{Z}+1/2\; .$$

    Compute the anticommutation relations of the modes ψ_r.

13. Let $\psi(z)$ be a dimension 1/2 free fermion field. Compute the first regular term $L_{-2}(\psi(z))$ (i.e., express it in terms of the field ψ) in the OPE

$$\mathcal{R}(T(w)\psi(z)) \;=\; \frac{1}{2}\frac{\psi(z)}{(w-z)^2} + \frac{\partial_z\psi(z)}{w-z} + L_{-2}(\psi(z)) + \ldots$$

    two different ways:

    (a) using $T(w) = \frac{1}{2} :\psi(w)\partial_w\psi(w):$ and calculating the OPE explicitly; and

    (b) using the null state predicted by the Kac formula.

14. We can treat the bc-system and the $\beta\gamma$-system in a unified fashion by defining a quantity ε (which we call the **signature**) that takes the value $+1$ for a bc-system and the value -1 for a $\beta\gamma$-system. Then the generic BC-system

$$S \;=\; \frac{1}{2\pi} \iint d^2z \,(B\overline{\partial}C + \overline{B}\partial\overline{C}) \;,$$

can give us either a bc-system or a $\beta\gamma$-system, depending on whether we have commutating or anticommuting fields (with ϵ identifying which situation we have).

Using this unified notation, compute, for both the bc-system and the $\beta\gamma$-system,

(a) the energy-momentum tensor $T(z)$; and

(b) the central charge c.

15. **A puzzle**: From the OPE $\mathcal{R}(T(z)T(w))$ show that

$$\langle\, \partial_z\Theta(z)T(w)\,\rangle \;=\; \frac{2c\pi}{3}\,\partial_z^2\partial_w\delta(z-w)\;.$$

This implies that $\Theta(z)$ cannot vanish identically for a CFT. However, recall that if $\Theta(z)$ is not identically zero the theory cannot be conformally invariant. How can these two statements both be true?

16. **The Liouville action**: In two dimensions, dimensional arguments require that the trace of the energy-momentum tensor of a theory S on a Riemann surface with scalar curvature R be given by

$$\Theta \;=\; a\,R + b\;, \tag{2.32}$$

where a and b are two constants (which can be computed exactly using well-known methods of QFT). The non-vanishing of this trace is known as the **conformal anomaly** (or **trace anomaly**).

Using equation (2.32), find an action $S_L = S\,[\sigma]$, such that the energy-momentum tensor of the action

$$S' \;=\; S + S_L$$

has vanishing trace. The field σ is called the **Liouville field** and S_L is called the **Liouville action**.

17. For a boson on a curved two-dimensional manifold M with action

$$S \;=\; \alpha \iint_M d^2\xi\,\sqrt{g}g^{ab}\partial_a\Phi\partial_b\Phi + \beta \iint_M d^2\xi\,\sqrt{g}R\,\Phi\;,$$

compute the energy-momentum tensor as defined by the equation (2.2).

2.3 SOLUTIONS

1. The solution of this problem requires the use of some topology — in particular, it makes use of the Chern (characteristic) classes for the Riemann surface M.

   The integral of the tangent bundle over M gives the Euler characteristic

   $$\chi(M) \;=\; 2\,(1-h) \;=\; \int_M c_1(TM)\ .$$

   The analogous integral of any other line bundle E over M must be an integer, as well,

   $$\int_M c_1(E) \;=\; n\ .$$

   Recall that a tensor on a manifold M is a section of the tangent bundle TM (a holomorphic tangent bundle in our case). Higher order tensors are sections of other bundles. If an object $T(z)$ belongs to a line bundle E of M formed as the m-th power of the holomorphic tangent bundle,

   $$T(w) \;=\; \left(\frac{dz}{dw}\right)^m T(z)\ , \quad m\in\mathbb{Q}\ ,$$

   then

   $$c_1(E) \;=\; m\,c_1(TM)\ .$$

   Integrating this relation over M, we find a relation that restricts the allowed values of n:

   $$n \;=\; 2\,m\,(1-h)\ ,$$

   Notice that, consequently, the line bundle with $m=1/2$ is always acceptable. This line bundle (which is, loosely speaking, the square root of TM) is called the **canonical bundle** of M, and it gives rise to fermions on the surface M.

   For other values of m, the corresponding bundle can be constructed only in special cases, namely if $h-1$ is a multiple of m. For example, for $m=1/(2k)$, $k\in\mathbb{Z}$, the bundle E can be constructed only on Riemann surfaces for which $h-1$ is a multiple of k.

2. Using the relation

   $$T(z)\phi(w) \;=\; \frac{\Delta}{(z-w)^2}\,\phi(w)+\frac{1}{z-w}\,\partial_w\phi(w)+\mathrm{reg}$$

for the OPE of a primary field, we find

$$T(z)\partial_w\phi(w) \;=\; \frac{2\Delta}{(z-w)^3}\,\phi(w) \;+\; \frac{\Delta+1}{(z-w)^2}\,\partial_w\phi(w) \;+\; \frac{1}{z-w}\,\partial_w(\partial_w\phi(w)) + \text{reg}\ .$$

A piece proportional to $\phi/(z-w)^3$ has appeared in the OPE, and therefore $\partial_w\phi$ is not primary. However, notice that if ϕ is a primary field of conformal weight zero, that is with $\Delta = 0$, then this problematic term drops out, and then in this case $\partial_w\phi$ *is* a primary field of conformal weight 1. (As an added exercise, compare this with what you know about the free massless boson.)

3. (a) We start by imposing the Jacobi identity for L_l, L_m, and L_n:

$$[L_l,[L_m,L_n]] + [L_n,[L_l,L_m]] + [L_m,[L_n,L_m]] \;=\; 0 \ .$$

Substituting the proposed algebra (2.30), we obtain the equation

$$\begin{aligned}(m-n)(l-m-n)L_{l+m+n} \;+\; (m-n)c_{l,m+n}\\ +\,(l-m)(n-l-m)L_{l+m+n} \;+\; (l-m)c_{n,n+l}\\ +\,(n-l)(m-n-l)L_{l+m+n} \;+\; (n-l)c_{m,n+l} \;&=\; 0\ .\end{aligned} \tag{2.33}$$

This is the main restriction on the form of c_{nm}. However, since

$$[L_n,L_m] \;=\; -\,[L_m,L_n]\ ,$$

there is an additional constraint on the c's, which is

$$c_{mn} \;=\; -\,c_{nm}\ .$$

(b) Our algebra is isomorphic to another one, which we find by making the transformation

$$L'_n \;=\; L_n \,+\, \alpha_n\ ,$$

for any n. We will choose specific values for the constants α_n as we proceed. Explicitly,

$$\left[L'_n,L'_m\right] \;=\; (n-m)L'_{n+m} \,+\, c'_{n+m}\ ,$$

where

$$c'_{nm} \;=\; c_{nm} - (n-m)\alpha_{n+m}\ . \tag{2.34}$$

We wish to set $c'_{-1,1} = 0$ and $c'_{n0} = c'_{0n} = 0$. To set $c'_{n0} = 0$, we must simply choose

$$\alpha_n = \frac{1}{n}c_{n0}\ , \qquad n \neq 0\ .$$

Of course, $c'_{00} = 0$ automatically, and once $c'_{n0} = 0$, then $c'_{0n} = 0$.

We still have α_0 to specify, and we can use this freedom to set $c'_{-1,1} = 0$; this is achieved by choosing

$$\alpha_0 = \frac{1}{2} c_{-1,1} \ .$$

Thus, by setting

$$\begin{aligned} \alpha_n &= \frac{1}{n}\, c_{n0} \ , \qquad n \neq 0 \ , \\ \alpha_0 &= \frac{1}{2}\, c_{1,-1} \ , \end{aligned}$$

the algebra of the L'_n has $c'_{n0} = c'_{0n} = 0 \ \ \forall n$ and $c'_{-1,1} = 0$.

(c) For $l = 0$, (2.33) applied to the L'_n gives

$$(m-n)\, c'_{0,m+n} \ - \ (n+m)\, c'_{nm} \ = \ 0 \ .$$

This is true if and only if

$$(n+m)\, c'_{nm} \ = \ 0 \ .$$

Therefore

$$c'_{nm} \ = \ 0 \ , \quad n+m \neq 0 \ ,$$

while c'_{nm} is left undetermined if $n+m = 0$. We can thus write

$$c'_{nm} \ \equiv \ c_n\, \delta_{n,-m} \ .$$

Now if we consider the case $l = -(m+1)$ and $n = 1$, (2.33) yields

$$(m-1)\, c'_{-(m+1),(m+1)} \ + \ (-2m-1)\, c'_{1,-1} \ + \ (m+2)\, c'_{m,-m} \ = \ 0 \ .$$

This in turn yields the recursion relation

$$c_{m+1} \ = \ \frac{m+2}{m-1}\, c_m \ ,$$

which implies that

$$c_{m+1} \ = \ c_2 \prod_{k=2}^{m} \frac{k+2}{k-1} \ = \ \frac{m(m+1)(m+2)}{6}\, c_2 \ .$$

Defining $c \equiv 2c_2$, we can write

$$c_m \ = \ \frac{m(m^2-1)}{12}\, c \ .$$

Our final result is thus

$$c'_{nm} \ = \ \frac{m(m^2-1)}{12}\, c\, \delta_{n,-m} \ .$$

4. At level 3, the descendant states are:

$$\begin{aligned} |\,i\,\rangle &\equiv L_{-3}\,|\,\Delta\,\rangle \\ |\,j\,\rangle &\equiv L_{-1}\,L_{-2}\,|\,\Delta\,\rangle \\ |\,k\,\rangle &\equiv L_{-1}^3\,|\,\Delta\,\rangle \end{aligned}$$

Let $|\,\chi\,\rangle$ be a null state constructed from them:

$$0 = |\,\chi\,\rangle \equiv \alpha\,|\,i\,\rangle + \beta\,|\,j\,\rangle + \gamma\,|\,k\,\rangle \ . \tag{2.35}$$

One of the coefficients α, β, and γ is irrelevant, but we are going to keep all three of them just for æsthetic reasons! By scalar multiplication of (2.35) with $|\,i\,\rangle$, $|\,j\,\rangle$, and $|\,k\,\rangle$, respectively, we get the following system of equations:

$$\left.\begin{aligned} \alpha\,\langle\,i\,|\,i\,\rangle + \beta\,\langle\,i\,|\,j\,\rangle + \gamma\,\langle\,i\,|\,k\,\rangle &= 0 \\ \alpha\,\langle\,j\,|\,i\,\rangle + \beta\,\langle\,j\,|\,j\,\rangle + \gamma\,\langle\,j\,|\,k\,\rangle &= 0 \\ \alpha\,\langle\,k\,|\,i\,\rangle + \beta\,\langle\,k\,|\,j\,\rangle + \gamma\,\langle\,k\,|\,k\,\rangle &= 0 \end{aligned}\right\} \tag{2.36}$$

Notice that we have a homogeneous linear system to be solved for α, β, and γ. Non-trivial solutions exist if the determinant of the coefficients is zero:

$$D \equiv \begin{vmatrix} \langle\,i\,|\,i\,\rangle & \langle\,i\,|\,j\,\rangle & \langle\,i\,|\,k\,\rangle \\ \langle\,j\,|\,i\,\rangle & \langle\,j\,|\,j\,\rangle & \langle\,j\,|\,k\,\rangle \\ \langle\,k\,|\,i\,\rangle & \langle\,k\,|\,j\,\rangle & \langle\,k\,|\,k\,\rangle \end{vmatrix} = 0 \ .$$

It is easy to calculate the elements of the above matrix using the Virasoro algebra commutators,

$$[L_n,\, L_m] = (n - m)\,L_{n+m} + \frac{c}{12}\,n\,(n^2 - 1)\,\delta_{n,-m} \ .$$

One finds the inner products take the following forms:

$$\begin{aligned} \langle\,i\,|\,i\,\rangle &= \langle\,\Delta\,|\; L_3\,L_{-3}\,|\,\Delta\,\rangle = \langle\,\Delta\,|\;[L_3,\,L_{-3}]\;|\,\Delta\,\rangle = 6\Delta + 2c\ , \\ \langle\,j\,|\,j\,\rangle &= \langle\,\Delta\,|\; L_2\,L_1\,L_{-1}\,L_{-2}\,|\,\Delta\,\rangle = \langle\,\Delta\,|\;[L_2\,L_1,\,L_{-1}\,L_{-2}]\;|\,\Delta\,\rangle \\ &= \ldots = 34\,\Delta + \Delta\,c + 8\,\Delta^2 + 2\,c\ , \\ \langle\,k\,|\,k\,\rangle &= \ldots = 48\,\Delta^3 + 64\,\Delta^2 + 20\,\Delta\ , \\ \langle\,i\,|\,j\,\rangle &= \langle\,j\,|\,i\,\rangle = 16\Delta + 2\,c\ , \\ \langle\,i\,|\,k\,\rangle &= \langle\,k\,|\,i\,\rangle = 24\,\Delta\ , \\ \langle\,j\,|\,k\,\rangle &= \langle\,k\,|\,j\,\rangle = 6\,\Delta\,(6\,\Delta + 3)\ . \end{aligned}$$

Now it is straightforward, though a little messy, to solve the system of equations (2.36). After some calculations, one finds

$$D = \Delta^2 (\Delta - \Delta_1^+)(\Delta - \Delta_2^+)(\Delta - \Delta_1^-)(\Delta - \Delta_2^-) \ ,$$

where

$$\Delta_1^\pm = \frac{1-c}{96}\left[\left(3 \pm \sqrt{\frac{25-c}{1-c}}\right)^2 - 4\right] = \frac{1}{16}\left[(5-c) \pm \sqrt{(1-c)(25-c)}\right] \ ,$$

$$\Delta_2^\pm = \frac{1-c}{96}\left[\left(4 \pm 2\sqrt{\frac{25-c}{1-c}}\right)^2 - 4\right] = \frac{1}{6}\left[(7-c) \pm \sqrt{(1-c)(25-c)}\right] \ .$$

Setting $D = 0$, we find that $\Delta = 0$, $\Delta_1^\pm$, or $\Delta_2^\pm$. The solution $\Delta = 0$ gives a null state at level 1, while $\Delta = \Delta_1^\pm$ are exactly the values that give a null state at level 2. Thus the solution $\Delta = \Delta_2^\pm$ is the one corresponding to the null state at level 3. Using these values and two of the three equations of the system (2.36), we can solve for β/α and γ/α (or for β and γ, if we accept $\alpha = 1$). We find:

$$\frac{\beta}{\alpha} = -\frac{2}{\Delta + 2} \ , \quad \frac{\gamma}{\alpha} = \frac{1}{(\Delta+1)(\Delta+2)} \ ,$$

so that the level 3 null state is

$$|\chi\rangle = \left[L_{-3} - \frac{2}{\Delta+2} L_{-1} L_{-2} + \frac{1}{(\Delta+1)(\Delta+2)} L_{-1}^3\right] |\Delta\rangle \ .$$

5. The boson propgator $G(x, x')$ is the solution of the differential equation

$$g\partial^\mu \partial_\mu G(x, x') = -2\pi \delta^{(2)}(x - x') \ . \tag{2.37}$$

One can proceed in several ways to solve this equation.

First Solution: The first way of calculating the boson propagator is by far the simplest one, once one realizes the analogies with Poisson's equation in electromagnetism. In particular, recall that Poisson's equation for the potential ϕ is

$$\nabla^2 \phi = -4\pi \rho \ , \tag{2.38}$$

where ρ is the electric charge density. Equation (2.37) becomes identical to (2.38) for a *linear* charge distribution of infinite length and density

$$\rho = -\frac{1}{2g}\delta^{(2)}(x - x') \ .$$

However, in this geometry, the symmetry of problem allows us to perform the calculation in a very simple way. Using Gauss's law, one can calculate the electric field at distance r from the infinite line, obtaining

$$E_r = \frac{1}{g}\frac{1}{r} .$$

From this, the calculation of the potential ϕ is straightforward, since $E_r = -\frac{\partial\phi}{\partial r}$, and thus one determines that

$$\phi(r) = -\frac{1}{g} \ln r + \text{constant} .$$

Using the convention that $\phi(r_0) = 0$, this fixes

$$\phi(r) = -\frac{1}{g} \ln \frac{r}{r_0} .$$

This is of course the result for the boson propagator, too. Noting that $r = |z|$, we finally get the desired result expressed in complex coordinates,

$$G(z, \overline{z}) = -\frac{1}{2g} \ln |z|^2 .$$

<u>Second Solution</u>: Even without the analogy to the electric field, one can still calculate the boson propagator quite easily by making use of the symmetries in Green's equation (2.37). First, observe that equation (2.37) is translationally and Lorentz invariant. This implies that the Green's function is a function of the radius r only,

$$G(x, x') = G(r) .$$

Writing equation (2.37) in polar coordinates,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 , \tag{2.39}$$

one immediately finds

$$G(r) = a \ln r + b ,$$

where a and b are two constants of integration. To calculate these constants, we need two boundary conditions.

Notice that equation (2.39) is singular for $r = 0$; therefore, we will derive a boundary condition to handle the singularity at this point. To this end, recall Green's Theorem on the plane,

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy ,$$

where $P(x,y)$ and $Q(x,y)$ are two functions with continuous derivatives and D is the region of the plane interior to the curve ∂D. For the case

$$Q = \frac{\partial G}{\partial x} , \qquad P = -\frac{\partial G}{\partial y} ,$$

Green's Theorem tell us that

$$\oint_{\partial D} \nabla G \cdot \vec{n}\, dl = \iint_D \nabla^2 G\, dS ,$$

where $\vec{n}$ is the outward-pointing normal vector on ∂D. Taking the domain D to be a circular region of radius $\varepsilon \to 0$, we see that

$$\left.\frac{dG}{dr}\right|_{r=\varepsilon} \cdot 2\pi\varepsilon \longrightarrow -\frac{2\pi}{g} .$$

Comparing this with

$$\frac{dG}{dr} = a\,\frac{1}{r} ,$$

we conclude that $a = -1/g$.

The constant b is calculated by introducing a normalization condition, which we choose arbitrarily to be $G(r_0) = 0$ for some distance r_0.

Thus, finally, we obtain

$$G(r) = -\frac{1}{g}\ln\frac{r}{r_0} ,$$

which, as we saw in the first solution to this problem, is exactly the desired propagator when put in complex form.

Third Solution: Our last way of calculating the boson propagator is simply to grind through the calculation. It is straightforward, in the sense that no analogy or trick is used; however, it does require a relatively higher level of comfort with the calculation of integrals.

The Fourier transform of $G(x-x')$ is given by

$$\tilde{G}(k) = \iint d^2x\, e^{-ik\cdot(x-x')}\, G(x-x') .$$

The inverse transformation is

$$G(x-x') = \iint \frac{d^2k}{(2\pi)^2}\, e^{+ik\cdot(x-x')}\, \tilde{G}(x-x') . \tag{2.40}$$

We also know that

$$\delta(x-x') = \iint \frac{d^2k}{(2\pi)^2} e^{+ik\cdot(x-x')} . \tag{2.41}$$

Substituting equations (2.40) and (2.41) into the definitions of the propagator, we find

$$\tilde{G}(k) \;=\; \frac{2\pi}{g}\,\frac{1}{k^2}\;.$$

Notice that this propagator exhibits the well-known $1/k^2$ behaviour for a massless boson in two dimensions. Now we can do the integrations in (2.40) using polar coordinates:

$$\begin{aligned}
\tilde{G}(x-x') \;&=\; \frac{2\pi}{g}\iint \frac{d^2k}{(2\pi)^2}\,\frac{e^{ik\cdot(x-x')}}{k^2}\\
&=\; \frac{1}{2\pi g}\int_0^{+\infty} dk \int_0^{2\pi} d\theta\, \frac{e^{ikr\cos\theta}}{k}\\
&=\; \frac{1}{g}\int_0^{+\infty} dk\, \frac{J_0(kr)}{k}\quad,
\end{aligned}\tag{2.42}$$

where we have set set $r \equiv x - x'$, and where J_0 is a Bessel function of the first kind. Since the small x behavior of $J_0(x)$ is

$$J_0(x) \;=\; 1 - \frac{x^2}{4} + \ldots \quad,$$

the integral in (2.42) diverges; therefore we regulate its infrared behavior introducing a reference point r_0:

$$G_{\mathrm{reg}}(r) \;=\; G(r) \;-\; G(r_0)\quad.$$

The regulated propagator is then

$$G_{\mathrm{reg}}(r) \;=\; \frac{1}{g}\int_0^{+\infty} dk\, \frac{J_0(kr) - J_0(kr_0)}{k}\;. \tag{2.43}$$

The integral in this equation can be evaluated in many ways. At the end of this problem, we present one method, which applied here shows that

$$\int_0^{+\infty} dk\, \frac{J_0(kr) \;-\; J_0(kr_0)}{k} \;=\; -\ln\frac{r}{r_0}\;,$$

thus leading once more to the result

$$G(z,\overline{z}) \;=\; -\frac{1}{2g}\,\ln|z|^2\;.$$

APPENDIX

The integral in (2.43) is a case of a broad category of integrals known as **Frullani integrals**, which are given by

$$I\,[f(x)] \equiv \int_0^{+\infty} dx\, \frac{f(bx) \;-\; f(ax)}{x}\quad.$$

We now evaluate the Frullani integral for functions $f(x)$ with properties similar to $J_0(x)$, i.e.

(i) $\lim\limits_{x\to+\infty} f(x) = \text{finite}$,

(ii) $f'(x)$ is continuous and integrable on $[0,+\infty)$.

The above two conditions imply

$$\int_0^{+\infty} dx\, f'(x) \;=\; f(+\infty) - f(0)\;.$$

We now define the integral

$$I(\alpha) \;=\; \int_0^{+\infty} dx\, f'(\alpha x)\;. \tag{2.44}$$

The integration can be trivially performed to give

$$I(\alpha) \;=\; \frac{f(+\infty) - f(0)}{\alpha}\;. \tag{2.45}$$

Now let us calculate the integral

$$\int_a^b d\alpha\, I(\alpha)$$

in two ways: first using the definition (2.44), and second based on the result (2.45).

The definition (2.44) gives

$$\begin{aligned}\int_a^b d\alpha\, I(\alpha) \;&=\; \int_a^b d\alpha \int_0^{+\infty} dx\, f'(\alpha x)\\ &=\; \int_0^{+\infty} dx \int_a^b d\alpha\, f'(\alpha x)\\ &=\; \int_0^{+\infty} dx\, \frac{f(bx) - f(ax)}{x}\;. \end{aligned}\tag{2.46}$$

Using equation (2.45), on the other hand, we find

$$\int_a^b d\alpha\, I(\alpha) \;=\; [f(+\infty) - f(0)] \int_a^b \frac{d\alpha}{\alpha} \;=\; [f(+\infty) - f(0)]\, \ln\frac{b}{a}\;. \tag{2.47}$$

Comparing the two results, (2.46) and (2.47), we conclude that

$$\int_0^{+\infty} dx\, \frac{f(bx) - f(ax)}{x} \;=\; [f(+\infty) - f(0)]\, \ln\frac{b}{a}\;.$$

---

6. Expanding the exponential in a power series, we have

$$\partial_z\phi(z)\;:e^{i\alpha\phi(w)}:=\;\partial_z\phi(z)\sum_{n=0}^{+\infty}\;:\frac{[i\alpha\phi(w)]^n}{n!}:\quad.$$

Using Wick's theorem [386], this becomes

$$\partial_z\phi(z)\;:e^{i\alpha\phi(w)}:\;=\;:\partial_z\phi(z)e^{i\alpha\phi(w)}:\;+\;\partial_z\phi(z)\phi(w)\sum_{n=1}^{+\infty}\frac{(i\alpha)^n\;:[\phi(w)]^{n-1}:}{n!}\,n\;.$$

Notice the factor n which appears as a result of all the possible contractions of $\partial_z\phi(z)$ with $\phi^n(w)$. Using (2.13), we can now write

$$
\begin{aligned}
\partial_z\phi(z)\ :e^{i\alpha\phi(w)}: \quad &= \quad :\partial_z\phi(z)e^{i\alpha\phi(w)}: \quad - \quad \frac{i\alpha}{2g}\,\partial_z\ln(z-w)\sum_{n=1}^{+\infty}:\frac{[i\alpha\phi(w)]^{n-1}}{(n-1)!}: \\
&= \quad :\partial_z\phi(z)e^{i\alpha\phi(w)}: \quad - \quad \frac{i\alpha}{2g}\,\frac{1}{z-w}\sum_{m=0}^{+\infty}:\frac{[i\alpha\phi(w)]^{m}}{m!}: \\
&= \quad :\partial_z\phi(z)e^{i\alpha\phi(w)}: \quad - \quad \frac{i\alpha}{2g}\,\frac{1}{z-w}\ :e^{i\alpha\phi(w)}: \ ,
\end{aligned}
$$

which is the desired result.

Having worked the simple case and mastered the ideas, we can now proceed to the general case. For simplicity, let us set

$$
A \ \equiv\ \partial_z\phi(z) \quad \text{and} \quad B \ \equiv\ i\alpha\phi(w)\ .
$$

According to Wick's theorem, the OPE

$$
\mathcal{R}\Big(:A^k::e^B:\Big) \quad = \quad :A^k:\sum_{n=0}^{+\infty}\frac{:B^n:}{n!}
$$

is calculated by adding all possible contributions for one contraction, two contractions, three contractions, etc., up to k contractions.

We notice that the term B^n can contribute l contractions if $n \geq l$. In particular, when the inequality $n \geq l$ is satisfied $:A^k::B^n:$ contributes

$$
(\underbracket{AB})^l\,\frac{k(k-1)\cdots(k-l+1)\,n(n-1)\cdots(n-l+1)}{l!} \quad = \quad (\underbracket{AB})^l\,\frac{k!n!}{(k-l)!l!(n-l)!}
$$

since there are $k(k-1)\cdots(k-l+1)$ ways to chose the l A's and $n(n-1)\cdots(n-l+1)$ ways to choose the l B's. The number is divided by $l!$ since the pairs $\underbracket{AB}$ are indistinguishable.

In this way, we find

$$
\begin{aligned}
\mathcal{R}\Big(:A^k::e^B:\Big) \quad = \quad :A^k\ e^B: \quad &+ \quad (\underbracket{AB})\ :A^{k-1}\sum_{n=1}^{+\infty}\frac{B^{n-1}:}{n!}\ \frac{n!\,k!}{(k-1)!\,1!(n-1)!} \\
&+ \quad (\underbracket{AB})^2\ :A^{k-2}\sum_{n=2}^{+\infty}\frac{B^{n-2}:}{n!}\ \frac{n!\,k!}{(k-2)!\,2!(n-2)!} \\
&+ \quad \cdots \\
&+ \quad (\underbracket{AB})^k\ :A^{k-k}\sum_{n=k}^{+\infty}\frac{B^{n-k}:}{n!}\ \frac{n!\,k!}{(k-k)!\,k!(n-k)!}
\end{aligned}
$$

$$
\begin{aligned}
&= \; :A^k\, e^B: \; + \; \sum_{l=1}^{k} \frac{(\underbracket{AB})^l}{l!} \; :A^{k-l} \sum_{n=l}^{+\infty} \frac{B^{n-l}:}{(n-l)!} \frac{k!}{(k-l)!} \\
&= \; :A^k\, e^B: \; + \; \sum_{l=1}^{k} \binom{k}{l} \, (\underbracket{AB})^l \; :A^{k-l} \sum_{m=0}^{+\infty} \frac{B^m:}{m!} \\
&= \; :A^k\, e^B: \; + \; \sum_{l=1}^{k} \binom{k}{l} \, (\underbracket{AB})^l \; :A^{k-l} e^B: \; ,
\end{aligned}
$$

or

$$
\mathcal{R}\Big(:A^k::e^B:\Big) \; = \; \sum_{l=0}^{k} \binom{k}{l} \, (\underbracket{AB})^l \; :A^{k-l} e^B: \; . \tag{2.48}
$$

Substituting the values of A and B in terms of the scalar fields, we arrive at our final result:

$$
\mathcal{R}\Big(:(\partial_z\phi(z))^k::e^{i\alpha\phi(w)}:\Big) \; = \; \sum_{l=0}^{k} \binom{k}{l} \frac{(-i\alpha/2g)^l}{(z-w)^l} \; :(\partial_z\phi(z))^{k-l}\, e^{i\alpha\phi(w)}: \; .
$$

---

7. We prove this identity employing the same method we used in the solution to the preceding problem. Using Wick's theorem, the initial product of the two exponentials

$$
:e^{i\alpha\phi(z)}::e^{i\beta\phi(w)}: \; = \; \sum_{n=0}^{+\infty} : \frac{[i\alpha\phi(z)]^n}{n!} : \sum_{n=0}^{+\infty} : \frac{[i\beta\phi(w)]^m}{m!} :
$$

equals a sum of terms, each of which takes into account all possible zero, single, double, triple, etc. contractions. In particular, the term with zero contractions is

$$
\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} : \frac{[i\alpha\phi(z)]^n \; [i\beta\phi(w)]^m}{n!m!} : \qquad ,
$$

the term with one contraction is

$$
\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} -\alpha\beta\underbracket{\phi(z)\phi}(w) \; : \frac{[i\alpha\phi(z)]^{n-1}\,[i\beta\phi(w)]^{m-1}}{n!\,m!} : \; \frac{n\,m}{1!} \qquad ,
$$

the term with two contractions is

$$
\sum_{n=2}^{+\infty}\sum_{m=2}^{+\infty} \Big[-\alpha\beta\underbracket{\phi(z)\phi}(w)\Big]^2 \; : \frac{[i\alpha\phi(z)]^{n-2}\; [i\beta\phi(w)]^{m-2}}{n!\,m!} : \; \frac{n(n-1)\,m(m-1)}{2!} \qquad ,
$$

and so on. The factors in the r.h.s. of the above expressions count all possible ways that the corresponding contractions can be performed. Therefore

$$
\begin{aligned}
: e^{i\alpha\phi(z)} :: e^{i\beta\phi(w)} : \;&=\; \sum_{n=0}^{+\infty} : \frac{[i\alpha\phi(z)]^n}{n!} : \sum_{n=0}^{+\infty} : \frac{[i\beta\phi(w)]^m}{m!} : \\
&= \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} : \frac{[i\alpha\phi(z)]^n\,[i\beta\phi(w)]^m}{n!m!} : \\
&\quad + \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} \frac{-\alpha\beta\phi(z)\phi(w)}{1!} : \frac{[i\alpha\phi(z)]^{n-1}\,[i\beta\phi(w)]^{m-1}}{(n-1)!\,(m-1)!} \\
&\quad + \sum_{n=2}^{+\infty}\sum_{m=2}^{+\infty} \frac{\left[-\alpha\beta\phi(z)\phi(w)\right]^2}{2!} : \frac{[i\alpha\phi(z)]^{n-2}\,[i\beta\phi(w)]^{m-2}}{(n-2)!\,(m-2)!} : + \cdots \\
&= \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} : \frac{[i\alpha\phi(z)]^n\,[i\beta\phi(w)]^m}{n!m!} : \\
&\quad + \frac{-\alpha\beta\phi(z)\phi(w)}{1!} \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} : \frac{[i\alpha\phi(z)]^n\,[i\beta\phi(w)]^m}{n!m!} : \\
&\quad + \frac{\left[-\alpha\beta\phi(z)\phi(w)\right]^2}{2!} : \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} : \frac{[i\alpha\phi(z)]^n\,[i\beta\phi(w)]^m}{n!m!} : + \cdots \\
&= \sum_{k=0}^{+\infty} \frac{\left[-\alpha\beta\phi(z)\phi(w)\right]^k}{k!} \left[\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} : \frac{[i\alpha\phi(z)]^n\,[i\beta\phi(w)]^m}{n!m!} :\right] .
\end{aligned}
$$

The last expression is just a product of exponentials, and thus this equation becomes

$$
\begin{aligned}
: e^{i\alpha\phi(z)} : \; : e^{i\beta\phi(w)} : \;&=\; \exp\Big(-\alpha\beta\,\phi(z)\phi(w)\Big) \; : e^{i\alpha\phi(z)}\, e^{i\beta\phi(w)} : \\
&=\; (z-w)^{\alpha\beta/2g} \; : e^{i\alpha\phi(z)}\, e^{i\beta\phi(w)} : \; .
\end{aligned}
$$

Another way to prove this result is by direct use of formula (2.48) with

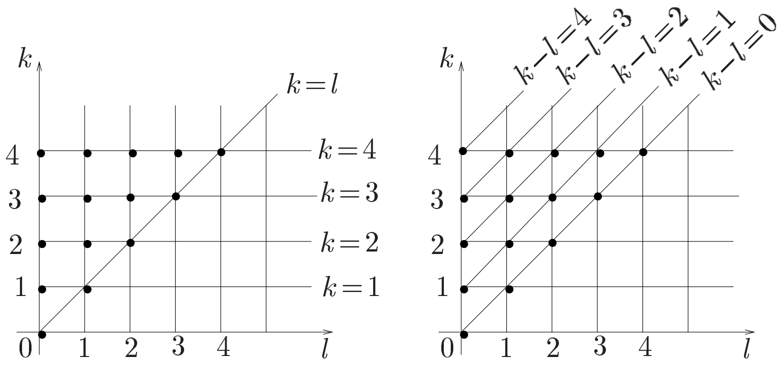
$$
A \;=\; i\alpha\phi(z)\;, \qquad B \;=\; i\beta\phi(w)\;.
$$

In this way, we find

$$
\begin{aligned}
\mathcal{R}(e^A e^B) \;&=\; \sum_{k=0}^{+\infty} \frac{: A^k e^B :}{k!} \\
&\overset{(2.48)}{=}\; \sum_{k=0}^{+\infty} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (AB)^l \; : A^{k-l} e^B :
\end{aligned}
$$

$$= \quad \sum_{k=0}^{+\infty} \sum_{l=0}^{k} \frac{(\underbrace{AB})^l}{l!} \; \frac{:A^{k-l}e^B:}{(k-l)!} \; .$$

Changing the summation indices from k, l to l, N, where $N = k - l$ (as is represented pictorially in the figure that follows), we can decouple the two sums in the series.



This re-writing yields

$$\mathcal{R}(e^A e^B) \quad = \quad \sum_{l=0}^{+\infty} \frac{(\underbrace{AB})^l}{l!} \sum_{N=0}^{+\infty} \frac{:A^N e^B:}{N!} \quad = \quad \exp\left(\underbrace{AB}\right) \; :e^A e^B: \; .$$

---

8. We begin by making a Laurent expansion for $i\partial_z \phi(z)$,

$$i\partial_z \phi(z) \;\; \equiv \;\; \sum_n \alpha_n z^{n-1} \; .$$

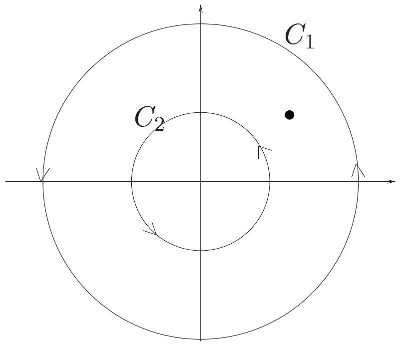
From the theory of complex variables, we have the relation

$$\alpha_n \;\; = \;\; \oint_C \frac{dz}{2\pi i} \, z^n \, i \, \partial\phi(z) \; ,$$

and thus

$$[\alpha_n, \alpha_m] \; = \; i^2 \left[\oint_{C(z)} \frac{dz}{2\pi i} \, , \, \oint_{C(w)} \frac{dw}{2\pi i} \right] \, z^n \, \partial_z \phi(z) \, w^m \, \partial_w \phi(w) \; .$$

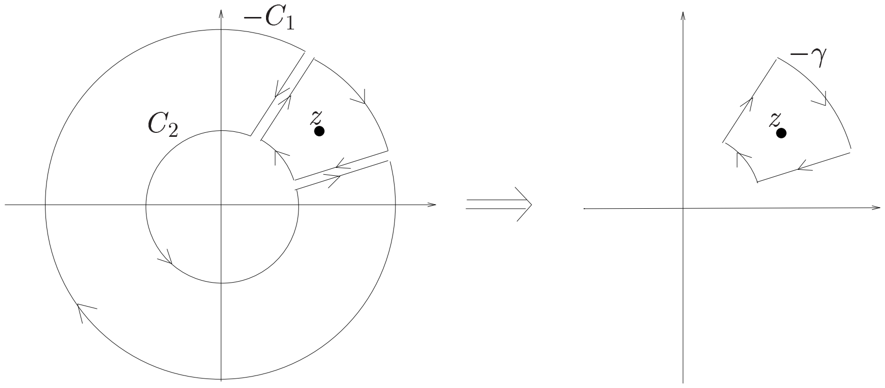
Notice that $C(z)$ and $C(w)$ must be closed curves around the origin of the complex plane; other than that, there is no other restriction on these curves. Therefore we can keep $C(z)$ fixed and take $C(w) = C_2$ for the first term of the commutator and



$C(w) = C_1$ for the second term (see the diagram). Doing so, the above equation becomes

$$
\begin{aligned}
[\alpha_n, \alpha_m] &= -\oint_{C(z)} \frac{dz}{2\pi i} \left(\oint_{C_2} - \oint_{C_1} \right) \frac{dw}{2\pi i}\, z^n w^m\, \partial_z \phi(z)\, \partial_w \phi(w) \\
&= -\oint_{C(z)} \frac{dz}{2\pi i} \oint_{\gamma} \frac{dw}{2\pi i}\, z^n w^m\, \mathcal{R}\left(\partial_z \phi(z) \partial_w \phi(w)\right) \ ,
\end{aligned}
$$

where $\mathcal{R}$ indicates radial ordering of the fields (analogous to temporal ordering) and the curve γ encircles the point z.



Since

$$
\mathcal{R}\left(\phi(z)\phi(w)\right) = -\frac{1}{2g} \ln(z-w) + \mathrm{reg} \ ,
$$

we see that

$$
\mathcal{R}\left(\partial_z \phi(z) \partial_w \phi(w)\right) = -\frac{1}{2g} \frac{1}{(z-w)^2} + \mathrm{reg} \ .
$$

and the commutation relation between two α's becomes:

$$
[\alpha_n, \alpha_m] = -\frac{1}{2g} \oint_{C(z)} \frac{dz}{2\pi i} \oint_{\gamma} \frac{dw}{2\pi i} \frac{z^n w^m}{(z-w)^2}
$$

$$
\begin{aligned}
&= -\frac{1}{2g}\oint_{C(z)} \frac{dz}{2\pi i}\, z^n\, m z^{m-1} \\
&= -\frac{m}{2g}\,\delta_{n,-m}\ ,
\end{aligned}
$$

or

$$
[\alpha_n, \alpha_m] \;=\; \frac{n}{2g}\,\delta_{n,-m}\ .
$$

9. Substituting the expression

$$
i\partial\phi(z) \;\equiv\; \sum_n \frac{\phi_n}{z^{n+1}}\ \ ,
$$

which defines the modes of the boson in the presence of the e_0 term in (2.23), we find:

$$
\begin{aligned}
T &= g\left(\sum_n \frac{\phi_n}{z^{n+1}}\right)^2 + e_0\,\partial\sum_n \frac{\phi_n}{z^{n+1}} \\
&= g\sum_{n,m}\frac{\phi_n\phi_m}{z^{n+m+2}} - e_0\sum_n \frac{(n+1)\,\phi_n}{z^{n+2}} \\
&= g\sum_{k,m}\frac{\phi_{k-m}\phi_m}{z^{k+2}} - e_0\sum_k \frac{(k+1)\,\phi_k}{z^{k+2}} \\
&= \sum_k z^{-(k+2)}\left[g\sum_m \phi_{k-m}\phi_m - e_0\,(k+1)\,\phi_k\right]\ .
\end{aligned}
$$

From the last expression, it is obvious that

$$
L_k \;=\; g\sum_m \phi_{k-m}\phi_m - e_0\,(k+1)\,\phi_k\ .
$$

The modes ϕ_n obey the commutation relations

$$
[\phi_n, \phi_m] \;=\; \frac{n}{2g}\,\delta_{n,-m}\ .
$$

Using these results,

$$
\begin{aligned}
L_0 &= g\sum_{m=-\infty}^{+\infty} \phi_{-m}\phi_m - e_0\ \phi_0 \\
&= g\sum_{m=1}^{+\infty}\phi_{-m}\phi_m + g\sum_{m=-1}^{-\infty}\phi_{-m}\phi_m + g\,\phi_0^2 - e_0\,\phi_0
\end{aligned}
$$

$$
\begin{aligned}
&= g \sum_{m=1}^{+\infty} \phi_{-m}\phi_m + g \sum_{m=1}^{+\infty} \phi_m\phi_{-m} + g\,\phi_0^2 - e_0\,\phi_0 \\
&= g \sum_{m=1}^{+\infty} \phi_{-m}\phi_m + g \sum_{m=1}^{+\infty} \left(\phi_{-m}\phi_m + \frac{m}{2g}\right) + g\,\phi_0^2 - e_0\,\phi_0 \\
&= 2g \sum_{m=1}^{+\infty} \phi_{-m}\phi_m + \frac{1}{2} \sum_{m=1}^{+\infty} m + g\,\phi_0^2 - e_0\,\phi_0 \ ,
\end{aligned}
$$

or, in other words,

$$
L_0 \ = \ 2g \sum_{m=1}^{+\infty} \phi_{-m}\phi_m + g\,\phi_0^2 - e_0\,\phi_0 + c_0 \ ,
$$

where $c_0 = \sum_m m/2$. The constant c_0 can be set to zero by normal ordering.

Comment: In some instances, we will need to introduce the ζ-function renormalization. Then

$$
c_0 \ = \ \frac{1}{2} \sum_{m=1}^{+\infty} m \ = \ \frac{1}{2}\,\zeta(-1) \ = \ -\,\frac{1}{24} \ .
$$

This last equation is explained in the appendix that follows.

APPENDIX

The sum $\sum_m m/2$ often is denoted by $\zeta(-1)/2$ in analogy with the ζ-function of number theory,

$$
\zeta(s) \ = \ \sum_{m=1}^{+\infty} \frac{1}{m^s} \ .
$$

As defined, this series converges for $Re(s) > 1$. The series does not converge for other values of s, but for these values, the function $\zeta(s)$ is defined uniquely by analytic continuation.

In the rest of the document, we will meet the ζ-function on several occasions. Here are two of the properties of the ζ-function:

$$
\begin{aligned}
\Gamma(s)\zeta(s) &= \int_0^{+\infty} \frac{x^{s-1}e^{-x}}{1-e^{-x}}\,dx \ , \quad \text{and} \\
\zeta(s) &= 2(2\pi)^{s-1}\,\Gamma(1-s)\,\sin\frac{\pi s}{2}\,\zeta(1-s) \ .
\end{aligned}
$$

Setting $s = -1$ in the second identity, and noticing that

$$
\zeta(2) \ = \ \sum_{n=1}^{+\infty} \frac{1}{n^2} \ = \ \frac{\pi^2}{6} \ ,
$$

we find

$$
\begin{aligned}
\zeta(-1) &= 2(2\pi)^{-2}\,\Gamma(2)\,\sin\frac{-\pi s}{2}\,\zeta(2) \\
&= \frac{2}{4\pi^2}\,1!\,(-1)\,\frac{\pi^2}{6} \ = \ -\,\frac{1}{12} \ .
\end{aligned}
$$

10. We consider the free boson defined by the Lagrange density

$$\mathcal{L} = \frac{1}{8\pi} (\partial_\mu \Phi)^2 \ ,$$

where Φ is compactified on a circle of radius R. From the Lagrangian, we derive the equation of motion

$$\partial_t^2 \Phi - \partial_\sigma^2 \Phi \ = \ 0 \ . \tag{2.49}$$

The canonical momentum is given by

$$\Pi \ = \ \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \ = \ \frac{1}{4\pi} \dot{\Phi} \ .$$

Φ and Π satisfy the canonical commutation relations

$$\begin{aligned} \big[\Phi(\sigma,t), \Phi(\sigma',t)\big] \ &= \ \big[\Pi(\sigma,t), \Pi(\sigma',t)\big] \ = \ 0 \ , \\ \big[\Phi(\sigma,t), \Pi(\sigma',t)\big] \ &= \ i\delta(\sigma - \sigma') \ . \end{aligned}$$

The Hamiltonian density for this model is

$$\mathcal{H} \ \equiv \ \Pi\dot{\Phi} - \mathcal{L} \ = \ \frac{1}{8\pi} \left[(\partial_t \Phi)^2 + (\partial_\sigma \Phi)^2 \right] \ . \tag{2.50}$$

As is conventional, to solve equation (2.49), we use the method of separation of variables, setting

$$\Phi(\sigma,t) \ = \ \Sigma(\sigma)\, T(t) \ .$$

Substituting this in (2.49), we find

$$\frac{\ddot{T}}{T} \ = \ \frac{\Sigma''}{\Sigma} \ = \ -k^2 \ ,$$

where k^2 is a constant. When $k \neq 0$, these last equations give

$$\begin{aligned} T(t) \ &= \ a\, e^{ikt} + b\, e^{-ikt} \ , \\ \Sigma(\sigma) \ &= \ c\, e^{ik\sigma} + d\, e^{-ik\sigma} \ , \end{aligned}$$

while for $k = 0$, these equations yield

$$\begin{aligned} T(t) \ &= \ a + b\, t \ , \\ \Sigma(\sigma) \ &= \ c + d\, \sigma \ . \end{aligned}$$

Therefore, a free boson can consist of

$$\Phi_0(\sigma,t) \ = \ \phi_0 + b_0\, \sigma + c_0\, t + d_0\, \sigma t \ ,$$

and

$$\Phi_k(\sigma,t) \;=\; \tilde{a}_{-k}\,e^{ik(\sigma+t)} + \tilde{a}_k\,e^{-ik(\sigma+t)} + a_k\,e^{ik(\sigma-t)} + a_{-k}\,e^{-ik(\sigma-t)}\;, \qquad k \neq 0\;.$$

The Hermiticity condition $\Phi = \Phi^\dagger$ requires

$$a_{-k} \;=\; a_k^\dagger\;, \qquad \tilde{a}_{-k} \;=\; \tilde{a}_k^\dagger\;,$$

and that ϕ_0, b_0, c_0, d_0, and k be real. The periodicity condition

$$\Phi(\sigma+\beta,t) \;=\; \Phi(\sigma,t) + 2\pi R\,w\;,$$

can be rewritten as

$$\begin{aligned}\Phi_0(\sigma+\beta,t) &= \Phi_0(\sigma,t) + 2\pi R\,w\;,\\ \Phi_k(\sigma+\beta,t) &= \Phi_k(\sigma,t)\;.\end{aligned}$$

The first equation implies that

$$d_0 \;=\; 0\;, \qquad b_0 \;=\; \frac{2\pi R}{\beta}\,w\;,$$

while the second imposes integer momentum modes:

$$k \;=\; \frac{2\pi n}{\beta}\;, \qquad n \in \mathbb{Z}^*\;.$$

Therefore

$$\begin{aligned}\Phi(\sigma,t) &= \Phi_0(\sigma,t) + \sum_{n\in\mathbb{Z}^*}\Phi_n(\sigma,t)\\ &= \phi_0 + \frac{2\pi R}{\beta}w\sigma + \frac{4\pi}{\beta}pt + i\sum_{n\neq 0}\left[\frac{\alpha_n}{n}\,e^{i\frac{2\pi}{\beta}n(\sigma-t)} - \frac{\overline{\alpha}_{-n}}{n}\,e^{i\frac{2\pi}{\beta}n(\sigma+t)}\right]\\ &= \phi_0 + \frac{2\pi R}{\beta}w\sigma + \frac{4\pi}{\beta}pt + i\sum_{n\neq 0}\left[\frac{\alpha_n}{n}\,e^{i\frac{2\pi}{\beta}n(\sigma-t)} + \frac{\overline{\alpha}_n}{n}\,e^{-i\frac{2\pi}{\beta}n(\sigma+t)}\right]\;,\end{aligned}$$

where we redefined

$$\alpha_n \;\equiv\; i\frac{a_n}{n}\;, \qquad \overline{\alpha}_n \;\equiv\; i\frac{\tilde{a}_{-n}}{-n}\;, \qquad c_0 \;\equiv\; \frac{4\pi}{\beta}p\;.$$

Differentiating w.r.t. time, we get

$$\Pi \;=\; \frac{1}{4\pi}\,\dot{\Phi}(\sigma,t) \;=\; \frac{1}{\beta}p + \frac{1}{2\beta}\sum_{n\neq 0}\left[\alpha_n\,e^{i\frac{2\pi}{\beta}n(\sigma-t)} - \overline{\alpha}_n\,e^{i\frac{2\pi}{\beta}n(\sigma+t)}\right]\;.$$

From the canonical commutation rules, we see that

$$\begin{aligned} [\phi_0, p] &= i \ , \\ [\alpha_n, \alpha_m] &= [\overline{\alpha}_n, \overline{\alpha}_m] \ = \ n\, \delta_{n+m,0} \ . \end{aligned}$$

The rest of the commutators vanish. Notice that p has the interpretation of momentum, and therefore has eigenvalues $p = n/R$.

After the analytic continuation $t \mapsto -i\tau$, we define

$$z \ \equiv \ e^{-i\frac{2\pi}{\beta}(\sigma+i\tau)} \ = \ e^{\frac{2\pi}{\beta}(-i\sigma+\tau)} \ , \qquad \overline{z} \ = \ e^{\frac{2\pi}{\beta}(i\sigma+\tau)} \ .$$

The series describing the boson thus gives

$$\begin{aligned} \Phi(z) &= \phi_0 + \frac{wR}{-2i}(\ln z - \ln \overline{z}) - ip(\ln z + \ln \overline{z}) + i \sum_{n\neq 0} \left[\frac{a_n}{n}\, z^{-n} + \frac{\overline{a}_n}{n}\, \overline{z}^{-n} \right] \\ &= \phi_0 + i \left(\frac{wR}{2} - p \right) \ln z + i \left(-\frac{wR}{2} - p \right) \ln \overline{z} + i \sum_{n\neq 0} \left[\frac{a_n}{n}\, z^{-n} - \frac{\overline{a}_{-n}}{n}\, \overline{z}^{-n} \right] \ . \end{aligned}$$

The boson field may be decomposed into two chiral components:

$$\begin{aligned} \phi(z) &= \frac{\phi_0}{2} - i\alpha_0 \, \ln z + i \sum_{n\neq 0} \frac{\alpha_n}{n z^n} \ , \\ \overline{\phi}(\overline{z}) &= \frac{\phi_0}{2} - i\overline{\alpha}_0 \, \ln \overline{z} + i \sum_{n\neq 0} \frac{\overline{\alpha}_n}{n \overline{z}^n} \ , \end{aligned}$$

where we have defined

$$\alpha_0 \ \equiv \ p - \frac{wR}{2} \quad \text{and} \quad \overline{\alpha}_0 \ \equiv \ p + \frac{wR}{2} \ .$$

Evaluating the expression (2.50), one determines that the Hamiltonian for the free compactified boson is

$$H \ = \ \frac{2\pi}{\beta} \left(L_0 + \overline{L}_0 \right) \ ,$$

where

$$\begin{aligned} L_0 &= \left(p - \frac{Rw}{2} \right)^2 + \sum_{n=1}^{+\infty} \alpha_{-n}\alpha_n - \frac{1}{24} \ , \\ L_0 &= \left(p + \frac{Rw}{2} \right)^2 + \sum_{n=1}^{+\infty} \overline{\alpha}_{-n}\overline{\alpha}_n - \frac{1}{24} \ . \end{aligned}$$

The vacuum state $|\,\emptyset; w, k\,\rangle$ is labeled by the winding number w and the momentum integer k. These are also heighest weight states with conformal weights

$$\begin{aligned}
\Delta_{kw} &= \frac{1}{2}\left(\frac{k}{R}-\frac{Rw}{2}\right)^2-\frac{1}{24}\ ,\\
\overline{\Delta}_{kw} &= \frac{1}{2}\left(\frac{k}{R}+\frac{Rw}{2}\right)^2-\frac{1}{24}\ .
\end{aligned}$$

11. The fermion propagator G is the solution of the equation

$$\overline{\partial}G = \frac{2\pi}{\lambda}\,\delta^{(2)}(z-w)\ .$$

Using the representation (1.19) of the δ-function

$$\delta^{(2)}(z-w) = \frac{1}{2\pi}\,\overline{\partial}\frac{1}{z-w}\ ,$$

the above equation reads

$$\overline{\partial}G = \frac{1}{\lambda}\,\overline{\partial}\frac{1}{z-w}\ ,$$

from which we can immediately infer that

$$G = \frac{1}{\lambda}\frac{1}{z-w}\ .$$

12. Working in the same way as in the case of the free boson, we define

$$i\psi(z) \equiv \sum_n \psi_n\, z^{-n-1/2}\ ,$$

for $n\in\mathbb{Z}+1/2$. Then

$$\psi_n = \oint\frac{dz}{2\pi i}\, z^{n-1/2}\, i\psi(z)\ .$$

We now recall that, due to the fermionic nature of the field ψ, changing the order of two ψ's gives an extra negative sign. Thus we can write the anticommutator in terms of contour integrals as follows:

$$\begin{aligned}
\{\psi_n,\psi_m\} &= i^2\left[\oint\frac{dz}{2\pi i},\oint\frac{dw}{2\pi i}\right] z^{n-1/2}\,\psi(z)\,w^{m-1/2}\,\psi(w)\\
&= \oint_{C(z)}\frac{dz}{2\pi i}\oint_{\gamma}\frac{dw}{2\pi i}\, z^{n-1/2}\,w^{m-1/2}\,\mathcal{R}\left(\psi(z)\psi(w)\right)\\
&= \oint_{C(z)}\frac{dz}{2\pi i}\oint_{\gamma}\frac{dw}{2\pi i}\, z^{n-1/2}\,w^{m-1/2}\,\frac{-1}{z-w}\ ,
\end{aligned}$$

which gives us

$$\{\psi_n, \psi_m\} \;=\; \delta_{n,-m}\ .$$

13. (a) Using Wick's theorem, we obtain

$$\begin{aligned}
\mathcal{R}\left(T(w)\psi(z)\right) &= \mathcal{R}\left(\frac{1}{2}:\psi(w)\partial_w\psi(w):\psi(z)\right) \\
&= \frac{1}{2}:\psi(w)\partial_w\psi(w)\psi(z): -\frac{1}{2}\,\psi(z)\psi(w)\,\partial_w\psi(w) \\
&\quad +\frac{1}{2}\,\partial_w\psi(w)\psi(z)\,\psi(w) \\
&= -\frac{1}{2}:\psi(w)\partial_w\psi(w)\,\psi(z): +\frac{1/2}{w-z}\,\partial_w\psi(w) + \frac{1/2}{(w-z)^2}\,\psi(w)\ .
\end{aligned}$$

Then Taylor expanding the field $\psi(w)$ around z gives

$$\begin{aligned}
\psi(w) &= \psi(z) + \partial_z\psi(z)(w-z) + \frac{1}{2}\partial_z^2\psi(z)(w-z)^2 + \cdots\ , \quad \text{and} \\
\partial_w\psi(w) &= \partial_z\psi(z) + \partial_z^2\psi(z)(w-z) + \cdots\ ,
\end{aligned}$$

and so we find

$$\begin{aligned}
\mathcal{R}\left(T(w)\psi(z)\right) &= -\frac{1}{2}:\left[\psi(z)+\partial_z\psi(z)(w-z)+\ldots\right]\left[\partial_z\psi(z)+\partial_z^2\psi(z)(w-z)+\ldots\right]\psi(z): \\
&\quad +\frac{1/2}{w-z}\left[\partial_z\psi(z)+\partial_z^2\psi(z)(w-z)+\ldots\right] \\
&\quad +\frac{1/2}{(w-z)^2}\left[\psi(z)\,+\,\partial_z\psi(z)(w-z)\,+\,\frac{1}{2!}\partial_z^2\psi(z)(w-z)^2\,+\,\ldots\right] \\
&= \frac{1/2}{(w-z)^2}\,\psi(z)\,+\,\frac{1}{w-z}\,\partial_z\psi(z)\,+\,\frac{3}{4}\,\partial_z^2\psi(z)\,+\,\ldots\ ,
\end{aligned}$$

where we have used the fact that

$$:\psi(z)\left(\partial_z\psi(z)\right)\psi(z): \;=\; 0\ .$$

From the result of the calculation above and the definition

$$\mathcal{R}\left(T(w)\psi(z)\right) \;=\; \sum_n \frac{L_n\psi(z)}{(w-z)^{n+2}}\ , \tag{2.51}$$

it is thus obvious that

$$L_{-2}\psi(z) \;=\; \frac{3}{4}\,\partial_z^2\psi(z)\ .$$

(b) Kac's formula on states

$$(L_{-2} - \frac{3}{4}L_{-1})\,|\,1/2\,\rangle \;=\; 0$$

can be written as a differential equation on fields:

$$L_{-2}(\psi(z)) \;-\; \frac{3}{4}L^2_{-1}(\psi(z)) \;=\; 0\ . \tag{2.52}$$

From the definition (2.51), we find that

$$L_{-n}(\psi(w)) \;=\; \oint \frac{dz}{2\pi i}\,(z-w)^{-n+1}\,\mathcal{R}\,(T(z)\psi(w))\ .$$

In particular

$$\begin{aligned} L_{-1}(\psi(w)) &= \oint \frac{dz}{2\pi i}\,\mathcal{R}\,(T(z)\psi(w)) \\ &= \oint \frac{dz}{2\pi i}\left[\frac{1/2}{(z-w)^2} + \frac{1}{z-w}\partial_z\psi(z) + \ldots\right]\ . \end{aligned}$$

Therefore

$$L_{-1}(\psi(w)) \;=\; \partial_w\psi(w)\ .$$

Then (2.52) gives

$$L_{-2}(\psi(z)) \;=\; \frac{3}{4}\,\partial_z^2\psi(z)\ .$$

14. (a) For a generic BC-system with signature ε, the OPEs between the two fields can be written as

$$\begin{aligned} B(z)C(w) &= \frac{1}{z-w}\ , \\ C(z)B(w) &= \frac{\varepsilon}{z-w}\ . \end{aligned}$$

The energy-momentum tensor may now be derived easily using the following argument. Since it is a field of weight 2, it should be a linear combination of all weight 2 products made out of B and C, and so it must be of the form

$$T(z) \;=\; \alpha_1\ :B(z)\,\partial_z C(z): + \alpha_2\ :\partial_z B(z)\,C(z):\ .$$

The coefficients α_1 and α_2 can be determined by demanding that $T(z)$ leads to the correct conformal weights for B and C. In particular, we must have

$$T(z)B(w) \;=\; \frac{j\,B(w)}{(z-w)^2} + \frac{\partial_w B(w)}{z-w} + \text{reg}\ . \tag{2.53}$$

On the other hand, using the expression for $T(z)$ in terms of the fields B and C, along with the OPEs for these fields, we see that

$$\begin{aligned}
T(z)B(w) &= [\alpha_1 \,:B(z)\,\partial_z C(z): + \alpha_2 \,:\partial_z B(z)\,C(z):]\,B(w) \\
&= \alpha_1\,B(z)\,\partial_z C(z)B(w) + +\alpha_2\,\partial_z B(z)\,C(z)B(w) + \text{reg} \\
&= \alpha_1\,B(z)\,\partial_z \frac{1}{z-w} + \alpha_2\,\frac{\partial_z B(z)}{z-w} + \text{reg} \\
&= -\alpha_1\,\frac{B(z)}{(z-w)^2} + \alpha_2\,\frac{\partial_z B(z)}{z-w} + \text{reg}\ .
\end{aligned}$$

We can now expand $B(z)$ and $\partial_z B(z)$ in Taylor series around w:

$$\begin{aligned}
T(z)B(w) &= -\alpha_1\,\frac{B(w) + (z-w)\,\partial_w B(w) + \mathcal{O}((z-w)^2)}{(z-w)^2} + \alpha_2\,\frac{\partial_w B(w) + \mathcal{O}(z-w)}{z-w} + \text{reg} \\
&= -\alpha_1\,\frac{B(w)}{(z-w)^2} + \frac{(-\alpha_1+\alpha_2)\partial_w B(w)}{z-w} + \text{reg}\ .
\end{aligned}$$

Comparing the last expression with (2.53), we see that $\alpha_1 = -j$ and $\alpha_2 = 1-j$. Therefore, the energy-momentum tensor is given by

$$T(z) = -j\,:B(z)\,\partial_z C(z): + (1-j)\,:\partial_z B(z)\,C(z):\ .$$

(b) To find the central charge c, we need to compute the OPE for $T(z)T(w)$, using the energy-momentum tensor we obtained in the previous part. In particular, it is enough to focus only on the most singular term of the OPE, the one that is obtained upon two contractions among the fields. The calculation is straightforward, and we present it here:

$$\begin{aligned}
T(z)T(w) &= [-j\,:B(z)\,\partial_z C(z): + (1-j)\,:\partial_z B(z)\,C(z):] \\
&\quad\ [-j\,:B(w)\,\partial_w C(w): + (1-j)\,:\partial_w B(w)\,C(w):] \\
&= +j^2\,:B(z)\,\partial_z C(z):\ :B(w)\,\partial_w C(w): \\
&\quad -j(1-j)\,:B(z)\,\partial_z C(z):\ :\partial_w B(w)\,C(w): \\
&\quad -j(1-j)\,:\partial_z B(z)\,C(z):\ :B(w)\,\partial_w C(w): \\
&\quad +(1-j)^2\,:\partial_z B(z)\,C(z):\ :\partial_w B(w)\,C(w): \\
&= j^2\,:B(z)\,\partial_z C(z):\ :B(w)\,\partial_w C(w): \\
&\quad -j(1-j)\,:B(z)\,\partial_z C(z):\ :\partial_w B(w)\,C(w): \\
&\quad -j(1-j)\,:\partial_z B(z)\,C(z):\ :B(w)\,\partial_w C(w): \\
&\quad +(1-j)^2\,:\partial_z B(z)\,C(z):\ :\partial_w B(w)\,C(w): + \cdots
\end{aligned}$$

$$
\begin{aligned}
&= j^2\,\partial_w \frac{1}{z-w}\,\partial_z \frac{\varepsilon}{z-w} - j(j-1)\,\frac{1}{z-w}\,\partial_w\partial_z\frac{\varepsilon}{z-w} \\
&\quad -j(j-1)\,\partial_w\partial_z\frac{1}{z-w}\,\frac{\varepsilon}{z-w} + (1-j)^2\,\partial_z\frac{1}{z-w}\,\partial_w\frac{\varepsilon}{z-w} + \cdots \\
&= j^2\,\frac{1}{(z-w)^2}\,\frac{-\varepsilon}{(z-w)^2} - j(j-1)\,\frac{1}{z-w}\,\frac{-2\varepsilon}{(z-w)^3} \\
&\quad -j(j-1)\,,\frac{-2}{(z-w)^3}\,\frac{\varepsilon}{z-w} + (j-1)^2\,\frac{1}{(z-w)^2}\,\frac{\varepsilon}{(z-w)^2} + \cdots \\
&= \varepsilon\,\frac{-j^2 + 2j(j-1) - (j-1)^2}{(z-w)^4} + \cdots \ .
\end{aligned}
$$

Thus we conclude that the central charge of the generic BC-system is

$$c \;=\; -\,\varepsilon\,(6j^2 - 6j + 1)\ .$$

15. Starting from the OPE

$$T(z_1)T(z_2) \;=\; \frac{c/2}{(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{\partial T(z_2)}{z_1-z_2} + \mathrm{reg}\ ,$$

we see that

$$\langle\, T(z,\overline{z})T(w,\overline{w})\,\rangle \;=\; \frac{c/2}{(z-w)^4}\ .$$

We rewrite this result in the form

$$\langle T(z)T(w)\rangle \;=\; -\,\frac{c}{12}\,\partial_z^2\partial_w\frac{1}{z-w}\ .$$

Differentiating with respect to $\overline{z}$, we get

$$
\begin{aligned}
\langle \partial_{\overline{z}} T(z)T(w)\rangle \;&=\; -\frac{c}{12}\,\partial_z^2\partial_w\partial_{\overline{z}}\frac{1}{z-w}\delta(z-w) \\
&=\; -\frac{c\pi}{6}\,\partial_z^2\partial_w\delta(z-w)\ ,
\end{aligned}
$$

where we have made use of equation (1.19). Using the conservation of the energy momentum tensor

$$\overline{\partial}T_{zz} + \partial T_{\overline{z}z} \;=\; \overline{\partial}T + \frac{1}{4}\,\partial\Theta \;=\; 0 \;\Rightarrow\; \overline{\partial}T \;=\; -\,\frac{1}{4}\,\partial\Theta\ ,$$

we finally find the relation sought:

$$\langle \partial_z\Theta(z)T(w)\rangle \;=\; \frac{2c\pi}{3}\,\partial_z^2\partial_w\delta(z-w)\ . \tag{2.54}$$

The confusion that creates the contradictory natury of equations (2.3) and (2.54) is eliminated if we say that the energy-momentum tensors that appear in the two equations are different! Therefore, the puzzle examined in the problem is only superficial, and arises by (bad?) notation.

Let us try to refine our notation. We call the energy-momentum tensor defined by (2.3) the **physical energy-momentum tensor** $T^{(phys)}_{\mu\nu}$. On the other hand, we call the tensor of equation (2.54) the **conformal energy-momentum tensor** $T^{(conf)}_{\mu\nu}$. How are the two tensors related? Simply, any acceptable conformally invariant theory is a sum of CFTs, each contributing its own *conformal* piece to the *physical* energy-momentum tensor:

$$T^{(phys)}_{\mu\nu} = \sum_{CFTs} T^{(conf)}_{\mu\nu} \ .$$

The total theory has $c = 0$, and therefore

$$\langle \partial_z \Theta(z) T(w) \rangle \ = \ 0$$

Thus no paradox is present.

16. The energy-momentum tensor (2.2), can be defined through

$$\delta S \ = \ -\frac{1}{4\pi} \iint d^2x \, \sqrt{g} \, T_{\mu\nu} \, \delta g^{\mu\nu} \ . \tag{2.55}$$

We can write $g_{\mu\nu}$ in terms of some reference metric $\tilde{g}_{\mu\nu}$ by defining

$$g_{\mu\nu} \ = \ e^{\sigma} \, \tilde{g}_{\mu\nu} \ \Rightarrow \ g^{\mu\nu} \ = \ e^{-\sigma} \, \tilde{g}^{\mu\nu} \ .$$

Then

$$\delta g^{\mu\nu} = -\delta\sigma \, g^{\mu\nu} \ ,$$

and equation (2.55) gives

$$\begin{aligned} \delta S \ &= \ \frac{1}{4\pi} \iint d^2x \, \sqrt{g} \, T_{\mu\nu} \, g^{\mu\nu} \, \delta\sigma \\ &= \ \frac{1}{4\pi} \iint d^2x \, \sqrt{g} \, \Theta \, \delta\sigma \ . \end{aligned}$$

Since the total theory S' has vanishing trace ($\Theta' = 0$), the equation just found, applied to the total action S', implies that $\delta S' = 0$. On the other hand, $\delta S' = \delta S + \delta S_L$. We thus conclude that

$$\begin{aligned} \delta S_L \ &= \ -\delta S \\ &= \ -\frac{1}{4\pi} \iint d^2x \, \sqrt{g} \, \Theta \, \delta\sigma \\ &= \ -\frac{1}{4\pi} \iint d^2x \, \sqrt{g} \, (aR + b) \, \delta\sigma \ . \end{aligned}$$

In terms of the reference metric $\tilde{g}_{\mu\nu}$ (see appendix):

$$\sqrt{g} \;=\; e^{\sigma}\sqrt{\tilde{g}}\,, \qquad R \;=\; e^{-\sigma}\left(\tilde{R}+\tilde{\Delta}\sigma\right)\,.$$

Then

$$\delta S_L \;=\; -\frac{1}{4\pi}\iint d^2x\,\sqrt{\tilde{g}}\left[a\left(\tilde{R}+\tilde{\Delta}\sigma\right)+b\,e^{\sigma}\right]\delta\sigma\,.$$

This result can be integrated easily:

$$S_L = -\frac{1}{4\pi}\iint d^2x\,\sqrt{\tilde{g}}\left[a\left(\tilde{R}\,\sigma+\frac{1}{2}\sigma\tilde{\Delta}\sigma\right)+b\,e^{\sigma}\right]\,.$$

This action is the Liouville action sought. Usually, the parametrization $\tilde{g}_{\mu\nu} = e^{\phi}\,\delta_{\mu\nu}$ is used. Then

$$\sqrt{\tilde{g}} \;=\; e^{\phi}\,, \quad \tilde{R} \;=\; -\,e^{-\phi}\,\partial_z\partial_{\overline{z}}\phi\,, \quad \tilde{\Delta}\sigma \;=\; -\,e^{-\phi}\,\partial_z\partial_{\overline{z}}\sigma\,,$$

and

$$S_L = \frac{1}{4\pi}\iint d^2x\,\left[a\left(\sigma\partial_z\partial_{\overline{z}}\phi+\sigma\partial_z\partial_{\overline{z}}\sigma\right)-b\,e^{\phi+\sigma}\right]\,.$$

APPENDIX

As we have proved in Exercise 6 of Chapter 1, the Riemann surfaces are conformally flat. A metric can then always be chosen such that

$$g_{\mu\nu} \;=\; \rho\,\delta_{\mu\nu}\,. \tag{2.56}$$

Using the equation

$$R \;=\; -\,\frac{2}{\sqrt{g_{11}g_{22}}}\left[\frac{\partial}{\partial x^1}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial x^1}\sqrt{g_{22}}\right)+\frac{\partial}{\partial x^2}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial x^2}\sqrt{g_{11}}\right)\right]\,,$$

which gives the scalar curvature of a 2-dimensional manifold, we can easily find that for the metric (2.56),

$$R \;=\; -\,\frac{1}{\rho}\,(\partial_1^2+\partial_2^2)\ln\rho \;=\; -\,\frac{1}{\rho}\,\partial_z\partial_{\overline{z}}\ln\rho\,.$$

Given a second metric

$$\tilde{g}_{\mu\nu} \;=\; \tilde{\rho}\,\delta_{\mu\nu}\,,$$

the corresponding scalar curvature would be

$$\tilde{R} \;=\; -\,\frac{1}{\tilde{\rho}}\,\partial_z\partial_{\overline{z}}\ln\tilde{\rho}\,.$$

We can easily find the relation between the two curvatures. If we set $\rho = e^{\sigma}\,\tilde{\rho}$, then

$$R \;=\; e^{-\sigma}\left(\tilde{R}-\frac{1}{\tilde{\rho}}\,\partial_z\partial_{\overline{z}}\sigma\right)\,.$$

Notice that the Laplacian of the metric $\tilde{g}_{\mu\nu}$ would be:

$$\tilde{\Delta}\sigma \;=\; -\,\frac{1}{\sqrt{\tilde{g}}}\,\partial_\mu\left(\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\sigma\right) \;=\; -\,\frac{1}{\tilde{\rho}}\,\partial_\mu\left(\tilde{\rho}\,\frac{1}{\tilde{\rho}}\delta^{\mu\nu}\,\partial_\nu\sigma\right) \;=\; -\,\frac{1}{\tilde{\rho}}\,(\partial_1^2+\partial_2^2)\sigma\,.$$

Therefore, we may write

$$R \;=\; e^{-\sigma}\left(\tilde{R}+\tilde{\Delta}\sigma\right)\,.$$

17. We need to calculate the variation of S with respect the variation of g_{ab}. Through the calculations, we shall make use of the well-known identities (for a reference on identities for metrics, Christoffel symbols, and related topics, see any standard reference on general relativity, such as [450, 501, 636]):

$$\begin{aligned}
\delta g_{mn} &= -g_{ma}g_{nb}\delta g^{ab} \ , \\
\delta g &= g\, g^{mn}\, \delta g_{mn} \ , \\
\delta\sqrt{g} &= \frac{1}{2}\sqrt{g}\, g^{mn}\, \delta g_{mn} \ , \\
\delta(\sqrt{g}g^{ab}) &= \sqrt{g}\,(\delta^a_m\delta^b_n - \frac{1}{2}\, g_{mn}g^{ab})\,\delta g^{mn} \ , \quad \text{and} \\
g^{mn}\delta g_{ms} &= -g_{ms}\delta g^{mn} \ .
\end{aligned}$$

Now we calculate δS, and get

$$\begin{aligned}
\delta S &= \alpha \iint_M d^2\xi\, \delta(\sqrt{g}\, g^{ab})\, \partial_a\Phi\partial_b\Phi + \beta \iint_M d^2\xi\, \delta(\sqrt{g}\, g^{ab}\, R_{ab})\, \Phi \\
&= \alpha \iint_M d^2\xi\, \delta(\sqrt{g}\, g^{ab})\, \partial_a\Phi\partial_b\Phi + \beta \iint_M d^2\xi\, \delta(\sqrt{g}\, g^{ab})\, R_{ab}\,\Phi + \beta \iint_M d^2\xi\, \sqrt{g}\, g^{ab}\,\delta R_{ab}\,\Phi \\
&= \alpha \iint_M d^2\xi\, \sqrt{g}\,(\delta^a_m\delta^b_n - \frac{1}{2}g_{mn}g^{ab})\,\partial_a\Phi\partial_b\Phi + \beta \iint_M d^2\xi\,(\delta^a_m\delta^b_n - \frac{1}{2}g_{mn}g^{ab})\, R_{ab}\,\Phi \\
&\qquad + \beta \iint_M d^2\xi\, \sqrt{g}\, g^{ab}\,\delta R_{ab}\,\Phi \\
&= \alpha \iint_M d^2\xi\, \sqrt{g}\,(\partial_m\Phi\partial_n\Phi - \frac{1}{2}g_{mn}g^{ab}\,\partial_a\Phi\partial_b\Phi)\,\delta g^{mn} + \beta \iint_M d^2\xi\,(R_{mn} - \frac{1}{2}g_{mn}R)\,\Phi\,\delta g^{mn} \\
&\qquad + \beta \iint_M d^2\xi\, \sqrt{g}\, g^{ab}\,\delta R_{ab}\,\Phi \ .
\end{aligned}$$

Therefore

$$T_{mn} = -4\pi\alpha(\partial_m\Phi\partial_n\Phi - \frac{1}{2}g_{mn}\,\partial_a\Phi\partial^a\Phi) - 4\pi\beta(R_{mn} - \frac{1}{2}g_{mn}R) - 4\pi\beta\,\frac{1}{\sqrt{g}}\frac{\delta A}{\delta g^{mn}} \ ,$$

where

$$A \equiv \iint_M d^2\xi\, \sqrt{g}\, g^{ab}\,\delta R_{ab}\,\Phi \ .$$

One must still rearrange the last term in T_{mn} in a convenient form. This requires some work. Before we undertake this task, the reader should keep in mind the following remarks:

- As well-known, the metric is covariantly constant, i.e., $g_{mn;r} = 0$. We can thus freely move it in and out of a covariant derivative.

- If A_a is a vector, then its divergence is given by

$$A^a{}_{;a} = \frac{1}{\sqrt{g}}\,\partial_a(\sqrt{g}A^a)\ . \tag{2.57}$$

- We will drop all terms of the form

$$\iint d^2\xi(\cdots)_{,n}$$

  since, by Gauss' theorem, they become boundary terms and thus vanish.

We now continue with the calculations.

Using the Palatini identity (2.62), we write

$$\begin{aligned} A &= \iint_M d^2\xi\,\sqrt{g}\,g^{mn}\,(\delta\Gamma^a_{ma;n} - \delta\Gamma^a_{mn;a})\,\Phi \\ &= \underbrace{\iint_M d^2\xi\,\sqrt{g}\,g^{mn}\,\delta\Gamma^a_{ma;n}\,\Phi}_{B} - \underbrace{\iint_M d^2\xi\,\sqrt{g}\,g^{mn}\,\delta\Gamma^a_{mn;a}\,\Phi}_{C}\ . \end{aligned}$$

Note that here and in what follows, the variation symbol "δ" acts only on the object immediately following it.

We begin re-arranging the integrand in the first term B:

$$\begin{aligned} B &= \iint_M d^2\xi\,\sqrt{g}\,g^{mn}\,\delta\Gamma^a_{ma;n}\,\Phi \\ &= \iint_M d^2\xi\,\sqrt{g}\,(g^{mn}\,\delta\Gamma^a_{ma})_{;n}\,\Phi \\ &\overset{(2.57)}{=} \iint_M d^2\xi\,(\sqrt{g}\,g^{mn}\,\delta\Gamma^a_{ma})_{,n}\,\Phi \\ &= \iint_M d^2\xi\,(\sqrt{g}\,g^{mn}\,\delta\Gamma^a_{ma}\,\Phi)_{,n} - \iint_M d^2\xi\,\sqrt{g}\,g^{mn}\,\delta\Gamma^a_{ma}\,\Phi_{,n} \\ &\overset{(2.58)}{=} -\iint_M d^2\xi\,\sqrt{g}\,g^{mn}\,\partial_m\delta(\ln\sqrt{g})\,\Phi_{,n} \\ &= -\iint_M d^2\xi\,\partial_m(\sqrt{g}\,g^{mn}\,\delta(\ln\sqrt{g})\,\Phi_{,n}) + \iint_M d\xi^1 d\xi^2\,\partial_m(\sqrt{g}\,g^{mn}\,\Phi_{,n})\,\delta(\ln\sqrt{g}) \\ &= \iint_M d^2\xi\,\partial_m(\sqrt{g}\,g^{mn}\,\Phi_{,n})\,\frac{\delta\sqrt{g}}{\sqrt{g}} \\ &\overset{(2.57)}{=} \iint_M d^2\xi\,\delta\sqrt{g}\,\Phi^{,m}{}_{;m} \\ &= -\frac{1}{2}\iint_M d^2\xi\,\sqrt{g}\,\delta g^{mn} g_{mn}\,\Phi^{,r}{}_{;r}\ . \end{aligned}$$

We now continue with the C term:

$$
\begin{aligned}
C &= \iint_M d^2\xi\,\sqrt{g}\,g^{mn}\,\delta\Gamma^a_{mn;a}\,\Phi \\
&= \iint_M d^2\xi\,\sqrt{g}\,(g^{mn}\,\delta\Gamma^a_{mn})_{;a}\,\Phi \\
&\overset{(2.57)}{=} \iint_M d^2\xi\,(\sqrt{g}\,g^{mn}\,\delta\Gamma^a_{mn})_{,a}\,\Phi \\
&= \iint_M d^2\xi\,\partial_a(\sqrt{g}\,g^{mn}\,\delta\Gamma^a_{mn}\,\Phi) - \iint_M d^2\xi\,\sqrt{g}\,g^{mn}\,\Phi_{,a}\,\delta\Gamma^a_{mn} \\
&= -\iint_M d^2\xi\,\sqrt{g}\,\Phi_{,a}\,g^{mn}\,\delta\Gamma^a_{mn} \\
&= \iint_M d^2\xi\,\sqrt{g}\,\Phi_{,a}\,\left[-\delta\,(g^{mn}\Gamma^a_{mn}) + \delta g^{mn}\Gamma^a_{mn}\right] \\
&\overset{(2.59)}{=} \iint_M d^2\xi\,\sqrt{g}\,\Phi_{,a}\,\delta\left(\frac{1}{\sqrt{g}}\partial_s(\sqrt{g}g^{as})\right) + \iint_M d^2\xi\,\sqrt{g}\,\Phi_{,a}\,\delta g^{mn}\Gamma^a_{mn} \\
&= \iint_M d^2\xi\,\sqrt{g}\,\Phi_{,a}\,\frac{\delta\sqrt{g}}{-g}\partial_s(\sqrt{g}g^{as}) + \iint_M d^2\xi\,\sqrt{g}\,\Phi_{,a}\,\frac{1}{\sqrt{g}}\partial_s\delta(\sqrt{g}g^{as}) + \iint_M d^2\xi\,\sqrt{g}\,\Phi_{,a}\,\delta g^{mn}\Gamma^a_{mn} \\
&= -\iint_M d^2\xi\,\Phi_{,a}\,\frac{\delta\sqrt{g}}{\sqrt{g}}\partial_s(\sqrt{g}g^{as}) + \iint_M d^2\xi\,\Phi_{,a}\,\partial_s\delta(\sqrt{g}g^{as}) + \iint_M d^2\xi\,\sqrt{g}\,\Phi_{,a}\,\delta g^{mn}\Gamma^a_{mn} \\
&= -\iint_M d^2\xi\,\Phi_{,a}\,\frac{\delta\sqrt{g}}{\sqrt{g}}\partial_s(\sqrt{g}g^{as}) + \iint_M d^2\xi\,\left[\partial_s\left(\Phi_{,a}\delta(\sqrt{g}g^{as})\right) - \partial_s\Phi_{,a}\delta(\sqrt{g}g^{as})\right] + \iint_M d^2\xi\,\sqrt{g}\,\Phi_{,a}\,\delta g^{mn}\Gamma^a_{mn} \\
&= -\iint_M d^2\xi\,\Phi_{,a}\,\frac{\delta\sqrt{g}}{\sqrt{g}}\partial_s(\sqrt{g}g^{as}) - \iint_M d^2\xi\,\partial_s\Phi_{,a}\delta(\sqrt{g}g^{as}) + \iint_M d^2\xi\,\sqrt{g}\,\Phi_{,a}\,\delta g^{mn}\Gamma^a_{mn} \\
&\overset{(2.59)}{=} \iint_M d^2\xi\,\Phi_{,a}\,\delta\sqrt{g}\,g^{mn}\Gamma^a_{mn} - \iint_M d^2\xi\,\partial_s\Phi_{,a}\delta(\sqrt{g}g^{as}) + \iint_M d^2\xi\,\sqrt{g}\,\Phi_{,a}\,\delta g^{mn}\Gamma^a_{mn} \\
&= \iint_M d^2\xi\,\Phi_{,a}\,\delta\sqrt{g}\,g^{mn}\Gamma^a_{mn} - \iint_M d^2\xi\,\partial_s\Phi_{,a}\,(\delta\sqrt{g}\,g^{as} + \sqrt{g}\,\delta g^{as}) + \iint_M d^2\xi\,\sqrt{g}\,\Phi_{,a}\,\delta g^{mn}\Gamma^a_{mn} \\
&= -\iint_M d^2\xi\,\delta\sqrt{g}\,g^{mn}\,(\partial_m\Phi_{,n} - \Gamma^a_{mn}\Phi_{,a}) - \iint_M d^2\xi\,\sqrt{g}\,\delta g^{mn}\,(\partial_m\Phi_{,n} - \Gamma^a_{mn}\Phi_{,a}) \\
&= -\iint_M d^2\xi\,\delta\sqrt{g}\,g^{mn}\,\Phi_{;nm} - \iint_M d^2\xi\,\sqrt{g}\,\delta g^{mn}\,\Phi_{;nm} \\
&= -\iint_M d^2\xi\,\delta\sqrt{g}\,\Phi^{;a}_{\ ;a} - \iint_M d^2\xi\,\sqrt{g}\,\delta g^{mn}\,\Phi_{;nm} \\
&= \iint_M d^2\xi\,\frac{1}{2}\sqrt{g}\,g_{mn}\,\delta g^{mn}\,\Phi^{;a}_{\ ;a} - \iint_M d^2\xi\,\sqrt{g}\,\delta g^{mn}\,\Phi_{;nm}\ .
\end{aligned}
$$

Combining the results for B and C we arrive at

$$A = \iint_M d^2\xi\,\sqrt{g}\,\delta g^{mn}\,\left(\Phi_{;nm} - g_{mn}\,\Phi^{;a}_{\ ;a}\right)\ .$$

Therefore

$$\frac{1}{\sqrt{g}}\frac{\delta A}{\delta g^{ij}} = \Phi_{;ik} - \Phi^{;a}_{\ ;a}\,g_{ik}\ ,$$

and the energy-momentum tensor is

$$T_{ij} = -4\pi\alpha\left(\partial_i\Phi\partial_j\Phi - \frac{1}{2}g_{ij}\partial_a\Phi\partial^b\Phi\right) - 4\pi\beta\left(R_{ij} - \frac{1}{2}g_{ij}R\right) - 4\pi\beta\left(\Phi_{;ij} - g_{ij}\,\Phi^{;a}_{\ ;a}\right)\ .$$

APPENDIX

For the Christoffel symbols, it is known that

$$\Gamma^a_{ma} = \partial_m \ln\sqrt{g} = \frac{\partial_m\sqrt{g}}{\sqrt{g}} \tag{2.58}$$

and

$$g^{jk}\,\Gamma^a_{jk} = -\frac{1}{\sqrt{g}}\,\partial_s(\sqrt{g}\,g^{as})\ . \tag{2.59}$$

Since the latter formula is not as well-known, we derive it here. Let A^{ij} be the cofactors of g_{ij}, i.e.,

$$g = g_{ij}A^{ij}$$

From this, we see that

$$\frac{\partial g}{\partial g_{ij}} = A^{ij} \Rightarrow g_{ij}\frac{\partial g}{\partial g_{ij}} = g_{ij}A^{ij} = g \Rightarrow \frac{\partial g}{\partial g_{ij}} = gg^{ij}\ .$$

On the other hand, from the chain rule,

$$\frac{\partial g}{\partial x^l} = \frac{\partial g}{\partial g_{ij}}\,\frac{\partial g_{ij}}{\partial x^l} = gg^{ij}\,\frac{\partial g_{ij}}{\partial x^l}\ .$$

Consequently,

$$\begin{aligned}
g^{jk}\Gamma^i_{jk} &= \frac{1}{2}\,g^{jk}g^{il}\,(\partial_j g_{lk} + \partial_k g_{lk} - \partial_l g_{jk}) \\
&= \frac{1}{2}\,g^{jk}g^{il}\,(2\partial_j g_{lk} - \partial_l g_{jk}) \\
&= g^{jk}g^{il}\,\partial_j g_{lk} - \frac{1}{2}\,g^{jk}g^{il}\,\partial_l g_{jk} \\
&= -\partial_j g^{jk}g^{il}\,g_{lk} - \frac{1}{2}\,g^{il}\,(g^{jk}\partial_l g_{jk}) \\
&= -\partial_j g^{ji} - \frac{1}{2g}\,g^{il}\,\partial_l g \\
&= -\frac{1}{\sqrt{g}}\,(\sqrt{g}\,\partial_j g^{ji} + \partial_j\sqrt{g}\,g^{ij}) \\
&= -\frac{1}{\sqrt{g}}\,\partial_j(\sqrt{g}\,g^{ij})\ .
\end{aligned}$$

For the Christoffel symbols

$$\Gamma^m_{ik} = \frac{1}{2}\,g^{mj}\,(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik})\ ,$$

the variation is

$$\begin{aligned}
\delta\Gamma^m_{ik} &= \frac{1}{2}\,\delta g^{mj}\,(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik}) + \frac{1}{2}g^{mj}\,(\partial_k\delta g_{ij} + \partial_i\delta g_{jk} - \partial_j\delta g_{ik}) \\
&= -\frac{1}{2}\,\delta g_{ln}\,g^{ml}\,g^{nj}\,(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik}) + \frac{1}{2}g^{mj}\,(\partial_k\delta g_{ij} + \partial_i\delta g_{jk} - \partial_j\delta g_{ik}) \\
&= -\delta g_{ln}\,g^{ml}\,\Gamma^n_{ik} + \frac{1}{2}g^{mj}\,(\partial_k\delta g_{ij} + \partial_i\delta g_{jk} - \partial_j\delta g_{ik}) \qquad (2.60) \\
&= \frac{1}{2}\,\delta g^{mj}\Big[\partial_k\delta g_{ij} - \Gamma^n_{ki}\,\delta g_{nj} - \Gamma^n_{kj}\,\delta g_{in} + \partial_i\delta g_{jk} - \Gamma^n_{ij}\,\delta g_{nk} - \Gamma^n_{ik}\,\delta g_{jn} \\
&\qquad - \partial_j\delta g_{ik} - \Gamma^n_{ji}\,\delta g_{nk} - \Gamma^n_{jk}\,\delta g_{in}\Big] \\
&= \frac{1}{2}\,g^{mj}\,(\delta g_{ij;k} + \delta g_{jk;i} - \delta g_{ik;j})\ . \qquad (2.61)
\end{aligned}$$

Finally, we study the variation of the Ricci tensor

$$-R_{ij} = \partial_a \Gamma^a_{ij} - \partial_j \Gamma^a_{ia} + \Gamma^b_{ij}\, \Gamma^a_{ba} - \Gamma^b_{ia}\, \Gamma^a_{bj} \ .$$

We have

$$\begin{aligned}
-\delta R_{ij} &= \delta\partial_a \Gamma^a_{ij} - \delta\partial_j \Gamma^a_{ia} + \delta\Gamma^b_{ij}\, \Gamma^a_{ba} + \Gamma^b_{ij}\, \delta\Gamma^a_{ba} - \delta\Gamma^b_{ia}\, \Gamma^a_{bj} - \Gamma^b_{ia}\, \delta\Gamma^a_{bj} \\
&= \partial_a \delta\Gamma^a_{ij} - \partial_j \delta\Gamma^a_{ia} + \delta\Gamma^b_{ij}\, \Gamma^a_{ba} + \Gamma^b_{ij}\, \delta\Gamma^a_{ba} - \delta\Gamma^b_{ia}\, \Gamma^a_{bj} - \Gamma^b_{ia}\, \delta\Gamma^a_{bj} \\
&= \quad \partial_a \delta\Gamma^a_{ij} - \Gamma^b_{ia}\, \delta\Gamma^a_{bj} - \Gamma^b_{ja}\, \delta\Gamma^a_{ib} + \Gamma^a_{ba}\, \delta\Gamma^b_{ij} \\
&\quad -\partial_j \delta\Gamma^a_{ia} + \Gamma^b_{ij}\, \delta\Gamma^a_{ba} + \Gamma^b_{ja}\, \delta\Gamma^a_{ib} - \Gamma^a_{bj}\, \delta\Gamma^b_{ia} \\
&= \delta\Gamma^a_{ij;a} - \delta\Gamma^a_{ia;j} \ .
\end{aligned} \tag{2.62}$$

This last formula is known as the **Palatini identity**.

Chapter 3

CORRELATORS IN CFT

References: The standard references for the material in this chapter are the same as in the previous chapter, namely [333, 186, 420]. Bosonization is summarized in [607], which also reprints many original papers. The application of bosonization to 2-dimensional QCD is reviewed in [290].

3.1 BRIEF SUMMARY

3.1.1 Computation of Correlators

In any QFT, the **Ward identities** can be used in the computation of the correlation functions

$$G = \langle\emptyset|\Phi(x_1)\cdots\Phi(x_N)|\emptyset\rangle \ .$$

In CFT, there are three holomorphic **conformal Ward identities**, given by

$$\langle\emptyset|\,[L_n, \Phi(x_1)\cdots\Phi(x_N)]\,|\emptyset\rangle \ = \ 0 \ , \qquad n = -1, 0, 1 \ ; \tag{3.1}$$

likewise, there are three antiholomorphic conformal Ward identities associated with the generators $\overline{L}_n, \ n = -1, 0, 1$.

Although the identities (3.1) are true for correlation functions of any fields, they are most useful for correlation functions of primary fields. In this case, the conformal Ward identities read

$$\begin{aligned}
\sum_{i=1}^{N} \partial_{z_i}\langle\emptyset|\Phi(x_1)\cdots\Phi(x_N)|\emptyset\rangle \ &= \ 0 \ , \\
\sum_{i=1}^{N} (z_i\partial_{z_i} + \Delta_i)\langle\emptyset|\Phi(x_1)\cdots\Phi(x_N)|\emptyset\rangle \ &= \ 0 \ , \\
\sum_{i=1}^{N} (z_i^2\partial_{z_i} + 2z_i\Delta_i)\langle\emptyset|\Phi(x_1)\cdots\Phi(x_N)|\emptyset\rangle \ &= \ 0 \ .
\end{aligned}$$

All correlation functions containing secondary fields can be obtained by the action of differential operators on correlation functions containing primary fields only. For example, suppose that in the correlation function

$$G = \langle \emptyset | \Phi_1(x_1) \cdots \Phi_{N-1}(x_{N-1}) \Phi_N(x_N) | \emptyset \rangle \ ,$$

all fields are primary except

$$\Phi_N(x_N) = L_{-k_1} \ldots L_{-k_l} \Phi(z_N, \overline{z}_N) \ .$$

Then

$$G = \mathcal{L}_{-k_1} \ldots \mathcal{L}_{-k_l} \langle \emptyset | \Phi_1(x_1) \cdots \Phi_{N-1}(x_{N-1}) \Phi(x_N) | \emptyset \rangle \ ,$$

where

$$\mathcal{L}_k \ = \ - \sum_{j=1}^{N-1} \left[\frac{(1-k)\Delta_j}{(z_j - z_N)^k} + \frac{1}{(z_j - z_N)^{k-1}} \frac{\partial}{\partial z_j} \right] \ .$$

One can perform similar calculations on correlation functions that contain more descendant fields.

3.1.2 The Bootstrap Approach

We can treat the set of OPEs of a QFT as the *fundamental* information of the theory. Then, knowing the OPEs, one can reconstruct the whole theory. This approach, which we discuss below, is called the **bootstrap approach**.

Let $\mathcal{A}$ be the space of all fields $\{\Phi_i\}$ of the QFT. This space can be decomposed, as usual, into a direct sum of two subspaces, a fermionic subspace $\mathcal{F}$ and a bosonic subspace $\mathcal{B}$:

$$\mathcal{A} \ = \ \mathcal{F} \oplus \mathcal{B} \ . \tag{3.2}$$

The fields that belong in these respective subspaces satisfy the following commutation and anti-commutation relations:

$$\Phi^{(B)}(x)\Phi^{(B)}(y) \ = \ \Phi^{(B)}(y)\Phi^{(B)}(x) \ , \tag{3.3}$$
$$\Phi^{(B)}(x)\Phi^{(F)}(y) \ = \ \Phi^{(F)}(y)\Phi^{(B)}(x) \ , \tag{3.4}$$
$$\Phi^{(F)}(x)\Phi^{(F)}(y) \ = \ -\Phi^{(F)}(y)\Phi^{(F)}(x) \ , \tag{3.5}$$

where $\Phi^{(F)}(x) \in \mathcal{F}$ and $\Phi^{(B)}(x) \in \mathcal{B}$. As a result, the selection rules in $\mathcal{A}$ are

$$\Phi^{(B)}\Phi^{(B)} \in \mathcal{B} \ ,$$
$$\Phi^{(F)}\Phi^{(F)} \in \mathcal{B} \ ,$$
$$\Phi^{(B)}\Phi^{(F)} \in \mathcal{F} \ .$$

The basic assumptions in this (non-Lagrangian) formulation are the following:

1. $\mathcal{A}$ is *complete*, i.e., it contains all fundamental fields, as well as all composite fields.

2. $\mathcal{A}$ is an *infinite dimensional* space that admits a *countable basis.*

3. $\mathcal{A}$ is an *associative algebra.*

We can now spell out explicitly the equations emerging from the previous assumptions. Let $\{A_a(x),\ a = 1, 2, \ldots\}$ be a basis in $\mathcal{A}$; notice that this is in concordance with the requirement that we have a countably infinite basis. Then any product $A_a(x)A_b(y)$ in the algebra can be decomposed in terms of the basis fields. It is therefore enough to consider the products[1] of the basis fields among themselves:

$$\mathcal{T}\Big(A_a(x)A_b(y)\Big) = \sum_k C^c_{ab}(x-y)\, A_c(y)\ , \tag{3.6}$$

where $C^c_{ab}(x-y)$ are c-numbers which are called **structure coefficients**.

In this approach to QFT, equation (3.6) and the commutation rules (3.3)-(3.5) are understood as constraints on the correlation functions. In particular, if $\Phi(x)$ is some field in the QFT, then

$$\langle\mathcal{T}\left(\Phi(x)A_a(x_1)A_b(x_2)\right)\rangle = \sum_k C^c_{ab}(x_1-x_2)\ \langle\mathcal{T}\left(\Phi(x)A_c(x_2)\right)\rangle\ .$$

Hence, equation (3.6) allows us to calculate all correlation functions of the theory recursively, reducing them ultimately to the 2-point correlation fuctions

$$\langle\mathcal{T}\left(A_a(x_1)A_b(x_2)\right)\rangle \equiv \Delta_{ab}(x_1, x_2)\ .$$

Obviously, then, if we know the structure coefficients, we know the QFT. In this way, **the problem of the classification of QFTs is reduced to the problem of classifying all possible structure coefficients**. One can use the associativity of the algebra $\mathcal{A}$ to aid in the calculation of the structure coefficients. The correlation function

$$\langle\mathcal{T}\left(A_a(x_1)A_b(x_2)A_c(x_3)A_d(x_4)\right)\rangle$$

can be calculated in two different ways by applying the expansion (3.6). One can take the OPEs of $A_a(x_1)A_b(x_2)$ and $A_c(x_3)A_d(x_4)$, or one can instead take the OPEs of $A_a(x_1)A_c(x_3)$ and $A_b(x_2)A_d(x_4)$. Consistency requires that these two computations lead to identical results. This is referred to as **crossing symmetry**. Mathematically, this takes the form

$$\sum_{nm} C^n_{ab}(x_1-x_2)\Delta_{nm}(x_2-x_4)C^m_{cd}(x_3-x_4) = \sum_{nm} C^n_{ac}(x_1-x_3)\Delta_{nm}(x_3-x_4)C^m_{bd}(x_2-x_4)\ .$$

This equation is known as the **bootstrap equation**, and finding the sets of structure coefficients that satisfy this equation would amount to a classification of QFTs. As mentioned earlier, this whole program is known as the bootstrap appraoch. Making progress using the bootstrap approach has proven to be very hard in more than two dimensions. However,

[1] Notice that, for convenience, we take the OPEs to be time-ordered.

in two dimensions it becomes easier to implement. For example, for 2-dimensional CFTs, conformal invariance severely restricts the 2-point and 3-point correlation functions; they must be of the form

$$\begin{aligned}\langle\, \Phi_i(z_i,\overline{z}_i)\Phi_j((z_j,\overline{z}_j)\,\rangle &= \frac{\delta_{ij}}{z_{ij}^{2\Delta_i}\,\overline{z}_{ij}^{2\overline{\Delta}_i}}\ , \quad \text{and}\\ \langle\, \Phi_i(z_i,\overline{z}_i)\Phi_j((z_j,\overline{z}_j)\Phi_k((z_k,\overline{z}_k)\,\rangle &= \frac{C_{ijk}}{z_{ij}^{\gamma_{ij}}\,\overline{z}_{ij}^{\overline{\gamma}_{ij}}\,z_{jk}^{\gamma_{jk}}\overline{z}_{jk}^{\overline{\gamma}_{jk}}\,z_{ki}^{\gamma_{ki}}\overline{z}_{ki}^{\overline{\gamma}_{ki}}}\ ,\end{aligned}$$

where $z_{ij} = z_i - z_j$ and $\gamma_{ij} = \Delta_i + \Delta_j - \Delta_k$ (and similarly for $\overline{z}_{ij}$ and $\overline{\gamma}_{ij}$). Only a set of **structure constants** in the OPEs remains undetermined:

$$\Phi_i(z_i,\overline{z}_i)\Phi_j(0,0) \;=\; \sum_k C_{ijk}\, z^{\Delta_k-\Delta_i-\Delta_j}\,\overline{z}^{\overline{\Delta}_k-\overline{\Delta}_i-\overline{\Delta}_j}\,[\Phi_k(0,0)]\ .$$

where the sum in r.h.s. is over the whole conformal tower. Correlation functions that contain descendants may be computed using the OPEs of $T(z)\Phi_i(w,\overline{w})$. Finally, the structure constants are the solutions of the bootstrap equation

$$\sum_p C_{ijp}\,C_{lmp}\,\mathcal{F}_{ij}^{lm}(p|x)\overline{\mathcal{F}}_{ij}^{lm}(p|\overline{x}) \;=\; \sum_q C_{ilq}\,C_{jmq}\,\mathcal{F}_{il}^{jm}(q|1-x)\overline{\mathcal{F}}_{il}^{jm}(q|1-\overline{x})\ ,$$

where $\mathcal{F}_{ij}^{lm}(p|x)\overline{\mathcal{F}}_{ij}^{lm}(p|\overline{x})$ is the contribution of the conformal family $[\Phi_p]$ to the 4-point correlation function.

3.1.3 Fusion Rules

The OPEs for any two fields of any conformal families are determined by the OPEs of the corresponding primaries. This fact allows one to write down the so-called **fusion rules**

$$[\Phi_i]\times[\Phi_j] \;=\; N_{ij}^k\;[\Phi_k]$$

as a shorthand for the OPEs of the conformal families. More concretely, the l.h.s. of the above equation represents the OPE between a field of the conformal family $[\Phi_i]$ and a field of the conformal family $[\Phi_j]$. Then the r.h.s. indicates which conformal families $[\Phi_k]$ may have their members appearing in the OPE.

The numbers N_{ij}^k in the fusion rules are integers that count the number of occurrences of the family $[\Phi_k]$ in the OPE. Said in another way, they count the number of independent ways to create the field Φ_k from the original fields Φ_i and Φ_j. Due to this interpretation, the numbers N_{ij}^k immediately satisfy the symmetry condition

$$N_{ij}^k \;=\; N_{ji}^k\ .$$

Also, due to the associativity of the algebra of the fields, they satisfy a quadratic constraint. Using the matrix notation $\boldsymbol{N}_i = \left[N_{ij}^k\right]$, this quadratic constraint takes the form

$$\boldsymbol{N}_i\,\boldsymbol{N}_l \;=\; \boldsymbol{N}_l\,\boldsymbol{N}_i\ .$$

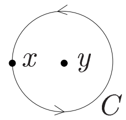
Therefore, the matrices $\boldsymbol{N}_i$ form a commutative associative representation of the fusion rules. If the theory contains an infinite number of fields, then these matrices are infinite dimensional. However, there is a special set of CFTs, known as **rational CFTs** (RCFTs), that contain only a finite number of primary fields. The MMs discussed in Chapter 2 are special cases of RCFTs. The RCFTs can be analyzed completely. In these theories, the matrices $\boldsymbol{N}_i$ can be diagonalized simultaneously, and their eigenvalues form 1-dimensional representations of the fusion rules.

3.1.4 Local vs. Non-Local Fields

The selections rules (3.3)-(3.5) are not the most general possible in two dimensions, as we saw in Chapter 1. Thus in two dimensions, it is appropriate to expand the space of fields $\mathcal{A}$ to include generalized selection rules.

In particular, let us denote by $\pi_{\mathcal{C}}(x)$ the operation of analytic continuation of the point x around the point y on the contour $\mathcal{C}$. In general,

$$A_a(\pi_{\mathcal{C}}(x))A_b(y) = R_{ab}^{\tilde{a}\tilde{b}}\, A_{\tilde{a}}(x)A_{\tilde{b}}(y) \ .$$



In the course of this collection of problems, we will find many realizations of this equation.

An interesting case that we shall study in a later chapter is the case of **para-fermions**:

$$A_a(\pi_C(x))A_b(y) \ = \ e^{2\pi i \varphi_{ab}}\, A_a(x)A_b(y) \ . \tag{3.7}$$

The phase φ_{ab} is a real number in the interval $[0,1)$; it is called the **exponent of relative locality** of the fields $A_a(x)$ and $A_b(x)$. The two fields $A_a(x)$ and $A_b(y)$ are called **mutually local** if $\varphi_{ab} = 0$; a field $A_a(x)$ is called **local** if $\varphi_{aa} = 0$. According to this definition, bosons are local fields, while a boson and a fermion are mutually local.

Let A_d be a field occurring in the OPE of A_a and A_b:

$$A_a(x)A_b(y) \ = \ \ldots + C_{ab}^{d}\, A_d + \ldots \ .$$

Then the exponent of relative locality of A_d with respect a field A_c is

$$\varphi_{dc} \ = \ \{\varphi_{ac} + \varphi_{bc}\} \ ,$$

where $\{r\}$ denotes the fractional part of r. From this result, we conclude that the correlation function

$$\langle A_d(x)A_c(y)\rangle$$

can be non-vanishing only if $\varphi_{dc} = 0$, that is, if

$$\varphi_{ac} \ = \ 1 - \varphi_{bc} \ . \tag{3.8}$$

Consider the special case that the algebra can be generated by the two fields ψ_1 and $\psi_1^\dagger \equiv \psi_{-1}$, with the additional condition, motivated by (3.8), that

$$\varphi_{1,1} \ = \ 1 - \varphi_{1,-1} \ \equiv \ \varphi \ .$$

In this model, we see that we can define a discrete charge Q such that the space $\mathcal{A}$ is decomposed into sectors of definite charge:

$$\mathcal{A} \;=\; \bigoplus_{n=-\infty}^{+\infty} \mathcal{A}_n \;, \quad Q\mathcal{A}_n \;=\; n\,\mathcal{A}_n \;. \tag{3.9}$$

In particular,

$$\psi_1 \in \mathcal{A}_1 \;, \quad I \in \mathcal{A}_0 \;, \quad \text{and} \quad \psi_1^\dagger \in \mathcal{A}_{-1} \;.$$

For two fields $\Phi_n \in \mathcal{A}_n$ and $\Phi_m \in \mathcal{A}_m$, the selection rule implies that

$$\Phi_n\,\Phi_m \in \mathcal{A}_{m+n} \;.$$

Consequently, the exponent of mutual locality of Φ_n and Φ_m is

$$\varphi_{nm} \;=\; \{nm\varphi\} \;,$$

since Φ_n is in the same charge sector as a product of n ψ_1 fields. Finally, the spins of the fields $\Phi_n \in \mathcal{A}_n$ are calculated trivially from the exponent γ_n. Noticing that

$$\Phi_n(\pi_{\mathcal{C}}(x))\Phi_n(y) \;=\; e^{2\pi i n^2\varphi}\,\Phi_n(x)\Phi_n(y) \;,$$

and comparing this result with the definition of Lorentz spin s, namely that under a rotation of $\theta = 2\pi$,

$$\Phi(x) \to e^{is\theta}\Phi(x) \;,$$

we conclude that

$$s_n \;=\; \frac{1}{2}\{n^2\varphi\} + \frac{1}{2}n_k \;=\; \frac{1}{2}\varphi_{nn} + \frac{1}{2}n_k \;,$$

where the n_k are integers not determined by the information we have specified.

If φ is rational, we can make a little more progress. Let us find $k, N \in \mathbb{N}$ such that $\varphi = k/N$. Then the decomposition (3.9) is finite since the exponents of mutual locality are periodic with period N:

$$\varphi_{n+N,m} \;=\; \{(n+N)m\varphi\} \;=\; \{nm\varphi + mk\} \;=\; \{nm\varphi\} \;=\; \varphi_{nm} \;.$$

In other words, if φ is rational, then the charge Q is a $\mathbb{Z}_N$ charge.

3.1.5 Bosonization

As has been explained in Chapter 1, in two dimensions, statistics is a matter of convention, and a map can be established between fields of different statistics. One often uses this, changing the initial representation to a bosonic representation, in order to calculate the correlation functions.

In the language of CFT, the free boson with $c = 1$ is equivalent to two spinors of $c = 1/2$. The bosonization map is established by

$$\psi_1 + i\psi_2 \;=\; \sqrt{2}\, e^{i\phi} \;.$$

This is only a special case of more general constructions.

The bc-system can be bosonized using a boson with background charge:

$$T(z) \;=\; -\,\frac{1}{2}\,(\partial\phi)^2 + ie_0\partial^2\phi \;.$$

The central charge of this bosonic system is

$$c \;=\; 1 - 12e_0^2 \;.$$

By adjusting the coefficient e_0, one can match the central charge of the bc-system.

For example, the case of $j = 2$ of the bc-system, which is of importance to string theory, is bosonized by a theory with $e_0 = 3/2$ boson. In this case

$$\begin{aligned} b(z) \;&=\; e^{-i\phi(z)} \;, \\ c(z) \;&=\; e^{i\phi(z)} \;. \end{aligned}$$

A $\beta\gamma$-system with $c = 11$ also appears in string theory. Its bosonization proceeds via a $j = 0$ bc-system of $c = -2$ and a boson with background charge of $c = 13$. The latter has a background charge of $e_0 = i$. The map is given by

$$\begin{aligned} \beta \;&=\; \partial b\, e^{-\phi} \;, \\ \gamma \;&=\; c\, e^{\phi} \;. \end{aligned}$$

3.2 EXERCISES

1. Given any four points x_i, x_j, x_k, x_l we can define their **cross-ratio** by

$$z_{ijkl} = \frac{|x_i - x_j|\,|x_k - x_l|}{|x_i - x_l|\,|x_k - x_j|} .$$

Clearly, the cross ratios are invariant under conformal transformations. Find the number of independent variables of the n-point correlation function $\langle\, \phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)\,\rangle$ for a conformally invariant theory in $D = 2$ dimensions by explicitly demonstrating how such a correlation fucntion can be expressed in terms of independent cross ratios.

2. (a) Explain as precisely as possible why, given the Virasoro algebra, we can write Ward identities only for the three generators L_{-1}, L_0, and L_1.

   (b) Derive the Ward identities of the operators L_{-1}, L_0, and L_1.

3. (a) A function $f(x_1, \dots, x_n)$ is called homogeneous of degree r if

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda^r f(x_1, \dots, x_n) . \tag{3.10}$$

For a homogeneous function, prove **Euler's identity**:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i = r\, f .$$

(b) Show that the Ward identity of L_0 for a correlation function $G^{(n)}$ can be translated to the following statement: $G^{(n)}$ is a homogeneous function of degree $\sum_{i=1}^{n} \Delta_i$.

4. Show that any correlation function in CFT has the form

$$G = \prod_{i=1}^{N} \prod_{j=i+1}^{N} z_{ij}^{-\gamma_{ij}}\, Y ,$$

where $z_{ij} = z_i - z_j$ and Y is a function of the cross-ratios (see the solution to Exercise 1). Find the equations satisfied by the γ_{ij}.

5. Show that it is always possible to choose the 2-point functions such that

$$\langle\, \phi_i(z_1)\phi_j(z_2)\,\rangle = \frac{\delta_{\Delta_i \Delta_j}}{(z_1 - z_2)^{2\Delta_i}} . \tag{3.11}$$

6. Let $\phi(z)$ be a field of conformal weight Δ with Fourier modes ϕ_n. Show that

$$\phi_n^\dagger = \phi_{-n} .$$

7. Show that a 3-point correlation function

$$\langle\, \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\,\rangle$$

is fully determined up to a constant. Then relate this constant to the OPE of the given fields.

8. Determine the functional form of an arbitrary 4-point correlation function, and show that it depends on an arbitrary function of a single variable.

9. For a free boson $\phi(z)$, show that the correlation function

$$\left\langle \prod_{j=1}^{n} e^{ia_j\phi(z_j)} \right\rangle$$

must vanish unless $\sum_{j=1}^{n} a_j = 0$.

10. Calculate the correlation function of the previous excercise.

11. (a) Derive the differential equation satisfied by the correlation function

$$\langle\, \sigma(z_1,\overline{z}_1)\sigma(z_2,\overline{z}_2)\ldots\sigma(z_{2M-1},\overline{z}_{2M-1})\sigma(z_{2M},\overline{z}_{2M})\,\rangle$$

of $2M$ spin fields in the Ising model.

(b) Solve the differential equation obtained in part (a) for the case of four spins and show that the solution can be expressed in terms of the hypergeometric function.

12. Consider the generic OPE

$$\mathcal{R}\left(\phi_n(z,\overline{z})\,\phi_m(0,0)\right) \;=\; \sum_p \sum_{\{k\}} \sum_{\{\overline{k}\}} c_{nm}^{p;\{k\}\{\overline{k}\}}\, \frac{\phi_p^{\{k\}\{\overline{k}\}}}{z^{\Delta_n+\Delta_m-\Delta_p\sum k_i}\,\overline{z}^{\overline{\Delta}_n+\overline{\Delta}_m-\overline{\Delta}_p-\sum \overline{k}_i}}\;,$$

with

$$c_{nm}^{p;\{k\}\{\overline{k}\}} \;=\; c_{nm}^{p}\,\beta_{nm}^{p;\{k\}}\,\overline{\beta}_{nm}^{p;\{\overline{k}\}}\;. \tag{3.12}$$

Describe a method that can be used to calculate the coefficients $c_{nm}^{p;\{k\}\{\overline{k}\}}$.

13. In the theory of open strings, the lowest order contribution to the tachyon-tachyon scattering amplitude is given by the CFT correlation function

$$A \;=\; \int dz_3\;\langle\, c(z_1)c(z_2)c(z_4)V_1V_2V_3V_4\,\rangle\;,$$

where $c(z)$ is a weight $j=2$ c-field of a bc-system (which in this context are ghosts), and

$$V_i \;=\; e^{ik_i^\mu X_\mu(z_i)}$$

is the vertex operator for a tachyon, which has $k^2=2$.

Compute the amplitude A in terms of standard functions of mathematical physics.

3.3 SOLUTIONS

1. Translation invariance requires that $G^{(n)}$ depend only on the differences $x_i - x_j$, $i < j$. Since
$$x_i - x_j \;=\; (x_i - x_1) - (x_j - x_1) \;,$$
$G^{(n)}$ depends only on the $n-1$ vectors $x_i - x_1$, $i = 2, 3, \ldots, n$, i.e. it depends on $2(n-1)$ variables.

   Furthermore, rotational invariance implies that $G^{(n)}$ depends only on the magnitudes r_{ij} of $x_i - x_j$, $\forall i, j$ and the relative orientations of these vectors, and not on their absoluate orientations. Since
$$r_{ij}^2 \;=\; r_{i1}^2 + r_{j1}^2 - 2 r_{i1} r_{j1} \cos\theta_{ij} \;,$$
where θ_{ij} is the angle between $x_i - x_1$ and $x_j - x_1$, and since in two dimensions, all these angles are determined once one knows θ_{i1}, the Green function can be written in terms of the $n-1$ magnitudes r_{i1} and the $n-2$ angles θ_{i2}. (One has $i > 1$ for the magnitudes, and $i > 2$ for the angles.) One can see this represented in figure 3.1.

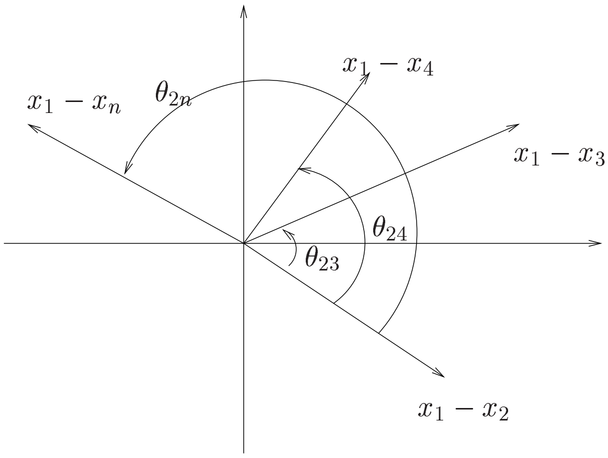


Figure 3.1: In $D = 2$ dimensions, knowlege of the angles θ_{2i}, $i = 3, 4, \ldots, n$ is enough to determine the relative orientation of the vectors $x_1 - x_i$.

   Continuing, scale invariance requires that $G^{(n)}$ depends only on ratios r_{ij}/r_{kl} and the angles (and in our case, we can represent the angles using magnitudes of vectors). Since any arbitrary ratio r_{ij}/r_{kl} can be written in the form
$$\frac{r_{ij}}{r_{kl}} \;=\; \frac{r_{ij}}{r_{12}} \frac{r_{12}}{r_{kl}} \;,$$
we can consider $G^{(n)}$ as a function of
$$u_{ij} \;=\; r_{ij}/r_{12} \;.$$

Obviously, we have now one variable less since $u_{12} = 1$. Therefore, so far, $G^{(n)}$ depends on $2(n-2)$ variables u_{1i}, u_{2i} $i = 3, 4, \ldots, n$.

Finally, special conformal invariance requires that $G^{(n)}$ depends only on the cross-ratios

$$\frac{r_{ij}r_{kl}}{r_{il}r_{kj}} = \frac{u_{ij}}{u_{il}} \left(\frac{u_{kj}}{u_{kl}}\right)^{-1} .$$

We can thus consider $G^{(n)}$ as a function of

$$\frac{u_{ij}}{u_{il}} .$$

Aparently we have $2(n-3)$ such quantities:

$$\frac{u_{1i}}{u_{13}} , \quad \frac{u_{2i}}{u_{23}} , \quad i = 4, \ldots, n .$$

Therefore, the number of independent variables the correlation function can depend upon is $2(n-3)$.

2. (a) Let G be a group of transformations with generators T_a, i.e., if $U \in G$ then $U = e^{i\alpha^a T_a}$ for some α^a.

The fields of the theory transform as

$$\phi \to \phi' = U\phi U^{-1} = e^{i\alpha T}\phi e^{-i\alpha T} .$$

For infinitesimal transformations, i.e., transformations with infinitesimal α, the fields transform as

$$\phi \to \phi' = \phi + i\alpha\,[T, \phi] .$$

The Ward identities are obtained from the group elements that leave the vacuum invariant:

$$\begin{aligned} U\,|\,\emptyset\,\rangle &= |\,\emptyset\,\rangle , \\ \langle\,\emptyset\,|\,U^{-1} &= \langle\,\emptyset\,| , \end{aligned}$$

which is equivalent to saying that the Ward identities are obtained using those generators of G that annihilate the vacuum state, i.e., those T_a such that

$$\langle\,\emptyset\,|\,T_a = T_a\,|\,\emptyset\,\rangle = 0 .$$

For such generators, the correlation function

$$G^{(n)} = \langle\,\emptyset\,|\,\phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n)\,|\,\emptyset\,\rangle ,$$

can be re-written as follows:

$$\begin{aligned} G^{(n)} &= \langle \emptyset \,|\, U^{-1}U\phi_1(x_1)U^{-1}U\phi_2(x_2)U^{-1}U\ldots UU^{-1}\phi_n(x_n)U^{-1}U \,|\, \emptyset \rangle \\ &= \langle \emptyset \,|\, (U\phi_1(x_1)U^{-1})(U\phi_2(x_2)U^{-1})\ldots(U\phi_n(x_n)U^{-1}) \,|\, \emptyset \rangle \\ &= \langle \emptyset \,|\, \phi_1'(x_1)\phi_2'(x_2)\ldots\phi_n'(x_n) \,|\, \emptyset \rangle \\ &\simeq \langle \emptyset \,|\, \phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n) \,|\, \emptyset \rangle + i\alpha^a \langle \emptyset \,|\, [T_a, \phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n)] \,|\, \emptyset \rangle \\ &= G^{(n)} + i\alpha^a \langle \emptyset \,|\, [T_a, \phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n)] \,|\, \emptyset \rangle \ . \end{aligned}$$

From the above result we infer that

$$\langle \emptyset \,|\, [T_a, \phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n)] \,|\, \emptyset \rangle \ = \ 0 \ ,$$

for all generators T_a that annihilate the vacuum.

Thus, given the Virasoro algebra, we will have a Ward identity for each generator that annihilates the vacuum. Therefore, we must first establish which generators do so. Not all Virasoro operators annihilate the vacuum, since if this were true, that is, if

$$L_n \,|\, \emptyset \rangle \ = \ 0 \ , \quad \forall n \ ,$$

then we would have

$$[L_n, L_m] \,|\, \emptyset \rangle \ = \ 0 \ , \quad \forall n, m \ ,$$

which is not true, because of the central charge in the algebra.

However, we do notice now that the central term cancels if $n, m \geq 0$. Thus we can consistently set

$$L_n \,|\, \emptyset \rangle \ = \ 0 \ , \quad \forall n \geq 0 \ .$$

Moreover, the central term still vanishes when L_{-1} is added to the list of operators that annihilate the vacuum, and so it is consistent to have

$$L_{-1} \,|\, \emptyset \rangle \ = \ 0 \ .$$

Using the Hermiticity condition $L_n^\dagger = L_{-n}$, we see that

$$\langle \emptyset \,|\, L_{-n} \ = \ 0 \ , \quad \forall n \geq -1 \ .$$

Therefore, we can see that the three operators

$$L_{-1} \ , L_0 \ , L_1$$

constitute the maximal set of Virasoro generators that can annihilate the vacuum from both the left and right.

(b) The Ward identities for a generic conformal field theory can now be derived in straightforward fashion. We have, for $j = -1$, 0, and 1,

$$\begin{aligned} 0 &= \langle\, \emptyset \,|\, [L_j, \phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n)] \,|\, \emptyset \,\rangle \\ &= \sum_{i=1}^{n} \langle\, \emptyset \,|\, \phi_1(x_1)\phi_2(x_2)\ldots[L_j,\phi_i]\ldots\phi_n(x_n) \,|\, \emptyset \,\rangle \\ &= \sum_{i=1}^{n} \langle\, \emptyset \,|\, \phi_1(x_1)\phi_2(x_2)\ldots\mathcal{L}_j\phi_i\ldots\phi_n(x_n) \,|\, \emptyset \,\rangle \ , \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}_{-1}\phi_i(x_i) &= \frac{\partial}{\partial z_i} \ , \\ \mathcal{L}_0\phi_i(x_i) &= z_i\frac{\partial}{\partial z_i} + \Delta_i \ , \\ \mathcal{L}_1\phi_i(x_i) &= z_i^2\frac{\partial}{\partial z_i} + 2x_i\Delta_i \ . \end{aligned}$$

3. (a) Differentiating the defining relation (3.10) of a homogeneous function with respect to λ, we find

$$\frac{d}{d\lambda}f(u_1,\ldots,u_n) \;=\; r\,\lambda^{r-1}\,f(x_1,\ldots,x_n) \ ,$$

where we have defined $u_i = \lambda x_i$. Setting $\lambda = 1$ gives

$$\frac{d}{d\lambda}f(u_1,\ldots,u_n)\Big|_{\lambda=1} \;=\; r\,f(x_1,\ldots,x_n) \ .$$

The derivative appearring in the l.h.s. is calculated using the chain rule:

$$\begin{aligned} \frac{d}{d\lambda}f(u_1,\ldots,u_n) &= \sum_{i=1}^{n}\frac{\partial f(u_1,\ldots,u_n)}{\partial u_i}\frac{du_i}{d\lambda} \\ &= \sum_{i=1}^{n}\frac{\partial f(u_1,\ldots,u_n)}{\partial u_i}\,x_i \ . \end{aligned}$$

When $\lambda = 1$, we have $u_i = x_i$, and therefore

$$\frac{d}{d\lambda}f(u_1,\ldots,u_n)\Big|_{\lambda=1} \;=\; \sum_{i=1}^{n}\frac{\partial f(x_1,\ldots,x_n)}{\partial x_i}\,x_i \ ,$$

thus completing the proof of Euler's identity.

(b) Recall now that any correlation function $G^{(n)}$ is a function of the $n(n-1)/2$ differences $z_{ij} = z_i - z_j$, $i < j$:

$$G = G(z_{ij}) \ .$$

It is now easy to see that the L_0 Ward identity takes the form

$$\sum_{i=1}^{n} \sum_{j>i}^{n} z_{ij} \frac{\partial G}{\partial z_{ij}} = - \sum_{i=1}^{n} \Delta_i G \ .$$

The desired result now follows by a direct comparison with Euler's equation.

---

4. The Ward identity of L_{-1} for the correlation function G is

$$\left(\frac{\partial}{\partial z_1} + \ldots + \frac{\partial}{\partial z_N} \right) G = 0 \ ,$$

which implies that G is a function of the differences z_{ij} only, i.e.,

$$G = G(z_{ij}) \ , \qquad i < j \ .$$

The L_0 Ward identity requires that G be a homogeneous function of degree $(-\sum_{i=1}^{N} \Delta_i)$. We thus write

$$G = \prod_{i=1}^{N} \prod_{j=i+1}^{N} z_{ij}^{-\gamma_{ij}} Y \ ,$$

with

$$\sum_{i=1}^{N} \sum_{j=i+1}^{N} \gamma_{ij} = \sum_{i=1}^{N} \Delta_i \ .$$

Therefore, Y is a homogeneous function of degree zero; it will be a function of the independent cross ratios.

Now, under the transformations

$$z \mapsto f(z) = \frac{az+b}{cz+d} \Rightarrow z_{ij} \mapsto \frac{z_{ij}}{(cz_i+d)(cz_j+d)} \ ,$$

the correlation function, as written above, transforms as

$$G \mapsto \prod_{i=1}^{N} \prod_{j=i+1}^{N} (cz_i+d)^{\gamma_{ij}} (cz_j+d)^{\gamma_{ij}} G \ .$$

This should be compared with the way the correlation should transform under these transformations:

$$G \mapsto \prod_{i=1}^{N} (cz_i + d)^{2\Delta_i}\, G \ .$$

Therefore, straighforwardly, we find that the numbers γ_{ij} must satisfy the equations:

$$\sum_j{}' \gamma_{ij} \ = \ 2\Delta_i \ ,$$

where we have defined $\gamma_{ij} = \gamma_{ji}$.

5. Let

$$G(z_1, z_2) \ = \ \langle\, \emptyset \,|\, \phi_1(z_1)\phi_2(z_2) \,|\, \emptyset \,\rangle \ .$$

Substituting in the Ward identity for L_{-1}, we find

$$\frac{\partial G}{\partial z_1} + \frac{\partial G}{\partial z_2} \ = \ 0 \ ,$$

which implies that G is a funtion only of the difference $z_1 - z_2$.

The L_0 Ward identity for G gives

$$z_1 \frac{\partial G}{\partial z_1} + z_2 \frac{\partial G}{\partial z_2} + (\Delta_1 + \Delta_2) G \ = \ 0 \ .$$

Setting $z = z_1 - z_2$, the above equation takes the simple form

$$z \frac{dG}{dz} + (\Delta_1 + \Delta_2) G \ = \ 0 \ ,$$

which gives

$$G(z) = \frac{A}{z^{\Delta_1 + \Delta_2}} \ ,$$

where A is a constant.

Finally, we make use of the L_1 Ward identity for G, which states that

$$z_1^2 \frac{\partial G}{\partial z_1} + z_2^2 \frac{\partial G}{\partial z_2} + (2z_1\Delta_1 + 2z_2\Delta_2) G \ = \ 0 \ .$$

Inserting the expression we had just found for G, we arrive at the algebraic equation

$$z\,(\Delta_1 - \Delta_2)\, A \ = \ 0 \ ,$$

from which we conclude that $A = 0$ if $\Delta_1 \neq \Delta_2$. When $\Delta_1 = \Delta_2$, we can rescale the fields in the theory such that $A = 1$. So, finally,

$$G(z) = \frac{1}{z^{2\Delta_1}} ,$$

which is the desired result.

Incidentally, let us make the following observation. For $z = re^{i\theta}$, the full 2-point correlation function takes the form (including the holomorphic and the antiholomorphic dependence)

$$G = \frac{e^{-2i\theta(\Delta-\overline{\Delta})}}{r^{2(\Delta_1+\overline{\Delta}_2)}} .$$

Under rescaling $r \to \lambda r$, the correlation function scales as

$$G \to \lambda^{-2(\Delta+\overline{\Delta})} G .$$

From this, we read the scaling dimension of the fields to be $h = \Delta + \overline{\Delta}$. Under rotations, $\theta \to \theta + \alpha$, the correlation function changes as

$$G \to e^{-2i\alpha(\Delta-\overline{\Delta})} G .$$

From this, the spin of the fields is seen to be $s = \Delta - \overline{\Delta}$.

---

6. The expansion of the field $\phi(z)$ in terms of its modes is

$$\phi(z) \;=\; \sum_n \phi_n\, z^{-n-\Delta} .$$

Then

$$\phi^\dagger(z) \;=\; \sum_n \phi_n^\dagger\, z^{-n-\Delta} . \tag{3.13}$$

The Hermitian conjugate $\phi^\dagger(z)$ of the field is constrained by the condition

$$\langle\, \phi|\phi\,\rangle \;=\; 1 .$$

This implies that

$$\langle\, \phi\,| \;\equiv\; \lim_{z\to 0} \langle\, \emptyset\,|\, \phi^\dagger(z) \;=\; \lim_{w\to\infty} \langle\, \emptyset\,|\, \phi(w) w^{2\Delta} . \tag{3.14}$$

To check this, notice that under the transformation $w = 1/z$,

$$\phi(w) \;=\; \left(\frac{dw}{dz}\right)^{-\Delta} \phi(1/z) \;=\; z^{-2\Delta}\, \phi(1/z) .$$

Then

$$\langle\, \phi|\phi\,\rangle \;=\; \lim_{z\to 0}\lim_{w\to\infty} \langle\, \emptyset \,|\, \phi(w) w^{2\Delta} \phi(z) \,|\, \emptyset\,\rangle \;,$$

or using (3.11)

$$\langle\, \phi|\phi\,\rangle \;=\; \lim_{z\to 0}\lim_{w\to\infty} \langle\, \emptyset \,|\, \frac{w^{2\Delta}}{(w-z)^{2\Delta}} \,|\, \emptyset\,\rangle \;=\; 1 \;.$$

Now, equation (3.14) implies

$$\lim_{z\to 0} \langle\, \emptyset \,|\, \phi^\dagger(z) \;=\; \lim_{z\to 0} \langle\, \emptyset \,|\, \phi(1/z) z^{-2\Delta} \;,$$

which in turn means that

$$\begin{aligned} \phi^\dagger(z) &= z^{-2\Delta}\, \phi(1/z) \\ &= z^{-2\Delta} \sum_n \phi_n\, z^{n+\Delta} \\ &= \sum_n \phi_n\, z^{n-\Delta} \\ &\overset{(n=-m)}{=} \sum_m \phi_{-m}\, z^{-m-\Delta} \;. \end{aligned} \tag{3.15}$$

Note that in the last line, we have relabeled using $m = -n$. Then, comparing equations (3.13) and (3.15), we arrive immediately at the result

$$\phi_n^\dagger \;=\; \phi_{-n} \;.$$

7. The Ward identity from L_{-1} for the correlation function G is

$$\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} \right) G \;=\; 0 \;,$$

which implies that G is a function of the differences $z_1 - z_2$, $z_2 - z_3$, and $z_3 - z_1$ only, i.e.,

$$G \;=\; G(z_1 - z_2,\; z_2 - z_3,\; z_3 - z_1) \;.$$

For simplicity, we will set

$$\alpha \;=\; z_1 - z_2 \;, \quad \beta \;=\; z_2 - z_3 \;, \quad \text{and} \quad \gamma \;=\; z_3 - z_1 \;,$$

so we can write G as a function $G(\alpha,\beta,\gamma)$. Subsituting this expression in the Ward identity for L_0, we find

$$\alpha\frac{\partial G}{\partial \alpha} + \beta\frac{\partial G}{\partial \beta} + \gamma\frac{\partial G}{\partial \gamma} \;=\; -\,(\Delta_1 + \Delta_2 + \Delta_3)\, G \;.$$

We can solve the last equation using the method of separation of variables:

$$G \;=\; A(\alpha)B(\beta)\Gamma(\gamma)\;.$$

Then

$$\frac{\alpha}{A}\frac{\partial A}{\partial\alpha}+\frac{\beta}{B}\frac{\partial B}{\partial\beta}+\frac{\gamma}{\Gamma}\frac{\partial\Gamma}{\partial\gamma} \;=\; -\left(\Delta_1+\Delta_2+\Delta_3\right)\;,$$

from which we conclude that

$$\begin{aligned}\frac{\alpha}{A}\frac{\partial A}{\partial\alpha} &= a\;,\\ \frac{\beta}{B}\frac{\partial B}{\partial\beta} &= b\;,\\ \frac{\gamma}{\Gamma}\frac{\partial\Gamma}{\partial\gamma} &= c\;,\end{aligned}$$

where a, b, and c are three constants such that

$$a+b+c \;=\; -\left(\Delta_1+\Delta_2+\Delta_3\right)\;.$$

Therefore

$$A \;=\; (\text{constant})\,\alpha^a\;,\quad B \;=\; (\text{constant})\,\beta^b\;,\quad \Gamma \;=\; (\text{constant})\,\gamma^c\;,$$

and so

$$G \;=\; (\text{constant})\,\alpha^a\,\beta^b\,\gamma^c\;.$$

Finally, to calculate the constants a, b, and c, we substitute the last expression in the Ward identity of L_1, which gives:

$$(a+c+2\Delta_1)z_1+(a+b+2\Delta_2)z_2+(b+c+2\Delta_3)z_3 \;=\; 0\;.$$

Since z_1, z_2, and z_3 are independent, we conclude that

$$a+c+2\Delta_1 \;=\; a+b+2\Delta_2 \;=\; b+c+2\Delta_3 \;=\; 0\;.$$

These algebraic equations are solved easily, yielding

$$a=\Delta_3-\Delta_1-\Delta_2\;,\quad b=\Delta_1-\Delta_2-\Delta_3\;,\quad \text{and}\quad c=\Delta_3-\Delta_1-\Delta_2\;.$$

So our final result reads

$$G \;=\; \frac{C_{ijk}}{(z_1-z_2)^{\Delta_1+\Delta_2-\Delta_3}\,(z_2-z_3)^{\Delta_2+\Delta_3-\Delta_1}\,(z_1-z_2)^{\Delta_1+\Delta_2-\Delta_3}}\;,$$

where C_{ijk} is a constant that depends only on the fields that enter in the correlation function. In fact, it follows that C_{ijk} is related to the OPE of the given fields by

$$\phi_i(z)\phi_j(w) \;=\; C_{ijk}\phi_k+\ldots$$

8. From the result of Exercise 4

$$G = \frac{1}{z_{12}^{\gamma_{12}} z_{13}^{\gamma_{13}} z_{14}^{\gamma_{14}} z_{23}^{\gamma_{23}} z_{24}^{\gamma_{24}} z_{34}^{\gamma_{34}}} Y(z_{ij}) ,$$

where

$$\begin{aligned} \gamma_{12} + \gamma_{13} + \gamma_{14} &= 2\Delta_1 , \\ \gamma_{21} + \gamma_{23} + \gamma_{24} &= 2\Delta_2 , \\ \gamma_{31} + \gamma_{32} + \gamma_{34} &= 2\Delta_3 , \\ \gamma_{41} + \gamma_{42} + \gamma_{43} &= 2\Delta_3 , \end{aligned}$$

and $\gamma_{ij} = \gamma_{ji}$.

The function Y must be homogeneous of degree 0; therefore, it can be written as a function of the cross ratios. There are only two independent cross ratios; without loss of generality, we choose

$$\frac{z_{12}\, z_{34}}{z_{13}\, z_{24}} \quad \text{and} \quad \frac{z_{14}\, z_{23}}{z_{13}\, z_{24}} .$$

It is a quick exercise to see that these two cross ratios sum to unity, that is, if we define

$$\frac{z_{12}\, z_{34}}{z_{13}\, z_{24}} \equiv x$$

then

$$\frac{z_{14}\, z_{23}}{z_{13}\, z_{24}} = 1 - x .$$

Thus, $Y = Y(x)$. The correlation function depends on the function $Y(x)$.

9. We define the chiral current

$$J(z) \equiv i\, \partial\phi(z) ,$$

which has conformal dimensions (1,0). Notice that

$$\langle J(z) J(0) \rangle = \frac{1}{z^2} ,$$

and therefore for large z,

$$J(z) \sim \frac{1}{z^2} .$$

We will use this information shortly.

Now observe that

$$
\begin{aligned}
\left\langle J(z) \prod_{j=1}^{n} e^{ia_j \phi(z_j)} \right\rangle &= \sum_k \left\langle J(z) e^{ia_k \phi(z_k)} \prod_{j \neq k} e^{ia_j \phi(z_j)} \right\rangle \\
&= \sum_k \frac{\alpha_k}{z - w_k} \left\langle \prod_{j=1}^{n} e^{ia_j \phi(z_j)} \right\rangle ,
\end{aligned}
$$

where we made use of Exercise 6 in Chapter 2.

For large z, the preceding result reads

$$
\begin{aligned}
0 \frac{1}{z} + \frac{1}{z^2} G + \mathcal{O}\left(\frac{1}{z^3}\right) &= \sum_k \frac{\alpha_k}{z} \left(1 - \frac{w_k}{z}\right)^{-1} G \\
&= \sum_k \frac{\alpha_k}{z} \left(1 - \frac{w_k}{z}\right)^{-1} G \\
&= \sum_k \frac{\alpha_k}{z} \left[1 + \frac{w_k}{z} + \left(\frac{w_k}{z}\right)^2 + \cdots \right] G \\
&= \frac{1}{z} \left(\sum_k \alpha_k\right) G + \mathcal{O}\left(\frac{1}{z^2}\right) .
\end{aligned}
$$

Comparing the two sides, we conclude that

$$
\left(\sum_k \alpha_k\right) G = 0 .
$$

Therefore G can only be non-vanishing when

$$
\sum_k \alpha_k = 0 .
$$

10. We first notice that

$$
\begin{aligned}
\partial_{z_i} G &= \left\langle e^{ia_1 \phi(z_1)} \cdots i a_1 : \partial_{z_i} \phi(z_i) e^{ia_i \phi(z_i)} : \cdots e^{ia_n \phi(z_n)} \right\rangle \\
&= a_i \left\langle e^{ia_1 \phi(z_1)} \cdots : J(z_i) e^{ia_i \phi(z_i)} : \cdots e^{ia_n \phi(z_n)} \right\rangle \\
&= \sum_{j \neq i} \frac{a_i a_j}{z_i - z_j} G .
\end{aligned}
$$

This first order differential equation is solved easily by multiplying both sides by

$$p \;=\; \exp\left(-\sum_{j\neq i}\int \frac{a_i a_j}{z_i - z_j}dz_i\right) \;=\; \prod_{j\neq i}(z_i - z_j)^{-a_i a_j}\ .$$

Then the differential equation reads

$$\partial_{z_i}(pG) \;=\; 0\ , \qquad \forall i\ ,$$

which gives

$$G \;=\; A\prod_{i<j}(z_i - z_j)^{a_i a_j}\ ,$$

where A is a constant. To calculate the constant, we consider the 2-point correlation function $\langle\, e^{ia\phi(z)}e^{ib\phi(w)}\,\rangle$, which can be calculated easily from standard identities in QFT:

$$\left\langle e^{ia\phi(z)}e^{ib\phi(w)}\right\rangle \;=\; e^{-ab\langle\,\phi(z)\phi(w)\,\rangle} \;=\; e^{ab\ln(z-w)} \;=\; (z-w)^{ab}\ .$$

Therefore, $A = 1$, and thus

$$G \;=\; \prod_{i<j}(z_i - z_j)^{a_i a_j}\ . \tag{3.16}$$

---

11. (a) Let $g^{(4)}$ be the holomorphic part of the correlation function. Given the state $|\,\sigma\,\rangle = \sigma(0)\,|\,\emptyset\,\rangle$, one can construct a null state

$$|\,\chi\,\rangle \;=\; \left(L_{-2} - \frac{4}{3}L_{-1}^2\right)|\,\sigma\,\rangle\ .$$

Then

$$\begin{aligned}
0 \;&=\; \langle\,\chi(z_1)\sigma(z_2)\ldots\sigma(z_{2M})\,\rangle\\
&=\; \left\langle\left(L_{-2} - \frac{4}{3}L_{-1}^2\right)\sigma(z_1)\sigma(z_2)\ldots\sigma(z_{2M})\right\rangle\\
&=\; \left(\mathcal{L}_{-2} - \frac{4}{3}\mathcal{L}_{-1}^2\right)\langle\,\sigma(z_1)\sigma(z_2)\ldots\sigma(z_{2M})\,\rangle\\
&\equiv\; \left(\mathcal{L}_{-2} - \frac{4}{3}\mathcal{L}_{-1}^2\right)g^{(4)}\ ,
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{L}_{-2} \;&=\; \sum_{j=2}^{2M}\left(\frac{1/16}{z_{1j}^2} + \frac{1}{z_{1j}}\frac{\partial}{\partial z_j}\right)\ \text{, and}\\
\mathcal{L}_{-1} \;&=\; -\sum_{j=2}^{2M}\frac{\partial}{\partial z_j}\ .
\end{aligned}$$

From the L_{-1} Ward identity, we know that

$$\sum_{j=1}^{2M} \frac{\partial g^{(4)}}{\partial z_j} = 0 \Rightarrow \frac{\partial g^{(4)}}{\partial z_1} = -\sum_{j=2}^{2M} \frac{\partial g^{(4)}}{\partial z_j} = \mathcal{L}_{-1} g^{(4)} \Rightarrow \frac{\partial}{\partial z_1} = \mathcal{L}_{-1} \; .$$

Inserting this last result in the previous expression, we find that

$$\frac{4}{3}\frac{\partial^2 g^{(4)}}{\partial z_1^2} - \sum_{j=2}^{2M} \left(\frac{1/16}{z_{1j}^2} + \frac{1}{z_{1j}} \frac{\partial}{\partial z_j} \right) g^{(4)} = 0 \; .$$

(b) Let

$$G^{(4)}(z_i, \overline{z}_i) \equiv \langle\, \sigma(z_1, \overline{z}_1)\, \sigma(z_2, \overline{z}_2)\, \sigma(z_3, \overline{z}_3)\, \sigma(z_4, \overline{z}_4) \,\rangle \; .$$

As we have discussed in Exercise 8, the conformal Ward identities determine the functional form of $G^{(4)}$ to be

$$G^{(4)}(z_i, \overline{z}_i) = \frac{1}{z_{12}^{\gamma_{12}}\, z_{13}^{\gamma_{13}}\, z_{14}^{\gamma_{14}}\, z_{23}^{\gamma_{23}}\, z_{24}^{\gamma_{24}}\, z_{34}^{\gamma_{34}}} \frac{1}{\overline{z}_{12}^{\overline{\gamma}_{12}}\, \overline{z}_{13}^{\overline{\gamma}_{13}}\, \overline{z}_{14}^{\overline{\gamma}_{14}}\, \overline{z}_{23}^{\overline{\gamma}_{23}}\, \overline{z}_{24}^{\overline{\gamma}_{24}}\, \overline{z}_{34}^{\overline{\gamma}_{34}}} Y(x, \overline{x}) \; ,$$

where $z_{ij} \equiv z_i - z_j$,

$$x \equiv \frac{z_{12}\, z_{34}}{z_{13}\, z_{24}} \; ,$$

and similar definitions for the conjugate (i.e., overbarred) quantities. In addition, we have

$$\begin{aligned} \sum_j{}' \gamma_{ij} &= 2\,\Delta_i = 1/8 \; , \text{and} \\ \sum_j{}' \overline{\gamma}_{ij} &= 2\,\overline{\Delta}_i = 1/8 \; . \end{aligned}$$

We can rewrite $G^{(4)}$ as

$$\begin{aligned} G^{(4)}(z_i, \overline{z}_i) &= \left(\frac{z_{13}\, z_{24}}{z_{12}\, z_{34}} \right)^{1/8} \frac{1}{z_{23}^{1/8}\; z_{41}^{1/8}} \left(\frac{\overline{z}_{13}\, \overline{z}_{24}}{\overline{z}_{12}\, \overline{z}_{34}} \right)^{1/8} \frac{1}{\overline{z}_{23}^{1/8}\; \overline{z}_{41}^{1/8}} F(x, \overline{x}) \\ &= \left(\frac{1}{x\, z_{23}\, z_{41}} \right)^{1/8} \left(\frac{1}{\overline{x}\, \overline{z}_{23}\, \overline{z}_{41}} \right)^{1/8} F(x, \overline{x}) \; . \end{aligned} \tag{3.17}$$

The holomorphic part $g^{(4)}$ of $G^{(4)}$ satisfies the differential equation

$$-\frac{4}{3}\frac{\partial^2 g^{(4)}}{\partial z_1^2} + \sum_{j=2}^{3} \left(\frac{1/16}{z_{1j}^2} + \frac{1}{z_{1j}} \frac{\partial}{\partial z_j} \right) g^{(4)} = 0 \; . \tag{3.18}$$

There is a similar equation for the antiholomorphic part $\overline{g}^{(4)}(\overline{x})$ of $G^{(4)}$.

If we use (3.17) to substitute for

$$g^{(4)} \;=\; (x\, z_{23}\, z_{41})^{-1/8}\, f(x)$$

in (3.18), we find the second order differential equation

$$x\,(1\,-\,x)\,\frac{d^2 f}{dx^2}\;+\;\left(\frac{1}{2}\,-\,x\right)\,\frac{df}{dx}\;+\;\frac{1}{16}\,f\;=\;0\;, \tag{3.19}$$

which is the differential equation sought.

(b) Compare[2] the last equation (3.19) with the well-known hypergeometric differential equation

$$x\,(1\,-\,x)\,\frac{d^2}{dx^2}\;+\;[c\,-\,(1\,+\,a\,+\,b)\,x]\,\frac{dF}{dx}\;-\;a\,b\,F\;=\;0\;.$$

We immediately see that (3.19) is a special case of the hypergeometric equation for which

$$c = 1/2 \quad \text{and} \quad a = -b = 1/4\;.$$

Therefore, the solution to our differential equation can be expressed in terms of the hypergeometric function by

$$\begin{aligned} f(x) \;&=\; A\,F(a,\,b;\,c;\,x)\;+\;B\,x^{1-c}\,F(a+1-c,\,b+1-c;\,2-c;\,x) \\ &=\; A\,F\left(\frac{1}{4},\,-\frac{1}{4};\,\frac{1}{2};\,x\right)\;+\;B\,x^{1/2}\,F\left(\frac{3}{4},\,\frac{1}{4};\,\frac{3}{2};\,x\right)\;, \end{aligned}$$

where A and B are constants. Recall that the hypergeometric function can be expanded as

$$F(a,\,b;\,c;\,x)\;\equiv\;\sum_{n=0}^{+\infty}\frac{1}{n!}\,\frac{(a)_n\,(b)_n}{(c)_n}\,z^n\;,$$

where $(a)_n$ is the **Barnes symbol**:

$$(a)_n\;\equiv\;a\,(a\,+\,1)\,\ldots\,(a\,+\,n\,-\,1)\;=\;\frac{\Gamma(a+n)}{\Gamma(a)}\;.$$

[2]One can solve (3.19) by making the substitution $x = \sin^2\theta$, in which case one finds

$$\frac{d^2 f}{d^2\theta} + \frac{1}{4}f \;=\; 0\;.$$

This equation has the obvious solution

$$f \;=\; A\cos\frac{\theta}{2} + B\sin\frac{\theta}{2}\;.$$

However, we present yet another, longer but completely direct approach to solve (3.19) since, if one does not know the final answer, it would require a terrible amount of imagination and experience to write down this transformation.

Two standard properties of the hypergeometric function [3] are

$$\begin{aligned} F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2\theta\right) &= \cos n\theta \ , \\ F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{3}{2}; \sin^2\theta\right) &= \frac{\sin n\theta}{n\sin\theta} \ . \end{aligned}$$

In particular, setting $n = 1/2$,

$$\begin{aligned} F\left(\frac{1}{4}, -\frac{1}{4}; \frac{1}{2}; \sin^2\theta\right) &= \cos\frac{\theta}{2} \ , \quad \text{and} \\ F\left(\frac{3}{4}, \frac{1}{4}; \frac{3}{2}; \sin^2\theta\right) &= \frac{1}{\cos\frac{\theta}{2}} \ . \end{aligned}$$

If we write

$$\sin^2\theta \equiv x \ ,$$

then

$$f(x) = A\cos\frac{\theta}{2} + B'\sin\frac{\theta}{2} \ .$$

Notice that

$$\begin{aligned} \cos\frac{\theta}{2} &= \sqrt{1 - \sin^2\frac{\theta}{2}} = \sqrt{1 - \frac{1-\cos\theta}{2}} = \frac{1}{\sqrt{2}}\sqrt{1 + \cos\theta} \\ &= \frac{1}{\sqrt{2}}\sqrt{1 + \sqrt{1 - \sin^2\theta}} = \frac{1}{\sqrt{2}}\sqrt{1 + \sqrt{1-x}} \end{aligned}$$

and

$$\sin\frac{\theta}{2} = \frac{1}{\sqrt{2}}\sqrt{1 - \sqrt{1-x}} \ .$$

Therefore

$$f(x) = \alpha f_1(x) + \beta f_2(x) \ ,$$

where

$$f_1(x) \equiv \sqrt{1 + \sqrt{1-x}} \ , \quad \text{and} \quad f_2(x) \equiv \sqrt{1 - \sqrt{1-x}} \ .$$

If we restore the antiholomorphic part, $G^{(4)}$ can be written as

$$G^{(4)}(z_i, \overline{z}_i) = \left|\frac{1}{x\, z_{23}\, z_{41}}\right|^{1/4} \sum_{i,j=1}^{2} \alpha_{ij}\, f_i(x)\, f_j(\overline{x}) \ .$$

Since the field $\sigma(z,\overline{z})$ is local, $G^{(4)}(z_i, \overline{z}_i)$ must be single-valued. This condition is better seen using the θ, $\overline{\theta}$ variables. Under the substitution

$$\theta \to -\theta \quad , \quad \overline{\theta} \to -\overline{\theta} \ , \tag{3.20}$$

$G^{(4)}(z_i, \overline{z}_i)$ is single-valued if there are not cross terms, *i.e.*

$$G^{(4)}(z_i,\, \overline{z}_i) \;=\; \left|\frac{1}{x\, z_{23}\, z_{41}}\right|^{1/4} \left(a\,|f_1(x)|^2 \,+\, b\,|f_2(x)|^2\right) \quad .$$

Actually, the substitution (3.20) takes into account the analytical continuation only around the point $(x = 0, \overline{x} = 0)$. However, the differential equation (3.19) has another singular point, namely $(x = 1, \overline{x} = 1)$, which corresponds to $(\theta = \pi/2, \overline{\theta} = \pi/2)$. Single-valuedness here implied that $a = b$, i.e. that

$$G^{(4)}(z_i,\, \overline{z}_i) \;=\; \left|\frac{1}{x\, z_{23}\, z_{41}}\right|^{1/4} a\,\left(|f_1(x)|^2 \,+\, |f_2(x)|^2\right) \;. \tag{3.21}$$

Finally, to calculate the coefficient a we proceed as follows: For $z_1 \simeq z_2$ and $z_3 \simeq z_4$, equation (3.21) gives

$$G^{(4)}(z_i,\, \overline{z}_i) \;\sim\; \frac{2a}{|z_{12}|^{1/4}\, |z_{34}|^{1/4}} \;. \tag{3.22}$$

On the other hand, using the OPE

$$\sigma(z_1,\, \overline{z}_1)\, \sigma(z_2,\, \overline{z}_2) \;\sim\; \frac{1}{|z_{12}|^2} \;,$$

we can find the singular part of $G^{(4)} = \langle\, \sigma\sigma\sigma\sigma\,\rangle$ independently, obtaining

$$G^{(4)}(z_i,\, \overline{z}_i) \;\sim\; \frac{1}{|z_{12}|^{1/4}\, |z_{34}|^{1/4}} \;. \tag{3.23}$$

Comparing (3.22) and (3.23), we conclude that $a = 1/2$. Thus the final result is

$$G^{(4)}(z_i,\, \overline{z}_i) \;=\; \frac{1}{2}\left|\frac{1}{x\, z_{23}\, z_{41}}\right|^{1/4} \left(|f_1(x)|^2 \,+\, |f_2(x)|^2\right) \;.$$

12. We can use the OPE in general to write

$$\phi_n(z, \overline{z})\, \phi_m(0,0) \;=\; \sum_p c^p_{nm}\, \chi^p_{nm}(z)\, \overline{\chi}^p_{nm}(\overline{z})\, z^{\Delta_p - \Delta_n - \Delta_m}\, \overline{z}^{\overline{\Delta}_p - \overline{\Delta}_n - \overline{\Delta}_m}\, \phi_p(0,0) \;,$$

where

$$\chi^p_{nm}(z) \;\equiv\; \sum_{\{k\}} \beta^{p_j\,\{k\}}_{nm}\, z^{\sum k_i}\, L_{-k_1} \ldots L_{-k_n} \;\cdots \tag{3.24}$$

In the following, we will drop the antiholomorphic part, and concentrate on

$$\phi_n(z)\,\phi_m(0) \;=\; \sum_p c^p_{nm}\,\chi^p_{nm}(z)\,z^{\Delta_p-\Delta_n-\Delta_m}\,\phi_p(0)\ . \tag{3.25}$$

Since

$$\phi_n(0)\,|\,\emptyset\,\rangle \;=\; |\,\Delta_n\,\rangle\ ,$$

applying (3.25) to the vacuum $|\,\emptyset\,\rangle$, we get

$$\phi_n(z)\,|\,\Delta_m\,\rangle = \sum_p c^p_{nm}\,\chi^p_{nm}(z)\,z^{\Delta_p-\Delta_n-\Delta_m}\,|\,\Delta_p\,\rangle\ . \tag{3.26}$$

We define

$$|\,z,\,\Delta_p;\,nm\,\rangle \equiv \chi^p_{nm}(z)\,|\,\Delta_p\,\rangle\ . \tag{3.27}$$

Then for any L_s, $s>0$ we get

$$\begin{aligned}
[L_s,\,\phi_n(z)]\,|\,\Delta_m\,\rangle \;&=\; L_s\phi_n(z)\,|\,\Delta_m\,\rangle - \phi_n(z)L_s\,|\,\Delta_m\,\rangle \;=\; L_s\phi_n(z)\,|\,\Delta_m\,\rangle\\
&= \sum_p c^p_{nm}\,z^{\Delta_p-\Delta_n-\Delta_m}\,L_s\,|\,z,\,\Delta_p;\,nm\,\rangle\\
\Rightarrow \left[z^{s+1}\partial_z \,+\, (s+1)\,\Delta_n\,z^s\right]\,\phi_n(z)\,|\,\Delta_m\,\rangle &= \sum_p c^p_{nm}\,z^{\Delta_p-\Delta_n-\Delta_m}\,L_s\,|\,z,\,\Delta_p;\,nm\,\rangle\\
\Rightarrow \left[z^{s+1}\partial_z \,+\, (s+1)\,\Delta_n\,z^s\right]\,&\sum_p c^p_{nm}\,z^{\Delta_p-\Delta_n-\Delta_m}\,|\,z,\,\Delta_p;\,nm\,\rangle\\
&= \sum_p c^p_{nm}\,z^{\Delta_p-\Delta_n-\Delta_m}\,L_s\,|\,z,\,\Delta_p;\,nm\,\rangle\\
\Rightarrow \left[z_s\,(\Delta_p \,-\, \Delta_n \,-\, \Delta_m \,+\, z^{s+1}\partial_z \,+\, (s+1)\,\Delta_n\,z^s\right]\,&|\,z,\,\Delta_p;\,nm\,\rangle\\
&= L_s\,|\,z,\,\Delta_p;\,nm\,\rangle\ .
\end{aligned} \tag{3.28}$$

These equations determine the state $|\,z,\,\Delta_p;\,nm\,\rangle$. Actually, the first two equations, with $s=1$ and $s=2$, are sufficient to determine the state (this arises from the fact that the L's have to obey the Virasoro algebra). If we could determine the action of L_s on $|\,z,\,\Delta_p;\,nm\,\rangle$ and solve (3.28), then we would be able to find the coefficients $\beta^{p;\,\{k\}}_{nm}$ using (3.24) and (3.27). Since we cannot proceed in this direct way, we instead introduce the Taylor expansion

$$|\,z,\,\Delta_p;\,nm\,\rangle \equiv \sum_{N=0}^{\infty} z^N\,|\,N,\,\Delta_p;\,nm\,\rangle\ . \tag{3.29}$$

We will show that the solution can now be approximated up to any chosen order. From the definitions (3.24), (3.27), and (3.29), we see that

$$L_0\,|\,N,\,\Delta_p;\,nm\,\rangle \;=\; (\Delta_p \,+\, N)\,|\,N,\,\Delta_p;\,nm\,\rangle\ .$$

Substituting (3.29) in (3.28), we find

$$[N + (s+1)\Delta_n + \Delta_p - \Delta_n - \Delta_m] \, | \, N, \Delta_p; nm \rangle = L_s \, | \, N, \Delta_p; nm \rangle \ .$$

Therefore

$$\begin{aligned} L_s \, | \, N, \Delta_p; nm \rangle &= 0 \ , \qquad N < s \\ L_s \, | \, N+s, \Delta_p; nm \rangle &= [N + s\Delta_n + \Delta_p - \Delta_m] \, | \, N, \Delta_p; nm \rangle \ . \end{aligned}$$

We have already mentioned that we need only the equations for $s = 1$ and $s = 2$, which are now

$$L_1 \, | \, N+1 \rangle = (N + \Delta_n + \Delta_p - \Delta_m) \, | \, N \rangle \ , \quad \text{and} \tag{3.30}$$

$$L_2 \, | \, N+2 \rangle = (N + 2\Delta_n + \Delta_p - \Delta_m) \, | \, N \rangle \ . \tag{3.31}$$

where we have simplified the notation for convenience.

Let us now compute the state $| \, z, \Delta_p; nm \rangle$ up to second order, i.e., determine the coefficients of the following expansion:

$$| \, z, \Delta_p; nm \rangle = \left[1 + z \, \beta_{nm}^{p;1} \, L_{-1} + z^2 \, (\beta_{nm}^{p;1,1} \, L_{-1}^2 + \beta_{nm}^{p;2} \, L_{-2}) + \ldots\right] | \, \Delta_p \rangle \ .$$

This amounts to writing the states $| \, N = 0 \rangle$, $| \, N = 1 \rangle$, and $| \, N = 2 \rangle$ as linear combinations of states from the Verma module of $| \, \Delta_p \rangle$. One finds

$$\begin{aligned} | \, N = 0 \rangle &= | \, \Delta_p \rangle \ , \\ | \, N = 1 \rangle &= \alpha L_{-1} \, | \, \Delta_p \rangle \ , \\ | \, N = 2 \rangle &= (\gamma L_{-1}^2 + \delta L_{-2}) \, | \, \Delta_p \rangle \ , \end{aligned}$$

where

$$\alpha \equiv \beta_{nm}^{p;1} \ , \quad \gamma \equiv \beta_{nm}^{p;1,1} \ , \quad \text{and} \quad \delta \equiv \beta_{nm}^{p;2} \ .$$

We present the calculation explicitly. Equation (3.30) for $N = 0$ implies

$$\begin{aligned} (\Delta_n + \Delta_p - \Delta_m) \, | \, \Delta_p \rangle &= L_1 \, | \, N = 1 \rangle \\ &= \alpha \, L_1 \, L_{-1} \, | \, \Delta_p \rangle \\ &= \alpha \, [L_1, L_{-1}] \, | \, \Delta_p \rangle \\ &= \alpha \, 2L_0 \, | \, \Delta_p \rangle \\ &= 2\alpha\Delta_p \, | \, \Delta_p \rangle \ . \end{aligned}$$

Therefore,

$$\alpha = \frac{\Delta_n + \Delta_p - \Delta_m}{2\Delta_p} \ .$$

From (3.31) for $N = 0$, we get

$$\begin{aligned}(2\Delta_n + \Delta_p - \Delta_m)\,|\,\Delta_p\,\rangle &= L_2\,|\,2\,\rangle \\ &= L_2\,(\gamma\,L_{-1}^2 + \delta\,L_{-2})\,|\,\Delta_p\,\rangle \\ &= (\gamma\,\left[L_2,\,L_{-1}^2\right] + \delta\,[L_2,\,L_{-2}])\,|\,\Delta_p\,\rangle \\ &= \left[\gamma\,6\Delta_p + \delta\,(4\Delta_p + \frac{c}{2})\right]\,|\,\Delta_p\,\rangle\ ,\end{aligned}$$

or, in other words,

$$6\Delta_p\,\gamma + \left(4\Delta_p + \frac{c}{2}\right)\,\delta\ =\ 2\Delta_n + \Delta_p - \Delta_m\ . \tag{3.32}$$

Again, from (3.30), but now for $N = 1$, we have

$$\begin{aligned}(\Delta_n + \Delta_p - \Delta_m + 1)\,|\,1\,\rangle &= L_1\,|\,2\,\rangle \\ \Rightarrow (\Delta_n + \Delta_p - \Delta_m + 1)\,\alpha\,L_{-1}\,|\,2\,\rangle &= (\gamma\,\left[L_1,\,L_{-1}^2\right] + \delta\,[L_1,\,L_{-2}])\,|\,\Delta_p\,\rangle \\ &= L_1(\gamma L_{-1}^2 + \delta L_{-2})\,|\,\Delta_p\,\rangle \\ &= [\gamma\,2(2\Delta_p + 1) + 3\delta)]\,L_{-1}\,|\,\Delta_p\,\rangle \\ \Rightarrow 2(2\Delta_p + 1)\,\gamma + 3\delta &= (\Delta_n + \Delta_p - \Delta_m + 1)\,\alpha\ . \end{aligned} \tag{3.33}$$

The system of (3.32) and (3.33) can be solved explicitly for γ and δ. This calculation gives

$$\begin{aligned}\gamma &= \frac{(4\Delta_p + \frac{c}{2})\,(\Delta_n + \Delta_p - \Delta_m + 1)\,(\Delta_n + \Delta_p - \Delta_m)\ -\ (6\Delta_n + 3\Delta_p - 3\Delta_m)\,2\Delta_p}{[2\Delta_p\,(8\Delta_p - 5)\ +\ c\,(1 + 2\Delta_p)]\ 2\Delta_p} \\ \delta &= \frac{(\Delta_m + \Delta_n - \Delta_p)\ +\ 2(\Delta_n\Delta_p + \Delta_m\Delta_p + 3\Delta_n\Delta_m) - (3\Delta_n^2 + 3\Delta_m^2 - \Delta_p^2)}{2\,\Delta_p\,(8\Delta_p - 5)\ +\ c\,(1 + 2\Delta_p)}\end{aligned}$$

Now it is clear how we can obtain all the coefficients $\beta_{nm}^{p;\{k\}}$ up to n-th order. All we have to do is to solve a linear algebraic system of $P(n)$-th order. Then we obtain the initial coefficients $c_{nm}^{p;\{k\}\{\bar{k}\}}$ by using equation (3.12). This method is straightforward in principle, but discouraging in practice!

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13. In the amplitude expressed as

$$A\ =\ \int dz_3\ \langle\, c(z_1)c(z_2)c(z_4)V_1V_2V_3V_4\,\rangle\ \ =\ \int dz_3\ \langle\, c(z_1)c(z_2)c(z_4)\,\rangle\ \langle\, V_1V_2V_3V_4\,\rangle\ ,$$

one can calculate the correlation functions inside the integral easily by using the bosonized version of $c(z)$ and the result of the problem 10. This enables us to re-write the amplitude as

$$A\ =\ \int dz_3\ z_{12}z_{14}z_{24}\prod_{i<j} e^{k_i\cdot k_j\,\ln z_{ij}}\ \ =\ \int dz_3\ z_{12}z_{14}z_{24}\prod_{i<j} z_{ij}^{k_i\cdot k_j}\ .$$

Since the result should not depend on the positions at which we have placed the ghosts, we can choose, without loss of generality,

$$z_1 \to \infty \ , \quad z_2 = 1 \ , \quad \text{and} \quad z_4 = 0 \ .$$

Then

$$\begin{aligned} P &\equiv z_{12}^{k_1\cdot k_2+1} z_{13}^{k_1\cdot k_3} z_{14}^{k_1\cdot k_4+1} z_{23}^{k_2\cdot k_3} z_{24}^{k_2\cdot k_4+1} z_{34}^{k_3\cdot k_4} \\ &\overset{z_2=1, z_4=0}{=} (z_1-1)^{k_1\cdot k_2+1} z_{13}^{k_1\cdot k_3} z_1^{k_1\cdot k_4+1} (1-z_3)^{k_2\cdot k_3} 1^{k_2\cdot k_4+1} z_3^{k_3\cdot k_4} \ . \end{aligned}$$

Now momentum conservation gives

$$k_3 \ = \ -\left(k_1 + k_2 + k_4\right) \ .$$

Using this value of k_3 in the exponent of z_{13} in P, we find

$$\begin{aligned} P &= (z_1-1)^{k_1\cdot k_2+1} z_{13}^{-k_1\cdot(k_1+k_2+k_4)} z_1^{k_1\cdot k_4+1} (1-z_3)^{k_2\cdot k_3} z_3^{k_3\cdot k_4} \\ &= \frac{(z_1-1)^{k_1\cdot k_2+1} z_1^{k_1\cdot k_4+1}}{z_{13}^{k_1^2+k_1\cdot k_2+k_1\cdot k_4}} (1-z_3)^{k_2\cdot k_3} z_3^{k_3\cdot k_4} \\ &\overset{z_1\to\infty}{=} (1-z_3)^{k_2\cdot k_3}\, z_3^{k_3\cdot k_4} \ , \end{aligned}$$

where we have used the tachyon condition $k_1^2 = 2$. As a result, we end up with the **Veneziano amplitude**, namely

$$A \ = \ \int_0^1 dz_3 \ (1-z_3)^{k_2\cdot k_3}\, z_3^{k_3\cdot k_4} \ = \ B(k_2\cdot k_3+1, k_3\cdot k_4+1) \ ,$$

where $B(x,y)$ stands for the well-known (beta) B-function.

Chapter 4

OTHER MODELS IN CFT

References: Orbifolds are reviewed in [333], affine algebras in [333, 297]. For information on the WZWN model, see [297, 629]. A review on the WZWN model from the functional integral point of view is [319].

4.1 BRIEF SUMMARY

4.1.1 Orbifolds

Let M be a manifold and Γ a discrete group acting on M, i.e., a map $\sigma : \Gamma \times M \to M$ is given. One generally writes this as a group element γ acting on a point x of the manifold. A point $x \in M$ is a fixed point of this map provided that there is a $\gamma_0 \in \Gamma$ such that $\gamma_0 \cdot x = x$ and $\gamma_0 \neq \mathrm{id}$. The orbit of x is the set $\Gamma \cdot x$ generated by the action of all the elements of Γ on x. Then the quotient space $O = M/\Gamma$ is the set of all distinct orbits of M under Γ. Such a quotient space is called an **orbifold** (from **orbi**t mani**fold**). When the action σ has no fixed points, then O is an ordinary manifold. When σ has fixed points, then O is not exactly an ordinary manifold, because of the behavior at the fixed points when the quotient space is constructed.

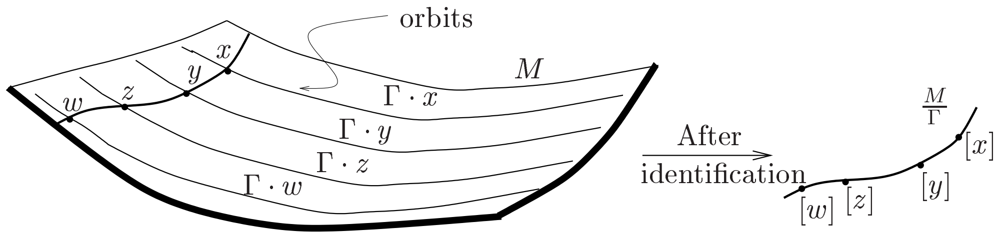

Figure 4.1: The quotient space of a manifold M by the group Γ. For ease of visualization, we have drawn continuous (instead of discrete) orbits.

The concept of an orbifold thus generalizes that of a manifold. This is relevant for our purposes, as one can define CFTs on orbifolds. In particular, given a modular invariant conformal field theory $\mathcal{T}$ that admits a discrete symmetry group Γ, one can construct an orbifold CFT (which is modular invariant) by taking the original CFT $\mathcal{T}$ and modding out by Γ.

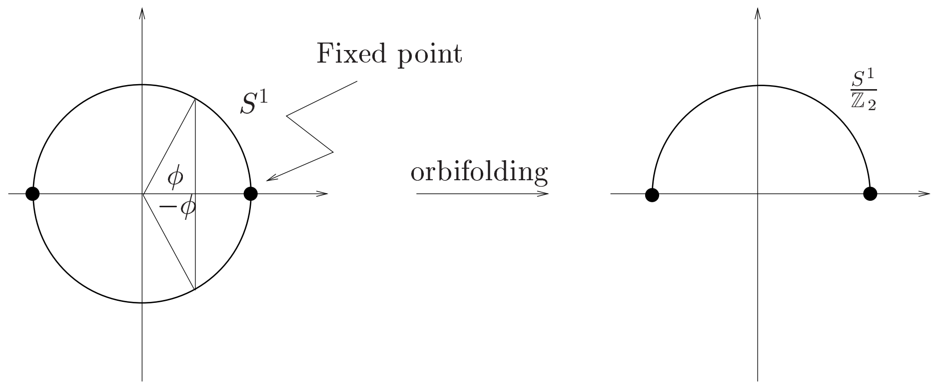

Figure 4.2: The orbifold obtained from the circle by modding out by a $\mathbb{Z}_2$ reflection group. The group $\mathbb{Z}_2$ identifies points that are symmetric with respect to the horizontal axis, and therefore the map has two fixed points. Compare this action with the action in $\mathbb{R}P^1 = S^1/\mathbb{Z}_2 \simeq S^1$, where $\mathbb{Z}_2$ identifies points symmetric with respect to origin of the coordinate system. In this case, the action on S^1 has no fixed points and the quotient space is a manifold, namely S^1.

One can adopt the notation

$$Z_{\mathcal{T}} = \square$$

to denote the partition function of $\mathcal{T}$. Then the partition function of $\mathcal{T}/\Gamma$ is given by

$$Z_{\mathcal{T}/\Gamma} = \frac{1}{|\Gamma|} \sum_{\substack{g,h\in\Gamma \\ gh=hg}} g\,\underset{h}{\square} \ .$$

4.1.2 WZWN Model

Given a group manifold G, we can define a corresponding σ-model, which is a theory with action

$$S_0 = \frac{k}{16\pi} \iint d^2x \,\mathrm{tr}\left(\partial_a g(x) \partial^a g^{-1}(x)\right) \ ,$$

where $g(x)$ is a field that takes its values in the group G. This action is not conformally invariant. However, one can make the theory conformally invariant by adding a **Wess-Zumino term**, leading to the action

$$S = S_0 + \frac{k}{24} \iiint d^3y\, f^{abc}\,\mathrm{tr}\left[\left(g(y)\partial_a g^{-1}(y)\right)\left(g(y)\partial_b g^{-1}(y)\right)\left(g(y)\partial_c g^{-1}(y)\right)\right] \ , \qquad (4.1)$$

where k is now an integer. The Wess-Zumino term, which has been added to the action, is topological. The integral is over a 3-dimensional space which has the physical two-dimensional space as its boundary; the Wess-Zumino term is the integral of a total derivative, and so ultimately only depends on the physical two-dimensional space, not on the particular extension into the three-dimensional space. Single-valuedness of the action leads to the requirement that k be an integer.

The model (4.1) has a local symmetry $\mathrm{G} \otimes \overline{\mathrm{G}}$ which acts on the fields as[1]

$$g(z,\overline{z}) \mapsto \Omega(z) g(z,\overline{z}) \overline{\Omega}(\overline{z}) \ , \qquad \Omega \in \mathrm{G},\ \overline{\Omega} \in \overline{\mathrm{G}} \ .$$

This symmetry is generated by the currents

$$\begin{aligned} J^a(z) &= -\frac{k}{2} \, (\partial^a g) \, g^{-1} \quad \text{and} \\ \overline{J}^a(z) &= -\frac{k}{2} \, g^{-1} \, (\partial^a g) \ . \end{aligned}$$

One has the associated OPEs

$$J^a(z) J^b(w) \ = \ \frac{k/2 \, \delta^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c \, J^c(w)}{z-w} + \text{reg} \ .$$

The modes J^a_n of $J^a(w)$,

$$J^a(z) \ = \ \sum_n \frac{J^a_n}{z^{n+1}} \ ,$$

satisfy an affine $\hat{\mathfrak{g}}$ Kac-Moody algebra with level k:

$$\left[J^a_m, J^b_n \right] = i \, f^{ab}{}_c \, J^c_{m+n} + \frac{k}{2} m \, \delta^a \, \delta_{m+n,0} \ .$$

The energy-momentum tensor is

$$T(z) = \frac{1}{k + \tilde{h}_{\mathfrak{g}}} \ \ddagger \, J_a(z) J_a(z) \, \ddagger \ ,$$

where $\ddagger \ \ddagger$ denotes normal ordering with respect to the modes of $J^a(z)$, and $\tilde{h}_{\mathfrak{g}}$ is the dual Coxeter number of $\hat{\mathfrak{g}}$. Unitarity of the model requires that k be a positive integer (just as single-valuedness of the action exponential does). From the OPE of the energy-momentum tensor, we find that

$$c \ = \ \frac{k \, \dim \mathfrak{g}}{k + \tilde{h}_{\mathfrak{g}}} \ .$$

The heighest weight representations at level k are labeled by a vector $\boldsymbol{p}$ in the set

$$\{ \boldsymbol{p} \in P \mid \boldsymbol{p} \cdot \boldsymbol{\omega}_i \geq 0 \ , \ \boldsymbol{p} \cdot \boldsymbol{\rho} \leq k \ \} \ ,$$

where P is the weight lattice of $\hat{\mathfrak{g}}$, $\{\boldsymbol{\omega}_i\}$ the corresponding fundamental weights, and $\boldsymbol{\rho}$ half the sum of the positive roots in $\hat{\mathfrak{g}}$. The primary fields have dimensions

$$\Delta_{\boldsymbol{p}} \ = \ \frac{\boldsymbol{p} \cdot (\boldsymbol{p} + 2\boldsymbol{\rho})}{k + \tilde{h}_{\mathfrak{g}}} \ .$$

[1]The groups G and $\overline{\mathrm{G}}$ are isomorphic; we use the overbar simply to indicate which side each group acts from.

4.1.3 Parafermions

The CFT UMM(3) is probably the simplest of the MMs; it includes a fermion field ψ with weight $(1/2, 0)$, a spin variable σ, and a disorder variable μ. Together with the energy-momentum tensor T, they build an algebra that enjoys a $\mathbb{Z}_2$ symmetry. This model is related to the Ising model of statistical physics.

One expects, correctly, that a similar construction is possible with other cyclic groups. Indeed, an algebra that enjoys an analogous $\mathbb{Z}_3$ symmetry is constructed as follows. We start with two fields ψ and $\psi^\dagger$ which both have weight $(2/3, 0)$; we call these **parafermion** fields. Although such objects are non-local, together with the energy-momentum tensor, they can be used to build a local CFT algebra:

$$\begin{aligned}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots \ , \\
T(z)\psi(w) &= \frac{(2/3)\psi(w)}{(z-w)^2} + \frac{\partial_w \psi(w)}{z-w} + \cdots \ , \\
\psi(z)\psi(w) &= \frac{a\psi(w)}{(z-w)^{2/3}} + \cdots \ , \\
\psi(z)\psi^\dagger(w) &= \frac{1}{(z-w)^{4/3}} \left(1 + \frac{8}{9c} T(w) + \cdots \right) \ .
\end{aligned}$$

Associativity fixes the values of the constants in this algebra to $c = 4/5$ and $a = 2/\sqrt{3}$. Just as the $\mathbb{Z}_2$ model is related to the Ising model, this model is related to the 3-state Potts model of statistical mechanics.

The above two cases are the first two models in the series of parafermionic models. The series is parametrized by the natural number $N \in \mathbb{N} \setminus \{0, 1\}$. The N-th model enjoys a $\mathbb{Z}_N$ symmetry similar to the previous cases. In particular, the N-th model contains $N-1$ parafermions $\psi_1, \ldots, \psi_{N-1}$ which are primary fields with conformal weights

$$\Delta_k = \frac{k(N-k)}{N} \ , \qquad k = 1, 2, \ldots, N-1 \ ,$$

which thus satisfy

$$T(z)\psi_k(w) = \frac{\Delta_k \psi_k(w)}{(z-w)^2} + \frac{\partial_w \psi_k(w)}{z-w} + \cdots \ .$$

These fields also obey the conjugation relation

$$\psi_k^\dagger = \psi_{N-k} \ .$$

The defining OPEs of the algebra are:

$$\begin{aligned}
\psi_k(z)\psi_{k'}(w) &= \frac{c_{kk'}}{(z-w)^{2kk'/N}} \left[\psi_{k+k'}(w) + \mathcal{O}(z-w) \right] \ , \quad k + k' < N \ , \\
\psi_k(z)\psi_{k'}^\dagger(w) &= \frac{c_{k,N-k'}}{(z-w)^{2k(N-k')/N}} \left[\psi_{k-k'}(w) + \mathcal{O}(z-w) \right] \ , \quad k < k' \ , \\
\psi_k(z)\psi_k^\dagger(w) &= \frac{1}{(z-w)^{2k(N-k)/N}} \left[I + \frac{2\Delta_k}{c} (z-w)^2 T(w) + \mathcal{O}\left((z-w)^3\right) \right] \ .
\end{aligned}$$

Demanding associativity determines the central charge; its value is

$$c = \frac{2(N-1)}{N+2} .$$

The model also contains a set of N spin fields σ with weights

$$\Delta(\sigma_k) = \frac{l(N-l)}{2N(N+2)} , \qquad l = 0, 1, \ldots, N-1 .$$

The theory includes N different sectors differentiated according to the modes A_n of ψ_1:

$$\psi_1 = \sum_n \frac{A_n}{z^{n+\frac{k(N-k)}{N}}} .$$

One of the most interesting results of the parafermionic models is their relation to the $\mathrm{SU}(2)_N$ WZWN model. One can show that (as we will discuss in the problems)

$$\boxed{\mathrm{SU}(2)_N \text{ WZWN model} \simeq \mathbb{Z}_N \text{ parafermion model} \otimes \text{free boson}} \tag{4.2}$$

The map between the two sides of this equivalence is given by:

$$\begin{aligned} J_+(z) &= \psi_1(z) :e^{i\sqrt{\frac{2}{N}}\phi(z)}: , \\ J_-(z) &= \psi_1^\dagger(z) :e^{-i\sqrt{\frac{2}{N}}\phi(z)}: , \\ H(z) &= \sqrt{N}\,\partial\phi(z) . \end{aligned}$$

The primary fields of the WZWN model $G_l(z,\overline{z})$ are constructed as products of the spin fields $\sigma_l(z,\overline{z})$ and the vertex operators of the free boson by

$$G_l(z,\overline{z}) = \sigma_l(z,\overline{z}) :e^{i\frac{l}{2\sqrt{N}}\phi(z)}: .$$

Therefore any correlation function of WZWN fields will factorize into a correlation fuction of spin fields in the parafermion model and a correlation function of vertex operators for a free boson:

$$\begin{aligned} \langle G_{l_1}(z_1,\overline{z}_1) G_{l_2}(z_2,\overline{z}_2) \ldots G_{l_n}(z_n,\overline{z}_n) \rangle &= \langle \sigma_{l_1}(z_1,\overline{z}_1)\sigma_{l_2}(z_2,\overline{z}_2) \ldots \sigma_{l_n}(z_n,\overline{z}_n) \rangle \\ &\quad \times \langle V_{l_1}(z_1,\overline{z}_1) V_{l_2}(z_2,\overline{z}_2) \ldots V_{l_n}(z_n,\overline{z}_n) \rangle . \end{aligned}$$

This identity can be used for the computation of all correlators in the parafermionic theory.

4.2 EXERCISES

1. We say that two elements $g, h \in \Gamma$ are conjugate, and write $g \sim h$, if there is a $k \in \Gamma$ such that $g = khk^{-1}$. The set

$$[g] \;\equiv\; \{h \in \Gamma \mid h \sim g\}$$

is called the **conjugacy class** of g. Let $N(g)$ be the set

$$N(g) \;\equiv\; \{h \in \Gamma \mid hg = gh\} \;.$$

This set is called the **stabilizer group** or the **isotropy group** of g.

Show that

$$|\Gamma| \;=\; |N(g)|\,|\,[g]\,| \;, \qquad \forall g \;,$$

where the notation $|S|$ stands for the number of elements in S. Then show that the partition function of an orbifold can be rewritten in the form:

$$Z_{\mathcal{T}/\Gamma} \;=\; \sum_{[h]\in\Gamma/\sim} \frac{1}{|\,[h]\,|} \sum_{g\in N(h)} {}_{g}\underset{[h]}{\square} \;.$$

2. The free boson and the SU(2) WZWN model at level 1 both have $c = 1$. In fact, these CFTs are equivalent, and we can express the currents of the WZWN model in terms of the free boson field.

(a) Find the bosonized expressions for the currents $J^{\pm}(z)$ and $H(z)$ such that their OPEs satisfy the SU(2) Kac-Moody algebra at level 1.

(b) The level 1 SU(2) WZWN model has a single primary field in the spin 1/2 representation. Find the bosonized expression for this field.

3. The SU(2) WZWN model at level 2 has $c = 3/2$, as does the CFT consisting of a free boson ϕ plus a free fermion ψ. As in the previous problem, one can show an equivalence between these two CFTs.

(a) For suitable constants a and b, one can represent the Kac-Moody currents in the form

$$J^{\pm}(z) \;=\; \psi\exp(\pm ia\phi) \quad\text{and}\quad H(z) \;=\; ib\,\partial_z\phi \;.$$

Find these constants.

(b) Find the representation of the primary fields (spin 1/2 and 1) of the WZWN theory in terms of the free boson and fermion.

4. Derive the following Ward identity (where Φ^j is a primary field of spin j):

$$\left(\frac{\partial}{\partial z_i} - \frac{1}{k+2}\sum_{j\neq i}^{N} \frac{t_i^a t_j^a}{z_i - z_j}\right)\left\langle\, \Phi^{j_1}(z_1)\ldots\Phi^{j_N}(z_N)\,\right\rangle \;=\; 0$$

by using the null state

$$\chi \ = \ \left(L_{-1} \ - \ \frac{1}{k+2} J^a_{-1} t^a \right) \Phi^j \ .$$

5. Using only the parafermion algebra, calculate a recursion relation for the correlation functions

$$\left\langle \psi_1(z_1) \dots \psi_1(z_n) \psi^\dagger(z_1') \dots \psi_1^\dagger(z_n') \right\rangle \ .$$

6. Using the map (4.2), rederive the recursion relation found in the previous problem.

7. Show that the $\mathbb{Z}_k$ parafermion algebra is only consistent with the conformal Ward identities if the central charge is given by $c = 2(k-1)/(k+2)$.

4.3 SOLUTIONS

1. (a) We can address the first part of the problem by studying the general case of Γ acting on the finite set M:

$$\Gamma \times M \to M \;\; : \;\; (g, x) \mapsto g \cdot x \; .$$

Let $\Gamma \cdot x$ be the orbit of $x \in M$, and let Γ_x be the isotropy group of x, i.e.,

$$\Gamma_x \;\; = \;\; \{g \in \Gamma \mid g \cdot x = x\} \; .$$

We will show that

$$|\Gamma| \;\; = \;\; |\Gamma_x| \, |\Gamma \cdot x| \; . \tag{4.3}$$

The group Γ has $|\Gamma|$ elements. When Γ acts on x, each element takes x to some point on its orbit. The action of Γ on x generates $|\Gamma|$ images of x. (Here, we are counting degeneracies; for example, if a point on the orbit is produced by the action of two distinct group elements, we count this as two images.)

We will compute this number again, but using another method, and then compare the two results. Let y be a point in the orbit $\Gamma \cdot x$. Then there exists an element $\gamma \in \Gamma$ such that

$$\gamma \cdot x \;\; = \;\; y \; .$$

Let $h \in \Gamma_x$, so that $h \cdot x = x$. Then the group element

$$g \;\; = \;\; \gamma h \in \Gamma \; ,$$

has the property

$$g \cdot x \;\; = \;\; (\gamma h) \cdot x \;\; = \;\; \gamma(h \cdot x) \;\; = \;\; \gamma \cdot x \;\; = \;\; y \; .$$

Therefore for each point y of the orbit $\Gamma \cdot x$, there are exactly $|\Gamma_x|$ elements of Γ that map x to y. Since the orbit has $|\Gamma \cdot x|$ *distinct* points, the point x has $|\Gamma_x| \, |\Gamma \cdot x|$ images under the action of the group. Equating our two results for this quantity gives us the desired identity (4.3). In the case that M is itself the group Γ, this becomes the result we wished to derive.

(b) Rewriting the partition function in the form asked is straightforward if we use the identity proved above:

$$\begin{aligned}
Z_{\mathcal{T}/\Gamma} \;\; &= \;\; \frac{1}{|\Gamma|} \sum_{\substack{g,h \in \Gamma \\ gh = hg}} \; g \underset{h}{\square} \\
&= \;\; \sum_{h \in \Gamma} \frac{1}{|N(h)| \, | \, [h] \, |} \sum_{g \in N(h)} \; g \underset{h}{\square} \\
&= \;\; \sum_{h \in \Gamma/\sim} \frac{1}{|N(h)| \, | \, [h] \, |} \, |N(h)| \sum_{g \in N(h)} \; g \underset{[h]}{\square} \\
&= \;\; \sum_{[h] \in \Gamma/\sim} \frac{1}{| \, [h] \, |} \sum_{g \in N(h)} \; g \underset{[h]}{\square} \; .
\end{aligned}$$

To go from the second to the third equality, we noticed that $N(h)$ depends only on the conjugacy class and not h itself.

2. (a) Using our general normalization

$$\langle\, \phi(z)\, \phi(w)\, \rangle \;=\; -\frac{1}{2g}\ln(z-w)\;,$$

we will prove that the fields

$$J^{\pm}(z) \;\equiv\; e^{\pm 2i\sqrt{g}\phi(z)} \quad \text{and} \quad H(z) \;\equiv\; 2i\sqrt{g}\,\partial_z\phi(z) \tag{4.4}$$

satisfy the defining OPEs of the $\mathfrak{su}(2)$ Kac-Moody algebra

$$J^a(z)\, J^b(w) \;=\; \frac{k\, q^{ab}}{(z\,-\,w)^2} + \frac{f^{ab}_{c}}{z\,-\,w}\, J^c(w) + \text{reg} \tag{4.5}$$

at level $k\,=\,1$. We use the following conventions for the structure constants:

$$\begin{array}{rcl} f^{0\,+}_{+} & = & -\,f^{+\,0}_{+} \;=\; f^{-\,0}_{-} \;=\; -\,f^{0\,-}_{-} \;=\; 2\;, \\ f^{+\,-}_{0} & = & -\,f^{-\,+}_{0} \;=\; 1\;. \end{array}$$

The Killing form, consequently, will be

$$\frac{1}{2}q^{0\,0} \;=\; q^{+\,-} \;=\; q^{-\,+} \;=\; 1\;.$$

We can write the current-current OPEs (4.5) explicitly for $k = 1$:

$$\begin{array}{rcl} J^+(z)\, J^+(w) & = & \text{reg}\;, \\ J^-(z)\, J^-(w) & = & \text{reg}\;, \\ H(z)\, H(w) & = & \dfrac{2}{(z-w)^2} + \text{reg}\;, \\ J^+(z)\, J^-(w) & = & \dfrac{1}{(z-w)^2} + \dfrac{1}{z\,-\,w}\, H(w) + \text{reg}\;, \\ H(z)\, J^{\pm}(w) & = & \dfrac{\pm\, 2}{z-w} + \text{reg}\;. \end{array}$$

We now check that the fields defined by (4.4) satisfy the previous OPEs.

$$\begin{array}{rcl} J^+(z)\, J^+(w) & = & :e^{2i\sqrt{g}\phi(z)}::e^{2i\sqrt{g}\phi(w)}: \\ & = & (z-w)^2\;:e^{4i\sqrt{g}\phi(z)}: + \cdots \;=\; \text{reg}\;, \\ J^-(z)\, J^-(w) & = & :e^{-2i\sqrt{g}\phi(z)}::e^{-2i\sqrt{g}\phi(w)}: \end{array}$$

$$
\begin{aligned}
&= (z-w)^2 \; : e^{-4i\sqrt{g}\phi(w)} : + \cdots \;=\; \text{reg} \;, \\
H(z)\,H(w) &= 2i\,\sqrt{g}\,\partial_z : \phi(z) : \, 2\,i\,\sqrt{g}\, : \partial_w \phi(w) : \\
&= -\,4g\,\partial_z\partial_w \phi(z)\phi(w) + \text{reg} \\
&= -\,4g\left(-\frac{1}{2g}\,\frac{1}{(z-w)^2}\right) + \text{reg} \\
&= \frac{2}{(z-w)^2} + \text{reg} \;, \\
J^+(z)\,J^-(w) &= : e^{2i\sqrt{g}\phi(z)} :: e^{-2i\sqrt{g}\phi(w)} : \\
&= \frac{1}{(z-w)^2} + \frac{2\,i\,\sqrt{g}\,\partial_w\,\phi(w)}{z-w} + \text{reg} \\
&= \frac{1}{(z-w)^2} + \frac{1}{z-w}\,H(w) + \text{reg} \;, \\
H(z)\,J^{\pm}(w) &= 2i\,\sqrt{g}\,\partial_z \phi(z)\, e^{\pm 2i\sqrt{g}\phi(z)} \\
&= 2i\,\sqrt{g}\,\frac{-i\,(\pm 2\sqrt{g})}{2g\,(z-w)}\, e^{\pm 2i\sqrt{g}\phi(w)} + \text{reg} \\
&= \frac{\pm 2\, e^{\pm 2i\sqrt{g}\phi(z)}}{z-w} + \text{reg} \\
&= \frac{\pm 2}{z\,-\,w}\, J^{\pm}(w) + \text{reg} \;.
\end{aligned}
$$

(b) Let $\Phi_m^{(1/2)}$, $m = \pm 1/2$, be the WZWN primary field. Using the general formula

$$
\Delta\left(\Phi^{(j)}\right) \;=\; \frac{j\,(j\,+\,1)}{k\,+\,2} \;,
$$

we see that

$$
\Delta\left(\Phi^{(1/2)}\right) \;=\; \frac{1}{4} \;.
$$

To find a bosonized expression for $\Phi_m^{(1/2)}$ we write

$$
\Phi_m^{(1/2)}(z) \;=\; e^{im\alpha\phi(z)} \;.
$$

The constant α is chosen so that

$$
\Delta\left(\Phi_m^{(1/2)}\right) \;=\; \frac{m^2\alpha^2}{4g} \;=\; \frac{1}{4} \quad\Rightarrow\quad \alpha \;=\; 2\,\sqrt{g} \;.
$$

We will show now explicitly that the expression

$$
\Phi_m^{(1/2)}(z) \;=\; e^{2i\sqrt{g}m\phi(z)}
$$

is a representation of the algebra $\mathfrak{su}(2)$. Recall that, in general, if $\phi_{(r)}$ transforms as a representation (r) of the algebra $\mathfrak{g}$, then

$$
J^{(a)}(z)\,\phi_{(r)}(w) \sim \frac{t^a_{(r)}}{z-w}\,\phi_{(r)}(w) \;,
$$

where the $t^a_{(r)}$ are representation matrices of the algebra $\mathfrak{g}$. In our case, it is easy to use the bosonic OPEs, as we did in part (a), to find:

$$J^+(z)\begin{bmatrix}\Phi^{(1/2)}_{-1/2}\\ \Phi^{(1/2)}_{+1/2}\end{bmatrix} \sim \frac{1}{z-w}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\Phi^{(1/2)}_{-1/2}\\ \Phi^{(1/2)}_{+1/2}\end{bmatrix},$$
$$J^-(z)\begin{bmatrix}\Phi^{(1/2)}_{-1/2}\\ \Phi^{(1/2)}_{+1/2}\end{bmatrix} \sim \frac{1}{z-w}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}\Phi^{(1/2)}_{-1/2}\\ \Phi^{(1/2)}_{+1/2}\end{bmatrix},$$
$$H(z)\begin{bmatrix}\Phi^{(1/2)}_{-1/2}\\ \Phi^{(1/2)}_{+1/2}\end{bmatrix} \sim \frac{1}{z-w}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\Phi^{(1/2)}_{-1/2}\\ \Phi^{(1/2)}_{+1/2}\end{bmatrix}.$$

---

3. (a) We will prove the map (4.2) for the general case. We thus seek a representation of the Kac-Moody algebra at a generic level of the form

$$\begin{aligned} J^+(z) &= a\,\psi\, e^{+i\alpha\phi(z)}\ ,\\ J^-(z) &= a\,\psi^\dagger\, e^{-i\alpha\phi(z)}\ ,\\ H(z) &= ib\,\partial_z\phi(z)\ . \end{aligned}$$

Without loss of generality, we choose ψ_1 as the representative of the parafermion fields. Then

$$\begin{aligned} J^+(z) &= a\,\psi_1\, e^{+i\alpha\phi(z)}\ ,\\ J^-(z) &= a\,\psi_1^\dagger\, e^{-i\alpha\phi(z)}\ . \end{aligned}$$

The parameter α is fixed by dimensional arguments:

$$1 = \Delta(J^\pm) = \Delta(\psi_1) + \Delta(e^{\pm i\alpha\phi(z)}) = \left(1-\frac{1}{N}\right) + \frac{\alpha^2}{4g}\ .$$

Hence

$$\alpha = 2\sqrt{\frac{g}{N}}\ .$$

The constants a and b cannot be fixed by dimensional arguments; instead, we use the requirement that $J^\pm(z)$ and $H(z)$ form a representation of the Kac-Moody algebra. Using Wick's theorem, we see that

$$\begin{aligned} J^+(z)J^-(w) &= a^2\ :\psi_1(z)e^{i\alpha\phi(z)}:\ \ :\psi_1(w)e^{-i\alpha\phi(w)}:\\ &= a^2\ \underbrace{\psi_1(z)\psi_1(w)}\ \underbrace{e^{i\alpha\phi(z)}e^{-i\alpha\phi(w)}} + \cdots\\ &= a^2\,(z-w)^{-2+2/N}\,(z-w)^{-2/N} + \cdots\\ &= \frac{a^2}{(z-w)^2} + \cdots\ . \end{aligned}$$

Comparing this with the OPE for the Kac-Moody algebra at level N,

$$J^+(z)J^-(w) \;=\; \frac{N}{(z-w)^2} + \cdots \ ,$$

we conclude that equivalence will require that $a = \sqrt{N}$. Similarly, the other OPEs require that $b = 2\sqrt{gN}$. Therefore, we arrive at the map

$$\begin{aligned} J^+(z) &= \sqrt{N}\,\psi_1\, e^{2i\sqrt{\frac{g}{N}}\phi(z)} \ , \\ J^-(z) &= \sqrt{N}\,\psi_1^\dagger\, e^{-2i\sqrt{\frac{g}{N}}\phi(z)} \ , \\ H(z) &= 2i\,\sqrt{gN}\,\partial_z\phi(z) \ , \end{aligned}$$

as given in the introduction of the model.

(b) Now we confine ourselves to the case of level $N = 2$. We have two primary fields, $\Phi_m^{(1)}(z)$ and $\Phi_m^{(1/2)}(z)$, to write in terms of a boson field $\phi(z)$, a fermion field $\psi(z)$, and a spin field $\sigma(z)$. We notice that the weights of the fields are rational numbers:

$$\Delta(\Phi^{(1)}) \;=\; \frac{1}{2} \quad \text{and} \quad \Delta(\Phi^{(1/2)}) \;=\; \frac{3}{16} \ .$$

Since

$$\Delta(\phi) \;=\; 0$$

and derivatives increase the dimension by one, in the bosonized form of the Φ's, the boson must enter in the expression in the form of a vertex operator. Thus we will have expressions of the form $e^{im\alpha\phi(z)}$, for suitable values of m. Taking into account that

$$\Delta(\psi) \;=\; \frac{1}{2} \quad \text{and} \quad \Delta(\sigma) \;=\; \frac{1}{16} \ ,$$

we see that the only possibility is

$$\Phi^{(1)}(z) \;\equiv\; \begin{bmatrix} \Phi_{-1}^{(1)}(z) \\ \Phi_0^{(1)}(z) \\ \Phi_1^{(1)}(z) \end{bmatrix} \;=\; \begin{bmatrix} e^{-i\sqrt{2g}\phi(z)} \\ \psi(z) \\ e^{i\sqrt{2g}\phi(z)} \end{bmatrix} \ ,$$

$$\Phi^{(1/2)}(z) \;\equiv\; \begin{bmatrix} \Phi_{-1/2}^{(1/2)}(z) \\ \Phi_{1/2}^{(1/2)}(z) \end{bmatrix} \;=\; \begin{bmatrix} \sigma(z)\, e^{-i\sqrt{\frac{g}{2}}\phi(z)} \\ \sigma(z)\, e^{i\sqrt{\frac{g}{2}}\phi(z)} \end{bmatrix} \ .$$

The reader may be surprised that dimensional analysis is so effective, so we think it useful to present the OPEs of the Φ's with the generators of the algebra; notice that the matrices which appear are indeed the 2-dimensional and 3-dimensional representations of $\mathfrak{su}(2)$ with which we are familiar from quantum mechanics. The OPEs are:

$$\begin{aligned}
J^+(z)\,\Phi^{(1)}(w) &\sim \frac{1}{z-w}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\Phi^{(1)}(w)\ , \\
J^-(z)\,\Phi^{(1)}(w) &\sim \frac{1}{z-w}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\Phi^{(1)}(w)\ , \\
H(z)\,\Phi^{(1)}(w) &\sim \frac{1}{z-w}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\Phi^{(1)}(w)\ , \\
J^+(z)\,\Phi^{(1/2)}(w) &\sim \frac{1}{z-w}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\Phi^{(1/2)}(w)\ , \\
J^-(z)\,\Phi^{(1/2)}(w) &\sim \frac{1}{z-w}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\Phi^{(1/2)}(w)\ , \\
H(z)\,\Phi^{(1/2)}(w) &\sim \frac{1}{z-w}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\Phi^{(1/2)}(w)\ .
\end{aligned}$$

We have given each of the above expressions up to a constant, the determination of which is left as an exercise to the reader.

4. In the following, the notation $\Phi^{(j)}_{(r)}$ stands for the primary field $\Phi^{(j)}(z)$ of spin j in the representation (r). Since

$$\chi \;=\; \left(L_{-1} - \frac{1}{k+2}\,J^a_{-1}\,t^a_{(r_i)}\right)\Phi^{(j_i)}_{(r_i)}(z_i) \;\equiv\; \mathcal{L}_{-1}\,\Phi^{(j_i)}_{(r_i)}(z_i)$$

is a null state,

$$\left\langle\left(L_{-1} - \frac{1}{k+2}\,J^a_{-1}\,t^a_{(r_i)}\right)\Phi^{(j_1)}_{(r_1)}(z_1)\,\ldots\,\Phi^{(j_i)}_{(r_i)}(z_i)\,\ldots\,\Phi^{(j_N)}_{(r_N)}(z_N)\right\rangle \;=\; 0\ , \qquad (4.6)$$

where the operator $\mathcal{L}_{-1}$ is acting on $\Phi^{(j_i)}_{(r_i)}(z_i)$. We have seen though that

$$\begin{aligned}
\left\langle\Phi^{(j_1)}_{(r_1)}(z_1)\,\ldots\,\mathcal{L}_{-1}\,\Phi^{(j_i)}_{(r_i)}(z_i)\,\ldots\,\Phi^{(j_N)}_{(r_N)}(z_N)\right\rangle &= -\sum_{n\neq i}^{N}\frac{\partial}{\partial z_n}\left\langle\Phi^{(j_1)}_{(r_1)}(z_1)\,\ldots\,\Phi^{(j_N)}_{(r_N)}(z_N)\right\rangle \\
&= \frac{\partial}{\partial z_i}\left\langle\Phi^{(j_1)}_{(r_1)}(z_1)\,\ldots\,\Phi^{(j_N)}_{(r_N)}(z_N)\right\rangle \qquad (4.7)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle \Phi^{(j_1)}_{(r_1)}(z_1)\,\ldots\,J^a_{-1}\,t^a(r_i)\,\Phi^{(j_i)}_{(r_i)}(z_i)\,\ldots\,\Phi^{(j_N)}_{(r_N)}(z_N)\right\rangle \\
&\quad = \left\langle \Phi^{(j_1)}_{(r_1)}(z_1)\,\ldots\,\oint \frac{dw}{2\pi i}\,\frac{J^a(w)}{w-z_i}\,t^a_{(r_i)}\,\Phi^{(j_i)}_{(r_i)}(z_i)\,\ldots\,\Phi^{(j_N)}_{(r_N)}(z_N)\right\rangle \\
&\quad = \oint \frac{dw}{2\pi i}\,\frac{1}{w-z_i}\,\left\langle J^a(w)\,\Phi^{(j_1)}_{(r_1)}(z_1)\,\ldots\,\Phi^{(j_N)}_{(r_N)}(z_N)\right\rangle\,t^a_{(r_i)} \\
&\quad = \oint \frac{dw}{w-z_i}\,\sum_{n\neq i}\frac{t^a_{(r_n)}}{w-z_n}\,\left\langle \Phi^{(j_1)}_{(r_1)}(z_1)\,\ldots\,\Phi^{(j_N)}_{(r_N)}(z_N)\right\rangle\,t^a_{(r_i)} \\
&\quad = \sum_{n\neq i}\frac{t^a_{(r_n)}\,t^a_{(r_i)}}{z_i-z_n}\,\left\langle \Phi^{(j_1)}_{(r_1)}(z_1)\,\ldots\,\Phi^{(j_N)}_{(r_N)}(z_N)\right\rangle\,. \qquad (4.8)
\end{aligned}$$

Using (4.6)–(4.8), we thus arrive at the equation

$$\left(\frac{\partial}{\partial z_i} - \frac{1}{k+2}\sum_{n\neq i}\frac{t^a_{(r_n)}\,t^a_{(r_i)}}{z_i-z_n}\right)\left\langle \Phi^{(j_1)}_{(r_1)}(z_1)\,\ldots\,\Phi^{(j_N)}_{(r_N)}(z_N)\right\rangle = 0\,.$$

5. We start by recalling the classic Mittag-Leffler theorem.

Theorem [Mittag-Leffler]
Let $f(z)$ be a complex function of a complex variable such that:

(i) $f(z)$ is analytic at $z = 0$;

(ii) $f(z)$ has only simple poles $a_1, a_2, \ldots$ on the finite z-plane (without loss of generality, these are given in order of increasing magnitude) with corresponding residues $b_1, b_2, \ldots$;

(iii) there exists a sequence of positive numbers R_N such that $R_N \to +\infty$ when $N \to +\infty$ and the circle $C_N = \{|z| = R_N\}$ does not pass through any pole of $f(z)$ for any values of N;

(iv) there exists a real number M such that $|f(z)| < M$ on the circles C_N for all N.

Then if z is not a pole of $f(z)$:

$$f(z) = f(0) + \sum_k b_k \left\{\frac{1}{z-a_k} + \frac{1}{a_k}\right\}\,.$$

The above formulation of the Mittag-Leffler theorem is simple, yet very useful. There are several other equivalent formulations. Here is an example taken from an introductory modern text [355] on complex variables:

Theorem [Mittag-Leffler]
Let $U \subseteq \mathbb{C}$ be any open set. Let $a_1, a_2, \ldots$ be a finite or countably infinite set of distinct

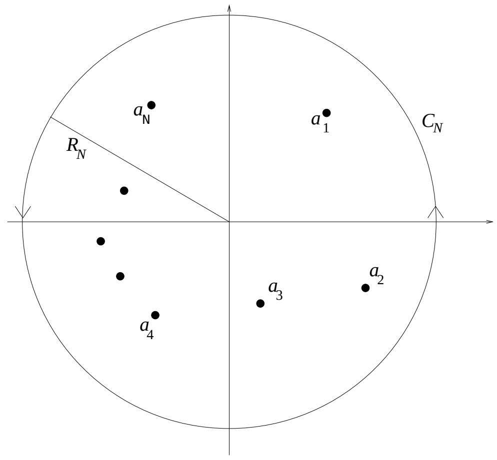

Figure 4.3: The set of poles $a_1, a_2, \ldots$ of a function $f(z)$ for which the Mittag-Leffler theorem is applicable may be finite or countably infinite. The circle C_N does not pass through any pole.

elements of U with no accumulation point in U. Suppose, for each j, that U_j is a neighborhood of a_j and $a_j \notin U_k$ if $k \neq j$. Further assume, for each j, that f_j is a meromorphic function defined on U_j with a pole at a_j and no other poles. Then there exists a meromorphic function f on U such that $f - f_j$ is holomorphic on U_j for every j and which has no poles other than those at the a_j.

Now let us state and prove[2] a 'generalized' Mittag-Leffler theorem.

Theorem
Let $f(z)$ be a complex function of a complex variable such that:

(i) $f(z)$ is analytic at $z = 0$;

(ii) $f(z)$ has only double poles a_1, a_2, …, on the finite z-plane with corresponding residues b_1, b_2, …. Furthermore, let c_1, c_2, … be the coefficients of the terms $1/(z-a_1)^2$, $1/(z-a_2)^2$, … in the Laurent expansion of $f(z)$. Without loss of generality we may assume that the poles are ordered with increasing magnitude;

(iii) there exists a sequence of positive numbers R_N such that $R_N \to +\infty$ when $N \to +\infty$ and the circle $C_N = \{|z| = R_N\}$ does not pass through any pole of $f(z)$ for any values of N;

(iv) there exists a real number M such that $|f(z)| < M$ on the circles C_N for all N.

[2]We rush to make the folowing comment: In textbooks of complex analysis, one never finds more than the Mittag-Leffler theorem. This is because this theorem is all one needs. If $f(z)$ has a double pole at $z = a$, then $(z - a)f(z)$ has a simple pole at $z = a$, and one can continue from this point. However, we will make use of the 'generalized' Mittag-Leffler theorem since it appears, at least superficially, simpler to apply in our case.

Then if z is not a pole of $f(z)$:

$$f(z) \;=\; f(0) + \sum_k b_k \left[\frac{1}{z - a_k} + \frac{1}{a_k}\right] + \sum_k c_k \left[\frac{1}{(z - a_k)^2} - \frac{1}{a_k^2}\right] . \tag{4.9}$$

Proof

We choose a point J which is not a pole of $f(z)$. Then the function

$$F(z) \;\equiv\; \frac{f(z)}{z - J}$$

has double poles at a_1, a_2, …, and a simple pole at J. By using the Residue Theorem, we can write

$$\oint_{C_N} \frac{dz}{2\pi i} \frac{f(z)}{z - J} = \operatorname*{Res}_{z=J} F(z) + \sum_{k=1}^{N} \operatorname*{Res}_{z=a_k} F(z) ,$$

where we have assumed that C_N contains the poles a_1, …, a_N. Now,

$$\begin{aligned}
\operatorname*{Res}_{z=a_k} F(z) \;&=\; \lim_{z\to a_k} \frac{d}{dz} (z - a_k)^2 F(z) \\
&=\; \lim_{z\to a_k} \left[\frac{1}{z - J}\frac{d}{dz} (z - a_k)^2 f(z) - \frac{(z - a_k)^2}{(z - J)^2} f(z)\right] \\
&=\; \frac{b_k}{a_k - J} - \frac{c_k}{(a_k - J)^2} ,
\end{aligned}$$

and

$$\operatorname*{Res}_{z=J} F(z) \;=\; f(J) .$$

Consequently,

$$\oint_{C_N} \frac{dz}{2\pi i} \frac{f(z)}{z - J} = f(J) + \sum_{k=1}^{N} \left[\frac{b_k}{a_k - J} - \frac{c_k}{(a_k - J)^2}\right] . \tag{4.10}$$

Since $f(z)$ is analytic at $z = 0$, setting $J = 0$ in the previous relation gives

$$\oint_{C_N} \frac{dz}{2\pi i} \frac{f(z)}{z} \;=\; f(0) + \sum_{k=1}^{N} \left[\frac{b_k}{a_k} - \frac{c_k}{a_k^2}\right] . \tag{4.11}$$

By subtracting (4.11) from (4.10), we get

$$\begin{aligned}
J\oint_{C_N} \frac{dz}{2\pi i} \frac{f(z)}{z\,(z - J)} \;=\; f(J) - f(0) \;&+\; \sum_{k=1}^{N} b_k \left[\frac{1}{a_k - J} - \frac{1}{a_k}\right] \\
&+\; \sum_{k=1}^{N} c_k \left[\frac{1}{a_k^2} - \frac{1}{(a_k - J)^2}\right] .
\end{aligned} \tag{4.12}$$

Let us notice now that

$$\left| \oint_{C_N} \frac{dz}{2\pi i} \, \frac{f(z)}{z\,(z\,-\,J)} \right| \;\leq\; \frac{M}{R_N\,(R_N\,-\,J)} \;\to\; 0 \quad , \qquad \text{as} \quad N \to +\infty \ .$$

This implies that, in the limit $N \to +\infty$, (4.12) gives our advertised formula, namely

$$f(J) \;=\; f(0) \,+\, \sum_k b_k \left[\frac{1}{J\,-\,a_k} \,+\, \frac{1}{a_k} \right] \,+\, \sum_k c_k \left[\frac{1}{(J\,-\,a_k)^2} \,-\, \frac{1}{a_k^2} \right] \ . \qquad \blacksquare$$

We can derive a corollary of the above result for a function $f(z)$ that vanishes asymptotically. Suppose

$$\lim_{|z| \to +\infty} f(z) = 0 \ .$$

Then taking the limit $|z| \to +\infty$ of (4.9), we find

$$0 \;=\; f(0) + \sum_k \frac{b_k}{a_k} - \sum_k \frac{c_k}{a_k^2} \ .$$

Using this result, (4.9) simplifies to

$$f(z) \;=\; \sum_k \frac{b_k}{z - a_k} + \sum_k \frac{c_k}{(z - a_k)^2} \ .$$

We will use the above result to derive a recursion relation for the function

$$g(z_1) \;\equiv\; \Big\langle\, \psi_1(z_1) \ldots \psi_1(z_n) \psi^\dagger(z_1') \ldots \psi_1^\dagger(z_n') \,\Big\rangle \ .$$

(We are suppressing the dependence of g on variables other than z_1 for simplicity, as we will be using g to define another function that does in fact depend only on z_1.) Now $g(z_1)$, as defined, has branch cuts, and the Generalized Mittag-Leffler Theorem we obtained above is not directly applicable. However, we can define a new function $f(z_1)$ which is a product of $g(z_1)$ and some factors that remove the branch cuts. Towards this end, we observe that the branch cuts in the variable z_1 come from the OPEs:

$$\begin{aligned}
\psi_1(z)\psi_1(z_i) \;&=\; \frac{c_{11}}{(z_1 - z_i)^{2/N}} \,\left[\psi_2(z_i) + \mathcal{O}(z_1 - z_i)\right] \quad \text{and} \\
\psi_1(z)\psi_1^\dagger(z_i') \;&=\; \frac{1}{(z_1 - z_i')^{2-2/N}} \,\left[1 + \mathcal{O}\left((z_1 - z_i)^2\right)\right] \ .
\end{aligned}$$

Therefore, the function $f(z_1)$ should be defined as

$$f(z_1) \;\equiv\; \left[\prod_{i=1}^{n} (z_1 - z_i)^{2/N}\right] \, \left[\prod_{i=1}^{n} (z_1 - z_i')^{-2/N}\right] \, g(z_1) \ .$$

The first factor removes any branch cuts or poles at the points $z_1 = z_i$. The second factor removes the branch cuts at the points $z_1 = z'_i$, but leaves double poles at these points. Moreover,

$$\psi_1(z) \overset{|z|\gg 1}{\simeq} \frac{1}{z^{2-2/N}} \overset{|z|\to+\infty}{\to} 0 \ .$$

This guarantees that $f(z_1) \to 0$ as $|z_1| \to +\infty$. Therefore, $f(z_1)$ can be expressed in the form

$$f(z_1) \ = \ \sum_k \frac{b_k}{z_1 - z'_k} + \sum_k \frac{c_k}{(z_1 - z'_k)^2} \ , \tag{4.13}$$

where $b_k = b_k(z_2, \ldots, z_n; z'_1, \ldots, z'_n)$ and $c_k = c_k(z_2, \ldots, z_n; z'_1, \ldots, z'_n)$. To finish the problem we must compute these functions. In fact, it is easily seen that

$$\begin{aligned} b_k \ &= \ \lim_{z_1 \to z'_k} \frac{d}{dz_1} \left[(z_1 - z'_k)^2 f(z_1) \right] \ , \quad \text{and} \\ c_k \ &= \ \lim_{z_1 \to z'_k} \left[(z_1 - z'_k)^2 f(z_1) \right] \ . \end{aligned}$$

In the limit $z_1 \to z'_k$, the contraction $\underbrace{\psi_1(z_1)\psi_1(z'_k)}$ will be dominant; therefore, the second of the above equations gives:

$$\begin{aligned} c_k \ &= \ \lim_{z_1 \to z'_k} (z_1 - z'_k)^2 \left[\prod_{i=2}^{n} (z_1 - z_i)^{2/N} \right] \left[\prod_{i=1}^{n} (z_1 - z'_i)^{-2/N} \right] \\ &\quad \times \left\langle \psi_1(z_1)\psi_2(z_2) \ldots \psi_1^\dagger(z'_1) \ldots \psi_1^\dagger(z'_k \ldots) \right\rangle \\ &= \ \lim_{z_1 \to z'_k} (z_1 - z'_k)^2 \left[\prod_{i=2}^{n} (z_1 - z_i)^{2/N} \right] \left[\prod_{i=1}^{n} (z_1 - z'_i)^{-2/N} \right] \frac{(-1)^{2(n+k-2)/N}}{(z_1 - z'_k)^{2-2/N}} \\ &\quad \times \left\langle \hat{\psi}_1(z_1)\psi_2(z_2) \ldots \psi_1^\dagger(z'_1) \ldots \hat{\psi}_1^\dagger(z'_k \ldots) \right\rangle \\ &= \ \lim_{z_1 \to z'_k} (-1)^{2(n+k-2)/N} \left[\prod_{i=2}^{n} (z_1 - z_i)^{2/N} \right] \left[\prod_{\substack{i=1 \\ i \neq k}}^{n} (z_1 - z'_i)^{-2/N} \right] \\ &\quad \times \left\langle \hat{\psi}_1(z_1)\psi_2(z_2) \ldots \psi_1^\dagger(z'_1) \ldots \hat{\psi}_1^\dagger(z'_k \ldots) \right\rangle \\ &= \ (-1)^{2(n+k-2)/N} \left[\prod_{i=2}^{n} (z'_k - z_i)^{2/N} \right] \left[\prod_{\substack{i=1 \\ i \neq k}}^{n} (z'_k - z'_i)^{-2/N} \right] \\ &\quad \times \left\langle \hat{\psi}_1(z_1)\psi_2(z_2) \ldots \psi_1^\dagger(z'_1) \ldots \hat{\psi}_1^\dagger(z'_k \ldots) \right\rangle \\ &= \ \left[\prod_{i=2}^{n} (z'_k - z_i)^{2/N} \right] \left[\prod_{i=1}^{k-1} (z'_i - z'_k)^{-2/N} \right] \left[\prod_{i=k+1}^{n} (z'_k - z'_i)^{-2/N} \right] \\ &\quad \times \left\langle \hat{\psi}_1(z_1)\psi_2(z_2) \ldots \psi_1^\dagger(z'_1) \ldots \hat{\psi}_1^\dagger(z'_k \ldots) \right\rangle \ , \end{aligned}$$

where the factors $(-1)^{2/N}$ came from the analytic continuation of a parafermion as one encircles another, i.e., from

$$\psi_1(\pi_C(x))\psi_1(y) \;=\; (-1)^{2/N}\,\psi_1(y)\psi_1(x)\ ,$$

and where we have used the notation that a caret ('hat') above a quantity indicates that this quantity is to be omitted. Analogous calculations determine also that

$$\begin{aligned} b_k \;&=\; \frac{d}{dz'_k}\left[\prod_{i=2}^{n}(z'_k - z_i)^{2/N}\right]\left[\prod_{i=1}^{k-1}(z'_i - z'_k)^{-2/N}\right]\left[\prod_{i=k+1}^{n}(z'_k - z'_i)^{-2/N}\right] \\ &\quad\times\left\langle \hat{\psi}_1(z_1)\psi_2(z_2)\ldots\psi_1^\dagger(z'_1)\ldots\hat{\psi}_1^\dagger(z'_k\ldots)\right\rangle \\ &=\; \frac{2}{N}\left[\sum_{i=2}^{n}\frac{1}{z'_k - z_i} - \sum_{i=1,i\neq k}\frac{1}{z'_k - z'_i}\right]\left[\prod_{i=2}^{n}(z_i - z'_k)^{2/N}\right]\left[\prod_{i=1}^{k-1}(z'_i - z'_k)^{-2/N}\right] \\ &\quad\times\left[\prod_{i=k+1}^{n}(z'_k - z'_i)^{-2/N}\right]\left\langle \hat{\psi}_1(z_1)\,\ldots\,\hat{\psi}_1^\dagger(z'_k)\right\rangle\ . \end{aligned}$$

Substituting these results for b_k and c_k in (4.13), we find the recursion relation first written by Zamolodchikov and Fateev:

$$\begin{aligned} \left\langle \psi_1(z_1)\,\ldots\,\psi_1(z_n)\,\psi_1^\dagger(z'_1)\,\ldots\,\psi_1^\dagger(z'_n)\right\rangle \;&=\; \left[\prod_{i=2}^{n}(z_1 - z_i)^{-2/N}\right]\left[\prod_{i=1}^{n}(z_1 - z'_i)^{2/N}\right] \\ &\times\sum_{k=1}^{n}\left[\frac{1}{(z_1 - z_k)^2} + \frac{2/N}{z_1 - z'_k}\left(\sum_{l=2}^{n}\frac{1}{z'_k - z_l} - \sum_{l=1,l\neq k}^{n}\frac{1}{z'_k - z_l}\right)\right] \\ &\times\left[\prod_{i=2}^{n}(z_i - z'_k)^{2/N}\right]\left[\prod_{i=1}^{k-1}(z'_i - z'_k)^{-2/N}\right]\left[\prod_{i=k+1}^{n}(z'_k - z'_i)^{-2/N}\right] \\ &\times\left\langle \hat{\psi}(z_1)\,\ldots\,\psi_1(z_n)\,\psi_1^\dagger(z'_1)\,\ldots\,\hat{\psi}_1^\dagger(z'_k)\,\ldots\,\psi_1^\dagger(z'_n)\right\rangle\ . \end{aligned}$$

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6. The mapping between the WZWN model and the parafermion model lets us write the currents as

$$\begin{aligned} J^+(z) \;&=\; \sqrt{N}\,\psi_1(z)\,e^{2i\sqrt{\frac{g}{N}}\,\phi(z)}\ , \\ J^-(z) \;&=\; \sqrt{N}\,\psi_1^\dagger(z)\,e^{-2i\sqrt{\frac{g}{N}}\,\phi(z)}\ , \quad \text{and} \\ H(z) \;&=\; 2i\sqrt{gN}\,\partial_z\,\phi(z)\ . \end{aligned}$$

Then the correlation function

$$\left\langle \psi_1(z_1)\cdots\psi_1(z_n)\,\psi_1^\dagger(z_1')\cdots\psi_1^\dagger(z_2') \right\rangle$$

can be computed from the identity

$$\begin{aligned}
\langle\, J^+(z_1)\cdots J^+(z_n)J^-(z_1')\cdots J^-(z_n')\,\rangle &= N^n \left\langle \psi_1(z_1)\cdots\psi_1(z_n)\,\psi_1^\dagger(z_1')\cdots\psi_1^\dagger(z_2') \right\rangle \\
&\times \left\langle e^{2i\sqrt{\frac{g}{N}}\,\phi(z_1)}\ldots e^{2i\sqrt{\frac{g}{N}}\,\phi(z_n)}e^{-2i\sqrt{\frac{g}{N}}\,\phi(z_1')}\ldots e^{-2i\sqrt{\frac{g}{N}}\,\phi(z_n')} \right\rangle \\
&= N^n \left\langle \psi_1(z_1)\cdots\psi_1(z_n)\,\psi_1^\dagger(z_1')\cdots\psi_1^\dagger(z_2') \right\rangle \\
&\times \left[\prod_{i<j}(z_i-z_j)^2\right]\left[\prod_{i<j}(z_i'-z_j')^2\right]\left[\prod_{i,j}(z_i-z_j')^{2/N}\right] .
\end{aligned}$$

The computation of the correlation function of the parafermions is thus reduced to a computation of a correlation function of currents in the WZWN model. For simplicity, we will concentrate on the 4-point correlation function for the remainder of this solution; the computation can be handled in the same way for higher point correlation functions. Therefore, let us consider the 4-point function

$$\langle\, J^+(z_1)J^+(z_2)J^-(z_1')J^-(z_2')\,\rangle \ .$$

First, we recognize that

$$\begin{aligned}
\langle\, J^+(z_1)\,J^+(z_2)\,J^-(z_1')\,J^-(z_2')\,\rangle &= \left\langle J^+(z_1)J^-(z_1')J^+(z_2)J^-(z_2') \right\rangle \\
&+ \left\langle J^+(z_1)\,J^-(z_2')J^+(z_2)J^-(z_1') \right\rangle \\
&= \left\langle J^+(z_1J^-(z_1')J^+(z_2)J^-(z_2') \right\rangle + (z_1' \leftrightarrow z_2') \ .
\end{aligned}$$

One then computes

$$\begin{aligned}
\left\langle J^+(z_1)J^-(z_1')J^+(z_2)J^-(z_2') \right\rangle &= \left\langle \left[\frac{N}{(z_1\,-\,z_1')^2}+\frac{H(z)}{z_1\,-\,z_1')}\right] J^+(z_2)\,J^-(z_2') \right\rangle \\
&= \frac{N}{(z_1-z_1')^2}\,\langle\, J^+(z_2)\,J^-(z_2')\,\rangle + \frac{1}{z_1-z_1'}\,\langle\, H(z_1')\,J^+(z_2)\,J^-(z_2')\,\rangle \\
&= \frac{N}{(z_1-z_1')^2}\,\langle\, J^+(z_2)\,J^-(z_2')\,\rangle \\
&+\frac{1}{z_1-z_1'}\left[\left\langle H(z_1')\,J^+(z_2)\,J^-(z_2') \right\rangle + \left\langle H(z_1')\,J^-(z_2')\,J^+(z_2) \right\rangle\right] \\
&= \frac{N}{(z_1\,-\,z_1')^2}\,\langle\, J^+(z_2)\,J^-(z_2')\,\rangle \\
&+\frac{1}{z_1\,-\,z_1'}\left[\frac{2}{z_1'\,-\,z_2}-\frac{2}{z_1'\,-\,z_2'}\right]\langle\, J^+(z_2)\,J^-(z_2')\,\rangle \\
&= N\left\{\frac{1}{(z_1\,-\,z_1')^2}+\frac{2/N}{z_1\,-\,z_1'}\left[\frac{1}{z_1'\,-\,z_2}-\frac{1}{z_1'\,-\,z_2')}\right]\right\}\langle\, J^+(z_2)\,J^-(z_2')\,\rangle \ .
\end{aligned}$$

Finally, we have

$$\left\langle J^+(z_2)\, J^-(z_2')\right\rangle \;=\; (z_2 - z_2')^{-2/N} \left\langle \psi_1(z_2)\, \psi_1^\dagger(z_2')\right\rangle \; .$$

Putting everything together, we obtain the relation

$$\begin{aligned}
&\left\langle \psi_1(z_1)\, \psi_1(z_2)\, \psi_1^\dagger(z_1')\, \psi_1^\dagger(z_2')\right\rangle \;= \\
&(z_1 - z_2)^{-2/N}\, (z_1 - z_1')^{2/N}\, (z_1 - z_2')^{2/N}\, (z_2 - z_1')^{2/N}\, (z_1' - z_2')^{-2/N} \\
&\quad\times \left[\frac{1}{(z_1 - z_1')^2} + \frac{2/N}{z_1 - z_1'}\left(\frac{1}{z_1' - z_2} - \frac{1}{z_1' - z_2'}\right)\right] \left\langle \psi_1(z_2)\, \psi_1^\dagger(z_2')\right\rangle \\
&+ (z_2 - z_2')^{2/N}\, (z_1' - z_2')^{-2/N} \\
&\quad\times \left[\frac{1}{(z_1 - z_1')^2} + \frac{2/N}{z_1 - z_2'}\left(\frac{1}{z_2' - z_2} - \frac{1}{z_2' - z_1'}\right)\right] \left\langle \psi_1(z_2)\, \psi_1^\dagger(z_1')\right\rangle
\end{aligned} \tag{4.14}$$

which is exactly the relation we found in the previous problem.

7. The central charge may be found using the 2-point correlation function of the energy-momentum tensor, since

$$\langle\, T(z)\, T(w)\,\rangle \;=\; \frac{c/2}{(z-w)^4}\; .$$

Therefore, let us first find an expression for the energy-momentum tensor in terms of the parafermions. To motivate the definition which is about to follow, let us think initially about the well-understood $\mathbb{Z}_2$ case. For this model, we have seen that[3]

$$T(z) \;=\; -\,\frac{1}{2}\; :\psi(z)\, \partial\psi(z): \; . \tag{4.15}$$

Notice that (4.15) can be written as

$$\begin{aligned}
T(z) \;&=\; -\frac{1}{2}\left[:\psi(z)\psi(z): \;+\; :\frac{1}{(z'-z)}\,\psi(z)\;(z'-z)\partial\psi(z):\right] \\
&=\; -\frac{1}{2}\lim_{z'\to z}\; :\frac{1}{(z'-z)}\,\psi(z)\left[\psi(z) \;+\; (z'-z)\partial\psi(z) \;+\; \mathcal{O}\left((z'-z)^2\right)\right]: \\
&=\; \frac{1}{2}\lim_{z'\to z}\frac{1}{(z-z')}\; :\psi(z)\psi(z'):\; .
\end{aligned} \tag{4.16}$$

Furthermore, since

$$\psi^\dagger \;=\; \psi\; ,$$

[3] The normalization in this problem is that of Zamolodchikov and Fateev.

equation (4.16) can be written in the more convenient (for our purposes) form

$$T(z) \;=\; \lim_{z'\to z} \;:\; \frac{1}{z-z'}\,\psi(z)\psi^\dagger(z') \;:\; . \tag{4.17}$$

This equation is very suggestive: the energy-momentum tensor for the free fermion can be extracted from the OPE of the fermion with its hermitian conjugate. We shall generalize equation (4.17) to the $\mathbb{Z}_N$ case. For any parafermion ψ_k, we have the OPE

$$\psi_k(z)\,\psi_k^\dagger(z') \;=\; \frac{1}{(z-z')^{2\Delta_k}}\left[1 \;+\; \alpha\,(z-z')\,\Psi_k^{(1)} \;+\; \beta\,(z-z')^2\,\Psi_k^{(2)} \;+\; \ldots\right] .$$

Obviously, the conformal dimensions of the fields $\Psi_k^{(1)}, \Psi_k^{(2)}, \ldots$ are

$$\begin{array}{rclcrcl} \Delta(\Psi_k^{(1)}) &=& 1 & , & \overline{\Delta}(\Psi_k^{(1)}) &=& 0\;, \\ \Delta(\Psi_k^{(2)}) &=& 2 & , & \overline{\Delta}(\Psi_k^{(2)}) &=& 0\;, \\ & \cdots & & , & & \cdots\;, & \end{array}$$

and therefore the corresponding spins are

$$s(\Psi_k^{(1)}) \;=\; 1\;,\; s(\Psi_k^{(2)}) \;=\; 2,\;\; \ldots\;.$$

A spin 1 field creates a U(1) symmetry (compare with QED), for which

$$\mathbb{Z}_N \subset U(1)\;.$$

Since our parafermion theory should not have this larger symmetry group, we impose the condition

$$\Psi_k^{(1)}(z) \;\equiv\; 0$$

on our algebra. On the other hand, there is only one spin 2 field in the theory, so we must identify this with the energy-momentum tensor, i.e.,

$$\Psi_k^{(2)}(z) \;\equiv\; T(z)\;.$$

Using the method we have described in Exercise 12 of Chapter 3, we find

$$\beta \;=\; \frac{2\Delta_k}{c}\;.$$

Thus, we define the energy-momentum tensor for the parafermion theories as follows:

$$\begin{aligned} T(z) \;&=\; \frac{1}{\beta}\lim_{z'\to z}\frac{1}{(z-z')^2}\left[-1 \;+\; (z-z')^{2\Delta_k}\,\psi_k(z)\,\psi_k^\dagger(z')\right] \\ &=\; \frac{c}{2\Delta_k}\;:\;\lim_{z'\to z}\frac{1}{(z-z')^{2-2\Delta_k}}\,\psi_k(z)\,\psi_k^\dagger(z')\;:\;. \end{aligned} \tag{4.18}$$

Given this equation for the energy-momentum tensor, we have the OPE

$$\begin{aligned}\langle\, T(z_1)T(z_2)\,\rangle \;&=\; \lim_{z_1'\to z_1}\lim_{z_2'\to z_2} \frac{c^2}{4\Delta_k^2}\,\frac{1}{(z_1-z_1')^2}\,\frac{1}{(z_2-z_2')^2} \\ &\quad\times\Big[-1 \;+\; (z_1-z_1')^{2\Delta_k}\,(z_2-z_2')^{2\Delta_k}\,\langle\,\psi_k(z_1)\,\psi_k^\dagger(z_1')\psi_k(z_2)\,\psi_k^\dagger(z_2')\,\rangle\Big]\;.\end{aligned}$$

This result is true for any parafermion ψ_k. However, since we have already studied the correlation functions

$$\langle\,\psi_1(z_1)\ldots\psi_1(z_n)\psi_1^\dagger(z_1')\ldots\psi_1^\dagger(z_n')\,\rangle\;,$$

we are going to use the fields ψ_1 and $\psi_1^\dagger$ to find c. In this case, then,

$$\begin{aligned}\langle\, T(z_1)T(z_2)\,\rangle \;&=\; \lim_{z_1'\to z_1}\lim_{z_2'\to z_2} \frac{c^2}{4\Delta_1^2}\,\frac{1}{(z_1-z_2')^2}\,\frac{1}{(z_2-z_2')^2} \\ &\quad\times\Big[-1 \;+\; (z_1-z_1')^{2\Delta_1}\,(z_2-z_2')^{2\Delta_1}\langle\,\psi_1(z_1)\,\psi_1^\dagger(z_1')\psi_1(z_2)\,\psi_1^\dagger(z_2')\,\rangle\Big] \\ &=\; \lim_{z_1'\to z_1}\lim_{z_2'\to z_2} \frac{c^2}{4\Delta_1^2}\,\frac{1}{(z_1-z_2')^2}\,\frac{1}{(z_2-z_2')^2} \\ &\quad\times\Big[-1 \;+\; (z_1-z_1')^{2\Delta_1}\,(z_2-z_2')^{2\Delta_1}\,(-1)^{\frac{2}{N}}\,G\Big]\;,\end{aligned}$$

where in the second equality we used the non-local behavior of parafermions

$$\psi_1(z_2)\psi_1^\dagger(z_1') \;=\; (-1)^{2\Delta_1}\,\psi_1^\dagger(z_1')\psi_1(z_2) \;=\; (-1)^{-2/N}\,\psi_1^\dagger(z_1')\psi_1(z_2)\;,$$

and then set

$$G \;\equiv\; \Big\langle\,\psi_1(z_1)\,\psi_1(z_2)\,\psi_1^\dagger(z_1')\,\psi_1^\dagger(z_2')\,\Big\rangle\;.$$

Inserting

$$\Big\langle\,\psi_1(z)\,\psi_1^\dagger(z')\,\Big\rangle \;=\; \frac{1}{(z-z')^{2\Delta_1}} \;=\; \frac{1}{(z-z')^{2-2/N}}$$

in equation (4.14), we get

$$\begin{aligned}G \;=\;& (z_1-z_2)^{-2/N}\,(z_1-z_1')^{2/N}\,(z_1-z_2')^{2/N}\,(z_2-z_1')^{2/N}\,(z_1'-z_2')^{-2/N}\,(z_2-z_2')^{2/N} \\ &\times\Bigg\{\bigg[\frac{1}{(z_1-z_1')^2}+\frac{2/N}{z_1-z_1'}\bigg(\frac{1}{z_1'-z_2}-\frac{1}{z_1'-z_2'}\bigg)\bigg]\,\frac{1}{(z_2-z_2')^2} \\ &\quad+\bigg[\frac{1}{(z_1-z_2')^2}+\frac{2/N}{z_1-z_2'}\bigg(\frac{1}{z_2'-z_2}-\frac{1}{z_2'-z_1'}\bigg)\bigg]\,\frac{1}{(z_2-z_2')^2}\Bigg\}\;.\end{aligned}$$

The calculations become more transparent if we use the variables

$$\xi \;\equiv\; \frac{(z_1'-z_2')(z_1-z_2)}{(z_1-z_2')(z_2-z_1')}\;,\quad \epsilon \;\equiv\; z_1-z_1'\;,\quad\text{and}\quad \eta \;\equiv\; z_2-z_2'\;.$$

The prefactor in the expression of G is then just $\xi^{-2/N}\,\epsilon^{2/N}\,\eta^{2/N}$. The quantity in the curly bracket can be writen in the form

$$\{\ldots\} \;=\; \frac{1}{\epsilon^2}\,\frac{1}{\eta^2}\,\left(a \,+\, b\,\xi \,+\, d\,\xi^2\right)\;,$$

with

$$a \;=\; b \;=\; \frac{2}{N}\,(N\,-\,1) \;=\; 2\,\Delta_1\;,\quad \text{and}\quad d \;=\; 1\;.$$

Then

$$G \;=\; \epsilon^{-2\Delta_1}\,\eta^{-2\Delta_1}\,\xi^{-2/N}\,\left(a \,+\, b\,\xi \,+\, d\,\xi^2\right)\;,$$

and

$$\langle\, T(z_1)T(z_2)\,\rangle \;=\; \lim_{\epsilon\to 0}\,\lim_{\eta\to 0}\,\frac{c^2}{4\Delta_1^2}\,\frac{1}{\epsilon^2}\,\frac{1}{\eta^2}\,[\,-1\,+\,(-1)^{\frac{2}{N}}\,\xi^{-\frac{2}{N}}\,(\,a\,+\,b\xi\,+\,d\xi^2\,)\,]\;. \quad (4.19)$$

If the limit in equation (4.19) exists when ϵ and η approach zero independently, it must also exist when $\epsilon = \eta$. In this case (see the appendix of this problem solution),

$$\xi \;=\; -\left[1\,+\,\frac{\epsilon^2}{y^2}\,+\,\frac{\epsilon^4}{y^4}\,+\,\mathcal{O}(\epsilon^6)\right]\;,$$

where $y = z_1' - z_2'$. Notice that because of the factor $1/\epsilon^4$ which appears in equation (4.19), in the limit $\epsilon \to 0$, terms of order $\mathcal{O}(\epsilon^k)$ for $k > 4$ do not contribute. Putting everything together, we find

$$\begin{aligned}
\langle\, T(z_1)T(z_2)\,\rangle \;&=\; \lim_{\epsilon\to 0}\,\frac{c^2}{4\Delta_1^2}\,\frac{1}{\epsilon^4}\left\{-1\,+\,(-1)^{\frac{2}{N}}\,(-1)^{-\frac{2}{N}}\left(1+\frac{\epsilon^2}{y^2}+\frac{\epsilon^4}{y^4}+\mathcal{O}(\epsilon^6)\right)^{-\frac{2}{N}}\right.\\
&\quad \left.\times\left[a-b\left(1+\frac{\epsilon^2}{y^2}+\frac{\epsilon^4}{y^4}+\mathcal{O}(\epsilon^6)\right)+d\left(1+\frac{\epsilon^2}{y^2}+\frac{\epsilon^4}{y^4}+\mathcal{O}(\epsilon^6)\right)^2\right]\right\}\\
&=\; \lim_{\epsilon\to 0}\,\frac{c^2}{4\Delta_1^2}\,\frac{1}{\epsilon^4}\left\{-1\,+\,\left(1\,-\,\frac{2}{N}\,\frac{\epsilon^2}{y^2}-\frac{2}{N}\frac{\epsilon^4}{y^4}+\frac{N+2}{N^2}\frac{\epsilon^4}{y^4}\right)\right.\\
&\quad \left.\times\left[a-b\left(1+\frac{\epsilon^2}{y^2}+\frac{\epsilon^4}{y^4}\right)+d\left(1+2\,\frac{\epsilon^2}{y^2}+2\,\frac{\epsilon^4}{y^4}+\frac{\epsilon^4}{y^4}\right)\right]\,+\,\mathcal{O}(\epsilon^6)\right\}\\
&=\; \lim_{\epsilon\to 0}\,\frac{c^2}{4\Delta_1^2}\,\frac{1}{\epsilon^4}\left\{(-1+a-b+d)+\frac{\epsilon^2}{y^2}\left[-b+2d-\frac{2}{N}\,(a-b+d)\right]\right.\\
&\quad \left.+\frac{\epsilon^4}{y^4}\left[-b+3d-\frac{2}{N}(-b+2d)+\left(\frac{-1}{N}+\frac{2}{N^2}\,(a-b+d)\right)\right]+\mathcal{O}(\epsilon^6)\right\}\\
&=\; \lim_{\epsilon\to 0}\,\frac{c^2}{4\Delta_1^2}\,\frac{1}{\epsilon^4}\left[\frac{\epsilon^4}{y^4}\,\frac{(N^2+N-2)}{N^2}\,+\,\mathcal{O}(\epsilon^6)\right]\;,
\end{aligned}$$

i.e.,

$$\langle\, T(z_1)T(z_2)\,\rangle \;=\; \frac{c^2}{4\Delta_1^2}\,\frac{(N^2+N-2)}{N^2}\,\frac{1}{y^4}\;,$$

which gives

$$\frac{c}{2} \;=\; \frac{c^2}{4\Delta_1^2}\,\frac{(N+2)(N-1)}{N^2} \;\Rightarrow\; c \;=\; \frac{2(N-1)}{N+2}\;.$$

APPENDIX

Considering the cross ratio ξ as a function of z_1 and z_2, we write down the Taylor expansion of ξ around the point $z_1 = z_1'$ and $z_2 = z_2'$:

$$\xi\,(z_1,\, z_2) \;=\; \sum_{n=0}^{+\infty} \frac{1}{n!}\,(\epsilon\,\partial_{z_1} \,+\, \eta\,\partial_{z_2})^n\;\xi(z_1',\, z_2')\;. \tag{4.20}$$

One can easily see that

$$\begin{aligned}
\partial_{z_1}\xi(z_1,\, z_2) \;&=\; \frac{(z_1' \,-\, z_2')\,(z_2 \,-\, z_2')}{(z_2 \,-\, z_1')\,(z_1 \,-\, z_2')^2} \quad \text{and} \\
\partial_{z_2}\xi(z_1,\, z_2) \;&=\; \frac{(z_1' \,-\, z_2')\,(z_2' \,-\, z_1)}{(z_1 \,-\, z_2')\,(z_2 \,-\, z_1')}\;,
\end{aligned}$$

and thus

$$\partial_{z_1}\xi(z_1',\, z_2') \;=\; \partial_{z_2}\xi(z_1',\, z_2') \;=\; 0\;.$$

We also notice that

$$\xi(z_1',\, z_2') \;=\; -1\;.$$

We can see further that

$$\begin{aligned}
\partial_{z_1}^k\,\xi(z_1,\, z_2) \;&=\; \frac{(z_1' \,-\, z_2')\,(z_2 \,-\, z_2')}{(z_2 \,-\, z_1')\,(z_1 \,-\, z_2')^{k+1}}\,(-1)^{k+1}\,k! \\
\Rightarrow\quad \partial_{z_2}\,\partial_{z_1}^k\,\xi(z_1,\, z_2) \;&=\; \frac{(z_1' \,-\, z_2')\,(z_2' \,-\, z_1')}{(z_2 \,-\, z_1')^2\,(z_1 \,-\, z_2')^{k+1}}\,(-1)^{k+1}\,k! \\
\Rightarrow\quad \partial_{z_2}^m\,\partial_{z_1}^k\,\xi(z_1,\, z_2) \;&=\; \frac{-(z_1' \,-\, z_2')^2}{(z_1 \,-\, z_2')^{k+1}\,(z_2 \,-\, z_1')^{m+1}}\,(-1)^{k+1}\,k!(-1)^{m+1}\,m! \\
\Rightarrow\quad \partial_{z_2}^m\,\partial_{z_1}^k\,\xi(z_1',\, z_2') \;&=\; \frac{k!\,m!\,(-1)^k}{(z_1' \,-\, z_2')^{k+m}}\quad,\quad mk \neq 0\;.
\end{aligned}$$

For $mk = 0$, we have

$$\partial_{z_2}^m\,\partial_{z_1}^k\,\xi(z_1',\, z_2') \;=\; 0\;,\quad mk \;=\; 0\;.$$

Thus, (4.20) is written

$$\begin{aligned}
\xi(z_1,\, z_2) \;&=\; \sum_{n=0}^{+\infty} \frac{1}{n!}\,(\epsilon\,\partial_{z_1} \,+\, \eta\,\partial_{z_2})^n\;\xi(z_1',\, z_2') \\
&=\; -1 \,+\, \sum_{n=2}^{+\infty} (\epsilon\,\partial_{z_1} \,+\, \eta\,\partial_{z_2})^n\;\xi(z_1',\, z_2') \\
&=\; -1 \,+\, \sum_{n=2}^{+\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k}\,\epsilon^k\,\eta^{n-k}\,\partial_{z_1}^k\,\partial_{z_2}^{n-k}\,\xi(z_1',\, z_2')
\end{aligned}$$

$$
\begin{aligned}
&= -1 + \sum_{n=2}^{+\infty} \frac{1}{n!} \sum_{k=1}^{n-1} \binom{n}{k} \epsilon^k \, \eta^{n-k} \, \partial_{z_1}^k \, \partial_{z_2}^{n-k} \, \xi(z_1', z_2') \\
&= -1 + \sum_{n=2}^{+\infty} \frac{1}{n!} \sum_{k=1}^{n-1} \frac{n!}{k! \, (n-k)!} \epsilon^k \, \eta^{n-k} \, \frac{k! \, (n-k)! \, (-1)^k}{(z_1' - z_2')^n} \\
&= -1 + \sum_{n=2}^{+\infty} \frac{1}{(z_1' - z_2')^n} \left[\sum_{k=1}^{n-1} (-1)^k \, \epsilon^k \, \eta^{n-k} \right] \\
&= -1 + \sum_{n=2}^{+\infty} \frac{1}{y^n} \left[\sum_{k=1}^{n-1} (-1)^k \, \epsilon^k \, \eta^{n-k} \right] .
\end{aligned}
$$

If $\epsilon = \eta$, then

$$
\sum_{k=1}^{n-1} (-1)^k \, \epsilon^k \, \eta^{n-k} = \epsilon^n \sum_{k=1}^{n-1} (-1)^k = \begin{cases} -\,\epsilon^n \ , & \text{if } n \text{ is even} \ , \\ 0 \ , & \text{if } n \text{ is odd} \ , \end{cases}
$$

and so

$$
\xi(z_1, z_2) \ = \ -1 - \sum_{k=1}^{+\infty} \left(\frac{\epsilon}{y} \right)^{2k} \ .
$$

Chapter 5

CONSTRUCTING NEW MODELS IN CFT

References: The Coulomb gas formulation (CGF) is reviewed in [223]. For discussions of the coset construction, see [344, 297]. These topics are also treated in the CFT textbooks [186, 420].

5.1 BRIEF SUMMARY

5.1.1 Coulomb Gas Construction or Formulation

The Coulomb Gas Construction (CGC) is a standard method of generating CFT minimal models which have particular symmetries, starting from a set of bosonic fields. Sometimes the models generated this way were previously known by other methods; in this case perhaps it makes more sense to refer to the CGF of these models, i.e., to view this as a bosonized description of these models. We will give an overview of the method using the MMs of Chapter 2.

Consider a free boson in the presence of a background charge, normalized such that

$$T \;=\; -\frac{1}{2}\,(\partial\phi)^2 + i\alpha_0\sqrt{2}\partial^2\phi \ .$$

Then the central charge is

$$c \;=\; 1 - 24\,\alpha_0^2 \ ,$$

and the conformal weight of the vertex operator

$$V_\alpha(z) \;=\; e^{i\sqrt{2}\alpha\phi(z)}$$

is given by

$$\Delta_\alpha \;=\; \alpha(\alpha - 2\alpha_0) \ .$$

For the theory of a boson with a background charge, there is a conserved U(1) current

$$J(z) \;=\; \frac{1}{\sqrt{2}}\,i\partial\phi(z) \ .$$

The corresponding charge Q measures the 'strength' of the vertex, and is given by

$$Q \;=\; \frac{1}{\sqrt{2}} \int \frac{dz}{2\pi i}\, i\partial\phi \; .$$

The vertex (2.17) has α units of charge:

$$[Q, V_\alpha] \;=\; \alpha\, V_\alpha \; .$$

Due to the background charge, a correlation function

$$\langle\, V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) ... V_{\alpha_n}(z_n)\,\rangle$$

vanishes by the $U(1)$ symmetry unless the net charge of the averaged operator is $2\alpha_0$, i.e., unless

$$\sum_{k=1}^{n} \alpha_k \;=\; 2\alpha_0 \; .$$

Correlation functions that do not satisfy this neutrality condition can be made non-vanishing using **screening charges**. A screening charge is an operator

$$S_\pm \;=\; \int \frac{dz}{2\pi i}\, J_\pm(z) \; ,$$

which has the effect of reducing the total charge without affecting the conformal properties of the function. The last condition requires

$$J_\pm(z) \;=\; e^{i\sqrt{2}\alpha_\pm \phi(z)} \; , \qquad \alpha_\pm \;=\; \alpha_0 \pm \sqrt{\alpha_0^2 + 1} \; .$$

Now, a correlation function that does not satisfy the neutrality condition, such as

$$\langle\, V_{2\alpha_0-\alpha}(z_1) V_\alpha(z_2) V_\alpha(z_3) V_\alpha(z_4)\,\rangle \; ,$$

may be transformed into one that does satisfy the neutrality condition by inserting a sufficient number of screening operators. Suppose, therefore, that we insert some screening operators, and construct the correlation function

$$\Big\langle\, V_{2\alpha_0-\alpha}(z_1) V_\alpha(z_2) V_\alpha(z_3) V_\alpha(z_4) S_+^{n-1} S_-^{m-1}\,\Big\rangle \; .$$

For a given n and m, this will satisfy the neutrality condition for certain values of the charge α; one finds that α must take one of the discrete values

$$\alpha_{mn} \;=\; \frac{1}{2}\,[(1-m)\alpha_- + (1-n)\alpha_+] \; , \qquad n, m \in \mathbb{N}^* \; . \tag{5.1}$$

The corresponding primary fields are then

$$V_{\alpha_{mn}}(z) \;=\; e^{i\sqrt{2}\alpha_{mn}\phi(z)} \;\equiv\; \phi_{mn}(z) \; , \tag{5.2}$$

with their conformal weights equal to

$$\Delta_{mn} \;=\; \frac{1}{4}\,\left[(\alpha_-\,m+\alpha_+\,n)^2-(\alpha_++\alpha_-)^2\right]\;. \tag{5.3}$$

The above construction does not impose any constraint on the allowed values of the central charge c, or equivalently, on the value of α_0. However, if we require that the number of primary fields be finite, say $1 \le n \le s-1$ and $1 \le m \le r-1$, where $r, s \in \mathbb{N} \smallsetminus \{0,1\}$, one must impose

$$\alpha_+ \;=\; \sqrt{\frac{r}{s}}\;, \qquad \alpha_- \;=\; -\sqrt{\frac{s}{r}}\;. \tag{5.4}$$

Obviously, r and s must be relatively prime, as we can always cancel their common factors. When the condition (5.4) is met, we recover the MMs of CFT we described in the previous chapter.

5.1.2 Coset Construction

As we have seen in the WZWN model, an affine algebra $\hat{\mathfrak{g}}$ generated by currents J_a that satisfy OPEs

$$J^a(z)J^b(w) \;=\; \frac{k/2\,\delta^{ab}}{(z-w)^2}+\frac{f^{ab}{}_c\,J^c(w)}{z-w}+\mathrm{reg}$$

gives rise to a Virasoro algebra with energy-momentum tensor

$$T(z)=\frac{1}{k+\tilde{h}_{\mathfrak{g}}}\;\ddagger\,J_a(z)J_a(z)\,\ddagger\;\;, \tag{5.5}$$

and central charge

$$c \;=\; \frac{k\,\dim\mathfrak{g}}{k+\tilde{h}_{\mathfrak{g}}}\;.$$

When T is written as in equation (5.5), we say that it has been expressed in the **Sugawara construction**.

Given a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, one can construct a new conformal theory which corresponds to the coset $\mathfrak{g}/\mathfrak{h}$ as follows. Let $T_{\mathfrak{g}}$ and $T_{\mathfrak{h}}$ be the energy-momentum tensors of the CFTs from the the Sugawara construction for $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Then their difference

$$T_{\mathfrak{g}/\mathfrak{h}} \;=\; T_{\mathfrak{g}}-T_{\mathfrak{h}}$$

commutes with the $\mathfrak{h}$ current algebra and the Sugawara construction on $\mathfrak{g}$, i.e.

$$\begin{aligned} T_{\mathfrak{g}/\mathfrak{h}}(z)J_A(w) \;&=\; \mathrm{reg}\;, \qquad \mathrm{A}=1,2,\ldots,\dim\mathfrak{h}\;,\\ T_{\mathfrak{g}/\mathfrak{h}}(z)T_{\mathfrak{g}}(w) \;&=\; \mathrm{reg}\;. \end{aligned}$$

The commutativity properties of the energy-momentum tensor $T_{\mathfrak{g}/\mathfrak{h}}$ imply that the Sugawara construction $T_{\mathfrak{g}} = T_{\mathfrak{g}/\mathfrak{h}} + T_{\mathfrak{h}}$ is a tensor product CFT, formed by tensoring the coset CFT with the CFT on $\mathfrak{h}$. The conformal anomaly of the coset CFT is

$$c_{\mathfrak{g}/\mathfrak{h}} \;=\; c_{\mathfrak{g}}-c_{\mathfrak{h}}\;.$$

5.2 EXERCISES

1. Consider the bosonic theory of Exercise 17 of Chapter 2, with $\beta = ie_0/4\pi$, defined on a Riemann surface M with no boundary and with genus h. Show that the correlation function

$$\left\langle \prod_i e^{i\alpha_i \Phi(\xi_i)} \right\rangle$$

can be non-vanishing only if

$$\sum_i \alpha_i \;=\; e_0\,(1-h) \ .$$

2. Consider the free boson theory with background charge. The energy momentum tensor is

$$T(z) \;=\; -g\,\partial_z\phi\partial_z\phi + ie_0\partial_z^2\phi \ .$$

Expand the boson field in modes according to $i\partial_z\phi = \sum_n \phi_n\, z^{n-1}$.

(a) Compute the commutation relations of the generators L_n with the modes ϕ_m.

(b) Find the $\phi_n^\dagger$.

(c) Define a $U(1)$ charge operator Q such that

$$[Q, V_\alpha(w)] \;=\; \alpha\, V_\alpha(w) \ .$$

in terms of the modes. Then show that for a generic operator O with charge q,

$$[Q, O] \;=\; q\, O \ ,$$

and that the expectation value $\langle\, O\,\rangle$ can be non-zero only if $q = 2e_0$.

3. Construct the screening charges for the theory of a free boson with a background charge. Do this for generic normalization.

4. One way to construct irreducible representations of the Virasoro algebra is as follows. A module $[\Phi_\Delta]$ is called degenerate if it contains at least one state $|\,\chi\,\rangle$ with the properties

$$\begin{aligned} L_n\,|\,\chi\,\rangle &= 0\ , \qquad n>0\ , \\ L_0\,|\,\chi\,\rangle &= (\Delta+N)\,|\,\chi\,\rangle\ , \end{aligned}$$

for some $N \in \mathbb{N}^*$, which is called the **degeneration level**. Such states, when found, must be set equal to zero (null states), as otherwise the representation would be reducible. This amounts to factorization of $[\Phi_\Delta]$ by $[\chi]$.

To construct null vectors for the module $[V_\alpha] = [V_{2\alpha_0-\alpha}]$, we start with a highest weight state $\left|\,V_{2\alpha_0-\alpha-n\alpha_\pm}\,\right\rangle$, and use screening operators to create another highest weight state $|\chi\rangle$, which has charge $2\alpha_0 - \alpha$ and weight $\Delta_\alpha + N$. Such a state, being in $[V_\alpha]$, must must be null, from which condition we can derive the allowed spectrum of charges and the degeneration level.

Use this method to derive the spectrum of the highest weight operators in the case of the CFT of a boson in the presence of a background charge.

5. With the help of the screening charges, derive the fusion relations of the MMs.

6. Derive a contour integral representation for the correlation function of Exercise 11 from Chapter 3 by using the CGF.

7. A **character** for a Verma module built over the field Φ_Δ with conformal weight Δ is defined by

$$\chi_\Delta(q) = q^{-c/24}\,\mathrm{tr}\left(q^{L_0}\right) \ .$$

Since for any state in the Verma module, the eigenvalue of L_0 has the form $\Delta + N$, the character takes the form

$$\chi_\Delta(q) = q^{\Delta - c/24} \sum_{N=0}^{+\infty} p(N)\, q^N \ .$$

The coefficient $p(N)$ in the expansion counts the number of states at level N.

(a) Show that if there are no null states, then

$$\chi_\Delta(q) = \frac{q^{\Delta-c/24}}{\prod_{n=1}^{+\infty}(1-q^n)} \ .$$

(b) Show that if there is a null state at level N, then

$$\chi_\Delta(q) = \frac{q^{\Delta-c/24}\,(1-q^N)}{\prod_{n=1}^{+\infty}(1-q^n)} \ .$$

(c) Explain how to generalize this to cases with null states within null states.

8. For the primary field Φ_{mn} of the minimal model $\mathrm{MM}(r,s)$, prove the **Rocha-Caridi formula**

$$\chi_{mn} = \frac{q^{-c/24}}{\prod_{n=1}^{+\infty}(1-q^n)} \sum_{k=-\infty}^{+\infty} \left(q^{a_{mn}(k)} - q^{b_{mn}(k)}\right) \ , \tag{5.6}$$

where

$$a_{mn}(k) = \frac{(2rsk + sn - mr)^2 - (r-s)^2}{4rs} \ , \quad \text{and}$$

$$b_{mn}(k) = \frac{(2rsk + sn + mr)^2 - (r-s)^2}{4rs} \ .$$

5.3 SOLUTIONS

1. Using the path integral representation, the correlation function given in the problem can be written as

$$\left\langle \prod_i e^{i\alpha_i \Phi(\xi_i)} \right\rangle \;=\; \int d\mu\,[\Phi]\; e^{-\alpha \int d^2\xi \sqrt{g} g^{ab}\partial_a\Phi\partial_b\Phi}\; e^{-i\frac{e_0}{2\pi}\int d^2\xi\sqrt{g}R\Phi} \prod_i e^{i\alpha_i\Phi(\xi_i)}\ .$$

This must be invariant under a shift in the bosonic field

$$\Phi \;\mapsto\; \Phi + \delta\ .$$

However, introducing this transformation in the path integral, we find

$$\left\langle \prod_i e^{i\alpha_i \Phi(\xi_i)} \right\rangle \;\mapsto\; \left\langle \prod_i e^{i\alpha_i \Phi(\xi_i)} \right\rangle e^{i\left(\sum_i \alpha_i - \frac{e_0}{2\pi}\int d^2\xi\sqrt{g}R\right)\delta}\ .$$

Thus the correlation function is invariant only if

$$\sum_i \alpha_i - \frac{e_0}{2\pi}\iint d^2\xi\sqrt{g}R \;=\; 0\ .$$

If this equation is not satisfied, then the correlation function vanishes.

On the other hand, the Gauss-Bonnet theorem states that for a manifold without boundary

$$\iint d^2\xi\sqrt{g}R \;=\; 4\pi\,(1-h)\ .$$

Thus, for the correlation function to be non-zero, it is necessary to have

$$\sum_i \alpha_i \;=\; 2e_0\,(1-h)\ .$$

2. (a) We define the modes of $T(z)$ and $\phi(z)$ using the standard relations

$$T(z) \;\equiv\; \sum_n L_n\, z^{-n-2}\ ,$$
$$i\,\partial_z\phi(z) \;\equiv\; \sum_n \phi_n\, z^{-n-1}\ .$$

From Cauchy's formula, we then have the explicit formulæ:

$$L_n \;=\; \oint \frac{dz}{2\pi i}\, z^{n+1}\, T(z)\ , \quad \text{and}$$
$$\phi_n \;=\; \oint \frac{dz}{2\pi i}\, z^{n}\, i\,\partial_z\phi(z)\ .$$

Repeating the standard argument, we find that

$$[L_n, L_m] \;=\; \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i}\, z^{n+1}\, w^m\, i\, \mathcal{R}\left(T(z)\, \partial_w \phi(w)\right) \;. \tag{5.7}$$

In order to proceed, we need to find the OPE $\mathcal{R}\left(T(z)\partial_w \phi(w)\right)$. Notice that

$$T(z) \;=\; T_{\rm old}(z) + ie_0\, \partial^2 \phi(z) \;.$$

Then we find that

$$\begin{aligned}
\mathcal{R}\left(T(z)\, \partial_w \phi(w)\right) \;&=\; \mathcal{R}\left(T_{\rm old}(z)\, \partial_w \phi(w)\right) + i\, e_0\, R\left(\partial_z^2 \phi(z)\, \partial_w \phi(w)\right) \\
&=\; \frac{\partial_w \phi(w)}{(z-w)^2} + \frac{\partial_w\left(\partial_w \phi(w)\right)}{z-w} - \frac{ie_0}{g}\, \frac{1}{(z-w)^3} + \text{reg} \;.
\end{aligned}$$

Substituting in (5.7),

$$[L_n, \phi_m] \;=\; -\, m\, \phi_{m+n} + \frac{e_0}{2g}\, n\,(n-1)\, \delta_{n,-m} \;. \tag{5.8}$$

The above relation is true for any n and m.

(b) Let

$$\phi(z) \;=\; x_0 - i\phi_0\, \ln z + i \sum_{n \neq 0} \frac{\phi_n}{n}\, z^{-n} \;,$$

for which the expansion of modes

$$i\partial\phi(z) \;=\; \sum_n \phi_n\, z^{-n-1}$$

follows. As we have seen, for a field $\Phi(z)$ with weight Δ, the adjoint is the original field evaluated at the point $1/z$ and multiplied by $z^{-2\Delta}$ (see the solution of Exercise 12 of Chapter 3):

$$\Phi^\dagger(z) \;=\; z^{-2\Delta}\, \Phi'\left(\frac{1}{z}\right) \;.$$

Therefore, for the bosonic field in the presence of background charge, recalling (2.26), we must have

$$\begin{aligned}
\phi^\dagger(z) \;&=\; \phi\left(\frac{1}{z}\right) + \frac{ie_0}{g}\, \ln\frac{1}{z} + \frac{ie_0}{2g}\, \ln(-1) \\
&=\; \left(x_0 + \frac{ie_0}{2g}\, \ln(-1)\right) + \left(-i\phi_0 + \frac{ie_0}{g}\right) \ln\frac{1}{z} + i \sum_{n\neq 0} \frac{\phi_n}{n}\, z^n \;.
\end{aligned}$$

Then

$$-i\partial\phi^\dagger(z) \;=\; \left(\phi_0 - \frac{e_0}{g}\right) \frac{1}{z} + \sum_{n\neq 0} \phi_{-n}\, z^{-n-1} \;.$$

From the last relation, we see that

$$\begin{aligned} \phi_0^\dagger &= \phi_0 - \frac{e_0}{g} \ , \qquad \text{and} \\ \phi_n^\dagger &= \phi_{-n} \ , \ \ n \neq 0 \ . \end{aligned}$$

(c) From the OPE

$$\mathcal{R}(\partial\phi(z) e^{ia\phi(w)}) \ = \ \frac{-ia/2g}{z-w}\, e^{ia\phi(w)} + \text{reg} \ ,$$

we see that the operator

$$\phi_0 \ \equiv \ \oint \frac{dz}{2\pi i}\, i\partial\phi(z) \ ,$$

satisfies the commutation relation

$$[\phi_0, V_a(w)] \ = \ \oint \frac{dz}{2\pi i}\, \mathcal{R}\left(i\partial\phi(z) V_a(w)\right) \ = \ \frac{a}{2g}\, V_a(w) \ .$$

This implies that the charge operator we are looking for is

$$Q \ \equiv \ 2g\, \phi_0 \ .$$

As we have seen in Exercise 9 of Chapter 2,

$$L_0 \ = \ 2g \sum_{n=1}^{+\infty} \phi_{-n}\phi_n + g\, \phi_0^2 - e_0\, \phi_0 \ .$$

The vacuum state $|\, \emptyset\, ;\, e_0 \,\rangle$ must be annihilated by this operator, so

$$0 \ = \ L_0 \,|\, \emptyset\, ;\, e_0 \,\rangle \ = \ (g x^2 - e_0 x)\, |\, \emptyset\, ;\, e_0 \,\rangle \ ,$$

where we also used the fact that for $n > 0$, $\phi_n \,|\, \emptyset\, ;\, e_0 \,\rangle = 0$, and that $\phi_0 \,|\, \emptyset\, ;\, e_0 \,\rangle = x \,|\, \emptyset\, ;\, e_0 \,\rangle$. The last equation implies that $x = e_0/g$.

The commutation relation

$$[Q,\, O] \ = \ q\, O$$

now implies

$$\begin{aligned} q\, \langle\, \emptyset\, ;\, e_0 \,|\, O \,|\, \emptyset\, ;\, e_0 \,\rangle \ &= \ 2g \,\langle\, \emptyset\, ;\, e_0 \,|\, \phi_0\, O \,|\, \emptyset\, ;\, e_0 \,\rangle - 2g \,\langle\, \emptyset\, ;\, e_0 \,|\, O\, \phi_0 \,|\, \emptyset\, ;\, e_0 \,\rangle \\ &= \ 2g \,\langle\, \emptyset\, ;\, e_0 \,|\left(\phi_0^\dagger + \frac{e_0}{g}\right) O \,|\, \emptyset\, ;\, e_0 \,\rangle - 2g \,\langle\, \emptyset\, ;\, e_0 \,|\, O\, \phi_0 \,|\, \emptyset\, ;\, e_0 \,\rangle \\ &= \ 2g \left(\frac{e_0}{g} + \frac{e_0}{g}\right) \langle\, \emptyset\, ;\, e_0 \,|\, O \,|\, \emptyset\, ;\, e_0 \,\rangle - 2g \frac{e_0}{g}\, \langle\, \emptyset\, ;\, e_0 \,|\, O \,|\, \emptyset\, ;\, e_0 \,\rangle \ , \end{aligned}$$

or

$$(q - 2e_0)\, \langle\, \emptyset\, ;\, e_0\, |\, O\, |\, \emptyset\, ;\, e_0\, \rangle \;=\; 0\ .$$

From this, it follows immediately that

$$q \neq 2e_0 \;\Rightarrow\; \langle\, O\, \rangle \;=\; 0\ ,$$

while $\langle\, O\, \rangle$ can be non-zero if and only if $q = 2e_0$.

3. A screening charge

$$S \;=\; \oint dz\, J(z)$$

should be an operator that does not change the conformal properties if inserted in the correlation functions. This implies that

$$\Delta_J \;=\; 1\ ,$$

since in this case S commutes with the Virasoro generators. To see this, we first notice that for unit conformal weight

$$\mathcal{R}\left(T(z)J(w)\right) \;=\; \partial_w \left(\frac{J(w)}{z-w}\right) + \text{reg}\ . \tag{5.9}$$

Then

$$\begin{aligned}
[L_n, J(z)] \;&=\; \oint \frac{dw}{2\pi i}\, w^{n+1}\, \mathcal{R}\left(T(w)J(z)\right) \;=\; \partial_z \left(T(z) \oint_z \frac{dw}{2\pi i}\, \frac{w^{n+1}}{w-z}\right) \\
&=\; \partial_z \left(z^{n+1} J(z)\right)\ , \quad \forall n\ .
\end{aligned}$$

For $n \geq -1$, the function $\partial_z(z^{n+1}J(z)) = (n+1)z^n J(z) + z^{n+1}\partial J(z)$ is analytic, and

$$[L_n, S] \;=\; \oint \frac{dz}{2\pi i}\, \partial_z \left(z^{n+1} J(z)\right) \;=\; 0\ . \tag{5.10}$$

This result says, in particular, that the screening operator S commutes with the generators L_{-1}, L_0, and L_1 which generate the global conformal group, as well as with all the L_k, $k > 0$, which annihilate the null states (see the next problem). What do you think happens for $n < -1$?

For a free boson, $J(z)$ may be of the form $\partial\phi$ or $e^{i\alpha\phi}$. The first choice gives us the U(1) charge

$$\oint \frac{dz}{2\pi i}\, i\partial\phi \;=\; \phi_0\ .$$

Therefore we consider the second possibility, $J(z) = e^{i\alpha\phi(z)}$. Then we must have

$$1 \;=\; \Delta_J \;=\; \frac{\alpha(\alpha - 2e_0)}{4g} \Rightarrow \alpha^2 - 2e_0\alpha - 4g \;=\; 0 \;,$$

the solution of which is

$$\alpha \;\equiv\; e_\pm \;=\; e_0 \pm \sqrt{e_0^2 + 4g} \;.$$

Therefore, there are two screening charges:

$$S_\pm \;=\; \oint dz\, e^{ie_\pm\phi} \;.$$

4. Given the vertex operator $V_{2\alpha_0-\alpha-n\alpha_+}(w)$, we can form the operator

$$\chi^+ = \oint_{C_1} \frac{dz_1}{2\pi i}\oint_{C_2}\frac{dz_2}{2\pi i}\cdots\oint_{C_n}\frac{dz_n}{2\pi i} J_+(z_1)J_+(z_2)\cdots J_+(z_n)V_{2\alpha_0-\alpha-n\alpha_+}(w) \;, \qquad (5.11)$$

carrying $2\alpha_0 - \alpha$ units of charge.

Using equation (5.10), we can show that the operator χ^+ defined above is a highest weight operator

$$\begin{aligned} L_0\chi^+ &= (\Delta_\alpha + N)\,\chi^+ \;, \\ L_k\chi^+ &= 0 \;, \qquad k > 0 \;, \end{aligned}$$

where Δ_α stands for the conformal weight of V_α and N is a number that will be computed below. To prove this, we examine the properties of the integrand in equation (5.11). Let us concentrate on the variable z_1 (similar considerations are valid for any of the variables). Using the identity

$$e^{i\sqrt{2}a\phi(z)}e^{ib\sqrt{2}\phi(w)} \;=\; (z-w)^{2ab}\; : e^{ia\sqrt{2}\phi(z)}e^{ib\sqrt{2}\phi(w)} : \;,$$

we see that the factor

$$(z_1 - z_2)^{2\alpha_+^2}(z_1 - z_3)^{2\alpha_+^2}\ldots(z_1 - z_n)^{2\alpha_+^2}(z_1 - w)^{2\alpha_+(2\alpha_0-\alpha-n\alpha_+)}$$

will apppear. The branch cuts are avoided if

$$2\alpha_+^2(n-1) + 2\alpha_+(2\alpha_0 - \alpha - n\alpha_+) \;=\; -m - 1 \;, \qquad (5.12)$$

where $m = 1, 2, \ldots$. The value $m = 0$ is excluded, as it gives $V_{2\alpha_0-\alpha}$. We thus have

$$\begin{aligned}
L_k\chi^+ &= \oint_{C_1}\frac{dz_1}{2\pi i}\oint_{C_2}\frac{dz_2}{2\pi i}\cdots\oint_{C_n}\frac{dz_n}{2\pi i}L_kJ_+(z_1)J_+(z_2)\cdots J_+(z_n)V_{2\alpha_0-\alpha-n\alpha_+}(w)\\
&= \oint_{C_1}\frac{dz_1}{2\pi i}\oint_{C_2}\frac{dz_2}{2\pi i}\cdots\oint_{C_n}\frac{dz_n}{2\pi i}J_+(z_1)L_kJ_+(z_2)\cdots J_+(z_n)V_{2\alpha_0-\alpha-n\alpha_+}(w)\\
&= \oint_{C_1}\frac{dz_1}{2\pi i}\oint_{C_2}\frac{dz_2}{2\pi i}\cdots\oint_{C_n}\frac{dz_n}{2\pi i}J_+(z_1)L_kJ_+(z_2)\cdots J_+(z_n)V_{2\alpha_0-\alpha-n\alpha_+}(w)\\
&= \cdots\cdots\\
&= \oint_{C_1}\frac{dz_1}{2\pi i}\oint_{C_2}\frac{dz_2}{2\pi i}\cdots\oint_{C_n}\frac{dz_n}{2\pi i}J_+(z_1)J_+(z_2)\cdots J_+(z_n)L_kV_{2\alpha_0-\alpha-n\alpha_+}(w)\\
&= \begin{cases} 0\ , & \text{if } k>0\ ,\\ \Delta_{2\alpha_0-\alpha-n\alpha_+}\chi^+\ , & \text{if } k=0\ ,\end{cases} .
\end{aligned}$$

We have thus proved that χ^+ is a highest weight operator and thus a null operator in $[V_\alpha]$.

The spectrum of the theory is determined directly from what has been written above. In particular, equation (5.12), when we use the facts that

$$2\alpha_0 \;=\; \alpha_+ + \alpha_- \quad \text{and} \quad \alpha_+\alpha_- \;=\; -1\ ,$$

gives

$$\alpha \;=\; \frac{1-m}{2}\alpha_-\ , \qquad m = 1, 2, \ldots\ .$$

We could repeat the same line of reasoning for the case of J_-. Doing so leads to the result

$$\alpha \;=\; \frac{1-n}{2}\alpha_+\ , \qquad n = 1, 2, \ldots\ .$$

Combining the two results, we have thus found that the spectrum of primary operators V_α is in one-to-one correspondence with the charges

$$\alpha_{mn} \;=\; \frac{1-m}{2}\alpha_- + \frac{1-n}{2}\alpha_+\ , \qquad m, n = 1, 2, \ldots\ .$$

To assist us later, we make some additional comments here. The charge $2\alpha_0 - \alpha_{mn}$ corresponds to $\alpha_{-m,-n}$, while the charge $2\alpha_0 - \alpha_{mn} - n\alpha_+$ corresponds to $\alpha_{-m,n}$. The conformal weights have the properties $\Delta_{mn} = \Delta_{-m,-n}$ and

$$\Delta_{-m,n} \;=\; \Delta_{mn} + mn\ .$$

From this we conclude that the module $[V_{mn}]$ admits a null state at level nm.

Finally, notice that if $\alpha_+ \;=\; \sqrt{r/s}$, then

$$\Delta_{mn} \;=\; \Delta_{r-m,s-n}\ .$$

This implies that the spectrum in this case is restricted to a finite number of primary fields with $m \leq r-1$ and $n \leq s-1$.

5. To study the fusion rules
$$[\phi_{m_1,n_1}] \times [\phi_{m_2,n_2}] \quad ,$$
we must study the 3-point correlation functions
$$\langle\, \phi_{m_1,n_1}\phi_{m_2,n_2}\phi_{m_3,n_3}\,\rangle \quad ,$$
and find for what combination of the indices these correlation functions do not vanish. According to the CGF, this correlation function is computed by
$$\left\langle\, V_{2\alpha_0-\alpha_1}V_{\alpha_2}V_{\alpha_3}S_+^{N_+}S_-^{N_-}\,\right\rangle \ , \tag{5.13}$$
where α_i is the charge corresponding to the pair of integers m_i and n_i. Charge neutrality in the last equation requires
$$(2\alpha_0-\alpha_1)+\alpha_2+\alpha_3+N_+\alpha_+ + N_-\alpha_- \;=\; 2\alpha_0 \ ,$$
or in other words,
$$-\alpha_1+\alpha_2+\alpha_3+N_+\alpha_+ + N_-\alpha_- \;=\; 0 \ .$$
Substituting the charges with their equivalent expressions from equation (5.1), we find
$$\left(\frac{1+m_1-m_2-m_3}{2}+N_+\right)\,\alpha_+ + \left(\frac{1+n_1-n_2-n_3}{2}+N_-\right)\,\alpha_- \;=\; 0 \ .$$
Let us consider first the cancellation of the α_+ charges. Since $N_+ \geq 0$, the quantity $1+m_1-m_2-m_3$ must be an even negative integer:
$$1+m_1-m_2-m_3 \;=\; \text{even} \ , \qquad 1+m_1-m_2-m_3 \leq 0 \ . \tag{5.14}$$
Instead of (5.13), one could have used either of the following two correlators:
$$\left\langle\, V_{\alpha_1}V_{2\alpha_0-\alpha_2}V_{\alpha_3}S_+^{M_+}S_-^{M_-}\,\right\rangle \ ,$$
$$\left\langle\, V_{\alpha_1}V_{\alpha_2}V_{2\alpha_0-\alpha_3}S_+^{L_+}S_-^{L_-}\,\right\rangle \ .$$
In these cases, one finds analogous constraints, respectively:
$$1-m_1+m_2-m_3 \;=\; \text{even} \ , \qquad 1-m_1+m_2-m_3 \leq 0 \ , \quad \text{or} \tag{5.15}$$
$$1-m_1-m_2+m_3 \;=\; \text{even} \ , \qquad 1-m_1-m_2+m_3 \leq 0 \ . \tag{5.16}$$
Adding the first relations of (5.14), (5.15), and (5.16) gives:
$$m_1+m_2+m_3 \;=\; \text{odd} \ .$$

The second relation of (5.16) implies that

$$m_3 \leq m_1 + m_2 - 1 \ ,$$

while the second relations of (5.14) and (5.15) imply that

$$\left.\begin{array}{l} m_3 \geq m_2 - m_1 + 1 \\ m_3 \geq m_1 - m_2 + 1 \end{array}\right\} \Rightarrow m_3 \geq |m_1 - m_2| + 1 \ .$$

Identical results are obtained for the second index using the cancellation of the α_- charges. We can thus write the fusion rules as follows:

$$[\phi_{m_1,n_1}] \times [\phi_{m_2,n_2}] \ = \sum_{\substack{m_3=|m_1-m_2|+1 \\ m_1+m_2+m_3=\text{odd}}}^{m_1+m_2-1} \ \sum_{\substack{n_3=|n_1-n_2|+1 \\ n_1+n_2+n_3=\text{odd}}}^{n_1+n_2-1} [\phi_{m_3,n_3}] \ .$$

For the MMs,

$$\Phi_{mn} \ = \ \Phi_{r-m,s-n} \quad ,$$

and therefore

$$[\phi_{m_1,n_1}] \times [\phi_{m_2,n_2}] \ = \ [\phi_{r-m_1,s-n_1}] \times [\phi_{r-m_2,s-n_2}] \ .$$

Applying the formula we found above, we see

$$\sum_{\substack{m_3=|m_1-m_2|+1 \\ m_1+m_2+m_3=\text{odd}}}^{m_1+m_2-1} \ \sum_{\substack{n_3=|n_1-n_2|+1 \\ n_1+n_2+n_3=\text{odd}}}^{n_1+n_2-1} [\phi_{m_3,n_3}] \ = \sum_{\substack{m_3=|m_1-m_2|+1 \\ m_1+m_2+m_3=\text{odd}}}^{2r-m_1-m_2-1} \ \sum_{\substack{n_3=|n_1-n_2|+1 \\ n_1+n_2+n_3=\text{odd}}}^{2s-n_1-n_2-1} [\phi_{m_3,n_3}] \ ,$$

or

$$\sum_{\substack{m_3>\min\{m_1+m_2-1,2r-m_1-m_2-1\} \\ m_1+m_2+m_3=\text{odd}}} \ \sum_{\substack{n_3>\min\{n_1+n_2-1,2s-n_1-n_2-1\} \\ n_1+n_2+n_3=\text{odd}}} [\phi_{m_3,n_3}] \ = \ 0 \ .$$

So, finally, the fusion rules for the MMs are

$$[\phi_{m_1,n_1}] \times [\phi_{m_2,n_2}] \ = \sum_{\substack{m_3=|m_1-m_2|+1 \\ m_1+m_2+m_3=\text{odd}}}^{\min\{m_1+m_2-1,2r-m_1-m_2-1\}} \ \sum_{\substack{n_3=|n_1-n_2|+1 \\ n_1+n_2+n_3=\text{odd}}}^{\min\{n_1+n_2-1,2s-n_1-n_2-1\}} [\phi_{m_3,n_3}] \ .$$

6. For the Ising model $c = 1/2$. This means that, using the CGF, one has $1-24\alpha_0^2 = 1/2$, which sets the value of the background charge to be

$$\alpha_0 \;=\; -\frac{1}{4\sqrt{3}} \;.$$

Then

$$\alpha_+ \;=\; \frac{\sqrt{3}}{2} \quad \text{and} \quad \alpha_- \;=\; -\frac{2}{\sqrt{3}} \;.$$

The field $\sigma(z)$ has conformal dimension $1/16$, and corresponds to ϕ_{12} with charge $\alpha_{12} = -\alpha_+/2 = -\sqrt{3}/4$.

Using the prescription of the CGF, the correlation function

$$G^{(4)}(z_i) \;=\; \langle\, \sigma(z_1)\, \sigma(z_2)\, \sigma(z_3)\, \sigma(z_4)\,\rangle$$

maps to the bosonic correlator

$$\begin{aligned} G^{(4)}(z_i) \;&=\; \langle\, \phi_{12}(z_1)\, \phi_{12}(z_2)\, \phi_{12}(z_3)\, \phi_{12}(z_4)\,\rangle \\ &=\; \oint_C dv\, \langle\, V_{\alpha_{12}}(z_1)\, V_{\alpha_{12}}(z_2)\, V_{\alpha_{12}}(z_3)\, V_{2\alpha_0-\alpha_{12}}(z_4)\, J_+(v)\,\rangle \;, \end{aligned}$$

which, upon using the identity

$$\left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \right\rangle \;=\; \prod_{i=1}^{n}\prod_{j=i+1}^{n} (z_i - z_j)^{2\alpha_i\alpha_j} \;,$$

takes the form

$$\begin{aligned} G^{(4)}(z_i) \;=\;& (z_1-z_2)^{2\alpha_{12}^2}\,(z_1-z_3)^{2\alpha_{12}^2}\,(z_1-z_4)^{2\alpha_{12}(2\alpha_0-\alpha_{12})} \\ &\times (z_2-z_3)^{2\alpha_{12}^2}\,(z_2-z_4)^{2\alpha_{12}(2\alpha_0-\alpha_{12})}\,(z_3-z_4)^{2\alpha_{12}(2\alpha_0-\alpha_{12})} \\ &\times \oint_C dv\,(z_1-v)^{2\alpha_{12}\alpha_+}\,(z_2-v)^{2\alpha_{12}\alpha_+}\,(z_3-v)^{2\alpha_{12}\alpha_+}\,(z_4-v)^{2\alpha_+(2\alpha_0-\alpha_{12})} \;. \end{aligned}$$

Recall from equation (3.17) that the (holomorphic) correlation function can be expressed as

$$G^{(4)} \;=\; (x z_{23} z_{41})^{-1/8}\, f(x) \;\Rightarrow\; f(x) \;=\; (x z_{23} z_{41})^{1/8}\, G^{(4)} \;.$$

We can take advantage of conformal invariance to fix the values of the points z_1, z_2, z_3, and z_4. We choose

$$z_1 \;=\; 0\;, \quad z_2 \;=\; x\;, \quad z_3 \;=\; 1\;, \quad \text{and} \quad z_4 \to \infty \;.$$

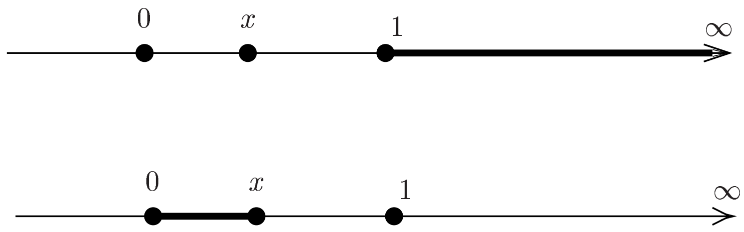
Then, using all of the above, we have

$$f(x) \;=\; x^{1/2}\,(1-x)^{1/2} \oint_C dv\, v^{\alpha}\,(x-v)^{\alpha}\,(1-v)^{\alpha} \;,$$

where

$$\alpha \;=\; 2\alpha_{12}\alpha_+ \;=\; -\,3/4\ .$$

Now we have to choose the contour C of integration; we have two independent choices, which are clearly depicted in the figure below:



Other choices are not independent. This means that we have two independent solutions:

$$f_1(x) \;=\; x^{1/2}\,(1-x)^{1/2}\int_1^\infty dv\, v^\alpha\,(x-v)^\alpha\,(1-v)^\alpha\ , \quad \text{and} \tag{5.17}$$

$$f_2(x) \;=\; x^{1/2}\,(1-x)^{1/2}\int_0^x dv\, v^\alpha\,(x-v)^\alpha\,(1-v)^\alpha\ . \tag{5.18}$$

The function $f(x)$ will be a linear combination of $f_1(x)$ and $f_2(x)$.

Now recall that the hypergeometric function has the following integral representation:

$$F\,(a,\,b;\,c;\,x) \;\equiv\; \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 du\, u^{b-1}\,(1\,-\,u)^{c-b-1}\,(1\,-\,xu)^{-a}\ , \tag{5.19}$$

for $|x| < 1$, $\mathrm{Re}\,c\, >\, \mathrm{Re}\,b\, >\, 0$. For other values, we can define F by analytic continuation.

By making the substitutions $v = 1/u$ and $v = xu$ in the integrals of equations (5.17) and (5.18), respectively, we find

$$f_1(x) \;=\; x^{1/2}\,(1-x)^{1/2}\int_0^1 du\, u^{-3\alpha-2}\,(1-xu)^\alpha\,(1-u)^\alpha\ , \quad \text{and}$$

$$f_2(x) \;=\; (1-x)^{1/2}\int_0^1 du\, u^\alpha\,(1-xu)^\alpha\,(1\,-\,u)^\alpha\ .$$

Using the representation of the hypergeometic function (5.19), we see

$$f_1(x) \;=\; x^{1/2}\,(1-x)^{1/2}\,F\,(-\alpha,\,-3\alpha\,-\,1;\,-2\alpha;\,x) \;=\; x^{1/2}\,(1-x)^{1/2}\,F\left(\frac{3}{4},\frac{5}{4};\frac{3}{2};x\right)\ ,$$

$$f_2(x) \;=\; (1-x)^{1/2}\,F\,(-\alpha,\,\alpha+1;\,2\alpha+2;\,z) \;=\; x^{1/2}\,(1-x)^{1/2}\,F\left(\frac{3}{4},\frac{1}{4};\frac{1}{2};x\right)\ ,$$

There is an identity for the hypergeometric function that is useful in the present context:

$$F\left(\frac{1}{2}+a,a;2a;x\right) \;=\; 2^{2a-1}\,(1-x)^{-1/2}\,\left(1+\sqrt{1-x}\right)^{1-2a} \;.$$

Substituting the values $a = 3/4$ and $\alpha = 1/4$ in this identity gives, respectively,

$$\begin{aligned} F\left(\frac{3}{4},\frac{5}{4};\frac{3}{2};x\right) &= \frac{1}{\sqrt{2}}\,(1-x)^{-1/2}\,\frac{1}{\sqrt{1+\sqrt{1-x}}}\;, \quad \text{and} \\ F\left(\frac{3}{4},\frac{1}{4};\frac{1}{2};x\right) &= \sqrt{2}\,(1-x)^{-1/2}\,\sqrt{1+\sqrt{1-x}}\;. \end{aligned}$$

Thus

$$\begin{aligned} f_1(x) &= \frac{1}{\sqrt{2}}\,\frac{\sqrt{x}}{\sqrt{1+\sqrt{1-x}}}\;, \\ f_2(x) &= \sqrt{2}\sqrt{1+\sqrt{1-x}}\;. \end{aligned}$$

The function $f_1(x)$ can easily be rewritten in an equivalent form:

$$\begin{aligned} f_1(x) &= \frac{1}{\sqrt{2}}\,\frac{\sqrt{x}}{\sqrt{1+\sqrt{1-x}}}\,\frac{\sqrt{1-\sqrt{1-x}}}{\sqrt{1-\sqrt{1-x}}} \;=\; \frac{1}{\sqrt{2}}\,\frac{\sqrt{x}\sqrt{1-\sqrt{1-x}}}{\sqrt{1-(1-x)}} \\ &= \frac{1}{\sqrt{2}}\sqrt{1-\sqrt{1-x}}\;. \end{aligned}$$

We have thus found the same functions as in Exercise 11 of Chapter 3.

7. (a) A state at level N has the form

$$L_{-N}^{n_N}\ldots L_{-2}^{n_2}L_{-1}^{n_1}\,|\,\Delta\,\rangle\;,$$

with the level N given by

$$N \;=\; N\,n_N + (N-1)\,n_{N-1} + \ldots + 2\,n_2 + n_1\;.$$

Therefore, there are as many states at level N as there are partitions $p(N)$ of N into k parts $(n_1, n_2, \ldots, n_k)$ with $k = 1, 2, \ldots, N$ according to

$$p(N) \;=\; \sum_{\substack{n_1,n_2,\ldots,n_N \\ n_1+2n_2+\ldots+Nn_N=N}} 1\;.$$

Consequently,

$$\begin{aligned}
\sum_{N=0}^{+\infty} p(N) q^N &= \sum_{N=0}^{+\infty} \sum_{\substack{n_1,n_2,\ldots,n_N \\ n_1+2n_2+\ldots+Nn_N=N}} q^{n_1+2n_2+\ldots+Nn_N} \\
&= \sum_{n_1,n_2,\ldots,n_N,\ldots} q^{n_1+2n_2+\ldots+Nn_N+\ldots} \\
&= \sum_{n_1=0}^{+\infty} q^{n_1} \sum_{n_2=0}^{+\infty} q^{2n_2} \cdots \sum_{n_N=0}^{+\infty} q^{Nn_N} \cdots \\
&= \frac{1}{1-q} \frac{1}{1-q^2} \frac{1}{1-q^3} \cdots \\
&= \frac{1}{\prod_{k=1}^{+\infty}(1-q^k)} \,.
\end{aligned}$$

Hence, without null states,

$$\chi_\Delta(q) = \frac{q^{\Delta-c/24}}{\prod_{k=1}^{+\infty}(1-q^k)} \,.$$

(b) We saw above that, without a null state, one has

$$\chi_{\text{naive}}(q) = \frac{q^{\Delta-c/24}}{\prod_{k=1}^{+\infty}(1-q^k)} \,.$$

However, once there are null states, this expression overcounts the states. To compensate, we must subtract the null states .

If there is a null state $|\,\Delta+N\,\rangle$ at level N, then all states of the form

$$L_{-k}^{n_k} \ldots L_{-2}^{n_2} L_{-1}^{n_1} \,|\,\Delta+N\,\rangle$$

are also null. They form a Verma module over $|\,\Delta+N\,\rangle$. Therefore the corresponding character is given by

$$\chi_{\text{null}}(q) = \frac{q^{\Delta+N-c/24}}{\prod_{k=1}^{+\infty}(1-q^k)} \,.$$

The correct counting of states is therefore given by

$$\chi_\Delta(q) = \chi_{\text{naive}} - \chi_{\text{null}}(q) = \frac{q^{\Delta-c/24}}{\prod_{k=1}^{+\infty}(1-q^k)}(1-q^N) \,.$$

(c) Referring to part (b), if the Verma module over the null state $|\,\Delta+N\,\rangle$ itself has a null state at level M, then we must first subtract the null states from this module to find the correct number of states that we should subtract from χ_{naive}. This gives

$$\chi_{\text{naive}} - \left[\chi_{\text{null}} - \chi_{\text{null of null}}\right] \,,$$

or

$$\frac{q^{\Delta - c/24}}{\prod_{k=1}^{+\infty}(1-q^k)}\left(1 - q^N + q^{N+M}\right) .$$

It now becomes obvious how to extend this in cases where the null states within the null states continue to a deep level, or even *ad infinitum*:

$$\chi_{\text{naive}} - \left\{\chi_{\text{null}} - \left[\chi_{\text{null of null}} - \left(\chi_{\text{null of null of null}} - (\quad \cdots \quad)\right)\right]\right\} ,$$

Also, it is straightforward to generalize these results to cases in which the Verma modules have more than one null state at each subtraction step, and which can then overlap (see problem 8).

---

8. Before we proceed to the solution, we introduce some notation to simplify our formulæ. Let

$$\begin{aligned} a(k) &\equiv \Delta_{m,n+2sk} = \Delta_{m-2rk,n} \quad , \quad \text{and} \\ b(k) &\equiv \Delta_{-m,n+2sk} = \Delta_{-m-2rk,n} \ , \end{aligned}$$

where we have also used the properties

$$\Delta_{mn} = \Delta_{r-m,s-n} = \Delta_{-m,-n} = \Delta_{r+m,s+n} .$$

We also let

$$[a(k)] = [V_{m,n+2sk}] \quad \text{and} \quad [b(k)] = [V_{-m,n+2sk}] .$$

To solve this problem, we directly apply the results from the previous problems. In Exercise 4, we saw that the module $[V_{mn}] = [a(0)]$ of the MMs admits a null state with weight $\Delta_{-m,n} = b(0)$. Because of the symmetry $\Delta_{mn} = \Delta_{r-m,s-n}$, the same module also admits a second null state with weight $\Delta_{-(r-m),s-n} = \Delta_{r-m,-s+n} = \Delta_{-m,-2s-n} = b(-1)$. Therefore, to compute the number of states for $[a(0)]$ we must subtract the states of $[b(0)]$ and $[b(-1)]$:

$$[a(0)] - [b(0)] - [b(-1)] .$$

However, this is not the end, since, by the same argument, the modules $[b(0)]$ and $[b(-1)]$ have null states themselves. Continuing this way, we discover an infinite tower of null states embedded within null states. Using simple arguments as above, we find the following embeddings:

- The module $[a(k)]\,,\ k > 0$ has null states with weights $b(k)$ and $b(-k-1)$.
- The module $[a(-k)]\,,\ k > 0$ has null states with weights $b(k)$ and $b(-k-1)$.

- The module $[b(k)]\,,\ k>0$ has null states with weights $a(k+1)$ and $a(-k-1)$.
- The module $[b(-k)]\,,\ k>0$ has null states with weights $a(k)$ and $a(-k)$.

The following diagram makes it easy to appreciate the series of embeddings:

$$
\begin{tikzcd}
 & b(0) \arrow[dl] & a(1) \arrow[l] \arrow[dl] & b(1) \arrow[l] \arrow[dl] & \cdots \arrow[l] \arrow[dl] \\
a(0) & & & & \\
 & b(-1) \arrow[ul] & a(-1) \arrow[l] \arrow[ul] & b(-2) \arrow[l] \arrow[ul] & \cdots \arrow[l] \arrow[ul]
\end{tikzcd}
$$

Since there is overlapping among the modules of the null states, we must be careful to subtract each null state only once. Notice, too, that the various modules are entering in the series with alternating signs:

$$
\begin{tikzcd}
 & b(0) \arrow[dl] & a(1) \arrow[l] & b(1) \arrow[l] & \cdots \arrow[l] \\
a(0) & & & & \\
 & b(-1) \arrow[ul] & a(-1) \arrow[l] & b(-2) \arrow[l] & \cdots \arrow[l] \\
+ & - & + & - &
\end{tikzcd}
$$

Therefore,

$$
\chi_{mn}(q) \;=\; \frac{q^{-c/24}}{\prod_{n=1}^{+\infty}(1-q^n)} \sum_{k\in\mathbb{Z}} \left(q^{a(k)} - q^{b(k)}\right) \;.
$$

The product $\prod_{n=1}^{+\infty}(1-q^n)$ appears often when one considers characters and related objects. It is standard and convenient to define the **Detekind η-function**

$$
\eta(q) \;\equiv\; q^{1/24} \prod_{n=1}^{+\infty} (1-q^n) \;,
$$

which we will encounter again often.

Chapter 6

MODULAR INVARIANCE

References: The mathematical literature on the topics covered in this chapter is vast. However, most of the main tools and techniques can also be found in physics-oriented papers and books, such as [94, 186, 333, 420]. A popular reference on Riemann surfaces is [249], and one on θ-functions is [509]. A recent short introduction to the theory of functions on compact surfaces is [438]. More results on algebraic geometry may be found in the classic book [357].

6.1 BRIEF SUMMARY

6.1.1 Riemann ϑ-function

On the sphere (or, equivalently, on the compactified plane), the monomials $z - z_i$ can be viewed as the fundamental building blocks out of which functions are built.[1] The analogous building blocks for functions defined on higher genus Riemann surfaces are the so-called ϑ-functions discussed in this chapter.

The **Riemann ϑ-function** is defined by the expression

$$\vartheta(u_1, u_2, \ldots, u_g|\tau_{ij}) \;\equiv\; \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \cdots \sum_{m_g=-\infty}^{+\infty} e^{2\pi i \sum\limits_{j=1}^{g} u_j m_j} \, e^{\pi i \sum\limits_{j,k=1}^{g} m_i \tau_{ij} m_j} \; .$$

The convergence of the Riemann ϑ-function is guaranteed if $\mathrm{Im}\tau_{ij} > 0$. Usually the following compact notation is used:

$$\vartheta(\boldsymbol{u}|\boldsymbol{\tau}) \;=\; \sum_{\boldsymbol{m}\in\mathbb{Z}^g} e^{2\pi i \boldsymbol{u}\boldsymbol{m}} \, e^{\pi i \boldsymbol{m}^T \boldsymbol{\tau}\boldsymbol{m}} \; , \tag{6.1}$$

where $\boldsymbol{u}$ and $\boldsymbol{m}$ are column matrices, and $\boldsymbol{\tau}$ is a $g \times g$ symmetric matrix:

$$\boldsymbol{u} \;\equiv\; [u_i] \; , \qquad \boldsymbol{m} \;\equiv\; [m_i] \; , \qquad \boldsymbol{\tau} \;\equiv\; [\tau_{ij}] \; .$$

[1] See, for example, the Mittag-Leffler theorem in Chapter 4.

When there is simply a dot product such as $u_i m_i$, to avoid clutter we will write this as $\boldsymbol{um}$ rather than the more proper $\boldsymbol{u}^T\boldsymbol{m}$. When the ϑ-function is related to a Riemann surface, the parameter g is the genus of the surface and $\boldsymbol{\tau}$ is the period matrix.

One can also introduce the Riemann **ϑ-function with characteristics**:

$$\vartheta\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}(\boldsymbol{u}|\boldsymbol{\tau}) \equiv \sum_{\boldsymbol{m}\in\mathbb{Z}^g} e^{2\pi i(\boldsymbol{m}+\boldsymbol{a})(\boldsymbol{u}+\boldsymbol{b})}\, e^{\pi i(\boldsymbol{m}+\boldsymbol{a})^T\boldsymbol{\tau}(\boldsymbol{m}+\boldsymbol{a})} \tag{6.2}$$

$$= e^{2\pi i\boldsymbol{a}(\boldsymbol{u}+\boldsymbol{b})}\, e^{\pi i\boldsymbol{a}^T\boldsymbol{\tau}\boldsymbol{a}}\, \vartheta(\boldsymbol{u}+\boldsymbol{\tau}\boldsymbol{a}+\boldsymbol{b}|\boldsymbol{\tau})\ , \tag{6.3}$$

where $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^g$.

For $g=1$, we introduce the simplified notation

$$q\equiv e^{2i\pi\tau}\ ,\qquad z\equiv e^{2i\pi u}\ .$$

In addition, in mathematical physics, the following notation is typically used:

$$\begin{aligned}
\vartheta_1(u|\tau) &\equiv \sum_{m=-\infty}^{+\infty}(-1)^m\, z^{m+1/2}\, q^{(m+1/2)^2/2} = -i\vartheta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(u|\tau)\ ;\\
\vartheta_2(u|\tau) &\equiv \sum_{m=-\infty}^{+\infty} z^{m+1/2}\, q^{(m+1/2)^2/2} = \vartheta\begin{bmatrix}1/2\\ 0\end{bmatrix}(u|\tau)\ ;\\
\vartheta_3(u|\tau) &\equiv \sum_{m=-\infty}^{+\infty} z^{m}\, q^{m^2/2} = \vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(u|\tau)\ ;\quad\text{and}\\
\vartheta_4(u|\tau) &\equiv \sum_{m=-\infty}^{+\infty}(-1)^m\, z^{m}\, q^{m^2/2} = \vartheta\begin{bmatrix}0\\ 1/2\end{bmatrix}(u|\tau)\ .
\end{aligned}$$

The above four functions are called the **Jacobi ϑ_i-functions** $(i=1,2,3,4)$.

In $\mathbb{C}^g$, we construct the lattice $L_{\boldsymbol{\tau}}$:

$$L_{\boldsymbol{\tau}} = \mathbb{Z}^g+\boldsymbol{\tau}\,\mathbb{Z}^g = \{\boldsymbol{r}\in\mathbb{C}^g \mid \exists \boldsymbol{l},\boldsymbol{n}\in\mathbb{Z}^g,\ \boldsymbol{r}=\boldsymbol{l}+\boldsymbol{\tau}\boldsymbol{n}\}\ .$$

The ϑ-function is completely characterized by its behavior under shifts in the above lattice:

$$\vartheta\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}(\boldsymbol{u}+\boldsymbol{\tau}\boldsymbol{n}+\boldsymbol{l}|\boldsymbol{\tau}) = e^{-\pi i\boldsymbol{n}_k^T\boldsymbol{\tau}\boldsymbol{n}}\, e^{-2\pi i\boldsymbol{n}(\boldsymbol{u}+\boldsymbol{b})}\, e^{2\pi i\boldsymbol{a}\boldsymbol{l}}\, \vartheta\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}(\boldsymbol{u}|\boldsymbol{\tau}) \tag{6.4}$$

In other words, the ϑ-function is a section of a (holomorphic line) bundle on the complex torus

$$J(\Sigma)=\frac{\mathbb{C}^g}{L_{\boldsymbol{\tau}}}\ .$$

$J(\Sigma)$ is called the **Jacobian** of the Riemann surface Σ.

One sees in addition that

$$\vartheta\begin{bmatrix}\boldsymbol{a}+\boldsymbol{n}\\ \boldsymbol{b}+\boldsymbol{l}\end{bmatrix}(\boldsymbol{u}|\boldsymbol{\tau}) = e^{2\pi i\boldsymbol{a}\boldsymbol{l}}\,\vartheta\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}(\boldsymbol{u}|\boldsymbol{\tau})\ . \tag{6.5}$$

6.1.2 The Modular Group

For a surface Σ, let $\mathrm{Diff}^+(\Sigma)$ be the group of all orientation preserving diffeomorphisms of Σ, and let $\mathrm{Diff}_0^+(\Sigma)$ be the (normal) subgroup of all such diffeomorphisms connected to the identity. The quotient group

$$\mathcal{M}(\Sigma) = \frac{\mathrm{Diff}^+(\Sigma)}{\mathrm{Diff}_0^+(\Sigma)}$$

is called the **modular group** (or **mapping class group**). A non-trivial class in $\mathcal{M}(\Sigma)$ (i.e., element of $\mathcal{M}(\Sigma)$) is called a **modular transformation**. For any such class, we can always choose a representative given by a **Dehn twist** D_γ defined as follows. Let γ be a loop on Σ. We now cut the surface Σ along the loop and twist one of the edges thus produced by 2π while leaving the other unaltered. Finally, we glue back the two edges. A complete set of generators is produced when we consider all Dehn twists along curves which wind around a single handle or at most two handles.

A matrix representation $M(D_\gamma)$ of the Dehn twists D_γ can be found by considering their action on the homology basis. The intersection matrix is invariant under diffeomorphisms, and so the action of $\mathcal{M}(\Sigma)$ on $H_1(\Sigma,\mathbb{Z})$ must also preserve the intersection matrix, and therefore $M(D_\gamma) \in \mathrm{Sp}(2g,\mathbb{Z})$. In fact, it is known that the matrices $M(D_\gamma)$ generate all of $\mathrm{Sp}(2g,\mathbb{Z})$.

A Dehn twist about a homologically trivial cycle does not affect the homology class of any curve[2]; it is thus represented by a unit matrix. All such twists generate a subgroup $\mathcal{T}(\Sigma)$ of $\mathcal{M}(\Sigma)$ known as the **Torelli group**. One can show that

$$\mathrm{Sp}(2g,\mathbb{Z}) = \frac{\mathcal{M}(\Sigma)}{\mathcal{T}(\Sigma)} .$$

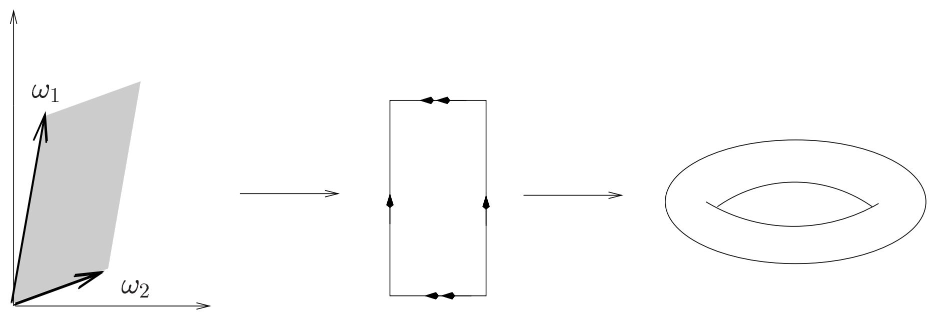


Figure 6.1: The torus is the quotient space of the plane divided by a 2-dimensional lattice.

[2]Be careful! This twist is still non-trivial.

6.1.3 Partition Functions and Modular Invariance

Mapping the plane into a cylinder, the dilation operators on the plane become the translation operators on the cylinder:

$$L_{-1}^{\rm cyl} = \frac{2\pi i}{\omega_1}\left(L_0^{\rm pl} - \frac{c}{24}\right) \quad \text{and} \quad \overline{L}_{-1}^{\rm cyl} = -\frac{2\pi i}{\overline{\omega}_1}\left(\overline{L}_0^{\rm pl} - \frac{c}{24}\right) .$$

The constant $-c/24$ arises as a Casimir effect due to the periodic boundary conditions. Assuming periodic boundary conditions on the bases of the cylinder, we create a toroidal geometry, with cycles ω_1 and ω_2. The partition function is thus given by

$$\begin{aligned} Z &= \operatorname{tr}\left(e^{\omega_2 L_{-1}^{\rm cyl} + \overline{\omega}_2 \overline{L}_{-1}^{\rm cyl}}\right) \\ &= e^{-2\pi i \frac{c}{24}\left(\frac{\omega_2}{\omega_1} + \frac{\overline{\omega}_2}{\overline{\omega}_1}\right)} \operatorname{tr}\left(e^{2\pi i \frac{\omega_2}{\omega_1} L_0^{\rm pl}} e^{2\pi i \frac{\overline{\omega}_2}{\overline{\omega}_1} \overline{L}_0^{\rm pl}}\right) . \end{aligned}$$

Defining

$$\tau \equiv \frac{\omega_2}{\omega_1} \quad \text{and} \quad q \equiv e^{2\pi i \tau} ,$$

and taking into account the decomposition of the Hilbert space into holomorphic and anti-holomorphic pieces, we can write the partition function as

$$Z = \sum_{\Delta, \overline{\Delta}} \mathcal{N}_{\Delta \overline{\Delta}} \, \chi_\Delta(\tau) \, \overline{\chi}_{\overline{\Delta}}(\tau) ,$$

where χ_Δ is the character for the Verma module built over the field with dimension Δ. The partition function contains the whole operator content of a model.

Mathematically speaking, the torus is the quotient of the complex plane divided by the lattice Λ generated by the vectors ω_1 and ω_2. Physical intuition suggests that the partition function may depend on the geometry of the torus parametrized by τ, but it should be invariant under different choices of the lattice basis ω_1, ω_2. In other words, the transformations

$$\begin{bmatrix} \omega_1' \\ \omega_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} , \qquad a, b, c, d \in \mathbb{Z} , \quad ad - bc = 1$$

should not affect the partition function. These transformations constitute the group $\mathrm{SL}(2, \mathbb{Z})$, and, since they do not change τ, we call this the modular group of the torus. The parameter τ is called the **modular parameter** (or simply the **modulus**) of the torus.

$\mathrm{SL}(2, \mathbb{Z})$ can be generated by the following two transformations of τ:

$$\begin{aligned} T : \tau &\mapsto \tau + 1 \Rightarrow (\omega_1, \omega_2) \mapsto (\omega_1, \omega_1 + \omega_2) , \\ S : \tau &\mapsto -\frac{1}{\tau} \Rightarrow (\omega_1, \omega_2) \mapsto (\omega_2, -\omega_1) . \end{aligned}$$

The requirement of modular invariance of the partition function reduces, therefore, to the conditions

$$Z(\tau+1) \;=\; Z(-1/\tau) \;=\; Z(\tau) \;.$$

Let $\chi(\tau)$ be a vector that contains the characters for all the representations present in a particular model. The partition function then takes the form

$$Z(\tau) \;=\; \chi^{\dagger}\mathcal{N}\chi \;.$$

Different models at fixed c corresponds to different matrices $\mathcal{N}$. Modular invariance implies

$$\mathcal{N} \;=\; T\mathcal{N}T^{\dagger} \quad \text{and} \quad \mathcal{N} \;=\; S\mathcal{N}S^{\dagger} \;.$$

The first equation simply requires $\Delta - \overline{\Delta} \in \mathbb{Z}$ — only integer spins are allowed. The second equation is a Diophantine equation that strongly restricts the allowed matrices $\mathcal{N}$.

There is a remarkable formula found by E. Verlinde and proved by Moore and Seiberg. The matrices $\boldsymbol{N}_i = \left[N_{ij}^{k}\right]$ that determine the fusion rules

$$[\phi_i] \times [\phi_j] \;=\; N_{ij}^{k}\,[\phi_k] \;,$$

can be written in terms of the modular S-matrix as follows:

$$N_{ij}^{k} \;=\; \sum_{n} \frac{S_j^{n}\,S_i^{n}\,S_n^{k}}{S_0^{k}} \;.$$

This equation is known as the **Verlinde formula**. A corollary of this formula is that the modular S-matrix diagonalizes the matrices $[\boldsymbol{N}_i]$:

$$\boldsymbol{N}_i \;=\; S\,\boldsymbol{D}_i\,S^{\dagger} \;,$$

where

$$\boldsymbol{D}_i \;=\; \mathrm{diag}\left[\frac{S_i^{n}}{S_0^{n}}\right] \;.$$

6.2 EXERCISES

1. (a) From the definition (6.2), prove equation (6.3) for the Riemann ϑ-function with characteristics.

   (b) Now use equation (6.3) to prove properties (6.4) and (6.5).

2. (a) For a function $g(x_1, x_2, ..., x_g)$ let

$$F_1 \equiv \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} \cdots \sum_{m_g=-\infty}^{+\infty} g(m_1, m_2, ..., m_g) \ , \quad \text{and}$$

$$F_2 \equiv \sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} \cdots \sum_{n_g=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{+\infty} dy_2 \ldots \int_{-\infty}^{+\infty} dy_g \, g(y_1, y_2, ..., y_g) \, e^{-2\pi i \sum_{j=1}^{g} n_j y_j} \ .$$

   Show that

$$F_1 = F_2 \ .$$

   This formula is known as the **Poisson resummation formula**.

   (b) Use the Poisson resummation formula to prove that the Riemann function (6.1) can also be written in the form:

$$\vartheta(\boldsymbol{u}|\boldsymbol{\tau}) = \frac{i^{g/2}}{\sqrt{|\det \boldsymbol{\tau}|}} e^{-\pi i \boldsymbol{u}^T \boldsymbol{\tau}^{-1} \boldsymbol{u}} \sum_{\boldsymbol{n}} e^{2\pi i \boldsymbol{n}^T \boldsymbol{\tau}^{-1} \boldsymbol{u}} \, e^{-\pi i \boldsymbol{n}^T \boldsymbol{\tau}^{-1} \boldsymbol{n}} \ .$$

3. Prove that Jacobi's ϑ_3-function has only one zero, which is located at $\frac{1}{2}\tau + \frac{1}{2}$.

4. In this problem, you are to prove **Jacobi's triple identity**

$$\prod_{n=1}^{+\infty} (1 - q^n)\,(1 + w\, q^{n-1/2})\,(1 + w^{-1}\, q^{n-1/2}) = \sum_{n=-\infty}^{+\infty} w^n\, q^{n^2/2} \ , \tag{6.6}$$

   for $|q| < 1$ and $w \neq 0$. You should do this two different ways:

   (a) Prove the result using physical arguments [333], starting by writing down the grand canonical partition function for a free system of fermion and antifermion oscillators.

   (b) Prove the results purely mathematically [37]. Define

$$P(q, w) \equiv \prod_{n=1}^{+\infty} (1 + w\, q^{n-1/2})\,(1 + w^{-1}\, q^{n-1/2}) \quad \text{and}$$

$$Q(q, w) \equiv \frac{1}{\prod_{n=1}^{+\infty} (1 - q^n)} \sum_{n=-\infty}^{+\infty} w^n\, q^{n^2/2} \ ,$$

and then prove that both satisfy the functional relation

$$X(q,w) = q^{1/2}\, w\, X(q,qw) \ .$$

Use this to establish (6.6).

5. Derive **Euler's pentagonal number theorem**

$$\prod_{k=1}^{+\infty} (1-q^k) \ = \ \sum_{k=-\infty}^{+\infty} (-1)^k \, q^{\frac{k(3k-1)}{2}} \ ,$$

using the following two methods:

(a) Use Jacobi's triple identity.

(b) Use the Rocha-Caridi formula for the minimal model with $p=2$ and $q=3$.

6. (a) Find an infinite product expression for each of the Jacobi ϑ_i-functions.

(b) Let

$$\theta_i(\tau) \ \equiv \ \vartheta_i(0|\tau) \ .$$

Show that

$$\eta^3(\tau) \ = \ \frac{1}{2}\,\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) \ .$$

7. Write down the matrix representation of the Dehn twists for a genus two surface.

8. Compute the partition function for a free boson. Then verify that it is modular invariant.

9. Compute the partition function for a compactified free boson, i.e., a free boson restricted to move on a circle. Then verify that this partition function is modular invariant.

10. (a) Find the characters χ_0, $\chi_{1/2}$, and $\chi_{1/16}$ of the free fermion theory by direct enumeration of the states.

(b) Determine the modular transformation matrix $\boldsymbol{S}$ acting on the character basis of part (a).

(c) Show that $\boldsymbol{S}$ diagonalizes the fusion rules of the theory.

11. Generalizing part (b) of Exercise 10, show that under the T and S transformations, the character vector of the MMs transforms as

$$T\chi(\tau) \ = \ \chi(\tau+1) \quad \text{and} \quad S\chi(\tau) \ = \ \chi(-1/\tau) \ ,$$

and derive the exact form of the matrices T and S.

12. For a theory with dyons (q, g) (i.e., a particle with electric charge q and magnetic charge g), the **Dirac-Schwinger-Zwanziger** (DSZ) **condition** says that the charges of any two dyons satisfy the quantization equation

$$q_1 g_2 - q_2 g_1 \;=\; 2\pi n_{12} \;, \qquad n_{12} \in \mathbb{Z} \;. \tag{6.7}$$

(a) Show that the general solution of this equation takes the form

$$\begin{aligned} q &= q_0 \left(n + \frac{\theta}{2\pi} m \right) \;, \\ g &= g_0 \, m \;, \end{aligned}$$

where $n, m \in \mathbb{Z}$; q_0 and g_0 are constants; and θ is a parameter with a value $0 \le \theta < 2\pi$.

(b) Show that the parameter θ is related to CP violation. In particular, show that for $\theta = 0$ and $\theta = \pi$, the theory is CP invariant, while for all other values of θ between 0 and 2π, the theory violates CP invariance.

(c) Rewrite the solution of part (a) in the form

$$q + ig \;=\; q_0 \, (n + m \, \tau) \;.$$

Explain how this form shows that the solutions to the DSZ condition (6.7) satisfy an $\mathrm{SL}(2, \mathbb{Z})$ invariance. Discuss the transformation properties of $q + ig$ under this group.

6.3 SOLUTIONS

1. (a) From the definition (6.2), we have:

$$\begin{aligned}
\vartheta\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}(\boldsymbol{u}|\boldsymbol{\tau}) &\equiv \sum_{\boldsymbol{m}} e^{2\pi i(\boldsymbol{m}+\boldsymbol{a})(\boldsymbol{u}+\boldsymbol{b})}\, e^{\pi i(\boldsymbol{m}+\boldsymbol{a})^T\boldsymbol{\tau}(\boldsymbol{m}+\boldsymbol{a})} \\
&= \sum_{\boldsymbol{m}} e^{2\pi i\boldsymbol{m}^T(\boldsymbol{u}+\boldsymbol{b})}\, e^{2\pi i\boldsymbol{a}(\boldsymbol{u}+\boldsymbol{b})}\, e^{\pi i\boldsymbol{m}^T\boldsymbol{\tau}\boldsymbol{m}}\, e^{\pi i\boldsymbol{m}^T\boldsymbol{\tau}\boldsymbol{a}}\, e^{\pi i\boldsymbol{a}^T\boldsymbol{\tau}\boldsymbol{m}}\, e^{\pi i\boldsymbol{a}^T\boldsymbol{\tau}\boldsymbol{a}} .
\end{aligned}$$

Since the matrix $\boldsymbol{\tau}$ is symmetric, one has

$$e^{\pi i\boldsymbol{m}^T\boldsymbol{\tau}\boldsymbol{a}}\, e^{\pi i\boldsymbol{a}^T\boldsymbol{\tau}\boldsymbol{m}} = e^{2\pi i\boldsymbol{m}^T\boldsymbol{\tau}\boldsymbol{a}} .$$

Hence,

$$\begin{aligned}
\vartheta\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}(\boldsymbol{u}|\boldsymbol{\tau}) &= \sum_{\boldsymbol{m}} e^{2\pi i\boldsymbol{m}^T(\boldsymbol{u}+\boldsymbol{b})}\, e^{2\pi i\boldsymbol{a}(\boldsymbol{u}+\boldsymbol{b})}\, e^{\pi i\boldsymbol{m}^T\boldsymbol{\tau}\boldsymbol{m}}\, e^{2\pi i\boldsymbol{m}^T\boldsymbol{\tau}\boldsymbol{a}}\, e^{\pi i\boldsymbol{a}^T\boldsymbol{\tau}\boldsymbol{a}} \\
&= e^{2\pi i\boldsymbol{a}(\boldsymbol{u}+\boldsymbol{b})}\, e^{\pi i\boldsymbol{a}^T\boldsymbol{\tau}\boldsymbol{a}} \sum_{\boldsymbol{m}} e^{2\pi i\boldsymbol{m}^T(\boldsymbol{u}+\boldsymbol{\tau}\boldsymbol{a}+\boldsymbol{b})}\, e^{\pi i\boldsymbol{m}^T\boldsymbol{\tau}\boldsymbol{m}} ,
\end{aligned}$$

which gives the formula sought:

$$\vartheta\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}(\boldsymbol{u}|\boldsymbol{\tau}) = e^{2\pi i\boldsymbol{a}(\boldsymbol{u}+\boldsymbol{b})}\, e^{\pi i\boldsymbol{a}^T\boldsymbol{\tau}\boldsymbol{a}}\, \vartheta(\boldsymbol{u}+\boldsymbol{\tau}\boldsymbol{a}+\boldsymbol{b}|\boldsymbol{\tau}) .$$

(b) Before we proceed to prove equations (6.4) and (6.5), we first obtain, as an intermediate step, an expression for $\vartheta(\boldsymbol{u}+\boldsymbol{\tau}\boldsymbol{n}+\boldsymbol{l}|\boldsymbol{\tau})$ in terms of $\vartheta(\boldsymbol{u}|\boldsymbol{\tau})$. One observes that

$$\begin{aligned}
\vartheta(\boldsymbol{u}+\boldsymbol{\tau}\boldsymbol{n}+\boldsymbol{l}|\boldsymbol{\tau}) &= \sum_{\boldsymbol{m}\in\mathbb{Z}^g} e^{2\pi i(\boldsymbol{u}+\boldsymbol{\tau}\boldsymbol{n}+\boldsymbol{l})\boldsymbol{m}}\, e^{\pi i\boldsymbol{m}^T\boldsymbol{\tau}\boldsymbol{m}} \\
&= \sum_{\boldsymbol{m}\in\mathbb{Z}^g} e^{2\pi i\boldsymbol{u}\boldsymbol{m}}\, e^{2\pi i(\boldsymbol{\tau}\boldsymbol{n})\boldsymbol{m}}\, e^{\pi i\boldsymbol{m}^T\boldsymbol{\tau}\boldsymbol{m}} ,
\end{aligned}$$

since the dot product $\boldsymbol{l}\boldsymbol{m}$ is an integer, i.e., $e^{2\pi i\boldsymbol{l}\boldsymbol{m}} = 1$. Now we make the change of variables $\boldsymbol{m} = \boldsymbol{k} - \boldsymbol{n}$, which yields

$$\begin{aligned}
\vartheta(\boldsymbol{u}+\boldsymbol{\tau}\boldsymbol{n}+\boldsymbol{l}|\boldsymbol{\tau}) &= \sum_{\boldsymbol{k}\in\mathbb{Z}^g} e^{2\pi i\boldsymbol{u}(\boldsymbol{k}-\boldsymbol{n})}\, e^{2\pi i(\boldsymbol{\tau}\boldsymbol{n})(\boldsymbol{k}-\boldsymbol{n})}\, e^{\pi i(\boldsymbol{k}-\boldsymbol{n})^T\boldsymbol{\tau}(\boldsymbol{k}-\boldsymbol{n})} \\
&= e^{-2\pi i\boldsymbol{u}\boldsymbol{n}}\, e^{-2\pi i(\boldsymbol{\tau}\boldsymbol{n})\boldsymbol{n}}\, e^{\pi i\boldsymbol{n}^T\boldsymbol{\tau}\boldsymbol{n}} \sum_{\boldsymbol{k}\in\mathbb{Z}^g} e^{2\pi i\boldsymbol{u}\boldsymbol{k}}\, e^{\pi i\boldsymbol{k}^T\boldsymbol{\tau}\boldsymbol{k}}\, e^{2\pi i(\boldsymbol{\tau}\boldsymbol{n})\boldsymbol{k}}\, e^{-\pi i\boldsymbol{n}^T\boldsymbol{\tau}\boldsymbol{k}}\, e^{-\pi i\boldsymbol{k}^T\boldsymbol{\tau}} \\
&= e^{-2\pi i\boldsymbol{u}\boldsymbol{n}}\, e^{-\pi i(\boldsymbol{\tau}\boldsymbol{n})\boldsymbol{n}} \sum_{\boldsymbol{k}\in\mathbb{Z}^g} e^{2\pi i\boldsymbol{u}\boldsymbol{k}}\, e^{\pi i\boldsymbol{k}^T\boldsymbol{\tau}\boldsymbol{k}} .
\end{aligned}$$

Thus,

$$\vartheta(\boldsymbol{u}+\boldsymbol{\tau n}+\boldsymbol{l}|\boldsymbol{\tau}) \;=\; e^{-2\pi i \boldsymbol{un}}\, e^{-\pi i(\boldsymbol{\tau n})\boldsymbol{n}}\, \vartheta(\boldsymbol{u}|\boldsymbol{\tau}) \;.$$

Now we are ready to prove formulæ(6.4) and (6.5). To do this, we will make use of equation (6.3) (which we proved in part (a) of this problem solution) and the equation we just derived. In particular,

$$\begin{aligned}
\vartheta\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}(\boldsymbol{u}+\boldsymbol{\tau n}+\boldsymbol{l}|\boldsymbol{\tau}) &= e^{2\pi i \boldsymbol{a}(\boldsymbol{u}+\boldsymbol{\tau n}+\boldsymbol{l}+\boldsymbol{b})}\, e^{\pi i \boldsymbol{\tau}^T \boldsymbol{\tau a}} \vartheta(\boldsymbol{u}+\boldsymbol{\tau n}+\boldsymbol{l}+\boldsymbol{\tau a}+\boldsymbol{b}|\boldsymbol{\tau}) \\
&= e^{2\pi i \boldsymbol{a}(\boldsymbol{u}+\boldsymbol{\tau n}+\boldsymbol{l}+\boldsymbol{b})}\, e^{\pi i \boldsymbol{\tau}^T \boldsymbol{\tau a}} \vartheta(\boldsymbol{u}+\boldsymbol{\tau \alpha}+\boldsymbol{b}|\boldsymbol{\tau})\, e^{-2\pi i(\boldsymbol{u}+\boldsymbol{\tau b})\boldsymbol{n}}\, e^{-\pi i(\boldsymbol{\tau n})\boldsymbol{n}} \\
&= e^{-\pi i(\boldsymbol{\tau n})\boldsymbol{n}}\, e^{-2\pi i(\boldsymbol{u}+\boldsymbol{b})\boldsymbol{n}}\, e^{2\pi i \boldsymbol{al}}\, e^{2\pi i \boldsymbol{a}(\boldsymbol{u}+\boldsymbol{b})}\, e^{\pi i \boldsymbol{a}^T \boldsymbol{\tau a}}\, \vartheta(\boldsymbol{u}+\boldsymbol{\tau a}+\boldsymbol{b}|\boldsymbol{\tau}) \\
&= e^{-\pi i(\boldsymbol{\tau n})\boldsymbol{n}}\, e^{-2\pi i(\boldsymbol{u}+\boldsymbol{b})\boldsymbol{n}}\, e^{2\pi i \boldsymbol{al}}\, \vartheta\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}(\boldsymbol{u}|\boldsymbol{\tau}) \;.
\end{aligned}$$

Additionally,

$$\begin{aligned}
\vartheta\begin{bmatrix}\boldsymbol{a}+\boldsymbol{n}\\ \boldsymbol{b}+\boldsymbol{l}\end{bmatrix}(\boldsymbol{u}|\boldsymbol{\tau}) &= e^{2\pi i(\boldsymbol{a}+\boldsymbol{n})(\boldsymbol{u}+\boldsymbol{b}+\boldsymbol{l})}\, e^{\pi i(\boldsymbol{a}+\boldsymbol{n})^T \boldsymbol{\tau}(\boldsymbol{a}+\boldsymbol{n})}\, \vartheta(\boldsymbol{u}+\boldsymbol{\tau a}+\boldsymbol{\tau n}+\boldsymbol{b}+\boldsymbol{l}|\boldsymbol{\tau}) \\
&= e^{2\pi i(\boldsymbol{a}+\boldsymbol{n})(\boldsymbol{u}+\boldsymbol{b}+\boldsymbol{l})}\, e^{\pi i(\boldsymbol{a}+\boldsymbol{n})^T \boldsymbol{\tau}(\boldsymbol{a}+\boldsymbol{n})}\, \vartheta(\boldsymbol{u}+\boldsymbol{\tau a}+\boldsymbol{b}|\boldsymbol{\tau})\, e^{-2\pi i(\boldsymbol{u}+\boldsymbol{\tau a}+\boldsymbol{b})\boldsymbol{n}}\, e^{-\pi i(\boldsymbol{\tau n})\boldsymbol{n}} \;.
\end{aligned}$$

Since $e^{2\pi i \boldsymbol{nl}} = 1$ and some of the exponentials cancel each other, we obtain, in the end,

$$\begin{aligned}
\vartheta\begin{bmatrix}\boldsymbol{a}+\boldsymbol{n}\\ \boldsymbol{b}+\boldsymbol{l}\end{bmatrix}(\boldsymbol{u}|\boldsymbol{\tau}) &= e^{2\pi i \boldsymbol{al}}\, e^{2\pi i \boldsymbol{a}(\boldsymbol{u}+\boldsymbol{b})}\, e^{\pi i \boldsymbol{a}^T \boldsymbol{\tau a}}\, \vartheta(\boldsymbol{u}+\boldsymbol{\tau a}+\boldsymbol{b}|\boldsymbol{\tau}) \\
&= e^{2\pi i \boldsymbol{al}}\, \vartheta\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}(\boldsymbol{u}|\boldsymbol{\tau}) \;.
\end{aligned}$$

---

2. (a) We define the function

$$f(\boldsymbol{x}) \;\equiv\; \sum_{\boldsymbol{m}\in\mathbb{Z}^g} g(\boldsymbol{x}+\boldsymbol{m}) \;. \tag{6.8}$$

Clearly, the function f is periodic, and so it can be expanded in a Fourier series. We thus write

$$f(\boldsymbol{x}) \;=\; \sum_{\boldsymbol{n}} a_{\boldsymbol{n}}\, e^{2\pi i \boldsymbol{b}\cdot\boldsymbol{x}} \;, \tag{6.9}$$

with

$$\begin{aligned} a_{\boldsymbol{n}} &= \int_0^1 f(\boldsymbol{y})\, e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \\ &= \sum_{\boldsymbol{m}} \int_0^1 g(\boldsymbol{y}+\boldsymbol{m})\, e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{y}}\, d\boldsymbol{y} \\ &= \sum_{\boldsymbol{m}} \int_{\boldsymbol{m}}^{\boldsymbol{m}+\boldsymbol{1}} g(\boldsymbol{w})\, e^{-2\pi i \boldsymbol{n}\cdot(\boldsymbol{w}-\boldsymbol{m})}\, d\boldsymbol{w}\ , \end{aligned}$$

where in the second equality we have used the definition of f, and in the third equality we have made the change of variables $\boldsymbol{w} = \boldsymbol{y} + \boldsymbol{m}$. In addition, $\mathbf{1}$ stands for a vector with all its components equal to 1. To proceed further, we notice that

$$e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{m}} = 1\ .$$

Then

$$\begin{aligned} a_{\boldsymbol{n}} &= \sum_{\boldsymbol{m}} \int_{\boldsymbol{m}}^{\boldsymbol{m}+\boldsymbol{1}} g(\boldsymbol{w})\, e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{w}}\, d\boldsymbol{w} \\ &= \sum_{\boldsymbol{m}} \int_{-\infty}^{+\infty} g(\boldsymbol{w})\, e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{w}}\, d\boldsymbol{w}\ . \end{aligned}$$

Inserting this result into the expression (6.9), we find

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{m}} e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{x}} \int_{-\infty}^{+\infty} g(\boldsymbol{w})\, e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{w}}\, d\boldsymbol{w}\ ,$$

or, in other words,

$$\sum_{\boldsymbol{m}} g(\boldsymbol{x}+\boldsymbol{m}) = \sum_{\boldsymbol{m}} e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{x}} \int_{-\infty}^{+\infty} g(\boldsymbol{w})\, e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{w}}\, d\boldsymbol{w}\ .$$

For the special case $\boldsymbol{x} = 0$, we find the Poisson resummation formula

$$\sum_{\boldsymbol{n}} g(\boldsymbol{m}) = \sum_{\boldsymbol{m}} \int_{-\infty}^{+\infty} g(\boldsymbol{w})\, e^{-2\pi i \boldsymbol{n}\cdot\boldsymbol{w}}\, d\boldsymbol{w}\ . \tag{6.10}$$

(b) Comparing the definition (6.1) of the ϑ-function with that of the auxiliary function (6.8) we used in part (a), we see that

$$g(\boldsymbol{u}) = e^{2\pi i \boldsymbol{u}\cdot\boldsymbol{m}}\, e^{\pi i \boldsymbol{m}^T \boldsymbol{\tau} \boldsymbol{m}}\ .$$

Inserting this in the identity (6.10), we can rewrite the ϑ-function as

$$\begin{aligned}
\vartheta(\boldsymbol{u}|\boldsymbol{\tau}) &= \sum_{\boldsymbol{n}} \int d\vec{y}\, g(\vec{y})\, e^{-2\pi i \vec{n}\cdot\vec{y}} \\
&= \sum_{\boldsymbol{n}} \int d\vec{y}\, e^{2\pi i \vec{y}\cdot\vec{n}}\, e^{\pi i \vec{n}\boldsymbol{\tau}\vec{n}}\, e^{-2\pi i \vec{n}\cdot\vec{y}} \\
&= \sum_{\boldsymbol{n}} \int d\vec{y}\, e^{\pi i \vec{n}\boldsymbol{\tau}\vec{n}} e^{-2\pi i(\vec{u}-\vec{n})\cdot\vec{y}} \\
&= \sum_{\boldsymbol{n}} \int d\vec{y}\, e^{-\pi \vec{N}\boldsymbol{T}\vec{N}} e^{2\pi(\vec{U}-\vec{N})\cdot\vec{y}} ,
\end{aligned}$$

where we have set

$$\vec{u} \equiv i\vec{U} , \quad \boldsymbol{n} \equiv i\boldsymbol{N} , \quad \text{and} \quad \boldsymbol{T} \equiv i\boldsymbol{\tau} .$$

The integral that has appeared is Gaussian, and so is easily evaluated according to the standard result

$$\int d\vec{y}\, e^{-\vec{y}\boldsymbol{A}\vec{y}} e^{\boldsymbol{B}\vec{y}} = \frac{\pi^{g/2}}{\sqrt{\det \boldsymbol{A}}} e^{\frac{1}{4}\boldsymbol{B}^T \boldsymbol{A}^{-1}\boldsymbol{B}} .$$

Therefore

$$\begin{aligned}
\vartheta(\boldsymbol{u}|\boldsymbol{\tau}) &= \sum_{\boldsymbol{n}} \frac{\pi^{g/2}}{\sqrt{\det(\pi\boldsymbol{T})}}\, e^{-\frac{1}{4}4\pi^2(\vec{U}-\vec{N})\frac{\boldsymbol{T}^{-1}}{\pi}(\vec{U}-\vec{N})} \\
&= \sum_{\boldsymbol{n}} \frac{1}{(-i)^{g/2}\sqrt{\det \boldsymbol{\tau}}}\, e^{-i\pi(\vec{u}-\vec{n})\boldsymbol{\tau}^{-1}(\vec{u}-\vec{n})} \\
&= \frac{i^{g/2}}{\sqrt{\det \boldsymbol{\tau}}} \sum_{\boldsymbol{n}} e^{-i\pi\vec{u}\boldsymbol{\tau}^{-1}\vec{u}} e^{+i\pi(\vec{u}\boldsymbol{\tau}^{-1}\vec{n}+\vec{n}\boldsymbol{\tau}^{-1}\vec{u})} e^{-i\pi\vec{n}\boldsymbol{\tau}^{-1}\vec{n}} \\
&= \frac{i^{g/2}}{\sqrt{\det \boldsymbol{\tau}}} e^{-i\pi\vec{u}\boldsymbol{\tau}^{-1}\vec{u}} \sum_{\boldsymbol{n}} e^{2i\pi\vec{n}\boldsymbol{\tau}^{-1}\vec{u}} e^{-i\pi\vec{n}\boldsymbol{\tau}^{-1}\vec{n}} ,
\end{aligned}$$

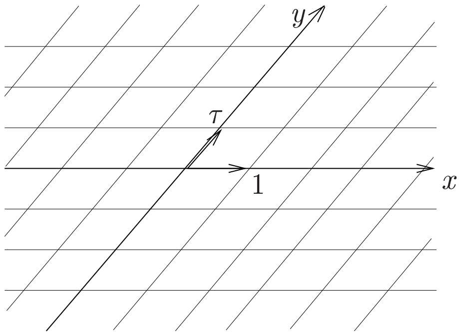
which completes the proof.

3. From the general properties of the ϑ-function, we can see that

$$\begin{aligned}
\vartheta_3(u+a\tau+b|\tau) &= e^{-\pi i a^2\tau}\, e^{-2\pi i a u}\, \vartheta_3(u|\tau) , \quad \text{and} \\
\vartheta_3(-u|\tau) &= \vartheta_3(u|\tau) ,
\end{aligned}$$

where $a, b \in \mathbb{Z}$. We can differentiate the first of these equations, which produces the identity

$$\vartheta_3'(u + a\tau + b|\tau) \;=\; e^{-\pi i a^2 \tau}\, e^{-2\pi i a u}\, \left[\vartheta_3'(u|\tau) - 2\pi i a\, \vartheta_3(u|\tau)\right] \;.$$



Using the above equations, we find that

$$\begin{aligned} \frac{\vartheta_3'(u+1)}{\vartheta_3(u+1|\tau)} &= \frac{\vartheta_3'(u)}{\vartheta_3(u|\tau)} \;, \quad \text{and} \\ \frac{\vartheta_3'(u+\tau)}{\vartheta_3(u+\tau|\tau)} &= \frac{\vartheta_3'(u)}{\vartheta_3(u|\tau)} - 2\pi i \;. \end{aligned}$$

Now from complex analysis, we know that

$$\begin{aligned} N_{roots} - N_{poles} &= \frac{1}{2\pi i} \oint_{unit\ cell} \frac{\vartheta_3'(u)}{\vartheta_3(u)}\, du \\ &= \frac{1}{2\pi i} \int_0^1 \frac{\vartheta_3'(x)}{\vartheta_3(x)}\, dx + \frac{1}{2\pi i} \int_0^\tau \frac{\vartheta_3'(1+iy)}{\vartheta_3(1+iy)}\, i dy \\ &\quad + \frac{1}{2\pi i} \int_1^0 \frac{\vartheta_3'(x+\tau)}{\vartheta_3(x+\tau)}\, dx + \frac{1}{2\pi i} \int_\tau^0 \frac{\vartheta_3'(iy)}{\vartheta_3(iy)}\, i dy \\ &= \frac{1}{2\pi i} \int_0^1 \frac{\vartheta_3'(x)}{\vartheta_3(x)}\, dx + \frac{1}{2\pi i} \int_0^\tau \frac{\vartheta_3'(iy)}{\vartheta_3(iy)}\, i dy \\ &\quad - \frac{1}{2\pi i} \int_0^1 \left(\frac{\vartheta_3'(x)}{\vartheta_3(x)} - 2\pi i \right) dx - \frac{1}{2\pi i} \int_0^\tau \frac{\vartheta_3'(iy)}{\vartheta_3(iy)}\, i dy \\ &= 1 \;. \end{aligned}$$

In the same way, we can show that

$$N_{poles} \;=\; \frac{1}{2\pi i} \oint_{unit\ cell} \vartheta_3(u) du \;=\; 0 \;.$$

Therefore, ϑ_3 has no poles and only one root in the unit cell.

From the properties of ϑ_3, we see that if u_0 is a root, so are $u_0 + a\tau + b$ and $-u_0$. This means that the set of all roots of ϑ_3 form a periodic set with respect to the lattice $\mathbb{Z} + \tau\mathbb{Z}$ and is also symmetric around 0. So symmetry dictates that the single root of ϑ_3 must be located at one of the points 0, 1/2, $\tau/2$, or $(\tau+1)/2$. This is the point $\frac{\tau+1}{2}$ as can be checked[3]explicitly from the definition of ϑ_3:

$$\vartheta_3\left(u + \frac{1}{2}\tau + \frac{1}{2}\right) \;=\; q^{-1/8} \sum_{m=-\infty}^{+\infty} (-)^m z^m q^{(m+1/2)^2/2}$$

[3]One can also verify explicitly that the other values, 0, 1/2, and $\tau/2$, render the ϑ_3-function non-vanishing.

which, for $u=0$, gives

$$\begin{aligned}
\vartheta_3\left(\frac{1}{2}\tau+\frac{1}{2}\right) &= q^{-1/8}\sum_{m=-\infty}^{+\infty}(-1)^m q^{(m+1/2)^2/2} \\
&= q^{-1/8}\sum_{m=-\infty}^{-1}(-1)^m q^{(m+1/2)^2/2}+q^{-1/8}\sum_{m=0}^{+\infty}(-1)^m q^{(m+1/2)^2/2} \\
&= q^{-1/8}\sum_{n=1}^{n=+\infty}(-1)^n q^{(n-1/2)^2/2}+q^{-1/8}\sum_{m=0}^{+\infty}(-1)^m q^{(m+1/2)^2/2} \\
&\overset{n=l+1}{=} -q^{-1/8}\sum_{l=0}^{l=+\infty}(-1)^l q^{(l+1/2)^2/2}+q^{-1/8}\sum_{m=0}^{+\infty}(-1)^m q^{(m+1/2)^2/2} \\
&= 0\ .
\end{aligned}$$

4. (a) The grand canonical partition function of the system is

$$\Xi=\sum_{\text{configurations}} e^{-\beta E+\mu\beta N}\ ,$$

where N is the total fermion number $N=N_f-N_{\bar f}$, which takes values in $\mathbb{Z}$. If n_k and $\bar n_k$ are the occupation numbers for the k-th fermion and antifermion modes,

In particular, from its definition, it is straightforward to see that

$$\vartheta_3(0|\tau)=\sum_{m=-\infty}^{+\infty} q^{m^2/2}\neq 0\ .$$

Then from the properties

$$\begin{aligned}
\vartheta_3(u+1/2|\tau) &= \vartheta_4(u|\tau)\ , \\
\vartheta_3(u+\tau/2|\tau) &= e^{-i\pi u}q^{-1/8}\vartheta_2(u|\tau)\ ,
\end{aligned}$$

for $u=0$ we find

$$\begin{aligned}
\vartheta_3(1/2|\tau) &= \vartheta_4(0|\tau)\ , \\
\vartheta_3(\tau/2|\tau) &= q^{-1/8}\vartheta_2(0|\tau)\ .
\end{aligned}$$

Using the definition of ϑ_2-function

$$\vartheta_3(\tau/2|\tau)=q^{-1/8}\sum_{m=-\infty}^{+\infty} q^{(m+1/2)^2/2}\neq 0\ .$$

Also, by consulting the results of problem 6 of this chapter, one can immediately see that $\vartheta_4(0|\tau)$ is non-zero, and thus $\vartheta_3(1/2|\tau)$ is non-zero.

respectively, then

$$N_f \;=\; \sum_k n_k \;, \quad \text{and} \quad N_{\bar f} \;=\; \sum_k \bar n_k \;.$$

The Hamiltonian of the system is

$$H = E_0 \sum_k \left(k - \frac{1}{2}\right)(n_k + \bar n_k) \;\;.$$

We will use the symbols w and q for the fugacity and 'fundamental' Boltzman weight, respectively, i.e.,

$$w \;\equiv\; e^{\beta\mu} \quad \text{and} \quad q \;\equiv\; e^{-\beta E_0} \;.$$

We now calculate the grand canonical partition function in two distinct ways. First, we write the grand canoncial partition function as a simple infinite product as follows:

$$\begin{aligned}
\Xi \;&=\; \sum_{\text{configurations}} q^{\sum_k \left(k-\frac{1}{2}\right)(n_k+\bar n_k)}\, w^{\sum_k (n_k+\bar n_k)} \\
&=\; \sum_{\text{configurations}} q^{\sum_k (k-1/2)\, n_k}\, w^{\sum_k n_k}\, q^{\sum_k (k-1/2)\, \bar n_k}\, w^{\sum_k \bar n_k} \\
&=\; \sum_{\{n_k\},\{\bar n_k\}} \prod_k \left(q^{k-1/2} w\right)^{n_k} \prod_k \left(q^{k-1/2} w^{-1}\right)^{\bar n_k} \\
&=\; \prod_{k=1}^{+\infty} \sum_{n_k=0}^{1} \left(q^{k-1/2} w\right)^{n_k} \sum_{\bar n_k=0}^{1} \left(q^{k-1/2} w^{-1}\right)^{\bar n_k} \\
&=\; \prod_{k=1}^{+\infty} \left(1 + q^{k-1/2} w\right) \left(1 + q^{k-1/2} w^{-1}\right) \;.
\end{aligned} \tag{6.11}$$

The second way we calculate the grand canoncial partition function begins by expressing Ξ as an infinite sum involving canonical partition functions. We see that

$$\begin{aligned}
\Xi \;&=\; \sum_{\text{configurations}} e^{-\beta E} w^N \\
&=\; \sum_{N=-\infty}^{+\infty} w^N \sum_{\text{configurations}}{}^{\prime} e^{-\beta E} \\
&\equiv\; \sum_{N=-\infty}^{+\infty} w^N\, Z_N(q) \;,
\end{aligned} \tag{6.12}$$

where $Z_N(q)$ is the partition function of the system for a fixed fermion number N. At fixed N, the energy E of the system is a sum of the energy E_N for the first N

levels filled plus the energy ΔE for zero fermion number excitations. The energy E_n is given by

$$E_N \;=\; E_0 \sum_{k=1}^{N} (k - 1/2) = \frac{E_0 \, N^2}{2} \;.$$

The zero fermion number excitations have equal numbers of fermions and anti-fermions, i.e., $n_k = \bar{n}_k$. In terms of n_k, then, we have

$$\Delta E \;=\; E_0 \sum_{k=1}^{m} k \, n_k \;,$$

up to some level m. However, there can be excitations for all possible values of m, thus implying that

$$\begin{aligned} Z_N(q) \;&=\; \sum_{\text{excitations}} e^{-\beta E_N} \, e^{-\beta \Delta E} \\ &=\; q^{\frac{N^2}{2}} \sum_{m=0}^{+\infty} q^{\sum\limits_{k=1}^{m} k \, n_k} \\ &=\; q^{\frac{N^2}{2}} \prod_{n=1}^{+\infty} \left(\sum_{k_n=0}^{+\infty} q^{n \, k_n} \right) \\ &=\; q^{\frac{N^2}{2}} \prod_{n=1}^{+\infty} \frac{1}{1 - q^n} \;. \end{aligned}$$

Substituting this in (6.12), we get

$$\Xi \;=\; \prod_{n=1}^{+\infty} \frac{1}{1 - q^n} \sum_{N=-\infty}^{+\infty} w^N \, q^{\frac{N^2}{2}} \;.$$

Combining this result with (6.11) proves Jacobi's triple identity.

(b) For $P(q, w)$ we have

$$\begin{aligned} P(q, qw) \;&=\; \prod_{n=1}^{+\infty} (1 + w q^{n+1-1/2})(1 + w^{-1} q^{n-1-1/2}) \\ &=\; \prod_{n=1}^{+\infty} (1 + w q^{n+1-1/2}) \prod_{n=1}^{+\infty} (1 + w^{-1} q^{n-1-1/2}) \;. \end{aligned}$$

In the first product, we make the change of variables $k = n + 1$, while in the second product, we make the change of variables $l = n - 1$. This yields

$$P(q, qw) \;=\; \prod_{k=2}^{+\infty} (1 + w q^{k-1/2}) \prod_{l=0}^{+\infty} (1 + w^{-1} q^{l-1/2})$$

$$
\begin{aligned}
&= \frac{1+w^{-1}q^{-1/2}}{1+wq^{1/2}} \prod_{k=1}^{+\infty}(1+wq^{k-1/2}) \prod_{l=1}^{+\infty}(1+w^{-1}q^{l-1/2}) \\
&= \frac{1+w^{-1}q^{-1/2}}{1+wq^{1/2}} P(q,w) = q^{-1/2}w^{-1}\, P(q,w) \ .
\end{aligned}
$$

Similarly, for $Q(q,w)$, we find

$$
\begin{aligned}
Q(q,qw) &= \frac{1}{\prod_{n=1}^{+\infty}(1-q^n)} \sum_{l=-\infty}^{+\infty} q^{l^2/2}\, q^l\, w^l \\
&= \frac{1}{\prod_{n=1}^{+\infty}(1-q^n)}\, q^{-1/2} \sum_{l=-\infty}^{+\infty} q^{(l+1)^2/2}\, w^l \ .
\end{aligned}
$$

Making the change of variables $k = l+1$, we finally arrive at the desired result, namely

$$
\begin{aligned}
Q(q,qw) &= q^{-1/2} \frac{1}{\prod_{n=1}^{+\infty}(1-q^n)} \sum_{k=-\infty}^{+\infty} q^{k^2/2}\, w^{k-1} \\
&= q^{-1/2}w^{-1}\, Q(q,w) \ .
\end{aligned}
$$

This last result shows that

$$
P(q,w) = \mathrm{const}\, Q(q,w) \ .
$$

To complete the proof, we must calculate this constant. This is simplified by comparing the coefficients of w^0.

To this end, we first make a small digression on partitions. Any partition of n into parts $n = p_1+p_2+\ldots+p_n$, $0 \leq p_1 \leq p_2 \leq \ldots \leq p_n \leq n$ can be rearranged in a matrix

$$
\begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & br \end{pmatrix} , \tag{6.13}
$$

called the **Frobenius symbol**, such that

$$
0 \leq a_1 < a_2 < \ldots a_r \ , \quad 0 \leq b_1 < b_2 < \ldots b_r \ , \quad n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_r \ .
$$

To understand how this map can be created, let us examine a specific example. Consider the following partition of the number 20:

$$
20 = 5+5+3+3+2+1+1 \ .
$$

This can be represented graphically:

$$
\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & & \\
\bullet & & & & \\
\bullet & & & & \\
\bullet & & & &
\end{array}
$$

Notice that the diagonal has $r = 3$ dots. We erase them. Then the rows to the right of the diagonal give a_1, a_2, and a_3, and the columns to the left of the diagonal give b_1, b_2, and b_3:

$$
\begin{array}{cccccccc}
 & * & * & * & * & & \leftarrow & a_1 \\
\star & & * & * & * & & \leftarrow & a_2 \\
\star & \star & & & & & \leftarrow & a_3 \\
\star & \star & \star & & & & & \\
\star & \star & & & & & & \\
\star & & & & & & & \\
\star & & & & & & & \\
 & & & & & & & \\
\uparrow & \uparrow & \uparrow & & & & & \\
b_1 & b_2 & b_3 & & & & &
\end{array}
$$

That is, we have rewritten the partition as

$$20 = 3 + (4 + 3 + 0) + (6 + 3 + 1) \ ,$$

giving us the Frobenius symbol

$$\begin{pmatrix} 4 & 3 & 0 \\ 6 & 3 & 1 \end{pmatrix} \ .$$

For each partition of n, there is exactly one Frobenius symbol. Turning this around, for each n, we can construct $p(n)$ Frobenius symbols, where as usual

$$\sum_{n=0}^{+\infty} p(n)\, q^n = \prod_{n=1}^{+\infty} \frac{1}{1 - q^n} \ .$$

Now, returning to calculating the coefficient of the constant term in $P(q, w)$, we observe that we get a contribution with weight 1 every time we have a product of r terms $wq^{n-1/2}$ with r terms $w^{-1}q^{m-1/2}$:

$$q^{-r+n_1+n_2+\ldots+n_r+m_1+m_2+\ldots+m_r} \ .$$

This term can be mapped to the Frobenius symbol

$$\begin{pmatrix} n_1 - 1 & n_2 - 1 & \dots & n_r - 1 \\ m_1 - 1 & m_2 - 1 & \dots & m_r - 1 \end{pmatrix} , \tag{6.14}$$

since the entries satisfy all the required properties. Therefore, the sum of contributions is exactly the number of Frobenius symbols. This determines that const=1, so that $P(q, \omega) = Q(q, \omega)$, and the proof is complete.

5. (a) In Jacobi's triple identity, we use $w = -p^{-1/2}$ and $q = p^3$ to obtain

$$\prod_{n=1}^{+\infty} (1 - p^{3n})\,(1 - p^{3n-2})\,(1 - p^{3n-1}) \;=\; \sum_{n=-\infty}^{+\infty} (-1)^n\, p^{-n/2}\, p^{3n^2/2} \,.$$

Since for every $k \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that

$$k = 3n \;, \quad k = 3n - 1 \;, \quad \text{or} \quad k = 3n - 2 \;,$$

when n runs over all $\mathbb{Z}$, k also runs over all $\mathbb{Z}$. Therefore, the product in the previous equation is exactly

$$\prod_{k=1}^{+\infty} (1 - q^n) \;.$$

Putting everything together, we arrive at Euler's pentagonal number theorem.

(b) For $p = 2$ and $q = 3$, the central charge is $c = 0$, and the model has only the identity operator. Therefore, $\chi_{1,1} = 1$. From the Rocha-Caridi formula

$$\chi_{1,1} = \frac{1}{\prod_{n=1}^{+\infty} (1 - q^n)} \sum_{k=-\infty}^{+\infty} \left(q^{6k^2+k} - q^{6k^2+5k+1} \right) \;.$$

From this we conclude that

$$\begin{aligned} \prod_{n=1}^{+\infty} (1 - q^n) \;&=\; \sum_{k=-\infty}^{+\infty} \left(q^{6k^2+k} - q^{6k^2+5k+1} \right) \\ &=\; \sum_{k=-\infty}^{+\infty} \left[(-1)^{2k}\, q^{\frac{3(2k)^2+(2k)}{2}} + (-1)^{2k-1}\, q^{\frac{3(2k-1)^2+(2k-1)}{2}} \right] \\ &=\; \sum_{l=-\infty}^{+\infty} (-1)^l\, q^{\frac{3l^2+l}{2}} \;, \end{aligned}$$

where to go from the first equality to the second, we just rewrite the various quantities in an equivalent form, and to go from the second equality to the third, we just notice that the two terms are the even and odd partial sums of one sum. Setting $l = -n$ in the last equation, we find again the advertised relation.

6. (a) In Jacobi's triple identity, we set $w = z$ or $w = -z$. In these two cases, we find, respectively,

$$\vartheta_3(u|\tau) = \prod_{n=1}^{+\infty}(1-q^n)(1+zq^{n-1/2})(1+z^{-1}q^{n-1/2}) \ , \quad \text{and}$$
$$\vartheta_4(u|\tau) = \prod_{n=1}^{+\infty}(1-q^n)(1-zq^{n-1/2})(1-z^{-1}q^{n-1/2}) \ .$$

For ϑ_2, we set $w = zq^{1/2}$ in Jacobi's triple identity, which gives

$$\sum_{n=-\infty}^{+\infty} z^n q^{\frac{n^2+n}{2}} = \prod_{n=1}^{+\infty}(1-q^n)(1+zq^n)(1+z^{-1}q^{n-1}) \ .$$

The l.h.s. can be rearranged to give

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} z^n q^{\frac{n^2+n}{2}} &= \sum_{n=-\infty}^{+\infty} z^n q^{\frac{(n+1/2)^2}{2}} q^{-1/8} \\ &= z^{-1/2}q^{-1/8} \sum_{n=-\infty}^{+\infty} z^{n+1/2} q^{\frac{(n+1/2)^2}{2}} \\ &= z^{-1/2}q^{-1/8}\, \vartheta_2(u|\tau) \ ,\end{aligned}$$

and therefore

$$\vartheta_2(u|\tau) = z^{1/2}q^{1/8} \prod_{n=1}^{+\infty}(1-q^n)(1+zq^n)(1+z^{-1}q^{n-1}) \ .$$

In the same way, setting $w = -zq^{1/2}$ in Jacobi's trible identity, we find that

$$\vartheta_1(u|\tau) = z^{1/2}q^{1/8} \prod_{n=1}^{+\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^{n-1}) \ .$$

(b) From the results of part (a), we have

$$\begin{aligned}\theta_2(\tau) &= q^{1/8} \prod_{n=1}^{+\infty}(1-q^n)(1+q^n)(1+q^{n-1}) \\ &= 2q^{1/8} \prod_{n=1}^{+\infty}(1-q^n)(1+q^n)^2 \ ; \qquad (6.15)\end{aligned}$$

$$
\begin{aligned}
\theta_3(\tau) &= \prod_{n=1}^{+\infty}(1-q^n)(1+q^{n-1/2})(1+q^{n-1/2}) \\
&= \prod_{n=1}^{+\infty}(1-q^n)(1+q^{n-1/2})^2 \ ; \quad \text{and} \qquad (6.16) \\
\theta_4(\tau) &= \prod_{n=1}^{+\infty}(1-q^n)(1-q^{n-1/2})(1-q^{n-1/2}) \\
&= \prod_{n=1}^{+\infty}(1-q^n)(1-q^{n-1/2})^2 \ . \qquad (6.17)
\end{aligned}
$$

Multiplying these results together, we find

$$
\begin{aligned}
\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) &= 2q^{1/8}\left[\prod_{n=1}^{+\infty}(1-q^n)\right]^3\left[\prod_{n=1}^{+\infty}(1+q^n)(1+q^{n-1/2})(1-q^{n-1/2})\right]^2 \\
&= 2\,\eta^3(\tau)\,f^2(q)\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
f(q) &= \prod_{n=1}^{+\infty}(1+q^n)(1+q^{n-1/2})(1-q^{n-1/2}) \\
&= \prod_{n=1}^{+\infty}(1+q^n)(1-q^{2n-1})\ . \qquad (6.18)
\end{aligned}
$$

To complete the proof, we must show that $f(q)=1$. Let us break the infinite product of terms $1+q^n$ into terms with odd and even powers. Then,

$$
\begin{aligned}
f(q) &= \prod_{n=1}^{+\infty}(1+q^{2n})(1+q^{2n-1})(1-q^{2n-1}) \\
&= \prod_{n=1}^{+\infty}(1+q^{2n})(1-q^{2(2n-1)}) \\
&\overset{(6.18)}{=} f(q^2)\ .
\end{aligned}
$$

From this identity we can compute the derivatives of $f(q)$ at $q=0$. These are

$$
\begin{aligned}
f'(q) &= 2qf'(q^2)\ , \\
f''(q) &= 2f'(q^2)+4q^2f''(q^2)\ , \\
f'''(q) &= 4qf''(q^2)+8qf''(q^2)+8q^3f'''(q^2)\ , \\
&\cdots\cdots
\end{aligned}
$$

from which we see that $0 = f'(0) = f''(0) = \ldots$. Then from the Taylor expansion

$$f(q) \;=\; f(0) + f'(0)\,q + \frac{f''(0)}{2!}\,q^2 + \cdots \;,$$

we find $f(q) = f(0) = 1$, which completes our proof.

7. For the canonical homology basis $\{a_1, a_2, b_1, b_2\}$, we consider the Dehn twists along the cycles a_1, a_2, b_1, b_2, and $a_1^{-1}a_2$.

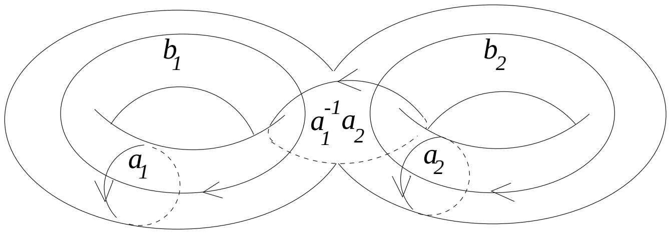


Figure 6.2: The cycles for a genus 2 Riemann surface.

After twisting along the cycle a_1 the new cycles will look as in figure 6.3.

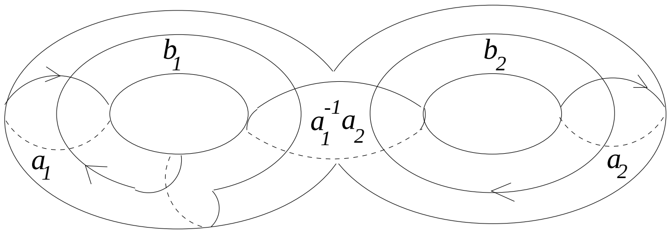


Figure 6.3: A Dehn twist around the a_1 cycle.

From this we conclude that

$$M(D_{a_1}) \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Similarly, we can find the following matrices:

$$M(D_{b_1}) \;=\; \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \qquad M(D_{b_2}) \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,$$

$$M(D_{a_2}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} , \qquad M(D_{a_1^{-1}a_2}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} .$$

8. The trace

$$Z = q^{-c/24}\,\overline{q}^{-c/24}\,\mathrm{tr}\left(q^{L_0}\overline{q}^{\overline{L}_0}\right)$$

is a product of independent contributions from different oscillators α_{-n} and momentum p. The contribution from the states

$$\left\{ |\,k,n\,\rangle \equiv \alpha_{-n}^{k}\,|\,\emptyset\,\rangle \ , \ k = 0,1,2,\dots \right\} ,$$

which have

$$L_0\,|\,k,n\,\rangle = kn\,|\,k,n\,\rangle ,$$

is

$$\sum_{k=0}^{+\infty} q^{kn} = 1 + q^n + q^{2n} + \dots = \frac{1}{1-q^n} .$$

The contribution from the states

$$\{|\,p\,\rangle \, , \ p \in \mathbb{R}\} ,$$

which have

$$L_0\,|\,p\,\rangle = \frac{p^2}{2}\,|\,p\,\rangle ,$$

is

$$\int_{-\infty}^{+\infty} dp\, q^{p^2/2}\,\overline{q}^{p^2/2} = \int_{-\infty}^{+\infty} dp\, e^{-2\pi \mathrm{Im}\tau\, p^2} = \sqrt{\frac{1}{2\mathrm{Im}\tau}} .$$

Therefore, putting all the factors together, the partition function for the free boson is

$$Z = \frac{1}{\sqrt{2\mathrm{Im}\tau}}\,\frac{1}{\eta(q)\eta(\overline{q})} ,$$

where $\eta(q)$ is the Dedekind η-function. Its modular properties (see the appendix) are

$$\begin{aligned} \eta(\tau+1) &= e^{i\pi/12}\,\eta(\tau) , \\ \eta(-1/\tau) &= \sqrt{-i\tau}\,\eta(\tau) . \end{aligned}$$

Therefore, under the T-transformation,

$$\begin{aligned} Z \mapsto Z' &= \frac{1}{\sqrt{2\mathrm{Im}(\tau+1)}}\,\frac{1}{\eta(\tau+1)\eta(\overline{\tau}+1)} \\ &= \frac{1}{\sqrt{2\mathrm{Im}\tau}}\,\frac{1}{e^{i\pi/12}\eta(\tau)\,e^{-i\pi/12}\eta(\overline{\tau})} \\ &= \frac{1}{\sqrt{2\mathrm{Im}\tau}}\,\frac{1}{\eta(\tau)\eta(\overline{\tau})} \;=\; Z\ . \end{aligned}$$

Likewise, under the S-transformation,

$$\begin{aligned} Z \mapsto Z' &= \frac{1}{\sqrt{2\mathrm{Im}\frac{-1}{\tau}}}\,\frac{1}{\eta(-1/\tau)\eta(-1/\overline{\tau})} \\ &= \frac{1}{\sqrt{2\mathrm{Im}\frac{-1}{\tau}}}\,\frac{1}{\eta(-1/\tau)\eta(-1/\overline{\tau})} \\ &= \frac{1}{\sqrt{\frac{\mathrm{Im}\tau}{\tau\overline{\tau}}}}\,\frac{1}{\sqrt{-i\tau}\eta(\tau)\,\sqrt{i\overline{\tau}}\eta(\overline{\tau})} \;=\; Z\ . \end{aligned}$$

APPENDIX

From the definition of the Dedekind η-function

$$\eta(q) \;\equiv\; q^{1/24}\prod_{n=1}^{+\infty}(1-q^n)\ ,$$

we see that under a T-transformation,

$$\eta(q) \mapsto q^{1/24}e^{i\pi/12}\prod_{n=1}^{+\infty}(1-e^{2\pi i n}q^n) \;=\; q^{1/24}e^{i\pi/12}\prod_{n=1}^{+\infty}(1-q^n) \;=\; e^{i\pi/12}\eta(q)\ .$$

To determine how η behaves under an S-transformation, we use the identity

$$\eta^3(\tau) \;=\; \frac{1}{2}\,\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)\ .$$

From the properties of the ϑ function, it is easy to show that

$$\theta_2(-1/\tau) \;=\; \sqrt{-i\tau}\,\theta_4(\tau)\ , \tag{6.19}$$
$$\theta_3(-1/\tau) \;=\; \sqrt{-i\tau}\,\theta_3(\tau)\ , \quad \text{and} \tag{6.20}$$
$$\theta_4(-1/\tau) \;=\; \sqrt{-i\tau}\,\theta_2(\tau)\ . \tag{6.21}$$

Thus we see that

$$\eta(-1/\tau) \;=\; \sqrt{-i\tau}\,\eta(\tau)\ . \tag{6.22}$$

9. The compactification of the boson on a circle of radius R has two consequences. First, the momenta p are quantized, restricted to the values

$$p = \frac{m}{R}, \quad m \in \mathbb{Z}.$$

Second, there are new winding states, because of the boundary conditions

$$\Phi(\sigma + 2\pi) \sim \Phi(\sigma) + 2\pi R w, \quad w \in \mathbb{Z}.$$

The boson decomposes into two chiral components:

$$\begin{aligned} \phi(z) &= x - ip \ln z + i \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n} \quad \text{and} \\ \overline{\phi}(\overline{z}) &= \overline{x} - i\overline{p} \ln \overline{z} + i \sum_{n \neq 0} \frac{\overline{\alpha}_n}{n} \overline{z}^{-n}. \end{aligned}$$

The eigenvalues of p and $\overline{p}$ are, respectively,

$$p = \frac{m}{R} + \frac{wR}{2} \quad \text{and} \quad \overline{p} = \frac{m}{R} - \frac{wR}{2}.$$

We see that the contribution of the bosonic modes in the present case is identical to the contribution of these modes in the previous problem. However, the integral over the momentum states becomes a discrete sum:

$$\int_{-\infty}^{+\infty} dp\, q^{p^2/2}\, \overline{q}^{p^2/2} \longrightarrow \sum_{m,w} q^{\left(\frac{m}{R}+\frac{wR}{2}\right)^2/2}\, \overline{q}^{\left(\frac{m}{R}-\frac{wR}{2}\right)^2/2}.$$

Therefore,

$$Z = \sqrt{2\,\mathrm{Im}\tau}\, Z_{\mathrm{boson}} \sum_{m,w} q^{\left(\frac{m}{R}+\frac{wR}{2}\right)^2/2}\, \overline{q}^{\left(\frac{m}{R}-\frac{wR}{2}\right)^2/2},$$

where Z_{boson} stands for the partition function of the uncompactified boson.

Under the T-transformation

$$\begin{aligned} Z \mapsto Z' &= \sqrt{2\,\mathrm{Im}\tau}\, Z_{\mathrm{boson}} \sum_{m,w} q^{\left(\frac{m}{R}+\frac{wR}{2}\right)^2/2}\, \overline{q}^{\left(\frac{m}{R}-\frac{wR}{2}\right)^2/2} e^{-i\pi\left(\frac{m}{R}+\frac{wR}{2}\right)/2} e^{i\pi\left(\frac{m}{R}-\frac{wR}{2}\right)/2} \\ &= \sqrt{2\,\mathrm{Im}\tau}\, Z_{\mathrm{boson}} \sum_{m,w} q^{\left(\frac{m}{R}+\frac{wR}{2}\right)^2/2}\, \overline{q}^{\left(\frac{m}{R}-\frac{wR}{2}\right)^2/2} e^{-2i\pi m w} \\ &= \sqrt{2\,\mathrm{Im}\tau}\, Z_{\mathrm{boson}} \sum_{m,w} q^{\left(\frac{m}{R}+\frac{wR}{2}\right)^2/2}\, \overline{q}^{\left(\frac{m}{R}-\frac{wR}{2}\right)^2/2} = Z. \end{aligned}$$

To study the transformation of Z under the S-transformation, we first rewrite Z in an equivalent form using the Poisson resummation formula. We see that we can write Z as

$$Z = \sqrt{2\,\mathrm{Im}\tau}\, Z_{\mathrm{boson}} \sum_{m,w} f(m,w) = \sqrt{2\,\mathrm{Im}\tau}\, Z_{\mathrm{boson}} \sum_{\vec{x}} \tilde{f}(\vec{x}),$$

where

$$
\begin{aligned}
f(m,w) &\equiv q^{\left(\frac{m}{R}+\frac{wR}{2}\right)/2} \\
&= \exp\left[-i\pi\tau\left(\frac{m^2}{R^2}+\frac{w^2R^2}{4}+wm\right)+i\pi\overline{\tau}\left(\frac{m^2}{R^2}+\frac{w^2R^2}{4}-wm\right)\right] \\
&= \exp\left[-i\pi\left((\tau-\overline{\tau})\frac{m^2}{R^2}+(\tau-\overline{\tau})\frac{w^2R^2}{4}+(\tau+\overline{\tau})wm\right)\right] \\
&= \exp\left[-i\pi\left(2i\mathrm{Im}\tau\,\frac{m^2}{R^2}+i\mathrm{Im}\tau\,\frac{w^2R^2}{2}+2\mathrm{Re}\tau\,wm\right)\right] \\
&= e^{-i\pi\vec{m}^TA(\tau)\vec{m}} ,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{f}(\vec{x}) &= \int d\vec{x}e^{-2\pi i\vec{x}\cdot\vec{m}}\,f(\vec{m}) = \int d\vec{x}e^{-2\pi i\vec{x}\cdot\vec{m}}\,e^{-\pi\vec{m}^TA\vec{m}} \\
&= \frac{\pi}{\sqrt{\det(\pi A)}}\,e^{-\frac{1}{4}(2\pi i\vec{x})^T(\pi A)^{-1}(2\pi i\vec{x})} \\
&= \frac{1}{\tau\overline{\tau}}\,e^{\pi\vec{x}^TA^{-1}(\tau)\vec{x}} ,
\end{aligned}
$$

where, for convenience, we have defined

$$
\vec{m} = \begin{bmatrix} m \\ w \end{bmatrix} \quad \text{and} \quad A(\tau) = \begin{bmatrix} \frac{2\mathrm{Im}\tau}{R^2} & -i\mathrm{Re}\tau \\ -i\mathrm{Re}\tau & \frac{R^2\mathrm{Im}\tau}{2} \end{bmatrix} ,
$$

and

$$
A^{-1}(\tau) = \begin{bmatrix} \frac{R^2\mathrm{Im}\tau}{2} & i\mathrm{Re}\tau \\ i\mathrm{Re}\tau & \frac{2\mathrm{Im}\tau}{R^2} \end{bmatrix} .
$$

Therefore, the partition function is

$$
Z = \frac{\sqrt{2\,\mathrm{Im}\tau}}{\sqrt{\tau\overline{\tau}}}\,Z_{\mathrm{boson}}(\tau)\sum_{\vec{x}} e^{\pi\vec{x}^TA^{-1}(\tau)\vec{x}} .
$$

Under an S-transformation, we now see that

$$
\begin{aligned}
Z \mapsto Z' &= \sqrt{2\,\mathrm{Im}\frac{-1}{\tau}}\,\sqrt{\tau\overline{\tau}}\,Z_{\mathrm{boson}}(\tau)\sum_{\vec{x}} e^{\pi\vec{x}^TA^{-1}(-1/\tau)\vec{x}} \\
&= \frac{\sqrt{2\,\mathrm{Im}\tau}}{\sqrt{\tau\overline{\tau}}}\,\sqrt{\tau\overline{\tau}}\,Z_{\mathrm{boson}}(\tau)\sum_{\vec{x}} e^{\pi\vec{x}^TPA^{-1}(\tau)P\vec{x}} \\
&= \sqrt{2\,\mathrm{Im}\tau}\,Z_{\mathrm{boson}}(\tau)\sum_{\vec{x}} e^{\pi\vec{x}^TPA^{-1}(\tau)P\vec{x}} ,
\end{aligned}
$$

where we have taken advantage of the fact that

$$\begin{aligned} A^{-1}(-1/\tau) &= \begin{bmatrix} \frac{R^2\mathrm{Im}\tau}{2} & -i\mathrm{Re}\tau \\ -i\mathrm{Re}\tau & \frac{2\mathrm{Im}\tau}{R^2} \end{bmatrix} \\ &= PA^{-1}(\tau)P\ , \qquad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\ . \end{aligned}$$

If we now make the change of variables $\vec{n} = P\vec{x}$ (which amounts simply to a permutation of the components of $\vec{x}$), then at last we see that

$$Z \mapsto Z' = \sqrt{2\,\mathrm{Im}\tau}\, Z_{\text{boson}}(\tau) \sum_{\vec{n}} e^{\pi \vec{n}^T A^{-1}(\tau)\vec{n}} = Z\ .$$

---

10. (a) The map

$$w \mapsto z = e^w$$

maps the w-cylinder to the z-plane. The fermion on the cylinder is

$$\psi_{\text{cyl}} = z^{1/2}\,\psi(z) = \sum_n e^{-nw}\,\psi_n\ .$$

After going around the cycle of the cylinder, $w \mapsto w + 2\pi i$, the fermion may return to its initial value or take an opposite value. Thus n can take values either in $\mathbb{Z}$ (the P sector, with periodic boundary conditions) or in $\mathbb{Z} + 1/2$ (the A sector, with antiperiodic boundary conditions). The L_0 operator is given by

$$L_0 = \sum_n n\psi_{-n}\psi_n + c_0\ ,$$

where $c_0 = 1/24$ in the P sector and $c_0 = -1/48$ in the A sector. (For the derivation of these values, see Exercise 4 of Chapter 7.)

Starting with the vacuum state $|\,\emptyset\,\rangle$, one can build a tower of states for the free fermion using the operators $\psi_{-n},\ n > 0$:

$$\psi_{-n_k} \ldots \psi_{-n_1}\,|\,\emptyset\,\rangle\ .$$

The calculation of $\mathrm{tr}q^{L_0}$ is elementary. First we notice that

$$\begin{aligned} \mathrm{tr}q^{L_0} &= q^{c_0}\,\mathrm{tr}\left(q^{\sum_{n>0} n\psi_{-n}\psi_n}\right) \\ &= q^{c_0}\,\mathrm{tr}\left(\bigotimes_{n>0} q^{n\psi_{-n}\psi_n}\right)\ . \end{aligned}$$

Now we notice that the subspace generated by the the n-mode has a very simple basis, given by

$$\{|\,\emptyset\,\rangle\,,\ \psi_{-n}\,|\,\emptyset\,\rangle\}\ .$$

As a result, the trace we compute above becomes

$$\begin{aligned}
\mathrm{tr} q^{L_0} &= q^{c_0}\,\mathrm{tr}\left(\bigotimes_{n>0}\begin{bmatrix}1 & 0\\ 0 & q^n\end{bmatrix}\right)\\
&= q^{c_0}\prod_{n>0}\mathrm{tr}\begin{bmatrix}1 & 0\\ 0 & q^n\end{bmatrix}\\
&= q^{c_0}\prod_{n>0}(1+q^n)\ .
\end{aligned}$$

There is a related trace that it is useful to compute. The operator $(-1)^F$ measures the number of fermions mod 2 in a state (it takes the value -1 for a fermionic state, and $+1$ for a bosonic state). Repeating the previous calculation, we find that

$$\begin{aligned}
\mathrm{tr}\left((-1)^F q^{L_0}\right) &= q^{c_0}\prod_{n>0}\mathrm{tr}\begin{bmatrix}1 & 0\\ 0 & -q^n\end{bmatrix}\\
&= q^{c_0}\prod_{n>0}(1-q^n)\ .
\end{aligned}$$

The traces we have found do not give immediately the characters of the primary fields. Let us study the A sector first. Table 6.1 gives some of the states of this sector. We see that the representation is actually the sum $[0]\oplus[1/2]$. The module $[0]$ contains states with an even number of fermions, while the module $[1/2]$ contains states with an odd number of fermions. In other words, the two modules can be separated by using the projection operators

$$P_{\pm} = \frac{1\pm(-1)^F}{2}\ .$$

Therefore

$$\begin{aligned}
\chi_0 &= \mathrm{tr}_{\mathrm{A}}\left(P_+ q^{L_0}\right)\\
&= \frac{1}{2}\,\mathrm{tr}_{\mathrm{A}} q^{L_0} + \frac{1}{2}\,\mathrm{tr}_{\mathrm{A}}\left((-1)^F q^{L_0}\right)\\
&= \frac{1}{2}\,q^{-1/48}\left(\prod_{n=1}^{+\infty}(1+q^{n-1/2}) + \prod_{n=1}^{+\infty}(1-q^{n-1/2})\right)\\
&= \frac{1}{2}\left[\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right]\ ,
\end{aligned}$$

L_0	state
0	$\vert\,\emptyset\,\rangle$
1/2	$\psi_{-1/2}\,\vert\,\emptyset\,\rangle$
3/2	$\psi_{-3/2}\,\vert\,\emptyset\,\rangle$
2	$\psi_{-3/2}\psi_{-1/2}\,\vert\,\emptyset\,\rangle$
5/2	$\psi_{-5/2}\,\vert\,\emptyset\,\rangle$
3	$\psi_{-5/2}\psi_{-1/2}\,\vert\,\emptyset\,\rangle$
7/2	$\psi_{-7/2}\,\vert\,\emptyset\,\rangle$

Table 6.1: The first states of the A sector of the free fermion.

and

$$\begin{aligned}
\chi_{1/2} &= \mathrm{tr}_{\mathrm{A}}\left(P_- q^{L_0}\right) \\
&= \frac{1}{2}\,\mathrm{tr}_{\mathrm{A}} q^{L_0} - \frac{1}{2}\,\mathrm{tr}_{\mathrm{A}}\left((-1)^F q^{L_0}\right) \\
&= q^{-1/48}\left(\prod_{n=1}^{+\infty}(1+q^{n-1/2}) - \prod_{n=1}^{+\infty}(1-q^{n-1/2})\right) \\
&= \frac{1}{2}\left[\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right] .
\end{aligned}$$

In the P sector, the anticommutation relations

$$\begin{aligned}
\{\psi_0, \psi_n\} &= 0\ , \quad n \neq 0\ , \\
\{\psi_0, \psi_0\} &= 1\ , \quad \text{and} \quad \{\psi_0, (-1)^F\} = 0
\end{aligned}$$

require two degenerate ground states $|\,1/16;\pm\,\rangle$. The modules built over them have, obviously, equal numbers of states, and therefore we can conclude immediately that

$$\begin{aligned}
\chi_{1/16} &= \mathrm{tr}_{\mathrm{P}} q^{L_0} \\
&= q^{1/24}\prod_{n=1}^{+\infty}(1-q^n) \\
&= \frac{1}{\sqrt{2}}\sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}} .
\end{aligned}$$

(b) By definition, when one performs the modular transformation

$$S : \tau \mapsto \tau' = -\frac{1}{\tau}\ ,$$

the characters transform according to a matrix $\boldsymbol{S}$ as

$$\chi_i(\tau) \;\mapsto\; \chi_i(\tau') \;=\; \boldsymbol{S}_{ij}\, \chi_j(\tau) \;.$$

Using the modular properties of the ϑ-function and η-function, given in equations (6.19)-(6.22), we find

$$\begin{aligned}
\chi_0(\tau) \;\mapsto\; \chi_0(\tau') \;&=\\
&= \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} \;+\; \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}\right)\\
&= \frac{1}{2}\left[\sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}} \;+\; \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} \;+\; \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) \;+\; \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} \;-\; \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right)\right]\\
&= \frac{1}{2}\left[\sqrt{2}\,\chi_{\frac{1}{16}}(\tau) \;+\; \chi_0(\tau) \;+\; \chi_{\frac{1}{2}}(\tau)\right] \;,
\end{aligned}$$

i.e.,

$$\chi_0(\tau) \;\mapsto\; \frac{1}{2}\,\chi_0(\tau) \;+\; \frac{1}{2}\,\chi_{\frac{1}{2}}(\tau) \;+\; \frac{1}{\sqrt{2}}\,\chi_{\frac{1}{16}}(\tau) \;. \tag{6.23}$$

In the same way, we find

$$\chi_{\frac{1}{2}}(\tau) \;\mapsto\; \frac{1}{2}\,\chi_0(\tau) \;+\; \frac{1}{2}\,\chi_{\frac{1}{2}}(\tau) \;-\; \frac{1}{\sqrt{2}}\,\chi_{\frac{1}{16}}(\tau) \;, \tag{6.24}$$

$$\chi_{\frac{1}{16}}(\tau) \;\mapsto\; \frac{1}{\sqrt{2}}\,\chi_0(\tau) \;-\; \frac{1}{\sqrt{2}}\,\chi_{\frac{1}{2}}(\tau) \;. \tag{6.25}$$

From equations (6.23), (6.24), and (6.25), we see that

$$\boldsymbol{S} \;=\; \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \;. \tag{6.26}$$

Notice that

$$\boldsymbol{S}^{-1} \;=\; \boldsymbol{S}^{\dagger} \;=\; \boldsymbol{S} \;.$$

Should we have expected this result?

(c) The fusion rules are written in general as

$$[\,\phi_i\,]\,[\,\phi_j\,] \;=\; N_{ij}^{k}\;[\,\phi_k\,] \;,$$

and the matrices to be diagonalized are

$$\boldsymbol{N}_i \;\equiv\; [\,N_{ij}^{k}\,] \;.$$

In the case of Ising model, the conformal families are

$$[\,\phi_i\,]\ :\ [\,1\,]\,,\ [\,\epsilon\,]\,,\quad \text{and}\quad [\,\sigma\,]\,,$$

and the non-trivial fusion rules are

$$\begin{aligned} [\,\epsilon\,]\ [\,\epsilon\,] &= [\,1\,]\ , \\ [\,\epsilon\,]\ [\,\sigma\,] &= [\,\sigma\,]\ ,\quad \text{and} \\ [\,\sigma\,]\ [\,\sigma\,] &= [1]\ +\ [\,\epsilon\,]\ . \end{aligned}$$

Hence, the corresponding $\boldsymbol{N}_i$ matrices are

$$\boldsymbol{N}_1 = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}\ ,\quad \boldsymbol{N}_\epsilon = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}\ ,\quad \text{and}\quad \boldsymbol{N}_\sigma = \begin{pmatrix} 0&0&1\\0&0&1\\1&1&0 \end{pmatrix}\ .$$

It is a trivial matter now to show that

$$\begin{aligned} \boldsymbol{S}\,\boldsymbol{N}_1\,\boldsymbol{S} &= \text{diag}(\,1,1,1\,)\ , \\ \boldsymbol{S}\,\boldsymbol{N}_\epsilon\,\boldsymbol{S} &= \text{diag}(\,1,1,-1\,)\ ,\quad \text{and} \\ \boldsymbol{S}\,\boldsymbol{N}_\sigma\,\boldsymbol{S} &= \text{diag}(\,\sqrt{2},-\sqrt{2},0\,)\ . \end{aligned}$$

11. Under the T transformation, $T:\tau\mapsto\tau+1$, the powers of the parameter q transform as $q^a\mapsto q^a\,e^{2\pi i a}$, and any character

$$\chi_\Delta = q^{\Delta-c/24}\sum_{N=0}^{+\infty} p(N)\,q^N$$

transforms as

$$\begin{aligned} \chi_\Delta(q)\mapsto\chi_\Delta(q') &= q^{\Delta-c/24}\,e^{2\pi i(\Delta-c/24)}\sum_{N=0}^{+\infty} p(N)\,q^N\,e^{2\pi i N} \\ &= q^{\Delta-c/24}\,e^{2\pi i(\Delta-c/24)}\sum_{N=0}^{+\infty} p(N)\,q^N \\ &= e^{2\pi i(\Delta-c/24)}\,\chi_\Delta(q)\ . \end{aligned}$$

This is a general result. In particular, for the minimal model MM(r,s), this becomes

$$\chi_\Delta(q)\overset{T}{\mapsto} e^{-2\pi i\left(\frac{1}{24}-\frac{(mr-ns)^2}{4rs}\right)}\,\chi_\Delta(q)\ .$$

To find the transformation of the characters under the S transformation, we write the characters in a more convenient form. Let

$$n_{\pm} \equiv sn \pm mr \quad \text{and} \quad N \equiv 2rs\ .$$

Then

$$\begin{aligned} a(k) &= \frac{(Nk+n_-)^2-(r-s)^2}{2N}\ , \quad \text{and} \\ b(k) &= \frac{(Nk+n_+)^2-(r-s)^2}{2N}\ . \end{aligned}$$

Thus

$$\chi(q) = \frac{1}{\eta(q)} \sum_{k=-\infty}^{+\infty} \left(q^{\frac{(Nk+n_-)^2}{2N}} - q^{\frac{(Nk+n_+)^2}{2N}} \right)\ ,$$

where, as usual,

$$\eta(q) = q^{-1/24} \prod_{n=1}^{+\infty} (1-q^n)\ .$$

Define the function

$$\Theta_{\lambda,N}(\tau) \equiv \sum_{k=-\infty}^{+\infty} e^{i\pi\tau\frac{(Nk+\lambda)^2}{N}}\ .$$

With this definition, we can write

$$\chi_{m,n}(q) = \frac{1}{\eta(q)} \left[\Theta_{n_-,N} - \Theta_{n_+,N} \right]\ .$$

It is now straightforward to work out the behavior of $\Theta_{\lambda,n}(\tau)$ under the S transformation:

$$\begin{aligned} \Theta_{\lambda,N}(\tau) \mapsto \Theta_{\lambda,N}(-\frac{1}{\tau}) &= \sum_{k=-\infty}^{+\infty} e^{-\frac{\pi i}{\tau}\frac{(Nk+\lambda)^2}{N}} \\ &\overset{(6.10)}{=} \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dy\, e^{-\frac{\pi i}{\tau}\frac{(Ny+\lambda)^2}{N}} e^{-2\pi iky} \\ &\overset{x=Ny+\lambda}{=} \frac{1}{N} \sum_{k=-\infty}^{+\infty} e^{\frac{2\pi ik\lambda}{N}} \int_{-\infty}^{+\infty} dx\, e^{-\frac{\pi i}{\tau}\frac{x^2}{N}} e^{-\frac{2\pi ikx}{N}} \\ &= \frac{1}{N} \sum_{k=-\infty}^{+\infty} e^{\frac{2\pi ik\lambda}{N}} \sqrt{\frac{\tau}{iN}}\, e^{\frac{\pi i\tau k^2}{N}}\ . \end{aligned}$$

To continue further, we observe that any integer k can be written in the form $k = mN + \lambda'$, where $\lambda' \in \{0, \ldots, N-1\}$. Then the previous expression can be written as

$$\begin{aligned}
\Theta_{\lambda,N}(\tau) \mapsto \Theta_{\lambda,N}(-\frac{1}{\tau}) &= \frac{1}{N} \sum_{m=-\infty}^{+\infty} \sum_{\lambda'=0}^{N} e^{2\pi i \frac{(mN+\lambda')\lambda}{N}} \sqrt{\frac{N\tau}{i}}\, e^{\pi i \tau \frac{(mN+\lambda')^2}{N}} \\
&= \sqrt{\frac{-i\tau}{N}} \sum_{\lambda'=0}^{N-1} e^{2\pi i \frac{\lambda'\lambda}{N}} \sum_{m=-\infty}^{+\infty} e^{\pi i \tau \frac{(mN+\lambda')^2}{N}} \\
&= \sqrt{\frac{-i\tau}{N}} \sum_{\lambda'=0}^{N-1} e^{2\pi i \frac{\lambda'\lambda}{N}}\, \Theta_{\lambda',N}(\tau) \ . \qquad (6.27)
\end{aligned}$$

Notice also that the function Θ defined above has the properties

$$\Theta_{-\lambda,N} = \Theta_{\lambda,N} = \Theta_{\lambda+N,N} \ . \qquad (6.28)$$

Using the property (6.27), we can write the transformation of the character under the S transformation in the form

$$\chi_{m,n}(q) \mapsto \chi_{m,n}(q') = \frac{1}{\sqrt{N}} \frac{1}{\eta(q)} \sum_{\lambda'=0}^{+\infty} \sum_{\lambda'=0}^{N-1} \left(e^{2\pi i \frac{\lambda' n_-}{N}} - e^{2\pi i \frac{\lambda' n_+}{N}} \right) \Theta_{\lambda',N}(\tau) \ .$$

Now, λ runs over $0, \ldots, 2rs-1$, or, equivalently, over $-rs, \ldots, rs$. Using the property (6.28), we can thus write

$$\begin{aligned}
\chi_{m,n}(q) \mapsto \chi_{m,n}(q') &= \frac{1}{\sqrt{N}} \frac{1}{\eta(q)} \sum_{\lambda'=1}^{rs} \left(e^{2\pi i \frac{\lambda' n_-}{N}} - e^{2\pi i \frac{\lambda' n_+}{N}} + e^{-2\pi i \frac{\lambda' n_-}{N}} - e^{-2\pi i \frac{\lambda' n_+}{N}} \right) \Theta_{\lambda',N}(\tau) \\
&= \frac{2}{\sqrt{N}} \frac{1}{\eta(q)} \sum_{\lambda'=1}^{rs} \left[\cos\left(2\pi \frac{\lambda' n_-}{N}\right) - \cos\left(2\pi \frac{\lambda' n_+}{N}\right) \right] \Theta_{\lambda',N}(\tau) \\
&= \frac{4}{\sqrt{N}} \frac{1}{\eta(q)} \sum_{\lambda'=1}^{rs} \left[\sin\left(\pi \frac{\lambda'(n_+ + n_-)}{N}\right) \sin\left(\pi \frac{\lambda'(n_+ - n_-)}{N}\right) \right] \Theta_{\lambda',N}(\tau) \\
&= \frac{4}{\sqrt{N}} \frac{1}{\eta(q)} \sum_{\lambda'=1}^{rs} \left[\sin\left(\pi \frac{\lambda' n}{r}\right) \sin\left(\pi \frac{\lambda' m}{s}\right) \right] \Theta_{\lambda',N}(\tau) \ ,
\end{aligned}$$

where we have used the identity

$$\cos A - \cos B = 2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \ ,$$

as well as the relationships

$$\frac{n_+ + n_-}{N} = \frac{n}{r} \quad \text{and} \quad \frac{n_+ - n_-}{N} = \frac{m}{s} \ .$$

Finally, we rewrite the sum over λ' as a sum over m' and n'. To do this, we recognize that λ' must take the values $n'_\pm = sn' \pm rm'$. Then

$$\begin{aligned}
\chi_{m,n}(q') &= \frac{4}{\sqrt{N}}\frac{1}{\eta(q)}\sum_{n'_-}\left[\sin\left(\pi\frac{n'_- n}{r}\right)\sin\left(\pi\frac{n'_- m}{s}\right)\right]\Theta_{n'_-,N}(\tau) \\
&+ \frac{4}{\sqrt{N}}\frac{1}{\eta(q)}\sum_{n'_+}\left[\sin\left(\pi\frac{n'_+ n}{r}\right)\sin\left(\pi\frac{n'_+ m}{s}\right)\right]\Theta_{n'_+,N}(\tau) \\
&= \frac{4}{\sqrt{N}}\frac{1}{\eta(q)}\sum_{n'_-}\sin\left(\pi\frac{nn's}{r}-\pi m'n\right)\sin\left(\pi n'm+\pi\frac{mm'r}{s}\right)\Theta_{n'_-,N}(\tau) \\
&+ \frac{4}{\sqrt{N}}\frac{1}{\eta(q)}\sum_{n'_+}\sin\left(\pi\frac{nn's}{r}+\pi m'n\right)\sin\left(\pi n'm-\pi\frac{mm'r}{s}\right)\Theta_{n'_+,N}(\tau) \\
&= \frac{4}{\sqrt{N}}\frac{1}{\eta(q)}\sum_{n'_-}(-)^{m'n}\sin\left(\pi\frac{nn's}{r}\right)(-)^{n'm}\sin\left(\pi\frac{mm'r}{s}\right)\Theta_{n'_-,N}(\tau) \\
&+ \frac{4}{\sqrt{N}}\frac{1}{\eta(q)}\sum_{n'_+}(-)\,(-)^{m'n}\sin\left(\pi\frac{nn's}{r}\right)(-)^{n'm}\sin\left(\pi\frac{mm'r}{s}\right)\Theta_{n'_+,N}(\tau) \\
&= \frac{4}{\sqrt{N}}\frac{1}{\eta(q)}\sum_{n',m'}(-)^{m'n+n'm}\sin\left(\pi\frac{nn's}{r}\right)\sin\left(\pi\frac{mm'r}{s}\right)\Big(\Theta_{n'_-,N}(\tau)-\Theta_{n'_+,N}(\tau)\Big) \\
&= \frac{2\sqrt{2}}{\sqrt{rs}}\sum_{n',m'}(-)^{m'n+n'm}\sin\left(\pi\frac{nn's}{r}\right)\sin\left(\pi\frac{mm'r}{s}\right)\chi_{m',n'}(q)\ .
\end{aligned}$$

---

12. (a) We can assume without loss of generality that there are states with electric charge but no magnetic charge. (We can always choose a basis in which this is true.) Let $q_0 > 0$ be the smallest such positive electric charge. There is guaranteed to be such a charge if there is magnetic charge, due to the DSZ quantization condition [198, 588, 678].

The DSZ condition for the electric charge q_0 (with no magnetic charge) and a dyon (q_i, g_i) implies that

$$q_0 g_i = 2\pi\, n_i \Rightarrow g_i = \frac{2\pi}{q_0} n_i\ ,$$

for some integer n_i. Therefore the smallest positive magnetic charge that can exist in this theory is

$$g_0 = \frac{2\pi}{q_0} m_0\ ,$$

where $m_0 = \min\{|n_j|\}_j$ and depends on the details of the theory.

Next, the DSZ condition for two dyons (q_1, g_0) and (q_2, g_0) implies

$$q_1 - q_2 = \frac{n}{m_0} q_0\ ,$$

for some integer n. The integer n must be a multiple of m_0, or else it would be possible to construct a state with electric charge between 0 and q_0. This analysis leads to the conclusion, then, that the *difference* of the electric charges is quantized, although the actual values of the electric charges may be arbitrary. We thus parametrize the electric charge as

$$q = q_0 n + \Theta .$$

It proves convenient to write $\Theta = q_0 \frac{\theta}{2\pi}$, i.e.,

$$q = q_0 \left(n + \frac{\theta}{2\pi} \right) .$$

Since $\theta \mapsto \theta + 2\pi$ is equivalent to $n \mapsto n+1$, we can take θ to be an angular variable with values in $[0, 2\pi)$.

We can repeat the previous argument for states of magnetic charge $m g_0$. Then the result is

$$q = q_0 \left(n + m \frac{\theta}{2\pi} \right) .$$

This is usually referred to as the **Witten effect** [644, 645].

(b) A CP transformation on the electric and magentic charges has the effect

$$(q, g) \mapsto (-q, g) .$$

Therefore, a CP invariant theory must have both states (q, g) and $(-q, g)$ in its spectrum. We then apply the DSZ condition for (q, g_0) and $(-q, g_0)$:

$$2 q g_0 = 2\pi n \Rightarrow 2 q_0 = q_0 n \Rightarrow q_0 \in \frac{1}{2}\mathbb{Z} .$$

Comparing this with the general solution of the DSZ condition, we see that CP invariant states require $\theta = 0$ or $\theta = \pi$. Any other value of θ signals CP violation.

(c) The general solution

$$\begin{aligned} q &= q_0 \left(n + q_0 \frac{\theta}{2\pi} m \right) , \\ g &= g_0 \, m , \end{aligned}$$

of the DSZ condition can also be written in the form

$$q + i g = q_0 \left(n + \theta \, \tau \right) ,$$

upon defining

$$\tau \equiv \frac{\theta}{2\pi} + i \, \frac{2\pi m_0}{q_0^2} .$$

Physical states $|\,n,m\,\rangle$ with electric and magnetic charges (q,g) are located on a discrete 2-dimensional lattice with periods q_0 and $q_0\tau$. We can change the basis of this lattice with an $\mathrm{SL}(2,\mathbb{Z})$ transformation

$$\tau \mapsto \frac{a\tau+b}{c\tau+d} \ , \qquad a,b,c,d \in \mathbb{Z}, \quad ad-bc=1 \ ,$$

without changing the physics. Under such a transformation, there is a corresponding transformation of the electric and magnetic charges, given by

$$\begin{bmatrix} n \\ m \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ \begin{bmatrix} n \\ m \end{bmatrix} \ .$$

The effect of the transformation on the charges can be studied by considering the effect of the two generating transformations T and S. The effect of $T : \tau \mapsto \tau+1$ is trivial, producing

$$\theta \mapsto \theta + 2\pi \Rightarrow |\,n,m\,\rangle \mapsto |\,n-m,m\,\rangle \ ,$$

The effect of $S : \tau \mapsto -1/\tau$ is less trivial — it interchanges the magnetic and electric roles! For example, if $\theta = 0$, then $|\,n,m\,\rangle \mapsto |\,m,-n\,\rangle$. This duality between electric and magnetic charges is the simplest example of a phenomenon that has become central to our understanding of string theory and related fields [26, 112, 173, 181, 235, 345, 349, 369, 387, 390, 391, 421, 430, 486, 605].

Bibliography

Warning! The bibliography contains some references for PART II. However, it is not yet complete. Please notify us for any missing references related to PART I only.

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