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Topics in String Theory and Quantum Gravity [1]

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Contents

General Introduction

The goal of this Les Houches school is to bring together the practitioners of different research lines concerned with the quantization of gravity ([1] and references therein). In this collection, the present lectures represent in part the point of view of string theorist (for details and references see for example [2, 3, 4, 5]). The most ambitious approach to the problem is certainly embodied by Superstring Theories where one obtains together with the gravitational field many other interactions with enough richness to account for many features of the Standard Model (SM). This is not an accident. One can consider String Theory as the culmination of several decades of effort dedicated to incorporate within the same framework gravity together with the other known interactions. After nearly a decade of renewed interest in String Theory it seems reasonable to recapitulate how this theory has come to be one of the leading candidates of the unification of all known interactions. We still do not know the basic physical or geometrical principles underlying String Theory. There has been some progress in the formulation of a String Field Theory (see the lecture by B. Zwiebach, [6] and references therein) but we are still far from having a complete theory of string fields. Progress has been made by following a number of consistency requirements which worked quite well in ordinary field theories, and in particular for the SM. These requirements when applied to the unification of gravity with the SM lead to very strong constraints which for the time being seem to be fulfilled only by String Theory.

One of the basic properties of the SM which supports many of its successes is its renormalizability, or equivalently, the fact that after a finite number of parameters are given, we have a rather successful machinery to calculate and explain a large number of high and low energy phenomena. This machinery is renormalizable Quantum Field Theory (QFT). This renormalizability, or rather, predictability of the SM is one of the desired features we would like to have in a quantum theory of gravity.

There are also several features of the SM that should be explained in any theory trying to go beyond. Namely, the chiral nature of the families of quarks and leptons with respect to the SM gauge group $SU(3) \times SU(2) \times U(1)$, the origin of the gauge interaction, the origin of the symmetry breaking, the vanishing of the cosmological constant etc. One of the more popular "Beyond the SM"avenues is the study of theories incorporating supersymmetry [7]. Supersymmetry is a central property of Superstring Theory. Unfortunately there is no evidence in present accelerators that supersymmetry is realized in Nature. However, if present or future accelerators would find evidence of supersymmetry partners of the known elementary particles, the temptation to extrapolate and to believe that Superstring Theory should play a central rôle in the unification of the known interactions would be nearly irresistible.

When one tries to look for a predictive theory containing Quantum Gravity together with the chiral structure of the low energy degrees of freedom, one seems to be left only with String Theory. These two requirements alone pose very stringent constraints on candidate Theories. As we will see in the next chapter various proposals including combinations of Supergravity [8] and Kaluza-Klein Theories [9] do not satisfy any of these requirements. If one consider only the quantization of the gravitational field independently of other interactions, there has been some substantial progress in the recent past which is reviewed in the lectures by A. Ashtekar.

The outline of these lectures is as follows. In Chapter 1 we present many of the approaches used to quantize gravity and to unify it with other interactions in the framework of Quantum Field Theory. We briefly analyze the difficulties one encounters when one tries to apply standard field theory techniques to perturbative computations including graviton loops. We also sumarize the approaches including Supergravity and Kaluza-Klein Theories. These theories are not only afflicted with uncontrollable infinities, but also by the fact that low energy chiral fermions are hard to obtain. We also present a rather brief study of the conceptual problems encountered in the quantization of fields in the presence of external gravitational backgrounds. We conclude the chapter with a collection of general remarks concerning the Euclidean and Hamiltonian approaches to the quantization of gravity. If taking seriously, the naïve approach of summing over all possible topologies and geometries meets with formidable mathematical difficulties. Furthermore, the use of semiclassical methods based on instantons, which could in principle provide some insights into quantum-gravitationally induced processes is also presented (and criticized). This chapter gives a rather negative portrait of the field theoretic attempts to understand Quantum Gravity, and needless to say, this view was not shared by many of the others lecturers.

In Chapter two we come to study the constraints imposed by the requirement of having consistent gauge and gravitational interactions between chiral fermions. This brings us to the analysis of Anomalies, both local and global. We present the general conceptual features of the computation of local anomalies in diverse dimensions; we present some efficient computational techniques, and apply them to few interesting examples, in particular the Green-Schwarz anomaly cancelation mechanism [96] which opened the way to the formulation of the Heterotic String [10]. We also present briefly the analysis of global anomalies, and in particular Witten's $SU(2)$ anomaly [11, 12]. In this chapter we want to exhibit how difficult it is to have anomaly-free chiral theories with gravitational and gauge interactions. In the Green-Schwarz mechanism for example we are left only with the possible gauge groups $SO(32)$, $E_8 \times E_8$ and $E_8 \times U(1)^{248}$. Thus, the spectrum of low energy excitations (massless states) is severely restricted by the requirement of absence of anomalies. Global anomalies (anomalies with respect to diffeomorphisms or gauge transformations not in the identity component) play a central rôle in determining the spectrum of possible Superstring Theories. We will analyze in detail in chapter four how the absence of global world-sheet diffeomorphism anomalies (modular invariance), a purely two-dimensional condition, restricts the space-time spectrum in String Theory.

In Chapter three we begin our study of String Theory. In this chapter we restrict for simplicity to the bosonic string. We present briefly the quantization of the bosonic string, the difference between critical and non-critical strings, and the appearance of a tachyon and a graviton in the spectrum in the critical dimension ($d = 26$). These two states are ubiquitous in critical bosonic string theories. We then present a general analysis of String Perturbation Theory. We use the operator formalism presented in [13], (other approaches can be found in [14, 15]). This approach emphasizes the geometrical nature of first quantized String Theory. String scattering amplitudes and physical state conditions are beautifully reflected in the properties of the moduli space of Riemann surfaces. Using this geometric formulation of string amplitudes provides us with a neat understanding of where infinities in string processes might come from. In particular, for the bosonic string it will become clear that the source of infinities is the presence of a tachyon in the spectrum. We have also included a brief section on space-duality to exhibit a property of strings not shared by field theories. This symmetry renders further evidence that in String Theory there is a fundamental length (even though the string interactions are purely local, something that one cannot have in field theory), and ultimately this fundamental length may be the basic reason for the finiteness of superstrings amplitudes. We also analyze how the Einstein equations follow from a two-dimensional requirement, conformal invariance, which provides the classical equations of motion.

In Chapter four we study fermionic and supersymmetric strings. We see how the tachyon is projected out using the GSO projection [130], and exhibit the geometric interpretation of this projection. This will make clear the interplay between modular invariance and the spectrum of the theory. The most remarkable example is the Heterotic String, which will be sumarized with special attention devoted to the rôle of modular invariance. We close this chapter with two topics: the first is a summary of the finite temperature of fermionic strings. Here again the modular group plays an interesting rôle. We present the issue of temperature duality (the canonical partition function at temperature T and $const/T$ is the same) and some of the riddles associated with this property. Second we give a brief status report on the finiteness of superstring perturbation theory and what is known about high order behavior of perturbation theory.

In Chapter five we summarize some of the actives areas of current research in String Theory and present the conclusions. In particular we summarize some of the present research on string or string inspired black hole physics.

There are obviously many aspect of String Theory which have been left out in these lectures. Most notably the subject of classical solutions to String Theory, which started with the Calabi-Yau [16] and orbifold solutions [17], and String Phenomenology [18] which have bloomed into rather large subjects have not been included. The aim of the school was centered upon the issues of Quantum Gravity, and it seems more reasonable to leave this subject to other ocassions (and to a more competent lecturer). Our aim was to show how far we have gone with a rather simple set of requirements. A lot of work is still necessary to clarify and explore many aspects of String Theory. Perhaps the most outstanding open problem in String Theory (as in any other theory of gravity) is the vanishing of the cosmological constant. String Theory does not provide us with any clue on this issue.

Needless to say, many attendants to the school did not share the moderately optimistic opinion of this lecturer. This provides a constant source of useful (and often entertaining) discussions. We still believe that many of the ingredients and properties of String Theory will be contained in the correct theory of Quantum Gravity. It is too early to tell whether this belief carries any truth.

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1 Field-theoretical approach to Quantum Gravity

General Relativity (GR) and Quantum Field Theory (QFT) are two of the most impressive achievements in XXth century Science. One of the first predictions of GR was the centennial precession of the orbit of Mercury. The relativistic contribution is (see for example [21])

$$\Delta\phi^{th} = 43.03 \quad seconds/century \tag{1}$$

while the observed value is

$$\Delta\phi^{obs} = 43.11 \pm 0.45 \quad seconds/century \tag{2}$$

Even gravitational radiation, which is also a prediction of GR, has been indirectly detected by the observation of the decay of the orbit of the binary pulsar PSR $1913 + 16$ [23].

On the other hand QFT allow us to calculate observables at microscopic scales, often with a very high precision. To give an example, the predicted value in Quantum Electrodynamics for the anomalous magnetic moment of the electron is [22]

$$a_e^{th} = 1.159652359(282) \times 10^{-12} \tag{3}$$

where uncertainities are given by those in the value of the fine structure constant. Experimentally it is found that:

$$a_e^{exp} = 1.159652410(200) \times 10^{-12} \tag{4}$$

To obtain a more complete fundamental picture of the physical world we would like to put together QFT and GR. There are still many obtacles to the achievement of this goal. It is also reasonably clear that such a synthesis should provide profound insights into many of the riddles of the Standard Model of Strong, Weak and Electromagnetic interactions. Most notably, the origin of the chiral nature of the families of quarks and leptons and possibly the origin of mass. When we study the running of the coupling constants for the $SU(3) \times SU(2) \times U(1)$ gauge group of the SM ($\alpha_3, \alpha_2, \alpha_1$ respectively), it is found that, extrapolating their present experimental value and making some assumption about the degrees of freedom between present experimental energies and the Planck scale (the desert hypothesis), they converge to a common value at a scale between $10^{16} - 10^{17}$ GeV where quantum gravitational effects can certainly not be neglected. Unification of GR and QFT is the most coveted Holy Grail in Theoretical Physics in the last quarter of the XXth century.

1.1 Linearized gravity

QFT has been succesfully applied to both strong and electroweak interaction, which are described in terms of local gauge field theories. The local gauge symmetry in the SM ensures that the theory is perturbatively renormalizable [24]; this means that there is an effective decoupling of the high energy degrees of freedom from the low energy predictions, so the infinities that appear in the perturbative expansion can be absorbed in redefinitions of the parameters of the theory that are experimentally measurable.

Gauge invariance+renormalizability seem to be two of the cornerstones of our understanding of present high energy physics. One then may wonder why not try the same program with GR. We begin with the Einstein-Hilbert action

$$S_{E-H} = -\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R \tag{5}$$

and make a decomposition of the metric into a background part $\hat{g}_{\mu\nu}$ and a dynamical part $h_{\mu\nu}$

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + \kappa h_{\mu\nu} \tag{6}$$

By doing so, 't Hooft and Veltman [25] showed that four-dimensional pure gravity has a finite one-loop S-matrix although off-shell it is not one-loop renormalizable. Considering gravity coupled to a scalar field, the one-loop counterterm of the lagrangian is the sum of two terms, the first one is proportional to the equation of motion and the second one is given by

$$\mathcal{L}_c^1 = \frac{1}{\epsilon}\sqrt{-\hat{g}}\frac{203}{80}\hat{R}^2 \tag{7}$$

where $\epsilon = 4-d$ is the usual parameter in dimensional regularization. In the case of pure gravity we should have $\hat{R}=0$ hence this counterterm vanishes and the theory is finite at one-loop level. However when considering the coupling to matter the counterterm is divergent (since now we have $\hat{R}\neq 0$).

The finiteness of the one-loop S-matrix for pure gravity was a rather surprising result, since Einstein theory is non-renormalizable by naïve power counting. At two loops the situation changes radically. Goroff and Sagnotti [26] were able to prove that Einstein pure gravity diverges at two loop level, and that the divergent part of the on-shell effective action is

$$\Gamma^{(2)} = \frac{209}{2880(4\pi)^4}\frac{1}{\epsilon}\int d^4x\sqrt{-g}R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta} \tag{8}$$

so the program that allowed us to construct a quantum field theory for both the strong and the electroweak interactions fails when applied to GR.

If we continue to maintain the notion of renormalizability (or predictability) should hold in the presence of gravitational interactions, this result imposes strong constraints on the possible quantum theories describing gravity. If this theory includes other fields, then there must be subtle cancellations between the different contributions. Following this line of thought the first proposed candidate was Supergravity (SUGRA) [8].

1.2 Supergravity

In SUGRA theories supersymmetry is a local symmetry. Since the commutator of two supercurrents generates the energy-momentum tensor, once supersymmetry is gauged we are inevitably led to a theory containing gravity. As in ordinary supersymmetry, in N=1 supergravity [28] each particle of mass m and spin s has associated with it another particle with the same mass and spin $s\pm\frac{1}{2}$. Thus the graviton which has spin 2 comes accompanied by the gravitino field with $s=\frac{3}{2}$. In principle one can also construct theories with extended supergravity, $N>1$. Since it is not known how to consistently quantize an interacting theory containing a massless spin $\frac{5}{2}$ field, the requirement that the highest helicity state be the graviton imposed the constraint that $N\leq 8$ [27].

The ultraviolet properties of perturbation theory improve with respect to those of pure gravity due to the cancelation of divergences between bosons and fermions in the same supermultiplet. Although SUGRA is again non-renormalizable by power counting, it is legitimate to ask whether for some $1 \leq N \leq 8$ the cancelation of infinities will render the S-matrix finite without the need to renormalize. Thus, even if the off-shell Green functions may be plaged with infinities, a presumably weaker requirement that renormalizability is to impose finiteness of the scattering amplitudes.

As an example, let us consider $N = 1$, $d = 4$ SUGRA described by the lagrangian [28]

$$\mathcal{L} = -\frac{1}{2\kappa^2} eR - \frac{1}{2}\epsilon^{\mu\nu\sigma\lambda}\bar{\psi}_\mu \gamma_5 \gamma_\nu D_\sigma \psi_\lambda \ , \tag{9}$$

where ψ_μ is the gravitino field, $e = det({e_\mu}^a)$ the determinant of the vierbein and D_μ is the covariant derivative

$$D_\mu = \partial_\mu + \frac{1}{2}{\omega_\mu}^{mn}\sigma_{mn} \ , \tag{10}$$

with ${\omega_\mu}^{mn}$ the spin connection and $\sigma_{mn} = \frac{1}{4}[\gamma_m, \gamma_n]$. The lagrangian $\mathcal{L}$ is invariant under the local supersymmetry transformations

$$\begin{aligned} \delta {e_\mu}^m &= \frac{1}{2}\kappa\bar{\epsilon}(x)\gamma^m\psi_\mu(x) \\ \delta{\omega_\mu}^{mn} &= 0 \\ \delta\psi_\mu &= D_\mu\epsilon(x) \end{aligned} \tag{11}$$

In the quantum theory, besides the graviton and the gravitino we have to include a coordinate spin-1 ghost and its corresponding partner, a spin-$\frac{1}{2}$ ghost field [29].

It can be shown that on-shell amplitudes in $N = 1$ SUGRA are finite up to two-loops order [30], since the possible counterterms in the action are zero on-shell. However at three-loop level there is a counterterm which does not vanish even on-shell [31] (see also the second reference in [30]). Unless the coefficient multiplying this counterterm is zero, $N = 1$ SUGRA would be three-loop divergent. This is still an open question. The computation to settle it is a formidable task that nobody has undertaken. However, since there exists a possible counterterm, one would tend to be rather pesimist about its outcome.

Since $N = 1$ SUGRA is expected to diverge at three-loops, one has to examine the case for a finite theory in N-extended supergravity theories. For some time $N = 8$ SUGRA in four dimensions was a candidate for a theory of "everything". This theory can be formulated as $N = 1$ SUGRA in eleven dimensions [32] which, after dimensional reduction, becomes $N = 8$ SUGRA in four dimensions [33]. In the eleven-dimensional theory we have an *elf-bein* ${e_\mu}^m$ together with the gravitino ψ_μ and an antisymmetric tensor of rank 3, $A_{\mu\nu\sigma}$ (it is easy to see that the number of bosonic and fermionic degrees of freedom is the same). The mechanism of dimensional reduction consists of making the fields depend only on the four-dimensional coordinates $x^1, \ldots, x^4$ and not on the eleven-dimensional ones $x^1, \ldots, x^{11}$. At the end we have the following spectrum in $d = 4$

# of fields	spin	
1	2	graviton
8	$\frac{3}{2}$	gravitinos
28	1	
56	$\frac{1}{2}$	
70	0	

It is not possible to construct SUGRA theories in dimensions $d > 11$ [34] since there is no possibility of matching bosonic and fermionic degrees of freedom. Moreover, could it be possible to construct simple SUGRA in $d > 11$ then by dimensional reduction we would be left with $N > 8$ SUGRA in $d = 4$, violating so the upper bound for the number of supersymmetric partners (see above).

Although, by symmetry arguments, $N = 8$ SUGRA is expected to be finite up to seven loops, for higher loops there are possible on-shell countertems [35]. This means that the theory could be divergent at that order if there is no a miraculous cancellation of the coefficients in front of the counterterms. It seems naïvely that supergravity is not the correct place to look for a finite quantum theory of gravity. In fact for N-extended SUGRA there are possible non-zero on-shell counterterms beyond $(N-1)$-loop order for $N \geq 3$ and beyond 2-loops order for $N = 1, 2$.

1.3 Kaluza-Klein theories

During the seventies and early eighties there was a good deal of activity in Kaluza-Klein theories. The main idea goes back to the 1920's when Kaluza and Klein [36] tried to unify Einstein's General Relativity and Maxwell theory by formulating GR in $M_4 \times S^1$, where M_4 is the ordinary 4-dimensional space-time and S^1 is a circle of very small radius (typically of the order of the Planck lenght $l_P \sim 1.6 \times 10^{-33}$ cm). By introducing coordinates (x^μ, ϕ) we make the Kaluza-Klein ansatz for the metric tensor in five dimensions

$$g_{AB}(x,\phi) = \begin{pmatrix} g_{\mu\nu}(x) & A_\mu(x) \\ A_\mu(x) & \sigma(x) \end{pmatrix} , \tag{12}$$

where we have retained only the zero-modes in the fifth coordinate ϕ (this is equivalent to restricting ourselves to massless models in M_4). Substituting (12) in the five-dimensional Einstein-Hilbert action we obtain the gravitational action in four dimensions plus the Maxwell kinetic term $(F_{\mu\nu})^2$ for the 4-dimensional vector field A_μ and the kinetic term for σ.

This scheme can be generalized to non-abelian gauge theories [37]. To do so we consider the Einstein-Hilbert action in a $4+n$-dimensional space-time $M_4 \times B$ where now B is a compact internal manifold having G as its group of isometries. Considering coordinates (x^μ, ϕ_i) ($\mu = 0, 1, 2, 3$ and $i = 1, \ldots, n$) the Kaluza-Klein ansatz for the metric tensor in $M_4 \times B$ is

$$g_{AB}(x,\phi) = \begin{pmatrix} g_{\mu\nu}(x) & \sum_{a=1}^{dimG} A_\mu^a(x)\xi_i^a(\phi) \\ \sum_{a=1}^{dimG} A_\mu^a(x)\xi_i^a(\phi) & \gamma_{ij}(\phi) \end{pmatrix} \tag{13}$$

where ξ_i^a $(a = 1, \ldots, dimG)$ are the Killing vectors associated with the symmetry group G

$$T^a : \phi_i \longrightarrow \phi_i + \xi_i^a(\phi)\epsilon^a \tag{14}$$

Substituting (13) in

$$S_{E-H} = \frac{-1}{2\kappa^2} \int d^{4+n}x\sqrt{-g}R^{(4+n)} \tag{15}$$

we obtain in particular the Einstein-Hilbert action in four dimensions as well as the Yang-Mills action $\sum_a (F^a_{\mu\nu})^2$, invariant under the gauge transformation

$$
\begin{aligned}
(x^\mu, \phi_i) &\longrightarrow \left(x^\mu, \phi_i + \sum_a \epsilon^a(x)\xi^a_i(\phi)\right) &(16)\\
A^a_\mu(x) &\longrightarrow A^a_\mu(x) + D_\mu \epsilon^a(x) &(17)
\end{aligned}
$$

where $\epsilon^a(x)$ are the gauge parameters and D_μ is the covariant derivative with respect to the gauge field A^a_μ.

So the main idea of Kaluza-Klein theories is to represent gauge invariance in four-dimensional space-time as resulting from the group of isometries of an internal n-dimensional manifold, a beautiful geometrical idea. Of course the problem now is to construct realistic Kaluza-Klein theories , i.e., theories with gauge group G containing the SM gauge group $SU(3) \times SU(2) \times U(1)$. This requirement imposes a lower bound on the dimension of the internal space [38]. The space of lowest dimension with symmetry group G is always a homogeneous space G/H with $H \subset G$ a maximal subgroup of G. In our case $G = SU(3) \times SU(2) \times U(1)$ and the largest subgroup of suitable dimension is $H = SU(2) \times U(1) \times U(1)$. This means that the minimal dimension of a manifold with symmetry group $SU(3) \times SU(2) \times U(1)$ is $12 - 5 = 7$, so the minimal dimension of the space-time is 11, exactly equal to the maximal dimension for supergravity (see above).

Nevertheless, Kaluza-Klein theories have serious problems. One of them is, being essentially Einstein gravity in dimensions higher than four, non-renormalizability. Perhaps their main difficulty is that they are unable to reproduce the chiral nature of the four-dimensional world. Consider a massless Dirac fermion in $4 + n$ dimensions. It obeys the Dirac equation

$$
\not{D}\psi = \sum_{i=1}^{4+n} \Gamma^i D_i \psi = 0, \qquad (18)
$$

which may be rewritten as

$$
\not{D}^{(4)}\psi + \not{D}^{(int)}\psi = 0 \qquad (19)
$$

where $\not{D}^{(4)} = \sum_{i=1}^{4} \Gamma^i D_i$ and $\not{D}^{(int)} = \sum_{i=5}^{4+n} \Gamma^i D_i$. It is easy to see from (19) that the eigenvalues of $\not{D}^{(int)}$ are the masses of the fermions in four dimensions. Since non-zero eigenvalues will be of the order of $1/R$ with R the typical length of the internal manifold ($\sim$ Planck length) we see that the phenomenologically relevant four-dimensional fermions will be the zero-modes of $\not{D}^{(int)}$. But in order to reproduce the SM we need these fermions to be chiral in four dimensions, i.e., the left and right handed representations of the gauge group should be different. This cannot be achieved by using Dirac fermions in $4+n$ dimensions, as can be seen by the following argument due to Lichnerowicz [39, 40]: if we square the Dirac operator $i\not{D}^{(int)}$ in the internal manifold we have (dropping the label *int* for simplicity)

$$
(i\not{D})^2 = -\sum_i D_i D^i + \frac{1}{4}R \qquad (20)
$$

where R is the scalar curvature. If $R > 0$ everywhere on the manifold B, and since $-\sum_i D_i D^i$ is positive defined, the Dirac operator cannot have zero-modes. Furthermore, a theorem by Lawson and Yau [41] states that on any compact space B with non-abelian symmetry group G, there is always a $G-$invariant metric of positive curvature. Thus, we cannot obtain massless fermions in four dimensions from Dirac fermions in $4+n$ dimensions. What can be said about Rarita-Schwinger fields? After all in $N=8$ SUGRA we start not with Dirac spinors but with Rarita-Schwinger fields in eleven dimensions, which after dimensional reduction give raise to four-dimensional Dirac spinors. The problem of obtaining four-dimensional chiral $\frac{1}{2}$ fermions from Rarita-Schwinger fields in higher dimension has been studied by Witten who also carried out the general analysis of whether chirality could be generated from Kaluza-Klein theories. We are essentially summarizing his arguments [40]. He showed by using Atiyah-Hirzebruch theorem [42] that the zero-modes of the Rarita-Schwinger operator on any homogeneous space G/H form a real representation leading to vector-like quantum numbers in four dimensions. It is then imposible to reproduce a chiral low-energy field theory in four dimensions using the Kaluza-Klein program.

The only way to avoid Witten's result without renouncing to Kaluza-Klein ideas is to consider internal manifolds that are neither coset spaces nor compact [43]. There is besides another way to bypass the problem of chiral fermions by including gauge fields in the higher-dimensional theory [44]. This last proposal, however, spoils the most relevant aim of Kaluza-Klein theories, namely, the unification of gravity with Yang-Mills theories in a geometrical setting. One needs other principles to justify the presence of gauge symmetries in higher dimensions. String theory is the only candidate which generates at the same time extra gauge symmetries, chirality, and a likely consistent quantum theory of the gravitational field.

1.4 Quantum field theory and classical gravity

Instead of trying to be so ambitious as to try to start from the beginning with Quantum Gravity, we can learn many lessons from the study of QFT in the presence of a classical gravitational background. This subject is reviewed in the lectures by R. Wald, and we will limit our considerations to some general remarks. QFT in this setting presents many interesting features in its own right [45]. Many properties of QFT which hold in Minkowski space either do not apply, or change radically on a curved backgroud. For example, the vacuum state in Minkowski QFT has several well-known properties:

- It is unique.
- There is a well defined concept of localized excitations (particles).
- It determines the symmetries of the world and their realizations.

In the presence of a gravitational field the vacuum state loses the absolute meaning it has in Minkowski space. The reason is that strong or rapidly varying gravitational fields can produce particles. It is easy to see that this makes the concept of particle ambiguous (observer-dependent); let us consider a region of space-time in which we have a very strong gravitational field (for example near the horizon of a black hole). A static observer will see, because of the strong gravity, that particles are created in pairs. On the other hand a free-falling observer, by the equivalence principle, will not observe (locally) any gravitational field, so for him there is no creation of particles at all. This means that what the free-falling observer sees as the vacuum state of his QFT is not a vacuum for the static observer.

There are a number of elements that we need in the construction of a QFT in a fixed background [46]:

- A Hilbert space of states $\mathcal{H}$.
- A classical space-time $(\mathcal{M}, g_{\mu\nu})$.
- Fields operators acting on $\mathcal{H}$.
- A set of canonical commutation relations obeyed by these fields.
- Wave equations.
- Rules for constructing "particles" observables and Fock basis of $\mathcal{H}$.
- Regularization and renormalization schemes to make expectation values like $\langle T_{\mu\nu}\rangle$ finite.

In constructing a QFT in curved background we face some difficult problems. First of all we are performing the quantization of the theory in a given background metric $g_{\mu\nu}$. It is natural to ask about the back-reaction of the QFT over the space-time geometry. To be more concrete, when solving the quantum theory we usually impose the consistency condition (in units in which $G = 1$)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}(R - 2\Lambda) = 8\pi \langle T_{\mu\nu}\rangle \tag{21}$$

In general our energy-momentum tensor will not verify (21), so it is necessary to proceed *à la* Hartree-Fock: we quantize the theory in the presence of a "test" background metric $g^{(0)}_{\mu\nu}$, and we evaluate the vacuum expectation value of the energy-momentum tensor $\langle T_{\mu\nu}\rangle$. Then we solve (21) with $\langle T_{\mu\nu}\rangle$ and so we obtain a first-corrected metric $g^{(1)}_{\mu\nu}$. Repeating the process we get a series of metrics $\{g^{(0)}_{\mu\nu}, g^{(1)}_{\mu\nu}, \ldots, g^{(N)}_{\mu\nu}, \ldots\}$ which, hopefully, should eventually converges to a self-consistent solution $g_{\mu\nu}$.

A second problem is that, in order to solve wave equations, we need our background space-time to be globally hyperbolic [47] (this means that there exist a global Cauchy hypersurface Σ). In the case in which we have singularities or Cauchy horizons we would need certain (unclear) boundary conditions.

In ordinary QFT the divergent vacuum energies of the fields can be substracted without any further problem since the substration amounts to a redefinition of the origin of energies. In the presence of gravity vacuum energies contributes to the value of the cosmological constant. As soon as we have symmetry breaking, the quantum contributions are unacceptably large. In view of this, the smallness of the cosmological constant is probably the most misterious fact which receives no explanation from any theory of Quantum Gravity, including String Theory.

Let us now consider the fields in a curved background. Since we have no invariant notion of time, it seems natural to work in the Heisenberg picture in which the states do not depend on time. The existence of a Cauchy hypersurface Σ allow us to define an inner product in the space of fields; for two classical fields ϕ_1, ϕ_2 we define their inner product (ϕ_1, ϕ_2) as

$$(\phi_1, \phi_2) = -\int_\Sigma d\Sigma^\mu J_\mu(\phi_1, \phi_2) \tag{22}$$

where $J_\mu(\phi_1, \phi_2)$ is given by

$$J_\mu(\phi_1, \phi_2) = i(\bar{\phi}_1 \nabla_\mu \phi_2 - \bar{\phi}_2 \nabla_\mu \phi_1) \tag{23}$$

for two scalar fields and

$$J_\mu(\psi_1, \psi_2) = \bar{\psi}_1 \gamma_\mu \psi_2 \tag{24}$$

if ψ_1, ψ_2 are spinor fields. By using the equations of motion it is easy to check that both currents are conserved ($\nabla_\mu J^\mu = 0$) so the product (22) does not depend on the particular choice of the Cauchy hypersurface (as long as the fields vanish at infinity).

For simplicity we are going to consider the case of a real massless scalar field $\phi(x)$ [49]. Let $\{u_i(x)\}$ be a complete set of solution of its wave equation $g^{\mu\nu}\nabla_\mu\nabla_\nu\phi(x) = 0$ whose general solution is

$$\phi(x) = \sum_i u_i(x) a_i + \bar{u}_i(x) \bar{a}_i \tag{25}$$

We assume that $\{u_i(x)\}$ is an orthonormal set with respect to the inner product (22). If $\xi^\mu(x)$ is a Killing field we can construct the quantity

$$H(\xi) = \int_\Sigma d\Sigma^\nu T_{\mu\nu} \xi^\mu, \tag{26}$$

where Σ is again a Cauchy hypersurface and $T_{\mu\nu}$ is the energy-momentum tensor for the scalar field. It is easy to evaluate the Poisson bracket of (26) with the field $\phi(x)$:

$$\{H(\xi), \phi(x)\} = \mathcal{L}_\xi \phi(x), \tag{27}$$

$\mathcal{L}_\xi$ is the Lie derivative with respect to the Killing vector ξ. Since $\mathcal{L}_\xi$ is an antihermitian operator with respect to the inner product, we will choose the $u_i(x)$'s to be eigenstates of this operator

$$\mathcal{L}_\xi u_i(x) = -i\kappa u_i(x), \tag{28}$$

with real κ. Then we can classify the modes $u_i(x)$ into two classes: if $\kappa > 0$ we will say that the associated mode is a positive frequency solution and a negative one if $\kappa < 0$. From (28) it is easy to see that $u_i(x)$ and $\bar{u}_i(x)$ both have the same frequencies but with opposite sign. Upon quantization we associate the creation operators with positive frequency solutions, thus

$$\hat{\phi}(x) = \sum_i u_i(x) a_i + \bar{u}_i(x) a_i^\dagger, \tag{29}$$

with the vacuum state defined by $a_i|0\rangle = 0$ for every a_i. It is important to note that the classification of the modes into positive and negative frequency solutions, and the definition of the vacuum state, was made with respect to a given Killing field $\xi^\mu(x)$. Once the choice of the Killing field is made one has introduced the notion of particle. A specially interesting case is when we consider so-called "sandwitch" space-times in which we have three regions: two of them M^- and M^+ are static and correspond respectively to the past and to the future. Between them we have the third region M^0 in which we have a time varying gravitational field. In this case we have different notions of particle appropiate to particular measurement processes at different times. Then the concept of particle will be observer dependent.

The ambiguity in the concept of particle steems from its global nature. We may speculate that since we always have (by the equivalence principle) free falling observers it should be possible to define particles only by the fact that they produce localized effects and by the dispersion relation $p^2 = m^2$. This might lead to a "bundle" of ground states with may not have global section in the presence of horizons.

We are going to illustrate the ambiguity in the concept of particle with a very well known example due to Unruh [48]. We consider a particle detector consisting of a quantum system which in absence of interaction with an external field is in its ground state. When it interacts the system get excited and it is no longer in the fundamental state. We can detect the presence of particles by observing if our detector is in an excited state.

Imagine the detector moving along a world line $x^\mu(\tau)$, where τ is the proper time. Suppose that it interacts with the scalar field $\phi(x)$ through the lagrangian

$$\mathcal{L}_{int} = cm(\tau)\phi[x^\mu(\tau)] \tag{30}$$

where $m(\tau)$ is the monopole moment of the detector. Furthermore, assume that we are in Minkowski space and that the field $\phi(x)$ is in its ground state $|0_M\rangle$. For a general trayectory the detector will not remain in its ground state with energy E_0, but will go to an excited state with $E > E_0$ while the field will be also in an excited state $|\psi\rangle$. For $|c| << 1$ the amplitude for this process to ocur is

$$\mathcal{A} = ic\langle E, \psi| \int_{-\infty}^{+\infty} d\tau\, m(\tau)\phi[x(\tau)]|0_M, E_0\rangle \tag{31}$$

$m(\tau)$ can be rewritten as:

$$m(\tau) = e^{iH_0\tau} m(0) e^{-iH_0\tau}, \tag{32}$$

with H_0 the hamiltonian of the detector. Since $H_0|E\rangle = E|E\rangle$. The amplitude $\mathcal{A}$ becomes,

$$\mathcal{A} = ic\langle E|m(0)|E_0\rangle \int_{-\infty}^{+\infty} d\tau\, e^{i(E-E_0)\tau}\langle\psi|\phi[x(\tau)]|0_M\rangle. \tag{33}$$

If $|\psi\rangle$ describes a state with one particle with momentum k, $|k\rangle$, using the decomposition of $\phi(x)$ in terms of Minkowski modes

$$\phi(x) = \int \frac{d^3k'}{\sqrt{16\pi^3\omega'}} \left[a_k^\dagger e^{-i\vec{k}'\cdot\vec{x}+i\omega' t} + h.c.\right], \tag{34}$$

with $\omega = \sqrt{k^2+m^2}$ we obtain

$$\langle k|\phi(x)|0\rangle = \frac{1}{\sqrt{2(2\pi)^3\omega}} e^{-i\vec{k}\cdot\vec{x}+i\omega t} \tag{35}$$

For an inertial observer, we have $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}\tau(1-v^2)^{-1/2}$, and performing the integration we obtain

$$\mathcal{A} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\vec{k}\cdot\vec{x}_0}\delta\left[E - E_0 + (\omega - \mathbf{k}\cdot\mathbf{v})(1-v^2)^{-1/2}\right] \tag{36}$$

but since $E > E_0$ and $\mathbf{k} \cdot \mathbf{v} \leq kv < \omega$, the argument of the δ-function is positive and the amplitude vanishes. For a more complicated trayectory $x^\mu(\tau)$ the probability transition to all E and all ψ is simply

$$\mathcal{P} = \sum_E |\langle E|m(0)|E_0\rangle|^2 \mathcal{F}(E - E_0) \tag{37}$$

with

$$\mathcal{F}(E - E_0) = \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' \, e^{-iE(\tau-\tau')} G^+[x(\tau), x(\tau')], \tag{38}$$

hence, the particle absortion per unit time is

$$\mathcal{W} = c^2 \sum_E |\langle E|m(0)|E_0\rangle|^2 \int_{-\infty}^{+\infty} d(\Delta\tau) e^{-i(E-E_0)\Delta\tau} G^+(\Delta\tau). \tag{39}$$

In the particular case of a massless scalar and a uniformly accelerated detector, the world line is given by

$$\begin{aligned} x &= y = 0 \\ z &= \alpha \cosh(\tau/\alpha) \\ t &= \alpha \sinh(\tau/\alpha) \end{aligned} \tag{40}$$

and the probability per unit time is

$$\mathcal{W} = \frac{c^2}{2\pi} \sum_E (E - E_0) \frac{|\langle E|m(0)|E_0\rangle|^2}{e^{2\pi\alpha(E-E_0)} - 1}. \tag{41}$$

Therefore a uniformly accelerated observer will see a flux of particles as if he were in a thermal bath of scalar massless particles at a temperature

$$T = \frac{1}{2\pi\alpha} \tag{42}$$

while the inertial observer will see that the field $\phi(x)$ is in its vacuum state $|0_M\rangle$ and consequently will not see any particle at all. This shows that when we depart from Minkowski space the concepts of vacuum and particles states become observer-dependent.

A general way to understand how the notion of no-particle state changes in curved backgrounds is to use Bogoliubov transformations (for more details and references see [46]). We have seen above that upon choosing a Killing field $\xi^\mu(x)$ we can make a decomposition of any solution of the wave equation in terms of a set of positive frequency solutions with respect to $\xi^\mu(x)$, $u_i(x)$ and their complex conjugates $\bar{u}_i(x)$. In many curved backgrounds there are no timelike Killing fields or the Killing fields are not timelike everywhere. In the first case we usually have to deal with vector fields that are asymptotically Killing fields and we have to use a S-matrix formulation.

Let us consider two complete sets of solutions of the wave equation for the massless scalar field $\{u_i(x)\}$ and $\{w_i(x)\}$. Associated with these sets we have the corresponding creation and annihilation sets of operators $\{a_i; a_i^\dagger\}$ and $\{b_i; b_j^\dagger\}$. Since both sets of modes are bases for the solutions of the wave equation, we can expand one set in terms of the other, namely

$$\begin{aligned} w_i(x) &= \sum_j \alpha_{ij} u_j(x) + \beta_{ij} \bar{u}_j(x) \\ \bar{w}_i(x) &= \sum_j \bar{\beta}_{ij} u_j(x) + \bar{\alpha}_{ij} \bar{u}_j(x) \end{aligned} \tag{43}$$

By general covariance we can write

$$\phi(x) = \sum_i a_i u_i(x) + a_i^\dagger \bar{u}_i(x) = \sum_i b_i w_i(x) + b_i^\dagger \bar{w}_i(x) \tag{44}$$

So both sets of creation-annihilation operators are related by the Bogoliubov transformations

$$\begin{aligned} b_i &= \sum_j \bar{\alpha}_{ij} a_j - \bar{\beta}_{ij} a_j^\dagger \\ b_i^\dagger &= \sum_j -\beta_{ij} a_j + \alpha_{ij} a_j^\dagger. \end{aligned} \tag{45}$$

We can apply these relations to the case with two asymptotically flat regions: *in* and *out*. We have the corresponding set of modes $u_i^{in}(x)$ and $u_i^{out}(x)$ related by

$$u_i^{out}(x) = \sum_j \alpha_{ij} u_j^{in}(x) + \beta_{ij} \bar{u}_j^{in}, \tag{46}$$

and by applying the previous result we obtain:

$$\begin{aligned} a_i^{out} &= \sum_j \bar{\alpha}_{ij} a_j^{in} - \bar{\beta}_{ij} a_j^{in\dagger} \\ b_i^{out\dagger} &= \sum_j -\beta_{ij} a_j^{in} + \alpha_{ij} a_j^{in\dagger}. \end{aligned} \tag{47}$$

The *in* and *out* vacuum states are defined respectively by

$$\begin{aligned} a_i^{in}|0_{in}\rangle &= 0 \qquad \forall i \\ a_i^{out}|0_{out}\rangle &= 0 \qquad \forall i\ , \end{aligned} \tag{48}$$

From (47) we see that when $\beta \neq 0$, the *in* vacuum is not a vacuum state for *out* particles:

$$n_i^{out} = \langle 0_{in}| a_i^{out\dagger} a_i^{out} |0_{in}\rangle = \sum_i |\beta_{ij}|^2 \tag{49}$$

Thus the total number of *out* particles in the *in* vacuum is given by

$$N^+ = \sum_j n_j^{out} = tr\, \beta\beta^\dagger \tag{50}$$

This show that if we consider that our system is in the vacuum state in the *in* region, in the *out* region an observer will see a total number of particles given by $tr\, \beta\beta^\dagger$. In practical cases the problem is to find the Bogoliubov transformation between the two asymptotic regions. Although in ordinary QFT we may also encounter Bogoliubov transformations they are ubiquitous in the presence of interesting gravitational backgrounds.

One of the most spectacular results which can be extracted from the study of quantum fields in the presence of the gravitational interaction was achieved by Hawking in 1975 [50][2]. He found that black holes emit particles as if they were black bodies at a temperature

$$T_{Hawking} = \frac{1}{8\pi M} \tag{51}$$

for the case of the Schwarzschild black hole with M being its mass. Since the temperature is inversely proportional to the mass and the total amount of energy radiated per unit time and per unit area is proportional to the fourth power of the temperature we find that, asuming the semiclassical approach all the way, the black hole will evaporate completely in a time proportional to M_0^3 with M_0 the initial mass of the black hole. It is also quite unintuitive thermodynamically that as the black hole radiates it heats up.

Black hole evaporation gives rise to serious conceptual problems (for a recent review see [51]). Hawking found that the radiation emited by a black hole is thermal; if this would exactly correct, it would mean that the radiation is in a mixed quantum state. In classical GR no information from inside the horizon can escape to the asymptotic region. It would seem natural in quantizing in the presence of a black hole to always take a trace over the states inside the horizon. This produces a mixed state. If we imagine a matter system in a pure state which collapse to form a black hole, and if we consider the complete evaporation process assuming the radiation to be exactly thermal, at the end we are left with a mixed state. In other words, although we know the initial state of the system we cannot predict what the final state would be. This clearly violates the laws of Quantum Mechanics.

Of course in the previous discussion it was assumed that the semiclassical derivation of the emission of particles is valid during the whole process of the black hole evaporation. When the mass of the black hole becomes close to the Planck mass the semiclassical description breaks down. The prediction of what happens after that point is impossible without a reliable theory of quantum gravity. One can imagine at least four possibilities [52]:

- The evaporations produces a naked singularity of negative mass which persists.
- The evaporation slows down and stops leaving a remnant black hole with a mass of the order of the Planck mass.
- The black hole evaporates completely but all the information about the initial state is encoded in the radiation that scapes to infinity.
- The black hole evaporates completely and takes with it all the information about its initial state and conserved quantities, except those coupled to long range fields (mass, charge and angular momentum).

[2]For details and references on Hawking radiation see the lectures by R. Wald.

There are some arguments [52, 51] in favour of the fourth posibility. In this case we are left with the puzzles pointed out before. Consider an asymptotic region in the past where our system is described by a density operator ρ_- defined in the Fock space of a free field and an asymptotic region in the future in which the system is described by a density operator ρ_+. Hawking [52] proposed to use a superscattering operator \$ relating the density operators in the two regions

$$\rho_+{}^A{}_B = \$^A{}_{BC}{}^D \rho_-{}^C{}_D. \tag{52}$$

We have to require the \$-operator to satisfy several conditions: first it has to map initial density operators of unit trace into final density operators which are positive, semi-definite and of unit trace. This implies that

$$\$^C{}_{CA}{}^B = \delta_A{}^B. \tag{53}$$

Moreover, \$ has to map hermitian operators into hermitian operators,

$$\$^A{}_{BC}{}^D = \$^B{}_{AC}{}^D. \tag{54}$$

In ordinary (Minkowski) QFT we have the S-operator which maps states in the asymptotic past into states in the asymptotic future. In this case the \$-operator can be factorized as the tensor product $S \otimes S^\dagger$

$$\$^A{}_{BC}{}^D = S^A{}_C \bar{S}_B{}^D \tag{55}$$

With this form for the superscattering operator, if the system is in a pure state, it will remain in a pure state, as it is required by the laws of Quantum Mechanics. This is not the case for a general \$. In fact it is possible to show that \$-operator has the form (55) as long as the Green's functions of the theory verify asymptotic completeness, that is to say, if the asymptotic *in* and *out* states span the Hilbert space of the theory. Hawking has argued that the axiom of asymptotic completeness is not verified in quantum gravity; this has the consequence that quantities that are not coupled to long range fields are not conserved (for example global $U(1)$ charges [52]). Further analysis and criticism of these arguments can be found in [53].

There has been some recent developments in the study of black hole evaporation. On the one hand, the classical no-hair theorems have been reanalyzed [54] and found to be modified substancially by Quantum Fields. Furthermore, some simple renormalizable theories of two-dimensional gravity coupled to dilatons, and containing many of the interesting features of four-dimensional black holes have recently been proposed [55]. In this models one can study the collapse of matter to form a black hole and its subsequent evaporation. Unfortunately, the final stages of this process are not yet accesible to the approximations used in [55].

1.5 Euclidean approach to Quantum Gravity

In ordinary QFT we are faced with the computation of path integrals of the form

$$Z = \int \mathcal{D}\phi \, e^{iS[\phi]} \tag{56}$$

with $S[\phi]$ the classical action. However, the oscillatory nature of the exponential in (56) makes the expression ill-defined. In this context the problem can be ameliorated by performing what is called a Wick rotation to euclidean space; this means that we will make an analytic continuation of real Minkowski time to pure imaginary values $t \to -it$ such that we can rewrite (56) as

$$Z_E = \int \mathcal{D}\phi \, e^{-S_E[\phi]} \tag{57}$$

where $S_E[\phi]$ is the Euclidean action. Since in most cases in QFT this euclidean action is positive definite (57) is now perfectly well defined because field configurations with large values of $S_E[\phi]$ are damped by the exponential factor. Once the path integral is evaluated in Euclidean space we return to Minkowski space by analytical continuation.

For the procedure we outlined above to be justified some conditions has to be met. In QFT the Wightman's axioms [57] guarantee the possibility of analytic continuation. Besides in QFT the continuation to the Euclidean space has many other applications. When we study quantum systems at finite temperature we need to calculate the canonical partition function $Z(\beta)$

$$Z(\beta) = tr \, e^{-\beta H} \tag{58}$$

with $\beta = 1/T$ and H the hamiltonian of the system. This trace can be represented as a path integral [58]. Using standard techniques one obtains a path integral in Euclidean space with time compactified to a circle whose length is the inverse temperature. The thermal partition function is [58]

$$Z(\beta) = \int_{\mathbf{R}^3 \times S^1_\beta} \mathcal{D}\phi \, e^{-S_E[\phi]} \tag{59}$$

For bosons we have to asume periodic boundary conditions in the compactified dimension. On the other hand if we have fermions they have to be antiperiodic. Euclidean field theory is also the arena in which Yang-Mills instantons are formulated [59].

Now we come to gravity. We may expect to find a similar formulation for the General Theory of Relativity, namely, to make an analytical continuation from the physical signature to Euclidean signature in which a path integral approach to quantum gravity would give better results than the methods described in previous sections; after calculating physical observables we would come back to the physical signature. However this is not so easy because we find difficulties from the very beginning: for curved spaces, even for globally hyperbolic ones, there are no axioms ensuring the analytic continuation in the time coordinate. Moreover, when we perform the continuation we find that singularities may disappear. This is the case for example of the Schwarzshild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left[\frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + \sin\theta d\phi^2)\right] \tag{60}$$

When we perform the analytical continuation into Euclidean space we miss the region inside the horizon $r < 2m$ and we are left only with the exterior part $r > 2m$ and the horizon in which the Killing vector ∂_t vanishes.

Even if we do not worry about this problem and continue ahead, we will soon be in trouble. Unlike the case of Yang-Mills theories, the gravitational action is unbounded from below; this is because the integrand is linear in the scalar curvature, so the action can be made arbitrarily negative. Let us consider the action for the Euclidean gravitational field coupled to matter in a four-dimensional riemannian manifold M with boundary ∂M (see [60] and references therein)

$$
\begin{aligned}
S &= -\frac{1}{16\pi G}\int_M d^4x\sqrt{g}(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{h}(K-K^0) \\
&- \int_M d^4x\sqrt{g}\,\mathcal{L}_{matt},
\end{aligned} \tag{61}
$$

where now K is the trace of the second fundamental form on ∂M (i.e., the extrinsic curvature) and h_{ab} is the metric on the three-dimensional boundary. The third term in the action represents the action for the matter fields. Let us write the metric g_{ab} in the form

$$
\tilde{g}_{ab} = \Omega^2 g_{ab} \tag{62}
$$

with $\Omega(x)$ a positive function. The scalar curvature changes according to

$$
\tilde{R} = \Omega^{-2}R - 6\Omega^{-3}\Box\Omega \ , \tag{63}
$$

and for the extrinsic curvature

$$
\tilde{K} = \Omega^{-1}K + 3\Omega^{-2}n^a\nabla_a\Omega \ , \tag{64}
$$

with n^a the unitary normal vector on ∂M. With this decompositions we can rewrite the gravitational part of (61) as

$$
\begin{aligned}
S &= -\frac{1}{16\pi G}\int_M d^4x\sqrt{g}\left[\Omega^2 R + 6g^{ab}\nabla_a\Omega\nabla_b\Omega - 2\Lambda\Omega^4\right] \\
&- \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{h}\Omega^2(K-K^0)
\end{aligned} \tag{65}
$$

It is easy to see that by considering a rapidly varying conformal factor $\Omega(x)$ we can make the action arbitrarily negative. This implies that the path integral over $\Omega(x)$ diverges since, in principle, we have to integrate over all possible conformal factors. Gibbons, Hawking and Perry [61] gave a prescription for the evaluation of the functional integral over the conformal factor: we separate all possible four-dimensional metrics into conformal classes and then we pick in each class one for which the scalar curvature vanishes. This could be accomplished for a given metric g_{ab} by solving

$$
(\Box - \frac{1}{6}R)\Omega(x) = 0 \tag{66}
$$

so $\Omega^2 g_{ab}$ has zero scalar curvature. Once this is achieved, we integrate over the conformal factor and over all conformal classes, and a particular analytic continuation of the integral over Ω is given to render it well defined. Since in this prescription we have to consider metrics with zero scalar curvature, the gravitational action is enterely governed by the surface term, and then by the boundary conditions over ∂M. Physical gravitational fields vanish asymptotically at infinity, so in physical cases we deal with metrics that are asymptotically flat. This fact led Gibbons, Hawking and Perry to formulate the positive action conjecture [61] according to which the gravitational action is non-negative for any asymptotically euclidean and positive defined metric with $R = 0$. This conjecture was finally proved in 1979 by Schoen and Yau [62].

Assume now that we have succesfully dealt with the unboundness of the gravitational action. Then, we can think about applying Polyakov's prescription, which has given such good results in two-dimensional gravity [63]. When we calculate the partition function (or any correlator) we have to sum over all topologies, which in the case of two-dimensional orientable surfaces are classified symply by their genus, hence

$$Z_{d=2} = \sum_{g=0}^{\infty} \int_{\mathcal{I}_g} \mathcal{D}g\, e^{-S_E[g]}, \tag{67}$$

where the path integral is performed at fixed topology. Then we would try to make a similar expansion in the case of four-dimensional gravity

$$Z_{d=4} = \sum_{topologies} \int_{fix.\,top.} \mathcal{D}g\, e^{-S_E[g]}. \tag{68}$$

However in trying to carry out this program in four dimensions lead us to an essential difficulty: there is no algorithmic way of deciding when any two four-dimensional manifolds are homeomorphic. We will briefly review the set of theorems which lead to this result (see [19] and references therein).

The starting point is the fact (Markov's theorem) that any finitely generated group G can be the fundamental group of a four-dimensional, smooth, compact, and connected manifold. That the group

$$G = \langle a_1, \ldots, a_n | R_1, \ldots, R_k \rangle, \tag{69}$$

is finitely generated means simply that it is generated by a finite number of elements $\{a_i\}$ which are subject to a finite number of relations $\{R_j\}$. The second important ingredient we need is a result first posed by Dehn and proved by Novikov known in group theory as the word problem: given a family of finitely generated groups $\{G_k\}$ with $k = 1, 2, \ldots$ there is no algorithmic way of distinguishing any trivial group in that family.

It is now easy to show the impossibility of classifying topologies in four dimensions. Let us construct for the family G_k a set of manifolds M_k such that $\pi_1(M_k) = G_k$ (which is possible by Markov's theorem). If we were able to say whether a manifold are homeomophic to one of the manifolds in the family we could say that their fundamental groups are equal. Thus by seeing what are the elements of $\{M_k\}$ which are homeomorphic to a given manifold M_0 with trivial fundamental group, we would be able to say what elements of $\{G_k\}$ are trivial; but that cannot be done.

Although the previous result tells us that it is impossible to classify four-dimensional topologies in general, we can try to classify the topologies in some subclass of four-dimensional manifolds. For example we can impose the restriction to the sector with trivial fundamental group $\pi_1(M) = 0$. Let us begin by defining the Kirby-Siebenman invariant for a manifold M: $\alpha(M) \in \mathbf{Z}_2$ such that $\alpha(M) = 0$ if the manifold $M \times S^1$ admits a smooth structure and $\alpha(M) = 1$ if it does not. A theorem by M. Freedman states that if M is a compact connected manifold with trivial fundamental group then it is classified by $H_2(M, \mathbf{Z})$, the second homology group and the Kirby-Siebenman invariant $\alpha(M)$.

Given a four-dimensional manifold we can define the intersection form ω as a symmetric bilinear form $\omega : H^2(M, \mathbf{Z}) \to \mathbf{Z}$ such that

$$\omega(\beta_1, \beta_2) = \int_M \beta_1 \wedge \beta_2 \tag{70}$$

This intersection form can be used to classify simply connected ($\pi_1(M) = 0$) compact four-dimensional manifolds into homeomorphism classes. Two of such manifolds are homotopy equivalent iff their intersection forms are isomorphic (Whitehead's theorem). We introduce the concept of signature $\sigma(\omega)$ for the intersection form ω as the number of its positive eigenvalues minus the number of its negative ones. It can be shown that whenever ω is even it is automatically divisible by eight. The concept of signature is important because there is a one-to-one correspondence between the set of simply-connected compact, four-dimensional manifolds and the pairs $\langle \omega, \alpha \rangle$, where α is the Kirby-Siebenman invariant defined earlier and if ω is even then $\omega/8 \equiv \alpha \; (mod\, 2)$ is satisfied. Thus, the pairs $\langle \omega, \alpha \rangle$ constitute the space of parameters for four-dimensional compact topological manifolds in the sector with $\pi_1(M) = 0$.

GR is invariant under diffeomorphisms; when performing the path integral at fixed topology we have to integrate over all diffeomorphism classes without counting each class more than once. So we have to classify four-dimensional manifolds into diffeomorphism classes in order to carry out the path integration. This can be done by using a theorem by Donaldson which states that given a closed four-dimensional, smooth and simply connected manifold with an intersection form ω which is positive definite, ω over the integers is equivalent to $1 + \ldots + 1$, i.e., the set of manifolds with intersection form ω does not exists smoothly for any non-zero positive definite ω. This implies the existence of different classes of differentiable structures which are classified in part by Donaldson's invariants. For example, there are exotic differentiable structures in $\mathbf{R}^4$, the so called $\mathbf{R}^4_{fake}$.

After this description of the problems we encountered in trying to use the Polyakov approach in four-dimensional gravity we have to conclude that it seems imposible to carry out this program completely. Nevertheless, one could, based in some yet unknown physical principle, impose some kind of restriction in the set of manifolds we integrate over; for example to integrate over simply connected manifolds ($\pi_1(M) = 0$). The previous arguments are too naïve. They are based on a semiclassical analysis of the problem of quantizing gravity. In the functional integral we will certainly have contributions coming from singular metrics. If we relax the conditions that the manifolds and geometries should be continuous and smooth, the objections raised above lose much of their strenght. What is not clear is the understanding of what physical conditions would imply the consideration of more general manifolds.

1.6 Canonical quantization of gravity

In this approach we restrict *ab initio* to space-times with a topology $\Sigma_3 \times \mathbf{R}$, where Σ_3 is a compact three-manifold; these hypersurfaces Σ_3 are interpreted as surfaces of constant time which "folliate" the whole space-time. Introducting this privileged time coordinate we can construct a hamiltonian and proceed to the canonical quantization of the space-time [3]. The corresponding Fock space would not be equivalent to that obtained from path-integral quantization, because

[3] A thorough presentation of canonical gravity is found in the lectures by A. Ashtekar in this volume.

in the second case we integrate also over four-dimensional manifolds which are not topologically $\Sigma_3 \times \mathbf{R}$. When canonically quantizing we are more interested on the classification of three-dimensional manifolds. Now the determination of the classical phase space is "simpler" as long as one believes Thurston's geometrization program [64]. In fact we can include every three-dimensional manifold in a list with a countable infinite number of parameters without leaving any three-manifold out of the list. However we cannot be completely happy because we may have the problem of an infinite overcounting.

There are several steps in the classification of three-dimensional manifolds. First we can consider genus g Heegard splitting (for an application of this technique see [65]) which consist of decomposing a three-manifold into two "handle bodies" of genus g by cutting along a Riemman surface Σ_g. After that the boundaries of the two parts are identified by a diffeomorphism not connected with the identity. A problem, nevertheless, arises when considering the fundamental group $\pi_1(M)$ of a three-dimensional manifold because although it is finitely generated there is no algorithmic way of knowing whether a given group is the fundamental group of a three-manifold.

We would like to find a set of topological invariants we could use to classify three-manifolds along the same lines as in two dimensions in where compact orientable manifolds are classified according to their genus. The search of these invariants in the three-dimensional case is a matter of current investigation (Jones-Witten invariants, Casson invariants...). An attempt to classify three-dimensional manifolds has been made by Thurston [64] based in the geometrization conjecture: every compact, orientable three-manifold can be cut by disjoint embedded two-spheres and tori into pieces which, after gluing three-balls to all boundary spheres, admit geometric structures. That the manifold admits a geometric structure simply means that for every pair of points $x, y \in M$ there are two isometric neighborhoods $U_x, U_y \subset M$.

One way in which we could get rid of these problems it to consider that at short distances we have a topological field theory [66]. Then, below some critical lenght, the theory is in a symmetrical phase in which it only depends on some well defined topological invariants so the phase space is perfectly well characterized. In the low energy phase the invariance under diffeomorphisms is spontaneously broken by the vacuum, which is only Lorentz invariant, and the graviton is the corresponding Goldstone boson. This idea is an old one and can be traced back to the early seventies [67].

The group of C^∞-diffeomorphisms of a smooth compact manifold M, Diff(M), has some peculiar properties [68]. It is a Lie group with its Lie algebra given by the set Vect(M) of smooth vector fields on M which close under Lie brackets. This group is very different from ordinary Lie groups because it is not possible in general to get any element of the connected component of Diff(M) as the image by the exponential map of an element of Vect(M). However there always exist a set $v_1, \ldots, v_k \in \mathrm{Vect}(M)$ such that any element $f \in \mathrm{Diff}(M)$ can be written as

$$f = EXP(v_1) \circ EXP(v_2) \circ \ldots \circ EXP(v_k) \tag{71}$$

We are interested in studying the topological structure of the set of all metrics modulo the group of diffeomorphism. The topology of this space is intimately connected to the topology of Diff(M). For instance

$$\pi_1[\mathrm{Metrics}(M)/\mathrm{Diff}(M)] \cong \pi_0[\mathrm{Diff}(M)], \tag{72}$$

that is to say, the symply connectness of Metrics(M)/Diff(M) is given by the connectness of Diff(M) and similarly for higher homotopy groups. Considering the group Diff(S^n) Smale proved that [69]

$$\pi_k[\mathrm{Diff}(S^n)] \cong \pi_k[O(n+1)] \tag{73}$$

for $n = 1, 2$ and for all k. Smale himself conjectured that this relation also holds for $n = 3$, namely

$$\pi_k[\mathrm{Diff}(S^3)] \cong \pi_k[O(4)] \tag{74}$$

This conjecture was finally proved by Hatcher [70].

As a final speculation, one can think that perhaps in the right theory one is allowed to make deformations wilder than homeomorphisms, so that there is some kind of quantum equivalence of otherwise distinct classical geometrical structures. This happens for example in String Theory in connection with duality relations [71] and mirror symmetries [72]. For example a string moving on a circle of radius R is equivalent to a string on a circle of radius $1/R$ (see sec. 3.6). One also finds that topologically very different Calabi-Yau manifolds (Ricci-flat compact Kähler manifolds) lead to the same physical theory (mirror symmetry). This is properly known to hold only for string theory. This short section has hopefully convinced the reader that a naïve analysis of the problem of canonical quantization of gravity is going to meet with rather severe mathematical difficulties if we insists on having smooth manifolds and geometries. Very important progress towards a proper understanding of canonical Quantum Gravity has been achieved with the work initiated by Ashtekar and coworkers [73].

1.7 Gravitational instantons

In Yang-Mills theory, instantons appear as classical solutions of the Euclidean field equations. From the point of view of Minkowsky space these solutions are interpreted as tunneling between vacuum states that are not topologically equivalent. If Euclidean gravity makes any sense we could expect to find gravitational instantons which would represent tunneling between inequivalent vacua. Some properties of these gravitational instantons are [74]:

- They describe gravitational fields which are localized in Euclidean space-time.
- These solutions approach an asymptotically locally Euclidean vacuum metric at infinity. This means that although the metric at infinity is locally flat, globally the space is not topologically equivalent to flat space-time. This is equivalent to the property of Yang-Mills self-dual solutions of being pure-gauge at infinity.
- They have non-trivial topological quantum numbers.

All these properties are very similar to those of the instantons we find in Yang-Mills theory [59]. The reason to expect to find self-dual solutions in four-dimensional Euclidean space is that the holonomy group is $SU(2) \times SU(2)$. Requiring the curvature to take values in only one of the $SU(2)$'s is the self-duality condition. If we look for regular manifolds with instanton-like solutions we find that the surface $K3$ is the only compact, regular and simply connected manifold without boundary which admits a non-trivial self-dual curvature. If we look for solutions with some singular points, the number of possible candidates is much bigger.

Let us illustrate the subject of gravitational instantons by considering some particular examples [74]: let us take the following ansatz for the metric

$$ds^2 = U(\mathbf{x})^{-1}(d\tau^2 + \omega \cdot d\mathbf{x}) + U(\mathbf{x})d\mathbf{x} \cdot d\mathbf{x} \tag{75}$$

which is self-dual as long as

$$\nabla \times \omega = \nabla U(\mathbf{x}) \tag{76}$$

The general solution can be written as

$$U(\mathbf{x}) = \epsilon + \sum_{i=1}^{k} \frac{2M_i}{|\mathbf{x} - \mathbf{x}_i|} \tag{77}$$

with ϵ an integration constant. In order to make (75) well behaved and free of singularities, we have to choose $M_i = M$ and to make τ periodic with period $8\pi M/k$.

Let us take now $\epsilon = 1$; in this case we have that the action is $S = 4\pi k M^2$. For the case $k = 1$ we have Euclidean Schwarzschild solution while for $k > 1$ it is not clear what physical situation requires this kind of instantons. They are known as multi-Taub-NUT metrics and they are assymptotically flat in the spatial directions $|\mathbf{x}| \to \infty$. Near spatial infinity these manifolds look like $S^2 \times (S^1/\mathbf{Z}_k)$ where S^2 has a radius growing like $|x|$ as $|x| \to \infty$ while $S^1/\mathbf{Z}_k$ approaches a fixed radius. The fiber identification implied by the $\mathbf{Z}_k$-modding makes the physical interpretation of these solutions very difficult. When $k = 1$, we have the only case of a gravitational instanton which has been used so far. It appears in the nucleation rate of black holes on a thermal gas of gravitons at a given temperature [75].

With the choice $\epsilon = 0$ we have that the action is zero for every value of k. For $k = 1$ we recover flat space-time and for $k = 2$ we have the Eguchi-Hanson metric in both cases modulo coordinate transformations. Asymptotically, these instantons look like $S^3/\mathbf{Z}_k$. Again their physical interpretation is unclear.

There is a very important difference between Yang-Mills and gravitational instantons. In the Yang-Mills case, instanton mediated amplitudes are suppresed by factors typically of the form $\exp\left(-4\pi^2/g^2\right)$ (the exponent is the instanton action). For the asymptotically locally euclidean instantons (like multi-Eguchi-Hanson instantons), their action vanishes, and therefore thery will not be suppressed by exponential factors. They are supressed only by powers due to the quantization of their zero modes and moduli. Unfortunately no physical process has been described which would require the use of these instantons.

This concludes our quick overview of some of the approaches used to describe Quantum Gravity in the context of QFT.

2 Consistency Conditions: Anomalies

2.1 Generalities about anomalies

In classical field theory it is well known that the presence of a continuous symmetry in the action leads to the existence of a conserved current associated with it (Noether's theorem).

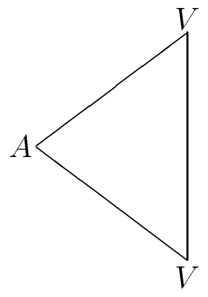
Nevertheless when quantizing the theory, classical symmetries may not be conserved, and their conservation equation may be modified by terms proportional to $\hbar$

$$\nabla_\mu \langle j^\mu(x) \rangle = O(\hbar) \tag{78}$$

The reason for this is simple. Currents $j^\mu(x)$ are composite operators and therefore are naively ill-defined, and we need to regularize them. The problem arises when the regularization methods are not compatible with the symmetries of the classical theory. In this case, after renormalization, the quantum theory does not necessarily recover the classical symmetries.

By definition an anomaly is a breakdown of a classical symmetry by quantum corrections. There are well known examples of anomalies in QFT, for example the massless ϕ^4 scalar field theory is invariant under scale transformations at the classical level. However, in the process of regularization and renormalization it is necessary to introduce an energy scale μ. As a result, not only the renormalized coupling constant λ_R is a function of this scale, but also the fields adquire anomalous dimensions.

Perhaps the most celebrated example of an anomaly in QFT is the Adler-Bell-Jackiw (ABJ) anomaly which arises when considering the decay $\pi^0 \to 2\gamma$ [76]. This comes from the anomalous term in the Ward identities associated with the triangle diagram with two vector currents and one axial vector current.

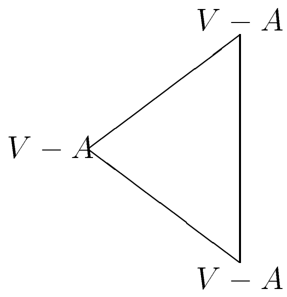


If one imposes both Bose symmetry and the conservation of vector currents, the result is that the axial vector current $A_\mu(x) = \bar{\psi}(x)\gamma_5\gamma_\mu\psi(x)$ is not conserved for a massless fermion, but instead we have

$$\partial_\mu A^\mu(x) = -\frac{\alpha\hbar}{16\pi^2}\epsilon_{\mu\nu\sigma\lambda}F^{\mu\nu}(x)F^{\sigma\lambda}(x), \tag{79}$$

$F^{\mu\nu}(x)$ is the electromagnetic field strength and $\epsilon_{\mu\nu\sigma\lambda}$ the completely antisymmetric Levi-Civita tensor. This anomaly is far from being dangerous, it affects a local current and in fact our understanding of the decay $\pi^0 \to 2\gamma$ is based on its existence [77].

Much more critical is the existence of anomalies in local gauge symmetries. The reason is that very important issues such as renormalizability or unitarity are based upon the preservation of the gauge symmetry at the quantum level. Thus the existence of gauge anomalies jeopardizes the consistency of the theory. The best example of this type is the SM based in the gauge group $SU(3) \times SU(2) \times U(1)$. As we will see below the first diagram in which an anomalous contribution could arise is the triangle diagram with three $V-A$ currents in the vertices coupled to gauge fields [78].



If after summing over all fermion species in the fermionic loop we obtain a non-zero contribution coming from this diagram we will have an anomalous Ward identity for $\langle (V-A)(V-A)(V-A)\rangle$. When applying Feynman rules to calculate a diagram with one fermionic loop and N vertices coupled to gauge fields we obtain an overall group theoretical factor of the form

$$\sum_{T_L} str(T_L^{a_1}\dots T_L^{a_N}) - \sum_{T_R} str(T_R^{a_1}\dots T_R^{a_N}), \tag{80}$$

T_L^a and T_R^a are respectively the generators of the representation carried by the left and right-handed fermions and str is the symmetrized trace (symmetrization is needed because of Bose symmetry). For non-chiral theories in which left and right-handed fermions transform under the same representation of the gauge group this factor is always zero, and consequently there is no anomaly. The problem arises when the theory is chiral (as the real world is) because in this case, unless we have a cancellation when summing over all fermions in the loop, the contribution of the diagram in non-zero.

Consider the SM. The quantum numbers of a single family with respect to $SU(3)\times SU(2)\times U(1)$ are

$$(3,2)_{1/6}\oplus(\bar{3},1)_{-2/3}\oplus(\bar{3},1)_{1/3}\oplus(1,2)_{-1/2}\oplus(1,1)_1\ , \tag{81}$$

where the first entry in the parenthesis indicates the representation of $SU(3)$, the second one that of $SU(2)$ and the subindex the weak hypercharge Y. All anomalies cancel for this representation; for example, it is easy to see that the contribution of the leptons to the anomaly is exactly canceled by the quarks (note that all the fermions in (81) are left-handed)

$$\begin{aligned}\sum_{leptons} Y_L^3 &= 2\times\left(-\frac{1}{2}\right)^3+1=\frac{3}{4}\\ \sum_{quarks} Y_L^3 &= 3\times\left[2\times\left(\frac{1}{6}\right)^3+\left(-\frac{2}{3}\right)^3+\frac{1}{3}\right]=-\frac{3}{4},\end{aligned} \tag{82}$$

All other combinations of the generators of $SU(3)\times SU(2)\times U(1)$ also vanish (as can be easily checked) and then the theory is anomaly free.

In general we expect gauge anomalies only in chiral theories (or equivalently in parity violating theories). This is because for non chiral theories it is always possible to construct a mass term in the lagrangian in a gauge invariant way. The existence of this mass term allows us to use of Pauli-Villars regularization, which does not break gauge invariance and thus the renormalized theory will be anomaly free.

2.2 Spinors in $2n$ dimensions

Since the existence of anomalies is intimately related to the presence chiral fermions it will be useful before plunging into the study of anomalies to review briefly the properties of spinors in $2n$ dimensions [4][79]. The starting point is the $2n$-dimensional Clifford algebra

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}, \tag{83}$$

with $a, b = 0, 1, \ldots, 2n-1$ and η_{ab} the flat metric with signature (s,t)

$$\eta_{ab} = diag\left((+1)^s, (-1)^t\right). \tag{84}$$

A Dirac spinor field is an object that under an infinitesimal transformation of $SO(t,s)$ it transforms according to

$$\delta\psi = -\frac{1}{2}\epsilon_{mn}\Sigma^{mn}\psi\ , \tag{85}$$

ϵ_{mn} are the parameters of the tranformation, and

$$\Sigma^{ab} = -\frac{1}{4}[\Gamma^m, \Gamma^n] \tag{86}$$

with

$$(\Gamma^1)^\dagger = \Gamma^1, \ldots, (\Gamma^t)^\dagger = \Gamma^t, (\Gamma^{t+1})^\dagger = -\Gamma^{t+1}, \ldots, (\Gamma^{t+s})^\dagger = -\Gamma^{t+s} \tag{87}$$

It is easy to check by using (83) that Σ^{ab} satisfies the commutation relations of the Lie algebra of $SO(t,s)$.

Since we are working in even dimension, we can define an element $\bar{\Gamma}$ of the Clifford algebra (the analog of the 4-dimensional γ_5)

$$\bar{\Gamma} = \alpha\Gamma^0\Gamma^1\ldots\Gamma^{2n-1} \tag{88}$$

satisfying the properties

$$\{\bar{\Gamma}, \Gamma^m\} = 0 \qquad\qquad m = 0, \ldots, 2n-1 \tag{89}$$

and

$$\bar{\Gamma}^2 = 1, \qquad \bar{\Gamma}^+ = \bar{\Gamma} \tag{90}$$

The last conditions can be translated into conditions over α

$$\alpha^2 = (-1)^{\frac{s-t}{2}} \qquad \alpha^* = (-1)^{\frac{s-t}{2}}\alpha \tag{91}$$

From (89) we deduce that $[\bar{\Gamma}, \Sigma^{ab}] = 0$. This, together with (90) suffices to construct the chiral projectors

$$P_\pm = \frac{1}{2}(1 \pm \bar{\Gamma}) \tag{92}$$

[4]Chiral theories exist only in even-dimensional spaces.

These two projectors $P_\pm$ split the representation Σ^{ab} into two irreducible representations of $SO(t,s)$

$$\psi_\pm = P_\pm \psi \tag{93}$$

where ψ_+ (ψ_-) is a positive (resp. negative) chirality Weyl spinor.

It is a known fact that the Clifford algebra (83) has a unique faithful representation of dimension 2^n. Thus, the representations supplied by

$$\Gamma^m, (\Gamma^m)^*, -(\Gamma^m)^*, (\Gamma^m)^T, -(\Gamma^m)^T, \tag{94}$$

are all equivalent. In particular there exists a matrix B such that

$$(\Sigma^{ab})^* = B\Sigma^{ab}B^{-1}. \tag{95}$$

We define the charge conjugate spinor by means of the antilinear operator C

$$\psi^c = C\psi \equiv B^{-1}\psi^*. \tag{96}$$

It is easy to check that ψ and ψ^c transform in the same way under $SO(s,t)$ transformations

$$\delta\psi^c = B^{-1}\delta\psi^* = -\frac{1}{2}\epsilon_{ab}[B^{-1}(\Sigma^{ab})^*B]B^{-1}\psi^* = -\frac{1}{2}\epsilon_{ab}\Sigma^{ab}\psi^c \tag{97}$$

Note that if ψ transforms under the gauge group (i.e. if it carries some gauge index) with the representations $R(G)$, the charge conjugate spinor transforms with the complex conjugate representation $\bar{R}(G)$.

In the case in which B can be found such that $C^2 = 1$ it is possible to construct two Majorana projectors

$$\frac{1}{2}(1 \pm C), \tag{98}$$

such that

$$\psi_M = \frac{1}{2}(1 + C), \tag{99}$$

and

$$\psi_{\bar{M}} = \frac{1}{2}(1 - C), \tag{100}$$

are Majorana and anti-Majorana fields.

Let us now see under what circunstances we can have Weyl and Majorana conditions at the same time. We calculate the commutator of $\bar{\Gamma}$ and C, the result being

$$C\bar{\Gamma} = (-1)^{\frac{s-t}{2}}\bar{\Gamma}C \tag{101}$$

This means that for $(s-t)/2$ odd the charge conjugate spinor has the opposite chirality of the original one, while for $(s-t)/2$ even we have the same chirality for both the spinor and its charge conjugated. If we consider now the case of Minkowski space, $s = 2n-1$, $t = 1$, it is easy to see that chirality and helicity are equivalent concepts. Indeed, the massless Dirac equation in momentum space can be written as

$$\Gamma^0\psi(p) = \Gamma^{2n-1}\psi(p) \tag{102}$$

where we have made use of the freedom we have to consider the particle moving along the $2n-1$-axis. By definition, the helicity operator in $2n$ dimensions is

$$h = \Sigma^{12}\Sigma^{34}\ldots\Sigma^{2n-3,2n-2} = \left(\frac{i}{2}\right)^{n-1}\Gamma^1\Gamma^2\ldots\Gamma^{2n-2} \tag{103}$$

By applying h to $\psi(p)$ we arrive at

$$h\psi(p) = \left(\frac{i}{2}\right)^{n-1}\Gamma^1\Gamma^2\ldots\Gamma^{2n-2}\psi(p) = \left(\frac{i}{2}\right)^{n-1}\bar{\Gamma}\psi(p) \tag{104}$$

making use of the Dirac equation (102). This means that in Minkowski space with $d = 4k$, charge conjugation flips helicity, while for $d = 4k+2$ it does not.

In order to further investigate the possibility of finding B such that $C^2 = 1$, we consider B as the matrix relating Γ^m to $-(\Gamma^m)^*$

$$-(\Gamma^m)^* = B\Gamma^m B^{-1}, \tag{105}$$

and we can write

$$\Gamma^m = -B^*(\Gamma^m)^*B^{-1} = (B^*B)\Gamma^m(B^*B)^{-1} \tag{106}$$

This implies that B^*B commutes with all the Γ's. Since they form an irreducible representation of the Clifford algebra (83), using Schur's lemma we can choose B satisfying

$$B^*B = \epsilon I \tag{107}$$

with $|\epsilon| = 1$. In fact, taking the complex conjugate of the last expression one can see (for example taking the trace) that ϵ must be real, so $\epsilon = \pm 1$. The standard charge conjugation matrix is the one relating Γ^m and $-(\Gamma^m)^T$

$$(\Gamma^m)^T = -C\Gamma^m C^{-1} \tag{108}$$

Since in Minkowski space $(\Gamma^m)^\dagger = \Gamma^0\Gamma^m\Gamma^0$ we can write, after normalizing properly

$$B^\dagger B = 1 \tag{109}$$

But, using of (107) we can finally arrive at

$$B^T = \epsilon B \tag{110}$$

and

$$C = B\Gamma^0 \tag{111}$$

To study the values of ϵ as a function of the dimension we will work in an explicit basis for $d = 2n$

$$\begin{aligned}
\Gamma^0 &= \sigma_x \otimes 1 \otimes \ldots \otimes 1 \\
\Gamma^1 &= i\sigma_y \otimes 1 \otimes \ldots \otimes 1 \\
\Gamma^2 &= i\sigma_3 \otimes \sigma_x \otimes \ldots \otimes 1 \\
\Gamma^3 &= i\sigma_3 \otimes \sigma_y \otimes \ldots \otimes 1 \\
&\vdots \\
\Gamma^{2n-2} &= i\overbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}^{n-1} \otimes \sigma_x \otimes 1 \otimes \ldots \otimes 1 \qquad (112)\\
\Gamma^{2n-1} &= i\sigma_3 \otimes \ldots \otimes \sigma_3 \otimes \sigma_y \otimes 1 \otimes \ldots \otimes 1 \qquad (113)
\end{aligned}$$

and

$$\bar{\Gamma} = \sigma_3 \otimes \sigma_3 \otimes \ldots \otimes \sigma_3 \tag{114}$$

where σ_x, σ_y and σ_3 are the 2-dimensional Pauli matrices. The charge conjugation operator becomes:

$$C = \sigma_y \otimes \sigma_x \otimes \sigma_y \otimes \ldots \tag{115}$$

the number of σ_y's being k if $d = 4k$ or $k+1$ if $d = 4k+2$. Thus we have the following values of ϵ

$$\epsilon = +1 \qquad\qquad d = 2, 4 \quad mod\ \ 8 \tag{116}$$

$$\epsilon = -1 \qquad\qquad d = 0, 6 \quad mod\ \ 8 \tag{117}$$

The final result is that the Majorana condition can only be imposed in $d = 2, 4\ \ mod\ \ 8$, since only in this case $C^2 = (BB^*)^{-1} = 1$. Moreover, we have found above that $[\bar{\Gamma}, C] = 0$ only if $d = 4k+2$, so Weyl and Majorana conditions can be simultaneously defined only in $d = 2\ \ mod\ \ 8$.

2.3 When can we expect to find anomalies?

Up to now we have met two kinds of anomalies. The first type is called singlet anomaly and consists of an anomaly in a global current (exemplified by the ABJ anomaly). As we said before the anomalies of this kind do not jeopardize the consistency of the theory and their appearance is crucial in order to explain some physical processes (for example the $\pi^0 \to 2\gamma$ decay). The second class of anomalies we mentioned are gauge anomalies which lead to a breakdown of gauge invariance in the quantum theory. Contrary to the first case these anomalies are very dangerous and its presence can render the theory inconsistent.

Besides these two kinds of anomalies there is a third type, the gravitational anomalies [80, 81]. They appear in theories coupled to gravity and they imply a failure of diffeomorphism invariance in the quantum theory. Unless the total anomaly cancels after summing over all fermion species, theories with this kind of anomalies cannot be consistently coupled to gravity.

Both, gauge and gravitational anomalies, can be further classified into two groups. The first one are the local gauge (gravitational) anomalies, in which the failure of gauge (diffeomorphism) invariance of the theory is restricted to transformations that belong to the connected component of the identity. They can appear in perturbation theory. This is the case for example of the potential gauge anomaly appearing in the standard model. Secondly we have global gauge (gravitational) anomalies. In this case, although the theory is invariant under infinitesimal gauge transformations (diffeomorphisms) it may not be so under transformations that cannot be continously joined to the identity. The most celebrated example of this kind is Witten's $SU(2)$ anomaly (see below).

We begin by analyzing how the properties of the quantum effective action action for fermions may affect the conservation of a current [81]. Define the one-loop fermion effective action by

$$e^{-\Gamma_{eff}(A_\mu)} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left[-\int d^d x\, \bar{\psi} i \not{D} \left(\frac{1-\bar{\Gamma}}{2}\right)\psi\right], \tag{118}$$

where $\not{D} = \Gamma^\mu(\partial_\mu + iA^a_\mu\lambda^a)$ is the covariant Dirac operator. Under an infinitesimal gauge tranformation of the gauge field

$$\delta A^a_\mu = D_\mu \epsilon^a \tag{119}$$

the effective action changes by

$$\delta\Gamma_{eff} = tr\int d^dx\,(D_\mu\epsilon^a)\frac{\delta\Gamma_{eff}}{\delta A^a_\mu} = -tr\int d^dx\,\epsilon^a D_\mu\left(\frac{\delta\Gamma_{eff}}{\delta A^a_\mu}\right)\ . \tag{120}$$

Now, by taking the functional derivative in both sides of (118) we find

$$\begin{aligned}\frac{\delta\Gamma_{eff}}{\delta A^a_\mu} &= -e^{\Gamma_{eff}(A^a_\mu)}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\left[\bar{\psi}\Gamma^\mu\lambda^a\left(\frac{1-\bar{\Gamma}}{2}\right)\psi\right]e^{-S(\psi,\bar{\psi},A)} \\ &= -\left\langle\bar{\psi}\Gamma^\mu\lambda^a\left(\frac{1-\bar{\Gamma}}{2}\right)\psi\right\rangle,\end{aligned} \tag{121}$$

finally, we can rewrite the variation of the effective action as

$$\begin{aligned}\delta\Gamma_{eff} &= tr\int d^dx\,\epsilon^a D_\mu\left\langle\bar{\psi}\Gamma^\mu\lambda^a\left(\frac{1-\bar{\Gamma}}{2}\right)\psi\right\rangle \\ &= tr\int d^dx\,\epsilon^a D_\mu\langle j^\mu_a\rangle.\end{aligned} \tag{122}$$

Thus, the breakdown in the conservation of the expectation value of the current $\langle j^\mu_a\rangle$ implies that the fermion effective action is not invariant under infinitesimal gauge tranformations.

A similar calculation can be done for gravitational anomalies. Now, we perform an infinitesimal coordinate tranformation $x^\mu \to x^\mu + \xi^\mu(x)$. The metric tensor transforms according to

$$\delta g_{\mu\nu} = D_\mu\xi_\nu + D_\nu\xi_\mu. \tag{123}$$

The variation of the fermion effective action is

$$\delta\Gamma_{eff} = \int d^dx\sqrt{g}\,\xi_\mu D_\nu\langle T^{\mu\nu}\rangle\ , \tag{124}$$

where $\langle T^{\mu\nu}\rangle$ is the expectation value of the energy-momentum tensor of the matter field and D_μ is the covariant derivative. Again we see that the failure in the conservation of the energy-momentum tensor makes the quantum theory non invariant under diffeomorphisms.

Anomalies are expected to occur only in parity violating theories; otherwise we can always construct a gauge invariant mass term and regularize the theory using the Pauli-Villars method. The anomalies are associated only with the imaginary part of the effective action. Take the fermions in a complex representation $R(G)$ of the gauge group. Since we cannot construct a gauge invariant mass term there is no evident way to regularize the theory preserving the symmetries. Thus the fermion effective action $\Gamma_R(A)$ is complex and not necessarily gauge invariant. Now, we can consider fermions in the complex conjugate representation $\bar{R}(G)$. In this case the discussion along the similar lines implies that the effective action $\Gamma_{\bar{R}}(A)$ (which is the complex conjugate of $\Gamma_R(A)$) is again potentially anomalous. If we consider fermions in the real representation $R(G)\oplus\bar{R}(G)$ it is possible to construct a gauge invariant mass term and the theory is anomaly free. Since the path integral is gaussian in the fermion fields, the effective action for this theory is simply $\Gamma_R(A)+\Gamma_{\bar{R}}(A) = 2\,Re\,\Gamma_R(A)$. So we conclude that the real part of $\Gamma_R(A)$ is gauge invariant or can be made gauge invariant by suitable local counterterms, and anomalies can only be present in the imaginary part of the fermion effective action.

The usual way in which we look for anomalies is by considering one loop diagrams coupled to external gauge fields or gravitons by means of chiral currents or energy-momentum tensors

$$\bar{\psi}T^a\Gamma_\mu P_+\psi \quad \leftrightarrow \quad A_\mu^a\,, \tag{125}$$

$$\bar{\psi}(D_\mu\Gamma_\nu + D_\nu\Gamma_\mu)\psi \quad \leftrightarrow \quad h_{\mu\nu}\,. \tag{126}$$

These amplitudes, which appear in the perturbative analysis of the Ward identities, have parity preserving and a parity violating parts; the anomaly can only come from the last one, since the parity preserving part is just like the effective action for Dirac fermions and there is no problem with its regularization.

Since it is impossible to define a parity violating amplitude satisfying all physical principles there are two standard definitions of the amplitude [82]. The first way, which is the hardest one from a computational point of view, is to define the parity violating part of the amplitude preserving Bose symmetry on the external lines. The anomaly so obtained is called the consistent anomaly because it satisfy the Wess-Zumino consistency condition [83].

The second alternative (the Adler-Rosenberg method [76, 84]) consists of considering the same diagram with a single axial vector current in one vertex and vector currents in the others. We then calculate the anomaly by imposing vector current conservation and Bose symmetry on the external vector lines. This form of the anomaly, called the covariant anomaly, does not satisfy the consistency conditions, but it is much easier to calculate. There is a standard formalism interpolating between consistent and covariant form of the anomaly [85, 79], so we can calculate the covariant anomaly (which is easier) and from it obtain the consistent form.

We now determine what diagrams are potentially anomalous. Let us consider a $k+1$ poligon with k external vector currents and one axial vector current. Since the amplitude is parity violating, it contains a Levi-Civitta tensor $\epsilon_{i_1\ldots i_{2n}}$ (we are in $2n$ dimensions). We want to check conservation of the axial vector current, so the polarization vector in the axial channel is proportional to the incoming momentum P_μ. Besides, we have the momenta in the other vertices $p_1,\ldots,p_k$ and the corresponding polarization vectors $\epsilon_1^\mu,\ldots,\epsilon_k^\mu$. We have to saturate the indices of the ϵ tensor and we have the momentum conservation law $P+\sum p_i = 0$. The only alternative is to take $k=n$ so the simplest diagram which is potentially anomalous is a $n+1$ polygon. In this case we can write the amplitude associated to this diagram as

$$A(p,\epsilon)\left[\sum_{T_L} str(T_L^{a_1}\ldots T_L^{a_{n+1}}) - \sum_{T_R} str(T_R^{a_1}\ldots T_R^{a_{n+1}})\right]\,. \tag{127}$$

Instead of calculating $str(T^{a_1}\ldots T^{a_{n+1}})$ it is easier, and completely equivalent, to calculate $tr\,H^{n+1}$ with H any element of the Lie algebra of the gauge group; for $d=4k$ we would have $tr\,H^{2k+1}$ and $tr\,H^{2k+2}$ for $d=4k+2$. Let us consider the case in which fermions are in a real or pseudoreal representation of the gauge group. In this case H satisfy

$$H^T = -S^{-1}HS\,, \tag{128}$$

with $S^T = \pm S$ ($+1$ for real representations and -1 for pseudoreal). Then for $d = 4k$ we have

$$tr\, H^{2k+1} = tr\,(H^{2k+1})^T = tr\,(-S^{-1}HS)^{2k+1} = -tr\, H^{2k+1} = 0\ . \tag{129}$$

So in $d = 4k$ there is no gauge anomaly if the gauge group has only real or pseudoreal representations. Thus we are sure that chiral theories with gauge groups $SO(2n+1)$, G_2, F_4, E_7 or E_8 are always anomaly free. Then the only dangerous possibilities are the gauge group to be $SO(2n)$, $SU(N)$ or E_6.

In the last paragraphs we studied gauge anomalies. For gravitational anomalies we have again that the first diagram from which we can expect an anomalous contribution is a polygon with $n+1$ vertices ($d = 2n$) [81]. The same reasoning as in the case of gauge anomalies applies, since the polarization vector of the graviton is a second rank symmetric tensor. We may also wonder when we can expect pure gravitational anomalies; for a given fermion ψ_+ in four dimensions, we have both particles and antiparticles inside the loop of the diagram carrying opposite helicity since $\{\bar{\Gamma}, C\} = 0$. But for the gravitons in the external lines, particles and antiparticles are not distinguishable and the coupling "looks" vector like. This is also applicable to $d = 4k$ in which the anticommutation of $\bar{\Gamma}$ and C also holds. The result is that in $4k$ dimensions we have no purely gravitational anomalies.

This is not the case in $d = 4k+2$. Now $[\bar{\Gamma}, C] = 0$ and particles have the same helicity as their antiparticles. The gravitational interaction is genuinely chiral and we can expect anomalies in the conservation of the energy-momentum tensor.

Pure gravitational anomalies can only be present in $4k+2$ dimensions. As an example let us compute the gravitational anomaly for a spin-1/2 fermion in two dimensions [81]. In the presence of a weak gravitational field $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, at linearized level the perturbation $h_{\mu\nu}$ couple to the fermion field through the fermion energy-momentum tensor by the lagrangian $\Delta\mathcal{L} = -\frac{1}{2}h^{\mu\nu}T_{\mu\nu}$. If we consider a chiral fermion obeying $\bar{\Gamma}\psi = -\psi$, the only non-vanishing component of $T_{\mu\nu}$ is T_{++}, and we are interested in the two-point function

$$U(p) = \int d^2x \langle\Omega| T\,(T_{++}(x)T_{++}(0))\ .|\Omega\rangle e^{ipx} \tag{130}$$

From naive conservation of $T_{\mu\nu}$, $\partial_- T_{++}(x) = 0$, we get the Ward identity

$$p_- U(p) = 0\ , \tag{131}$$

implying $U(p) = 0$ for all $p_- \neq 0$. But by analyticity we will have $U(p) = 0$ for all p_-, and this is impossible for the two-point function of a hermitian operator, so we must have an anomaly in the Ward identity. This anomaly can be explicitly calculated from fig. 1 [81], and the result is

$$p_- U(p) = \frac{i}{24\pi} p_+^3\ . \tag{132}$$

With the value of $U(p)$ we can construct the effective action to second order in the perturbation $h_{\mu\nu}$ simply by coupling each vertex in the figure to $-\frac{1}{2}h_{--}$

$$S_{eff}(h_{\mu\nu}) = -\frac{1}{192\pi}\int d^2p \frac{p_+^3}{p_-} h_{--}(p)h_{--}(-p)\ , \tag{133}$$

Figure 1: The gravitational anomaly in two dimensions

where $h_{--}(p)$ is the Fourier transform of $h_{--}(x)$. Now we can test the invariance of this effective action under coordinate transformations and verify that it is not; we still have the freedom of adding local terms to the effective action in order to make it diffemorphism invariant. Considering the most general possible counterterm with the proper dimension

$$\Delta S = \int d^2p \left[A p_+^2 h_{--}(p) h_{+-}(-p) + B p_+ p_- h_{+-}(p) h_{+-}(-p) \right. \\ \left. + C p_+ p_- h_{++}(p) h_{--}(-p) + D p_-^2 h_{++}(p) h_{+-}(-p) \right] , \tag{134}$$

it can be checked that there is no way of choosing the constants in order to make $S_{eff} + \Delta S$ invariant under coordinate transformations.

The situation is quite different if we consider Dirac fermions in $1+1$ dimensions. In this case, to get the fermion effective action, we have only to add to (133) its parity transformed

$$S_{eff} = -\frac{1}{192\pi} \int d^2p \left[\frac{p_+^3}{p_-} h_{--}(p) h_{--}(-p) + \frac{p_-^3}{p_+} h_{++}(p) h_{++}(-p) \right] . \tag{135}$$

This action is non-invariant under general coordinate transformations, but now we can add local couterterms to obtain an invariant action

$$\bar{S} = -\frac{1}{192\pi} \int d^2p \frac{R(p)R(-p)}{p_+ p_-} , \tag{136}$$

where $R(p) = p_+^2 h_{--} + p_-^2 h_{++} - 2 p_+ p_- h_{+-}$ is the linearized form of the scalar curvature.

From this result we can obtain the trace anomaly of the energy momentum tensor. As we indicated T_{+-} vanishes classically. The result of this is that h_{+-} does not appear in the effective action (135). However in order to preserve diffeomorphism invariance we have to add counterterms that contain h_{+-}, so we get a non-zero expectation value for $T_{+-}(p)$ given by

$$\langle 2T_{+-}(p) \rangle = -2 \frac{\delta \bar{S}}{\delta h_{+-}(-p)} = -\frac{1}{24\pi} R(p) , \tag{137}$$

which means that classical conformal invariance is broken by quantum corrections.

2.4 The Atiyah-Singer Index Theorem and the computation of anomalies

In the last two decades, sophisticated technology borrowed from topology has been applied to the computation of anomalies. A specially useful tool has been the Atiyah-Singer index theorem [86, 87] and its generalization, the Atiyah-Singer index theorem [88] for families of elliptic operators. By applying the first one it is possible to compute singlet anomalies, while the use of the second one allows us to calculate both local and global anomalies.

In what follows we will not review the whole technology for computing anomalies, which is available in a number of reviews [85, 79]. Instead we will present the basic results and how to use them. Since, as we will see, the problem of computing anomalies is intimately related to the index theorem for the Dirac operator (as well as that for Rarita-Schwinger operator), we start by reviewing the main features of the index theorem for the Dirac operator. Given an operator $\mathcal{O}$ we define the index as the difference between the dimension of the kernel of $\mathcal{O}$ and that of its adjoint

$$ind\, \mathcal{O} = dim\, ker\, \mathcal{O} - dim\, ker\, \mathcal{O}^\dagger \; . \tag{138}$$

The Atiyah-Singer index theorem relates the index of an elliptic operator with the topological properties of the manifold in which it is defined. We will use this theorem in the particular case in which $\mathcal{O}$ is the Weyl operator $D_+ = i\not{D} P_+$.

Let M be a manifold and let $S_+ \otimes E$ and $S_- \otimes E$ be two vector bundles over M. $S_\pm$ is the space of spinors of positive and negative chirality respectively and E is the space carrying the representation of the gauge group. Let $A = A_\mu dx^\mu$ be the connection 1-form over E, which transforms under a gauge tranformation g as

$$A \to g^{-1}(a + d)g \; , \tag{139}$$

and associated with it the curvature F

$$F = dA + A^2 \; , \tag{140}$$

where we assume wedge product between forms (i.e. $A \wedge A = A^2$). In the presence of gravity we have, besides the gauge bundle, the vector bundle of orthonormal or coordinates frames. In the first case we will work with the spin connection $\omega = \omega_\mu dx^\mu$ and in the second one with the Christoffel connection $\Gamma = \Gamma_\mu dx^\mu$; the curvature associated R has an expression similar to that of the gauge curvature $R = d\omega + \omega^2$. The Dirac operator $\not{D}$ is given explicitly by

$$\not{D} = \Gamma^\mu(\partial_\mu + \frac{1}{2}\omega_{\mu\, ab}\Sigma^{ab} + A_\mu^a T^a) \; , \tag{141}$$

In terms of (141) we define the Weyl operators $D_\pm = i\not{D} P_\pm$ mapping $S_\pm \otimes E$ into $S_\mp \otimes E$

$$D_\pm : S_\pm \otimes E \longrightarrow S_\mp \otimes E \; , \tag{142}$$

and verify $D_+^\dagger = D_-$. In a base in which $\bar{\Gamma}$ is diagonal the Dirac operator can be written as

$$\not{D} = \left(\begin{array}{cc} 0 & D_- \\ D_+ & 0 \end{array} \right) \; . \tag{143}$$

Now, we can define the index of D_+ as

$$ind\, D_+ = dim\, ker\, D_+ - dim\, ker\, D_- \; . \tag{144}$$

$ind\, D_+$ is a topological invariant; using the Atiyah-Singer index theorem we can express it in terms of the topological invariants of the bundle. Before doing so we have to introduce some elementary notions from the theory of characteristic classes [85, 79, 89]. Let Ω be some matrix-valued two-form taking values in the Lie algebra of some group G and let $P(\Omega)$ be an invariant polynomial, i.e., a polynomial in Ω satisfying

$$P(g^{-1}\Omega g) = P(g) \tag{145}$$

for any element g of G. Let us consider the particular case in which Ω is either the gauge field strength tensor F or the curvature R. In this case any invariant polynomial $P(\Omega)$ satisfies the following properties:

- $P(\Omega)$ is a closed form, i.e., $dP(\Omega) = 0$.
- The integrals of $P(\Omega)$ are topological invariants.

To prove these two properties it is enough to consider monomials of the form $P_m(\Omega) = tr(\Omega^m)$ since these are the building blocks of any invariant polynomial. It is straighforward to show that $dP(\Omega) = 0$.

$$dP_m = m\, tr(d\Omega\, \Omega^{m-1}) = m\, tr(D\Omega\, \Omega^{m-1}) = 0\ , \tag{146}$$

where D is the covariant derivative and we have used the Bianchi identity

$$D\Omega = d\Omega + \Omega\omega - \omega\Omega = 0 \tag{147}$$

with ω the connection 1-form.

The second statement is not as evident as the first one. We have to prove that the integral of the polynomial P_m over M is independent of the connection ω. Consider a one-parameter family of connections interpolating between two given connections ω_0 and ω_1

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0)\ , \tag{148}$$

with $0 \leq t \leq 1$, and the family of curvatures associated to them

$$\Omega_t = d\omega_t + \omega_t^2\ . \tag{149}$$

Now we evaluate the partial derivative of Ω_t with respect to the parameter t (we denote this derivative by a dot)

$$\begin{aligned} \frac{\partial P_m(\Omega_t)}{\partial t} &= m\, tr\left(\dot{\Omega}_t\, \Omega_t^{m-1}\right) = -m\, tr\left(D_t\dot{\omega}_t\Omega_t^{m-1}\right) = \\ & m\, d\, tr\left(\dot{\omega}_t\Omega_t^{m-1}\right)\ , \end{aligned} \tag{150}$$

since we have that

$$\dot{\Omega}_t = d\dot{\omega}_t + \dot{\omega}_t\omega_t + \omega_t\dot{\omega}_t\ . \tag{151}$$

Now, by integrating (150) with respect to t from 0 to 1 we obtain

$$P_m(\Omega_1) - P_m(\Omega_0) = m\, d \int dt\, tr(\dot{\omega}_t\Omega_t^{m-1}) = dQ^0_{2n-1}(\omega_t, \Omega_t)\ . \tag{152}$$

where Q^0_{2n-1} is called the Chern-Simons form. If we now integrate over any closed $2m$-dimensional submanifold $M_{2m} \subset M$ we have

$$\int_{M_{2m}} P_m(\Omega_1) = \int_{M_{2m}} P_m(\Omega_0) \tag{153}$$

Then the integral of $P_m(\Omega)$ over M_{2m} is independent of the connection, so it is a topological invariant and the second statement is proven . Moreover, this integral is also invariant under deformations of M_{2m}. Let us take a deformation $M^{'}_{2m}$ of M_{2m} and let B_{2m+1} be a manifold such that $\partial B_{2m+1} = M_{2m} - M^{'}_{2m}$. In this case, since $dP_m(\Omega) = 0$ we have, using Stokes theorem, that

$$0 = \int_{B_{2m+1}} dP_m(\Omega) = \int_{M_{2m}} P_m(\Omega) - \int_{M^{'}_{2m}} P_m(\Omega). \tag{154}$$

We now introduce some well known classes, in term of which the index of the Weyl operator will be expressed. First we take a bundle with group $U(n)$, so the curvature Ω is a hermitian matrix of two forms. The the total Chern class is defined as

$$c(\Omega) = det\left(1 + \frac{i}{2\pi}\Omega\right) \ . \tag{155}$$

This total class can be expanded as a sum of the form

$$c(\Omega) = 1 + c_1(\Omega) + c_2(\Omega) + \ldots \ , \tag{156}$$

where $c_i(\Omega)$ is a $2i$-form called the i-th Chern class. To determine the precise form of the Chern classes we use the following trick: since the matrix Ω is hermitian we can formally diagonalize it and write

$$\frac{i}{2\pi}\Omega = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \ , \tag{157}$$

where the eigenvalues x_i are 2-forms. Then the total Chern class becomes

$$c(\Omega) = \prod_{i=1}^{n}(1 + x_i) = 1 + \sum_{i=1}^{n} x_i + \sum_{i<j} x_i x_j + \ldots + \prod_{i=1}^{n} x_i \ . \tag{158}$$

Now, we can identify the terms in the expansion as

$$\begin{aligned} c_1(\Omega) &= \sum_{i=1}^{n} x_i = \frac{i}{2\pi} tr\,\Omega \\ c_2(\Omega) &= \sum_{i<j} x_i x_j = \left(\frac{i}{2\pi}\right)^2 \left[(tr\,\Omega)^2 - tr\,(\Omega^2)\right] \\ &\vdots \\ c_n &= \prod_{i=1}^{n} x_i = det\left(\frac{i}{2\pi}\Omega\right) \ , \end{aligned} \tag{159}$$

and we express the Chern classes in terms of Ω. The total Chern class has good properties with respect to the direct sum (Whitney sum) of bundles. If E and F are two $U(n)$ bundles; the total Chern character of the direct sum $E \oplus F$ is

$$c(E \oplus F) = c(E)c(F). \tag{160}$$

This behavior of the total Chern character with respect to direct sums is not maintained for tensor products. We can define another useful polynomial, the Chern character, associated with a $U(n)$ bundle by

$$ch(\Omega) = tr\, e^{\frac{i}{2\pi}\Omega} = ch_0(\Omega) + ch_1(\Omega) + \dots\ , \tag{161}$$

where $ch_i(\Omega)$ is i-th Chern character. The identifications of these Chern characters follows from expanding the exponential:

$$tr\, e^{\frac{i}{2\pi}\Omega} = \sum_{k=0}^{\infty} \frac{1}{n!} tr\, \left(\frac{i}{2\pi}\Omega\right)^k. \tag{162}$$

This sum contains always a finite number of terms since Ω^k is a $2k$-form and then $2k \leq dim\, M$. We obtain the j-th Chern character as

$$\begin{aligned} ch_0(\Omega) &= 2n & (163)\\ ch_j(\Omega) &= \frac{1}{j!}\left(\frac{i}{2\pi}\right)^j tr\, \Omega^j, \qquad 2 \leq 2j \leq dim\, M. & (164) \end{aligned}$$

The total Chern character enjoys simple properties with respect to both direct sums and tensor products of bundles

$$\begin{aligned} ch(E \oplus F) &= ch(E) + ch(F) \\ ch(E \otimes F) &= ch(E) ch(F). \end{aligned} \tag{165}$$

Up to now we have defined invariant polynomials for $U(n)$ bundles in which the curvature is a hermitian matrix of 2-forms. Let us now consider the case in which the group bundle is $SO(n)$, as in Riemannian geometry. We define the total Potrjagin class as:

$$p(\Omega) = det\left(1 + \frac{1}{2\pi}\Omega\right) = 1 + p_1(\Omega) + p_2(\Omega) + \dots\ , \tag{166}$$

with $p_i(\Omega)$ the i-th Potrjagin class. The curvature Ω is an antisymmetric matrix of 2-forms and it cannot be diagonalized by a similarity transformation but it can be always brought to the skew-diagonal form

$$\frac{1}{2\pi}\Omega = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{167}$$

The antisymmetry of Ω implies that the only non-zero polynomials in Ω are of even degree; thus $p_j(\Omega)$ is a $4j$-form. The total Potrjagin class can be written in terms of the formal eigenvalues as

$$p(\Omega) = \prod_i (1 + x_i^2) = 1 + \sum_i x_i^2 + \sum_{i<j} x_i^2 x_j^2 + \dots \tag{168}$$

We are now ready to formulate the Atiyah-Singer index theorem for the Weyl operator

$$D_+ = i\Gamma^\mu (\partial_\mu + \frac{1}{2}\omega_{\mu\, ab}\Sigma^{ab} + A_\mu) P_+ \, . \tag{169}$$

The theorem states that the index of D_+ is given by

$$ind\, D_+ = \int_M \left[\hat{A}(M) ch(F)\right]_{vol} \tag{170}$$

where $\hat{A}(M)$, called the Dirac genus of M (or the A-roof genus), is a polynomial in the 2-forms x_i, and the subindex vol indicates that we only retain from the product the volume form. The Dirac genus is given by

$$\hat{A}(M) = \prod_a \frac{x_a/2}{\sinh{(x_a/2)}} \, . \tag{171}$$

It can be rewritten in terms of the invariant polynomials $tr(R^m)$ with R the curvature form of M

$$\hat{A}(M) = 1 + \frac{1}{(4\pi)^2}\frac{1}{12} tr\, R^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{288}(tr\, R^2)^2 + \frac{1}{360} tr\, R^4\right] + \dots \tag{172}$$

This equation, together with the expansion of the total Chern character (161) and (164), allows us to write explicitly the index of D_+, for example, in $d = 4$ dimensions

$$ind\, D_+ = \frac{1}{(2\pi)^2}\int_M \left(\frac{i^2}{2} tr\, F^2 + \frac{r}{48} tr\, R^2\right) \qquad\qquad d = 4 \tag{173}$$

with r the dimension of the vector bundle.

The index of the Rarita-Schwinger operator is also of interest in the computation of anomalies. Now we have to be careful because ghost fields are needed in order to remove unphysical degrees of freedom. The constraint

$$k_\mu \psi^\mu = 0, \tag{174}$$

together with the invariance under

$$\psi^\mu \to \psi^\mu + k^\mu \chi, \tag{175}$$

remove two spin-$\frac{1}{2}$ degrees of freedom of the same chirality, while the constraint

$$\Gamma^\mu \psi_\mu = 0, \tag{176}$$

removes one spin-$\frac{1}{2}$ degree of freedom with the opposite chirality. The final form of the index theorem for this operator is obtained by substracting from the contribution of a $\frac{1}{2}$-spinor with a vector index the contribution of a $\frac{1}{2}$-spinor, namely

$$ind \not{D}_{3/2} = \int_M \left[\hat{A}(M)(tr\, e^{iR/2\pi} - 1) ch(F)\right]_{vol} . \tag{177}$$

Finally, when we study anomalies in ten dimensions (for example in the case of the low-energy field theory corresponding to superstrings) we may expect anomalies coming from antisymmetric tensor fields whose field strength is self-dual. The index theorem for these fields is

$$ind\ (iD_A) = \frac{1}{4}\int_M [L(M)]_{vol}\ , \tag{178}$$

where $L(M)$ is the Hirzebruch polynomial defined by

$$L(M) \equiv 2^n \prod_a \frac{x_a/2}{\tanh{(x_a/2)}}\ . \tag{179}$$

All the technology we have introduced can be directly applied to the computation of anomalies. We are going to begin by showing how, by using Atiyah-Singer index theorem, we can obtain the singlet chiral $U(1)$ anomaly of sec. 2.1. To do that we use Fujikawa's approach [90]. The one-loop effective action for a Dirac spinor in Euclidean space is

$$e^{-\Gamma_{eff}(A)} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp{(-\int \sqrt{g}\, d^{2n}x\, \bar{\psi}\, i\not{D}\psi)} \tag{180}$$

with $\not{D} = \Gamma^\mu(\partial_\mu + A_\mu)$. In Euclidean space we have to consider ψ and $\bar{\psi}$ as independet variables. The action is classically invariant under the global infinitesimal chiral transformation

$$\begin{aligned} \delta\psi &= i\alpha\bar{\Gamma}\ , \psi \\ \delta\bar{\psi} &= i\alpha\bar{\psi}\bar{\Gamma}\ . \end{aligned} \tag{181}$$

By Noether's theorem, we have a classically conserved current

$$j_5^\mu = \bar{\psi}\bar{\Gamma}\Gamma^\mu\psi\ . \tag{182}$$

In the quantum theory if we look at the expression for the fermion effective action (180) we see that the path integral is invariant under (182) if the action and the integration measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ are both invariant. If we make a chiral transformations (182) taking the parameter α as a function of x the action varies according to

$$\int (dx)\, \bar{\psi}\, \not{D}\, \psi \longrightarrow \int (dx)\, \bar{\psi}\, \not{D}\, \psi + \int (dx)\, \alpha(x) \nabla_\mu j_5^\mu\ , \tag{183}$$

(dx) we denotes the whole volume element and $\nabla_\mu = \partial_\mu + \omega_\mu$ is the covariant derivative. We need to compute as well the jacobian $\mathcal{J}$ induced in the integration measure by the transformation $\psi \to \psi + \delta\psi$ (and the corresponding one for $\bar{\psi}$)

$$\begin{aligned} &\int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(\int (dx)\, \bar{\psi}\, i\not{D}\, \psi\right) \\ =\ &\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, \mathcal{J}\ \exp\left(\int (dx)\, \bar{\psi}\, i\not{D}\, \psi + \int (dx)\, \alpha(x)\nabla_\mu j_5^\mu\right)\ . \end{aligned} \tag{184}$$

Were the jacobian be equal to unity, by simply expanding the exponential to first order in $\alpha(x)$, we would obtain the conservation of the axial current $\langle \nabla_\mu j_5^\mu \rangle = 0$. However, this is not necessarily so. To compute $\mathcal{J}$, we expand ψ and $\bar{\psi}$ in terms of the eigenfunctions of the Dirac operator

$$i\not{D}\psi_n = \lambda_n \psi_n \tag{185}$$

as

$$\psi = \sum_n a_n \psi_n \ , \tag{186}$$

$$\bar{\psi} = \sum_n \bar{b}_n \psi_n^\dagger \ , \tag{187}$$

where a_n and b_n are Grassmann parameters. Now the integration measure is simply $\prod_n d\bar{b}_n da_n$ and the action can be rewritten as

$$\int (dx)\, \bar{\psi}\, i\not{D}\, \psi = \sum_n \lambda_n \bar{b}_n a_n \ . \tag{188}$$

The number of positive eigenvalues of $\not{D}$ equals the number of negative ones; since $\{\not{D}, \bar{\Gamma}\} = 0$ for any eigenfunction ψ with positive eigenvalue λ, $\bar{\Gamma}\psi$ is such that $\not{D}\bar{\Gamma}\psi = -\lambda\bar{\Gamma}\psi$. This implies that the only asymmetry in the spectrum of the Dirac operator is restricted to the zero modes. There is no difficulty in computing the jacobian in the basis chosen. The result is

$$\mathcal{J} = \exp\left(-2i \sum_n \langle \psi_n | \alpha(x) \bar{\Gamma} | \psi_n \rangle\right) \simeq 1 - 2i \sum_n \langle \psi_n | \alpha(x) \bar{\Gamma} | \psi_n \rangle \ , \tag{189}$$

where the inner product is defined in the usual way

$$\langle \psi | \alpha(x) \bar{\Gamma} | \psi \rangle = \int (dx) \alpha(x) \psi^\dagger(x) \bar{\Gamma} \psi(x). \tag{190}$$

This jacobian is unfortunately infinite and it is neccesary to regularize it. This can be accomplished by introducing a gaussian cut-off

$$\begin{aligned} & -2i \sum_n \int (dx) \alpha(x) \psi_n^\dagger(x) \bar{\Gamma} \psi_n(x) e^{-\frac{\lambda_n^2}{M^2}} \\ = \ & -2i \sum_n \langle \psi_n | \alpha(x) \bar{\Gamma} e^{-\frac{\not{D}^2}{M^2}} | \psi_n \rangle \ . \end{aligned} \tag{191}$$

Since the non-zero eigenvalues of the Dirac operator are paired, ψ_n and $\bar{\Gamma}\psi_n$ have opposite eigenvalues and therefore

$$\langle \psi_n | \bar{\Gamma} | \psi_n \rangle = 0 \tag{192}$$

provided $i\not{D}\psi_n \neq 0$. In the limit $\partial_\mu \alpha(x) \to 0$, (191) receives contributions only from the zero mode sector,

$$\alpha \lim_{M\to\infty} \langle \psi_n | \bar{\Gamma} e^{-\frac{\not{D}^2}{M^2}} | \psi_n \rangle = \alpha \sum_{zero\ modes} \langle \psi_n | \bar{\Gamma} | \psi_n \rangle \tag{193}$$

but this last sum equals the number of zero modes of D_+ minus the number of zero modes of D_-, that is, the index of D_+. So the regularized jacobian is simply the index of the Weyl operator, and we can write the final expression of the integrated axial anomaly as

$$\int (dx) \langle \nabla_\mu j_5^\mu \rangle = 2 \int_M [\hat{A}(M) ch(F)]_{vol} \tag{194}$$

where we have used the Atiyah-Singer index theorem stated previously. This expression gives the form of the axial anomaly in $2n$ dimensions in the presence of both gauge and gravitational fields. In particular, in four dimensions, using the expressions for the Dirac genus and the Chern character we obtain

$$\int_M \langle \nabla_\mu j_5^\mu \rangle = \frac{1}{(2\pi)^2} \int_M \left(-tr\, F^2 + \frac{r}{24}\, tr\, R^2 \right) \tag{195}$$

which agrees with (79) in the flat space-time limit ($R = 0$).

We are now ready to face the computation of local gauge and gravitational anomalies using the Atiyah-Singer index theorem for families of operators. Once again the fermion effective action is

$$e^{-\Gamma_{eff}(A)} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int d^{2n}x\, \bar{\psi}\, iD_+\, \psi\right) , \tag{196}$$

$D_+ = \not{D} P_+$. There is a problem in defining this path integral, since as we saw D_+ does not map a space into itself, but

$$D_+ : S_+ \otimes E \longrightarrow S_- \otimes E \tag{197}$$

This means that we do not have a well posed eigenvalue problem $D_+\psi = \lambda\psi$ and then there is no natural definition of the path integral in terms of the determinant of D_+. We can define a new operator $i\hat{D}$ with a well defined eigenvalue problem and whose determinant can be identified with the path integral

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int d^{2n}x\, \bar{\psi}\, iD_+\, \psi\right) = det\, \hat{D} \tag{198}$$

where some suitable regularization of the determinant is assumed. Moreover, since we know that the anomaly comes from the imaginary part of the effective action (see above), $|det\, \hat{D}|$ has to be gauge invariant. We also have to make sure that the perturbative expasion for $\hat{D}$ is the same as that for D_+.

We will work with the operator $\hat{D}$ [91] defined by

$$\hat{D} = \begin{pmatrix} 0 & D_+ \\ \not{\partial}_- & 0 \end{pmatrix} \tag{199}$$

which satisfies the properties listed. For example, the modulus of $det\, \hat{D}$ is given by

$$(det\, \hat{D})^2 = det\, (\hat{D}^\dagger D) = det(\not{\partial}_+ \not{\partial}_-)\, det(D_+ D_-) \tag{200}$$

where $det\, (\not{\partial}_+ \not{\partial}_-)$ does not depend of the gauge fields and therefore is gauge invariant. Since $det\, (D_+ D_-)$ equals $det\, i\not{D}$ (see above), it is also gauge invariant.

The computation of anomalies is carried out by relating them to the Atiyah-Singer index theorem for a Dirac operator in $2n+2$ dimensions [91]. We compactify effectively the space-time to S^4 by choosing appropiate boundary conditions. Let us consider the one parameter family of gauge transformations $g(\theta, x) : S^1 \times S^{2n} \to G$ subject to the boundary conditions $g(0, x) = g(2\pi, x) = 1$. Associated with them we can define the family of connections

$$A^\theta = g^{-1}(\theta, x)(A + d)g(\theta, x) , \tag{201}$$

Figure 2: The disc $\mathcal{D}$ which parametrizes the two-parameter family of gauge fields $A^{t,\theta}$ with polar coordinates t, θ

where A is a reference connection chosen in such a way that the associated Dirac operator has no zero modes. Defining the corresponding set of operators $\hat{D}(A^\theta)$ and using the fact that $|det\, \hat{D}(A^\theta)|$ is invariant, we may write

$$det\, \hat{D}(A^\theta) = \left[det\, i\not{D}(A)\right]^{\frac{1}{2}} e^{iw(A,\theta)}, \tag{202}$$

i.e., the anomaly is in the phase of the determinant. But we have identified the determinant of $\hat{D}$ with the exponential of minus the effective action, so we have that, under an infinitesimal variation of the parameter θ, the effective action behaves according to

$$\delta\Gamma_{eff}(A) = -i\frac{\partial w(A,\theta)}{\partial\theta}\delta\theta\ , \tag{203}$$

Thus we have related the gauge anomaly with the derivative of $w(\theta, A)$. The exponential of this function defines a map $\exp\left[iw(\theta, A)\right] : S^1 \to S^1$, whose winding number is given by

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta\, \frac{\partial w(\theta, A)}{\partial\theta} = m\ , \tag{204}$$

and we can identify the anomaly with the winding number density.

This winding number density is obtained from the index theorem. Define the two-parameter family of connections:

$$A^{t,\theta} = tA^\theta \tag{205}$$

with $t \in [0,1]$. This family defines in the space of all connections a disc $\mathcal{D}$ whose boundary is the circle $\{A^\theta\}$. Now, the determinant $det\, D(A^{t,\theta})$ may vanish since $A^{t,\theta}$ and A are no longer related by a gauge transformation for $0 \leq t < 1$. By deforming the boundary S^1 of the disc towards the interior the total winding number is equal to the signed sum of the winding numbers associated with each of the internal zeroes of the determinant. Moreover, it is possible to show that these winding numbers can only be ± 1 and coincide with the chirality of the eigenfunction of the Dirac operator whose eigenvalue vanishes at that point [91] (fig. 2). These arguments allow us to identify the winding number with the index of the Dirac operator in $S^{2n} \times \mathcal{D}$

$$ind\, \not{D}_{2n+2} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \frac{\partial w(\theta, A)}{\partial\theta}\ . \tag{206}$$

Since we are interested in relating the anomaly with the topology of our bundle we have to make use of the index theorem. Now our manifold has a boundary and we cannot use the

Atiyah-Singer index theorem but its generalization to manifolds with boundary, the Atiyah-Pattodi-Singer index theorem [87]. However we can proceed in a different way. We construct a G-bundle over $S^2 \times S^{2n}$ in the following way. Divide S^2 into two parts corresponding to the two hemispheres S^2_+ and S^2_-. In the upper hemisphere, with coordinates (t, θ), we take the gauge field to be

$$\mathcal{A}(x,t,\theta) = A^{t,\theta} + tg^{-1}d_\theta g \tag{207}$$

while in the lower hemisphere, now with coordinates (s, θ), the gauge field will be simply $\mathcal{A}(x,s,\theta) = A(x)$. By d_θ we denote the exterior differentiation with respect to θ, i.e. $d_\theta = d\theta\partial/\partial\theta$. At the equator $s = t = 1$ both connections are related by a gauge transformation as required. The Atiyah-Singer index theorem is thus applied to $S^2 \times S^{2n}$

$$\begin{aligned} ind[i\not{D}_{2n+1}(\mathcal{A})] &= \int_{S^2\times S^{2n}} [ch(\mathcal{F})]_{vol} \\ &= \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1} \int_{S^2\times S^{2n}} tr\left(\mathcal{F}^{n+1}\right), \end{aligned} \tag{208}$$

where $\mathcal{F}$ is the curvature associated with $\mathcal{A}$

$$\mathcal{F} = (d + d_\theta + d_t)\mathcal{A} + \mathcal{A}^2 , \tag{209}$$

and the definition of d_t is analogous to that of d_θ. Next we split this integral into two integrals over $S^2_\pm \times S^{2n}$. In each one, from (152) setting $\omega_0 = 0$, we see that $tr\,\mathcal{F}^{n+1}$ can be written as $dQ_{2n+1}(\mathcal{A})$ with $Q_{2n+1}(\mathcal{A})$ the $2n+1$-Chern-Simons form. Then by applying Stokes theorem we can write the index of $\not{D}_{2n+2}$ as the difference of two integrals over $S^1 \times S^{2n}$

$$\begin{aligned} ind[i\not{D}_{2n+1}(\mathcal{A})] &= \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1} \left[\int_{S^1\times S^{2n}} Q_{2n+1}(\mathcal{A})|_{t=1}\right. \\ &- \left.\int_{S^1\times S^{2n}} Q_{2n+1}(\mathcal{A})|_{s=1}\right] . \end{aligned} \tag{210}$$

To connect this expression with the anomaly we need the local winding density $d_\theta w(\theta, A)$. First of all let us notice that the second integral does not contribute to the anomaly, because we need a $d\theta$ component to saturate the integral over S^1. Then we can write

$$ind[i\not{D}_{2n+1}(\mathcal{A})] = \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1} \int_{S^1\times S^{2n}} Q_{2n+1}(A^\theta + \hat{v}, F^\theta) \tag{211}$$

where $\hat{v} = g^{-1}d_\theta g$ and F^θ is the gauge curvature associated with A^θ. From here it is now straightforward to obtain the local winding density as

$$id_\theta w(A,\theta) = \frac{i^{n+2}}{(2\pi)^n(n+1)!}\int_{S^n} Q^1_{2n}(\hat{v}, A^\theta, F^\theta) \tag{212}$$

where $Q^1_{2n}(\hat{v}, A^\theta, F^\theta)$ is the term of $Q_{2n+1}(\hat{v}, A^\theta, F^\theta)$ linear in $\hat{v}$. This form of the anomaly verifies the Wess-Zumino consistency conditions. We can also derive the descent equations of

Stora and Zumino [92]. We consider the case where we have a family of gauge transformations $g(\theta^\alpha, x)$ parametrized by a collection of angles θ^α. Defining

$$\begin{aligned} \bar{A} &= g^{-1}(A+d)g \\ \hat{v} &= g^{-1}\delta g \ , \end{aligned} \tag{213}$$

with δ the differential with respect to the parameters θ^α, it is possible to show that (see for example [85])

$$(d+\delta)Q_{2n+1}(\bar{A}+\hat{v},\bar{F}) = dQ_{2n+1}(\bar{A},\bar{F}) \ , \tag{214}$$

where $\bar{F}$ is the curvature associated with $\bar{A}$. Now we expand the Chern-Simons form $Q_{2n+1}(\bar{A}+\hat{v},\bar{F})$ in powers of $\hat{v}$

$$Q_{2n+1}(\bar{A}+\hat{v},\bar{F}) = Q^0_{2n+1}(\bar{A},\bar{F}) + Q^1_{2n}(\hat{v},\bar{A},\bar{F}) + \ldots + Q^{2n+1}_0 \tag{215}$$

where the superscript of the Q's indicates the number of powers of $\hat{v}$. Then, by substitutying this expansion into (214) we get the Stora-Zumino descent equations

$$\begin{aligned} \delta Q^0_{2n+1} + dQ^1_{2n} &= 0 \\ \delta Q^1_{2n} + dQ^2_{2n-1} &= 0 \\ &\vdots \\ \delta Q^{2n}_1 + dQ^{2n+1}_0 &= 0 \\ \delta Q^{2n+1}_0 = 0 \ , & \end{aligned} \tag{216}$$

the second equality being equivalent to the Wess-Zumino consistency condition.

We have applied index theorem to the computation of gauge anomalies. There, instead of applying the Atiyah-Patodi-Singer [87] index theorem for manifold with boundary we constructed a G-bundle over a closed manifold and from it obtained the form of the anomaly. We would like to find a similar procedure to compute gravitational and mixed anomalies. To do that, we will introduce the Atiyah-Singer index theorem for families of elliptic operators [88].

Let us consider a family of elliptic operators D_p with p lying in some manifold P. The index of D_p over the manifold P is constant, since it is invariant under deformations, as long as the operator varies smoothly over P. Nonetheless this is not true for both $dim\,ker\,D_p$ and $dim\,ker\,D_p^\dagger$ separately and then $ker\,D_p$ and $ker\,D_p^\dagger$ are not necessarily well defined vector bundles over the parameter space P. However, in the context of the K-theory for P [93] we can have a well defined vector bundle over P by defining the vector bundle $Ind\,D_p$ as

$$Ind\,D_p = ker\,D_p \ominus ker\,D_p^\dagger \tag{217}$$

where the ordinary index for D_p is given by $ch_0(Ind\,D_p)$. Using the Atiyah-Singer index theorem for families of operators we may write [94]

$$ch(Ind\,D) = \int_{M_{2n}} \hat{A}(Z)ch(V) \tag{218}$$

where Z is a fiber space with base P and fiber M_{2n}, and V is the vector bundle. In the case in which we are dealing with pure gravity, V is of purely geometrical origin; for example for a spin-$\frac{1}{2}$ fermion $V = 0$ and for a Rarita-Schwinger field $V = TM_{2n}$, the tangent bundle of M_{2n}. In the case of mixed anomalies V also contains the gauge bundle. In fact, when we made the computation of gauge anomalies, we used a version of the Atiyah-Singer index theorem for a family of operators with $M_{2n} = S^{2n}$, $P = S^2$ and the vector bundle was constructed using the gauge transformation $g(\theta, x)$.

When the theory we are interested in is in presence of a gravitational field, we have to be sure that the quantum theory is invariant under the symmetries of the classical theory, i.e., coordinate diffeomorphisms and local Lorentz transformations. In principle, we then may expect two kind of anomalies, Einstein anomalies and Lorentz anomalies. In the first case the breakdown of diffeomorphism invariance is translated into a non-conservation of the energy-momentum tensor

$$\langle \nabla_{\mu\nu} T^{\mu} \rangle \neq 0 \tag{219}$$

while in the second case, since the variation of the effective action under a local Lorentz transformation $\delta e^a_{\ \mu} = \alpha^a_b e^b_{\ \mu}$ is

$$\delta_\alpha \Gamma_{eff} = -\int dx\, \alpha^a_b e^b_{\ \mu} \frac{\delta \Gamma_{eff}}{\delta e^a_{\ \mu}} = -\int dx\, e\, \alpha^{ab} \langle T_{ab} \rangle \tag{220}$$

we have that, being $\alpha_{(ab)} = 0$, $\delta_\alpha \Gamma_{eff} \neq 0$ implies $\langle T_{[ab]} \rangle \neq 0$. However it can be shown that both types of anomalies (Eintein and Lorentz) [82, 85] are related. In fact, by adding a local term to the action, it is always possible to switch from one type of anomaly to the other; we usually prefer to think in terms of Einstein anomalies.

After this remark we give the prescription for the computation of gravitational anomalies. Following the same argument that gave us the gauge anomaly, the local winding number density is obtained from the characteristic polynomial (218) for a suitable $2n+2$-dimensional Dirac operator. The characteristic polynomial for the fields that contribute to the anomaly are the following. For spin-$\frac{1}{2}$ fermions

$$I_{1/2} = \prod_i \frac{x_i/2}{\sinh(x_i/2)} \,. \tag{221}$$

For Rarita-Schwinger fermions

$$I_{3/2} = \prod_i \frac{x_i/2}{\sinh(x_i/2)} \left[tr\left(e^{R/2\pi} - 1\right) + 4k - 3 \right] \,. \tag{222}$$

And finally for the self-dual antisymmetric tensor

$$I_{s.d.} = -\frac{1}{8} \prod_i \frac{x_i}{\tanh x_i} \,. \tag{223}$$

The minus sign in the last case indicates the bosonic nature of the antisymmetric field. Here x_i are the skew-eigenvalues of $R/2\pi$, where now the indices a, b in $R^a_{\ b}$ are those appropiated for $2n$ dimensions but the form part of R may contain components in the extra dimensions. If we are interested in computing mixed anomalies we have to multiply these polynomials by $ch(\mathcal{F})$ where $\mathcal{F}$ is the 2n+2-dimensional gauge curvature.

2.5 Examples: Green-Schwarz cancellation mechanism and Witten's $SU(2)$ global anomaly

As an application of all this machinery we present the non-trivial cancellation of gravitational anomalies for some supergravity theories [81]. In $d = 2$ dimensions the relevant characteristic polynomials of order 4 are

$$\begin{aligned} I_{1/2} &= -\frac{1}{24}p_1 \\ I_{3/2} &= \frac{23}{24}p_1 \\ I_{s.d.} &= -\frac{1}{24}p_1 \ , \end{aligned} \tag{224}$$

and there are many ways of cancelling the anomaly. Let us note that $I_{1/2} = I_{s.d.}$ indicates the fact that a positive chirality fermion is equivalent to a right-moving scalar because of two-dimensional bosonization [95]. In $d = 6$,

$$I_{1/2} = \frac{1}{5760}(7p_1^2 - 4p_2) \tag{225}$$

$$I_{3/2} = \frac{1}{5760}(275p_1^2 - 980p_2) \tag{226}$$

$$I_{s.d.} = \frac{1}{5760}(16p_1^2 - 112p_2) \ . \tag{227}$$

The minimal cancellation ocurs for

$$21I_{1/2} - I_{3/2} + 8I_{s.d.} = 0 \tag{228}$$

so we must have 21 Weyl spin-$\frac{1}{2}$ fermions, 1 gravitino with the opposite helicity and 8 self-dual antisymmetric tensors as a multiplet free from gravitational anomalies. Finally in $d = 10$ we have

$$I_{1/2} = \frac{1}{967680}(-31p_1^3 + 44p_1p_2 - 16p_3) \tag{229}$$

$$I_{3/2} = \frac{1}{967680}(225p_1^3 - 1620p_1p_2 + 7920p_3) \tag{230}$$

$$I_{s.d.} = \frac{1}{967680}(-256p_1^3 + 1664p_1p_2 - 7936p_3) \tag{231}$$

and the minimal solution is found to be [81]

$$-I_{1/2} + I_{3/2} + I_{s.d.} = 0 \tag{232}$$

Thus the gravitational anomaly cancels for a spin-$\frac{1}{2}$ Weyl fermion, a gravitino field of opposite chirality, and a self-dual tensor field. These fields appear in the $N = 2$ chiral supergravity multiplet in $d = 10$ [81].

To conclude with the subject of local anomalies we present the cancelation of anomalies for type-I superstring found by Green and Schwarz [96] (we follow the presentation in [97]).

The low energy field theory for type-I superstrings is $N=1$ super Yang-Mills coupled to $N=1$ SUGRA in ten dimensions. The $N=1$ SUGRA multipled contains the graviton $e_\mu{}^a$, the left-handed Weyl-Majorana gravitino ψ^μ, the right-handed Weyl-Majorana fermion λ, the antisymmetric tensor $B_{\mu\nu}$ and the dilaton Φ. The super Yang-Mills multiplet has the ten-dimensional gluon A^a_μ and the Weyl-Majorana gluino λ^a in the adjoint representation of the gauge group G. Shifting all gravitational anomalies to the local-Lorentz transformations we get the 12-form characterizing the anomaly is:

$$
\begin{aligned}
I_{12} &= -\frac{1}{15} tr\, F^6 + \frac{1}{24} tr\, F^4\, tr\, R^2 \\
&- \frac{1}{960} tr\, F^2 \left[5(tr\, R^2)^2 + 4 tr\, R^4\right] + \frac{N-496}{7560} tr\, R^6 \\
&+ \left(\frac{N-496}{5760} + \frac{1}{8}\right) tr\, R^4\, tr\, R^2 + \left(\frac{N-496}{13824} + \frac{1}{32}\right)(tr\, R^2)^3 \,,
\end{aligned}
\tag{233}
$$

where the trace of the gauge curvature is in the adjoint representation and N is the dimension of the gauge group. From this polynomial we obtain the complete set of anomalies for the low-energy field theory of type-I superstrings.

To describe the cancellation of anomalies discovered by Green and Schwarz, we note, first of all, that if we consider the leading terms with $tr\, F^6$ and $tr\, R^6$, the anomalies they originate cannot be cancelled by the addition of local counterterms to the effective action. In order to cancel the anomaly coming from the leading terms the gauge group is restricted to have dimension $N = 496$ and such that in the adjoint representation $tr\, F^6$ is not an independent Casimir. This could be accomplished for example if

$$
tr\, F^6 = \alpha\, tr\, F^2\, tr\, F^4 + \beta (tr\, F^2)^3 \tag{234}
$$

The crucial point is that all anomalies can be cancelled provided I_{12} factorizes according to

$$
I_{12} = (tr\, R^2 + k\, tr\, F^2) I_8 \tag{235}
$$

where I_8 is an invariant 8-form constructed in terms of F and R. It can be shown that this factorization is possible if $k = -1/30$ and

$$
tr\, F^6 = \frac{1}{48} tr\, F^2\, tr\, F^4 - \frac{1}{14400}(tr\, F^2)^3 \tag{236}
$$

so I_8 is equal to

$$
\begin{aligned}
I_8 &= \frac{1}{24} tr\, F^4 - \frac{1}{7200}(tr\, F^2)^2 - \frac{1}{240} tr\, F^2\, tr\, R^2 \\
&+ \frac{1}{8} tr\, R^4 + \frac{1}{32}(tr\, R^2)^2 \,.
\end{aligned}
\tag{237}
$$

All these conditions (factorization and $N = 496$) can be fulfilled if and only if the gauge group is $SO(32)$, $E_8 \times E_8$ or $E_8 \times U(1)^{248}$. The most important feature of Green and Schwarz result is that by simply imposing the cancellation of both gauge and gravitational anomalies we have narrowed down the possible gauge groups to just three possible choices. Soon after the Green-Schwarz anomaly cancellation appeared, the heterotic string [10] was formulated. Although we have heterotic strings for both $SO(32)$ and $E_8 \times E_8$, it is the latter which seems more promising in making contact with the SM at low energies.

After checking that all local anomalies cancel we still have to make sure that the theory is not afflicted by global (gauge or gravitational) anomalies. Global anomalies lead to a breakdown of gauge (diffeomorphism) invariance under transformations that do not lie in the connected component of the identity of the symmetry group. The best known example was discovered by Witten [11] for gauge theories with an odd number of left-handed $SU(2)$ doublets. We briefly review this case.

The relevant mathematical requirement for the existence of Witten's anomaly is the fact that the fourth homotopy group of the gauge group is non trivial. This only happens for $Sp(n)$, $\pi_4[Sp(n)] = \mathbf{Z}_2$. In particular $Sp(1) = SU(2)$, and we have

$$\pi_4[SU(2)] = \mathbf{Z}_2 \tag{238}$$

Impose boundary conditions to effectively compactify space-time to S^4. The gauge trasformations $U(x)$ are mappings $U : S^4 \to SU(2)$ with are classified by $\pi_4[SU(2)]$. Since this homotopy group is non-trivial we have transformations that cannot be continously deformed to the identity. To make the discussion concrete imagine that we have one $SU(2)$ left-handed doublet. Since for a theory with a doublet of Dirac fermions the path integral is given by

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\int d^4x\, \bar{\psi} i \not{D} \psi} = det\,(i\not{D})\,. \tag{239}$$

In our case the path integral will be equal to the square root of the determinant of $\not{D}$

$$\int (\mathcal{D}\psi \mathcal{D}\bar{\psi})_{Weyl} e^{-\int d^4x\, \bar{\psi} i \not{D} \psi} = [det\,(i\not{D})]^{\frac{1}{2}}\,. \tag{240}$$

The problem is then to assign a definite sign to the square root. Since we are only interested in relative signs we may, for example, choose the positive branch of the square root. However, for this choice to be consistent we must have that the sign is preserved by gauge transformations; this is evidently true for infinitesimal gauge transformations, but for the topologically non-trivial ones,

$$[det\, i\not{D}(A_\mu)]^{\frac{1}{2}} = -[det\, i\not{D}(A_\mu^U)]^{\frac{1}{2}}\,. \tag{241}$$

This spells trouble, because A_μ and A_μ^U can be continously connected, and we have no consistent way of excluding A_μ^U from the path integral. It follows that the partition function is zero because the contribution coming from A_μ is exactly cancelled by that from A_μ^U.

We may understand (241) in terms of eigenvalue flows. We know that the number of positive and negative non-zero modes of $i\not{D}$ are equal. Then we may define $[det\, i\not{D}]^{\frac{1}{2}}$ as the regularized product of the positives eigenvalues of the Dirac operator

$$[det\, i\not{D}]^{\frac{1}{2}} = \prod_{\lambda_n > 0} \lambda_n\,. \tag{242}$$

Since we can continuously interpolate between A_μ and A_μ^U, consider the family of connections $A_\mu^t = tA_\mu^U + (1-t)A_\mu$ with $t \in [0,1]$. We can study the flow of the eigenvalues of $i\not{D}(A_\mu^t)$ as a function of t. The condition implying that the determinant changes sign in passing from A_μ to A_μ^U is that the number of eigenvalues $\lambda_n(t)$ passing from positive to negative values is odd as we go from $t=0$ to $t=1$. This result follows from a slightly exotic version of the Atiyah-Singer index theorem, called the mod two index theorem, to a certain five-dimensional Dirac operator. The five-dimensional Dirac operator of interest is

$$\mathcal{D} = i\bar{\Gamma}\frac{\partial}{\partial t} + \not{D}[A(t)] \ . \tag{243}$$

The interpolating parameter between $A_\mu(x)$ and $A_\mu(x)^U$ plays the rôle of the fifth coordinate. The operator $\mathcal{D}$ is real and antisymmetric, hence its non-zero eigenvalues are purely imaginary and come in complex conjugate pairs, and the number of zero modes is well defined modulo two. In our case the five-dimensional space is $S^4 \times \mathbf{R}$, $A_\tau = 0$, and $A_\mu(x, \tau = -\infty) = A_\mu(x)$, $A_\mu(x, \tau = +\infty) = A_\mu^U(x)$. An application of the mod 2 index theorem led Witten to conclude that for this configuration the index is one, this means that as we interpolate between A_μ and A_μ^U there is a level crossing of the four-dimensional Dirac operator. From (242) it is the clear that we cannot assign a definite sign to the square root and we cannot implement gauge invariance. Thus $SU(2)$ gauge theory with an odd number of fermion doublets is afflicted by a global anomaly and the physical Hilbert space is empty.

3 String Theory I. Bosonic String

The lesson to be extracted from our journey in chapter one through the various proposals of how to quantize gravity is that all of them are afflicted by problems. The situation is somewhat remminiscent of Fermi's theory of the weak interactions. It was able to explain a variety of low energy phenomena, but it failed to be renormalizable and it also contained unitarity violations at high energy. The cure of its problems came with the SM, based on completely different dynamical principles. It seems unreasonable to treat Einstein's Theory of Relativity to be a fundamental theory all the way from cosmological scales down to the Planck length. There are more than sixty orders of magnitude in between. It seems more appropiate (specially in view of the difficulties described) to think of GR as an effective theory, and we should strive to identify the appropiate framework capable of encompassing a quantum theory of space-time which in the long distance limit agrees with Einstein's theory. It is clear from previous discussions that such a theory will provide profound insights into many of the riddles of the SM, like the origin of mass, symmetry breaking and the chiral nature of quarks and leptons. We have seen in our brief study of anomalies that as soon as the space-time dimension goes beyond four, the constraints imposed by the existence of a rich chiral structure together with reasonable gauge interactions are extremely restricted, and we were led for instance in $d = 10$ to the low energy spectrum of the heterotic string [10].

So far the only candidate which incorporates gravity with other interactions at high energies is String Theory. There are still many puzzles and difficulties to be resolved in String Theory, but one should also mention that many of the problems encountered in the treatment of Quantum Gravity in terms of local field theories disappear in this context. There are strong arguments (and explicit computations) showing the ultraviolet finiteness of superstring theories. There are quite interesting results (duality, finite temperature properties; see below) which indicate that String Theory defined on spaces of different sizes (an even different topologies) are completely equivalent. For instance a string propagating on a circle of radius R is equivalent to one propagating on a circle of radius $1/R$ (in string units, with the Regge slope set equal to one).

String Theory contain a fundamental length, and still have local interactions. This is something which cannot be achieved in terms of local field theories. We also know (see below) from the heterotic string that one can in principle accomodate in this framework the quantum numbers and interactions of the SM. Our knowledge of string theory is unfortunately too rudimentary to have a glimpse on its physical basis. However, the results obtained so far make us optimistic. It is likely that many of the new ingredients brought in by String Theory will appear in the correct theory of Quantum Gravity.

We begin in this chapter our exploration of the formulation and properties of String Theory. We will focus on general properties and on the formalism which has been developed in order to compute string amplitudes to any order of perturbation theory. Along the way we will encounter how the Einstein equations appear in String Theory, we will study the simplest form of duality, and we will also describe the geometrical interpretation of string infinities and physical states. Our study will center on the bosonic string for the time being. This makes the discussion simpler. In the next chapter we will briefly review the properties of fermionic strings.

3.1 Bosonic String

We review the basic features of the bosonic string [2, 3, 5]. We start with the case of a free relativistic point particle in Minkowski space. The action for such a system is constructed by taking the simplest Lorentz invariant we can associate with the particle's trajectory, that is, the length of its world-line. Then if the particle moves in Minkowski space along the world-line $x^\mu(s)$ with s being the proper time, the action functional is given by

$$S[x^\mu(s)] = -m \int_a^b ds \sqrt{\eta_{\mu\nu} \dot{x}^\mu(s) \dot{x}^\nu(s)} \, . \tag{244}$$

When dealing with strings we have a one-dimensional object, whose points we parametrize by a coordinate σ running from 0 to π, in a d-dimensional Minkowski space (the target space). Its time evolution is represented by a two-dimensional surface, the world-sheet of the string. This two-dimensional surface is parametrized by two coordinates (σ, τ), the first one being that introduced to label the points on the string and τ playing the role of the proper time in the case of the point particle. The embedding of the world-sheet in Minkowski space is described by d functions $X^\mu(\sigma, \tau)$ which form a vector under Lorentz transformations in the target space.

Now we construct the action functional for the bosonic string by similarity with the case of the relativistic particle: we will the action to be proportional to the area of the world-sheet swept out by the string. The metric induced on the world-sheet by that of the target space is

$$\Gamma_{ab} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} \tag{245}$$

where $a = 0, 1$ and $(\sigma^0, \sigma^1) = (\tau, \sigma)$, and the induced area equals the integral of the square root of minus the determinant of Γ_{ab}. The action functional for the strings is taken to be

$$S[X^\mu] = T \int d^2\sigma \sqrt{-det\left[\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^a}\frac{\partial X^\nu}{\partial\sigma^b}\right]} \tag{246}$$

where T is a constant, the string tension. It is the mass or energy per unit length. This form of the action was first proposed by Y. Nambu and T. Goto [98] in the early seventies and it is known as the Nambu-Goto action. By using the expression of the Minkowski metric and the definitions

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial\tau} \qquad\qquad X^{'\mu} = \frac{\partial X^\mu}{\partial\sigma} \tag{247}$$

we can rewrite Nambu-Goto action as

$$S[X^\mu(\tau,\sigma)] = \int d\tau d\sigma \sqrt{\dot{X}^2 X'^2 - (\dot{X}\cdot X')^2}\,. \tag{248}$$

Although (248) has a very simple physical interpretation its form is not very pleasant specially because of the presence of the square root. We would like to find another action classically equivalent to (248) but without its unpleasant features. The idea of using auxiliary fields to simplify the action was introduced by Brink, DiVechia and Howe [99] and by Deser and Zumino [100]. Now, besides the embeddings $X^\mu(\tau,\sigma)$ we introduce a new independent field $g_{ab}(\tau,\sigma)$ which is the intrinsic metric on the world-sheet. The simplified action is

$$S_P[X^\mu, g_{ab}] = -\frac{T}{2}\int d^2\sigma\sqrt{-g}\, g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \tag{249}$$

This form of the action was used by Polyakov [63] to formulate String Theory in terms of path integrals.

It is worth noticing that there is a counterpart of the action (249) in the case of the relativistic particle. We can introduce an auxiliary field $e(s)$, which plays the role of the intrinsic metric on the world-line, and write the action

$$S = \frac{1}{2}\int ds \left(\frac{\dot{x}^2(s)}{e(s)} - m^2 e(s)\right) \tag{250}$$

which is classically equivalent to (244). This action has the supplementary advantage that it can be applied to the massless case in which (244) is not defined.

The string tension T can be rewritten in a more useful way by defining a new constant $\alpha^{'}$ (which was called the Regge slope in the times of the dual models of hadronic resonances [101]) defined by

$$T = \frac{1}{2\pi\alpha^{'}} \tag{251}$$

Neither T nor $\alpha^{'}$ affect in any way the classical string solutions, since they are overall factor in the action. In the quantum theory, in a path integral approach, each configuration is weighted by

$$\exp\,[i\frac{T}{\hbar}S^{'}]\,, \tag{252}$$

where $S^{'}$ is the action divided by T. Then from $\alpha^{'}$ and $\hbar$ we can construct a constant with dimensions of length as (in units in which $c = 1$)

$$\lambda_S = \sqrt{\alpha' \hbar}\ , \tag{253}$$

which is the fundamental length in String Theory. This, together with the vacuum expectation value of the dilaton (which give us the coupling constant of the string) determines all couplings in String Theory.

The action (249) has a number of symmetries. First of all it is invariant under world-sheet diffeomorphism $\sigma^a \to \sigma^a + \xi^a$

$$\begin{aligned} \delta X^\mu &= \xi^a \partial_a X^\mu \\ \delta g_{ab} &= \nabla_a \xi_b + \nabla_b \xi_a\ . \end{aligned} \tag{254}$$

Secondly we have invariance under Weyl rescalings of the metric

$$\begin{aligned} \delta g_{ab} &= \delta\Omega(\sigma) g_{ab} \\ \delta X^\mu &= 0\ . \end{aligned} \tag{255}$$

Finally the action is invariant under Poincaré transformations of the target space (we are taking the target space to be Minkowski space)

$$\begin{aligned} \delta X^\mu &= a^\mu_{\ \nu} X^\nu + b^\mu \qquad a_{(\mu\nu)} = 0 \\ \delta g_{ab} &= 0\ . \end{aligned} \tag{256}$$

Invariance under Weyl transformations implies that the two-dimensional classical field theory described by (249) is a conformal field theory. As we will see the preservation of this property after quantization lead to severe constraints on the theory.

The classical equations of motion for (249) give the free field equations for X^μ

$$\nabla_\sigma \nabla^\sigma X^\mu = 0\ , \tag{257}$$

together with the equations of motion for g_{ab}. The two-dimensional energy-momentum tensor T_{ab} is given by:

$$T_{ab} = -\frac{2}{T}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{ab}} \tag{258}$$

and the variation of (249) with respect to g_{ab} yields:

$$T_{ab} = \partial_a X \cdot \partial_b X - \frac{1}{2} g_{ab} g^{cd} \partial_c X \cdot \partial_d X = 0\ . \tag{259}$$

The energy-momentum tensor of the two-dimensional field theory described by the action (249) vanishes identically. As a subproduct of these equation we have that the energy-momentum tensor is traceless; this is a consequence of the invariance of the action under Weyl rescalings of the metric. From these two equations it is easy to see that, whenever the constraint on the energy-momentum tensor (259) is satisfied, (249) and the Nambu-Goto action are classically equivalent; this does not necessarily imply the equivalence of both theories at the quantum level.

The invariances of the action implies that we have a lot of freedom in choosing a gauge for the intrinsic metric g_{ab}. For example, by making a suitable choice of local coordinates, we can write (locally) the metric in the form

$$g_{ab} = e^{\phi(\sigma)}\eta_{ab} \, , \tag{260}$$

with η_{ab} the flat Minkowski metric on the world-sheet. This is the conformal gauge.

We now analyze a free propagating closed string. First we impose the periodicity condition of the bosonic coordinates $X^\mu(\sigma, \tau)$

$$X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma) \, . \tag{261}$$

The world-sheet is a cylinder $S^1 \times \mathbf{R}$ parametrized by the coordinates $0 \leq \sigma < \pi$ and $-\infty < \tau < \infty$. By choosing the conformal gauge and using light-cone coordinates in the world-sheet $\sigma^\pm = \tau \pm \sigma$ we can write the action as

$$S = -\frac{T}{2}\int d^2\sigma \partial_+ X^\mu \partial_- X^\nu \eta_{\mu\nu} \, . \tag{262}$$

If we make the Wick rotation $\tau \to -i\tau$, we can define the complex coordinates

$$\begin{aligned} w &= 2(\tau - i\sigma) \, , \\ \bar{w} &= 2(\tau + i\sigma) \, , \end{aligned} \tag{263}$$

It is possible now to transform the cylinder into the punctured complex plane $\mathbf{C}^* = \mathbf{C} - \{0\}$ by the conformal transformation

$$z = e^w \tag{264}$$

The field equations for X^μ are

$$\partial_z \partial_{\bar{z}} X^\mu(z, \bar{z}) = 0 \tag{265}$$

and

$$\begin{aligned} &\partial_z X \cdot \partial_z X = 0 \\ &\partial_{\bar{z}} X \cdot \partial_{\bar{z}} X = 0 \end{aligned} \tag{266}$$

for the constraints. Equation (265) can be solved by noting that the most general solution is a sum of a holomorphic and an antiholomorphic part

$$X^\mu(z, \bar{z}) = X^\mu(z) + X^\mu(\bar{z}) \, , \tag{267}$$

Therefore the solution to (265) can be expressed as a Laurent series:

$$X^\mu(z, \bar{z}) = q^\mu - \frac{i}{4}p^\mu \log|z|^2 + \frac{i}{2}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}z^{-n} + \frac{i}{2}\sum_{n\neq 0}\frac{\bar{\alpha}_n^\mu}{n}\bar{z}^{-n} \, . \tag{268}$$

In general there are three possible approaches to the quantization of the string.

a) Light-cone gauge. We take in the target space a light-cone frame, i.e., coordinates $(X^{\pm}, X^i)$ with $X^{\pm} = X^0 \pm X^{d-1}$ and $i = 1, \ldots, d-2$. Next we use the freedom we still have in the conformal gauge to set (see [2] for details)

$$X^+ = q^+ + p^+ \tau \tag{269}$$

In this gauge it is possible to show that the constrains can be explicitly solved and that by substituying them into the solution of (265) we eliminate all the unphysical degrees of freedom so the theory depends only on the transverse modes. However, it is necessary to test that the quantum theory preserves the Lorentz invariance of the classical action. To verify this we check whether the conserved charges $\{J^{\mu\nu}\}$ associated with the Lorentz invariance satisfy the Lorentz algebra. For example classically we would have $[J^{i-}, J^{j-}] = 0$. In the quantum theory on the other hand, the result is [2]

$$[J^{i-}, J^{j-}] = -\frac{1}{(p^+)^2} \sum_{m=1}^{\infty} \Delta_m (\alpha^i_{-m} \alpha^j_m - \alpha^j_{-m} \alpha^i_m) \tag{270}$$

where the coefficient Δ_m is given by

$$\Delta_m = m \left(\frac{26-d}{12} \right) + \frac{1}{m} \left(\frac{26-d}{12} + 2(1-b) \right) \tag{271}$$

and b is a normal ordering constant that appears because of the ambiguity in the order of the operators (see also below). We see that Lorentz invariance is only recovered if we take $d = 26$ and $b = 1$. Hence this procedure is only consistent in the critical dimension $d = 26$.

b)Old covariant approach. Consider the components of the energy-momentun tensor in the complex coordinate basis $(z, \bar{z})$. Energy-momentum conservation and tracelessness imply,

$$\partial_z T_{\bar{z}\bar{z}} = \partial_{\bar{z}} T_{zz} = 0 \; . \tag{272}$$

We can Laurent expand around $z = \bar{z} = 0$

$$\begin{aligned} T_{zz}(z) &= \sum_{n \in \mathbf{Z}} L_n z^{-n-2} \; , \\ T_{\bar{z}\bar{z}}(\bar{z}) &= \sum_{n \in \mathbf{Z}} \bar{L}_n z^{-n-2} \; , \end{aligned} \tag{273}$$

where $L_n, \bar{L}_n$ are called the Virasoro generators and they satisfy $L_n^* = L_{-n}$ (and the same for barred quantities). Using the expansion (268) and the canonical Poisson brackets of the classical theory, we obtain the algebra of the Virasoro generators

$$\begin{aligned} \{L_n, L_m\} &= -i(n-m) L_{n+m} \; , \\ \{\bar{L}_n, \bar{L}_m\} &= -i(n-m) \bar{L}_{n+m} \; , \\ \{L_n, \bar{L}_m\} &= 0 \; . \end{aligned} \tag{274}$$

After quantization, we obtain that the previous algebra is not exactly recovered as the algebra of the operators $L_n, \bar{L}_m$. Instead we get a central term in the commutation relations

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{d}{12}m(m^2-1) \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{d}{12}m(m^2-1) \\
[L_m, \bar{L}_n] &= 0
\end{aligned}
\tag{275}
$$

In general, whenever we have a classical system with a set of first class constrains $\{\phi_i\}$, physical states are defined as those that are annihilated by the operators $\{\hat{\phi}_i\}$. When we try to quantize the bosonic string this procedure is not available, since the gravitational anomaly in 2 dimensions (see above) makes its impossible to impose $\hat{T}_{ab}|phys\rangle = 0$ and the best we can do is to use Gupta-Bleuler method [22]. It is possible to show that the presence of the gravitational anomaly in two dimensions and the existence of a central term in the Virasoro algebra are equivalent.

Since we cannot impose that physical states are annihilated by all the Virasoro generators, we proceed in a way similar to the Gupta-Bleuler quantization of the electromagnetic field. The physical states are annihilated by the positive frequency part of the energy-momentum tensor, namely

$$
\begin{aligned}
L_n|phys\rangle &= \bar{L}_n|phys\rangle = 0\,, \qquad n > 0 \\
(L_0 - b)|phys\rangle &= (\bar{L}_0 - b)|phys\rangle = 0\,,
\end{aligned}
\tag{276}
$$

where b is a normal ordering constant that we have to include because of the ambiguity in the order of the operators a_n, $\bar{a}_n$ in the definition of L_0 and $\bar{L}_0$. As in the Gupta-Bleuler method, we have to ensure that the spectrum of the theory has no ghost states, no physical states $|\chi\rangle$ with negative norm $\langle\chi|\chi\rangle < 0$. Brower, Goddard and Thorn [103] proved a no-ghost theorem in which they showed that there are ghost states as long as $d > 26$. However, if $d = 26$ and $b = 1$ or $d < 26$ and $b < 1$ the spectrum is always ghost-free. This result implies an upper bound on the dimensionality of the target space.

c)Modern covariant quantization. In this approach we begin with the path integral [63, 102]

$$
Z = \int \mathcal{D}g\mathcal{D}X \exp\left[\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-g}\, g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\right]. \tag{277}
$$

In order to perform the path integration without overcounting it is necessary to make a gauge fixing. For example we may use the conformal gauge

$$
g_{ab} = e^{\phi(\sigma)}\hat{g}_{ab} \tag{278}
$$

where $\hat{g}_{ab}$ is a fiducial metric. Since the action is also invariant under diffeomorphisms we have to integrate not over all $\hat{g}_{ab}$ but only over those that are not related by a world-sheet diffeomorphism. In fact it can be shown that the gauge slice we have to choose is a finite-dimensional space [104], the moduli space of Riemann surfaces.

The gauge fixing with respect to the group of diffeomorphisms is made using Fadeev-Popov ghosts. After factoring out the volume of the group of diffeomorphisms we are left with:

$$Z = \int d\mu\, \mathcal{D}_{e^\phi \hat{g}} X \mathcal{D}_{e^\phi \hat{g}} \phi \mathcal{D}_{e^\phi \hat{g}} b \mathcal{D}_{e^\phi \hat{g}} c \, \exp\left\{-S_P[X,\hat{g}] - S_{gh}[b,c,\hat{g}]\right\} \tag{279}$$

where $\mathcal{D}_{e^\phi \hat{g}}$ indicates that the integration measure is evaluated for the original metric g_{ab}, $d\mu$ is the measure in the space of fiducial metrics (moduli space), and S_P and S_{gh} are respectively the string action and the action for the ghost fields

$$S_{gh}[b,c,\hat{g}] = \frac{i}{\pi\alpha'} \int d^2\sigma \sqrt{\hat{g}} \hat{g}^{ab} b_{bc} \nabla_a c^c \; . \tag{280}$$

We would like to write the integration measures with respect to the fiducial metric, in order to make explicit the dependence in the field $\phi(\sigma)$. This can be done for the X^μ fields [105]

$$\mathcal{D}_{e^\phi \hat{g}} X = e^{(d/48\pi) S_L[\phi,\hat{g}]} \mathcal{D}_{\hat{g}} X \; , \tag{281}$$

and for the ghost fields

$$\mathcal{D}_{e^\phi \hat{g}} b \mathcal{D}_{e^\phi \hat{g}} c = e^{(-26/48\pi) S_L[\phi,\hat{g}]} \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \; , \tag{282}$$

where $S_L[\phi,\hat{g}]$ is the Liouville action for the conformal factor

$$S_L[\phi,\hat{g}] = \int d^2\sigma \sqrt{\hat{g}} \left(\frac{1}{2} \hat{g}^{ab} \partial_a \phi \partial_b \phi + R\phi + \mu e^\phi \right) \; . \tag{283}$$

We finally arrive at

$$\begin{aligned} Z &= \int d\mu\, \mathcal{D}_{e^\phi \hat{g}} \phi \mathcal{D}_{\hat{g}} X \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \\ &\times \; \exp\left\{ -S_P[X,\hat{g}] - S_{gh}[b,c,\hat{g}] - \frac{26-d}{48\pi} S_L[\phi,\hat{g}] \right\} \end{aligned}$$

When $d \neq 26$ the conformal mode of the metric (the Liouville field) becomes a dynamical field. Hence the realization of the conformal symmetry is very different depending on whether $d = 26$ or $d \neq 26$. Non-critical strings are those with $d \neq 26$, and the Liouville field is necessary in order to satisfy the requirement of conformal invariance. When $d = 26$, the Liouville field decouples from the action and we can absorbe the integration over ϕ in the normalization of the path integral. This is the case for critical ($d = 26$) strings. The only remainder of the integration over metrics in this case is the integration over moduli $d\mu$:

$$Z = \int d\mu\, \mathcal{D}_{\hat{g}} X \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \, \exp\left\{-S_P[X,\hat{g}] - S_{gh}[b,c,\hat{g}]\right\} \tag{284}$$

Under some plausible assumptions we can extract the dependence of the measure on the conformal factor:

$$\begin{aligned} Z &= \int \mathcal{D}_{\hat{g}} \phi \mathcal{D}_{\hat{g}} X \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \\ &\times \; \exp\left\{ -S_P[X,\hat{g}] - S_{gh}[b,c,\hat{g}] - \frac{25-d}{48\pi} S'_L[\phi,\hat{g}] \right\} \end{aligned} \tag{285}$$

where $S_L^{'}[\phi, \hat{g}]$ is given by

$$S_L^{'}[\phi, \hat{g}] = \int d^2\sigma \sqrt{\hat{g}} \left(\frac{1}{2} \hat{g}^{ab} \partial_a \phi \partial_b \phi + R\phi + \mu e^{\gamma\phi} \right) \tag{286}$$

with

$$\gamma = \frac{25 - d - \sqrt{(25-d)(1-d)}}{12} \tag{287}$$

This expression was obtained first by David and Distler and Kawai [106] by imposing a general ansatz for the jacobian and requiring the resulting theory to be well defined. It is clear that the results obtained look reasonable only for $d < 1$. For $d > 1$ the naïve arguments in [106] break down. For $d \geq 26$ the formulae may still hold as long as we think of the Liouville field as a time variable.

In the critical dimension we are left with a two-dimensional quantum field theory of d scalar fields and the reparametrization ghosts. The system of *X-fields + ghosts* preserves conformal invariance at the quantum level. This can be seen by constructing the Virasoro generators for the ghost fields which satisfy the algebra

$$[L_m^{gh}, L_n^{gh}] = (m-n)L_{m+n}^{gh} + \frac{1}{6}m(1-13m^2)\delta_{m+n,0} \tag{288}$$

and correspondingly for $\bar{L}_m^{gh}$. Considering now the total Virasoro generator (matter+ghosts)

$$\begin{aligned} L_m^{tot} &= L_m + L_m^{gh} - b\delta_{m,0} \\ \bar{L}_m^{tot} &= \bar{L}_m + \bar{L}_m^{gh} - b\delta_{m,0} \end{aligned} \tag{289}$$

we find that they satisfy the classical Virasoro algebra without central term

$$\begin{aligned} [L_m^{tot}, L_n^{tot}] &= (m-n)L_{m+n}^{tot} \\ [\bar{L}_m^{tot}, \bar{L}_n^{tot}] &= (m-n)\bar{L}_{m+n}^{tot} \end{aligned} \tag{290}$$

so our quantum theory is invariant under the two-dimensional conformal group.

The handling of ghosts fields makes BRST formalism [2] specially well suited. We define the BRST operator from the matter and ghost fields satisfying $Q^2 = 0$. Then physical states are defined as those in the cohomolgy of the Q operator with minimum ghost number.

The quantum field theory on the world-sheet describing the embedding in the target space has the special property of having a traceless energy-momentum tensor; in other words, it is a two-dimensional conformal field theory. Up to now we have worked with a flat target space metric. We could instead start with a general target space with metric $G_{\mu\nu}$. Since to construct a string theory we need a two-dimensional conformal field theory, we demand that the β-function $\beta_{\mu\nu}(G)$ associated with the quantum field theory on the world-sheet vanishes (as it is required for conformal invariance). Then we would obtain constrains on the background metric in the form $\beta_{\mu\nu}(G) = 0$. These equations, in the lowest order in powers of $\alpha^{'}$, will give us the Einstein equations for $G_{\mu\nu}$ (see below).

3.2 Conformal Field Theory

As we have seen, conformal field theories (CFT) are the starting point for the construction of classical string models. We now review the basic properties of these theories. Two-dimensional CFT's were studied in a seminal paper by Belavin, Polyakov and Zamolodchikov [107] which has become the basic reference on the subject (for a general review see also [108]).

Consider a field theory in d dimensions and its conserved energy-momentum tensor $T_{\mu\nu}$. For each vector field $\xi^\mu(x)$ we can construct the current $j_\nu = \xi^\mu(x)T_{\mu\nu}$ which will be conserved ($\partial_\mu j^\mu = 0$) provided $\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = 0$ since

$$\partial_\mu j^\mu = \partial_\mu\xi_\nu T^{\mu\nu} = \frac{1}{2}(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu)T^{\mu\nu} \ . \tag{291}$$

If we take the generator of dilatations $\xi^\mu(x) = x^\mu$ we would have that, by appliying (291), the corresponding current is conserved if the energy-momentum tensor is traceless. This conservation implies that the theory is invariant under scale transformations.

A conformal trasformation is a change of coordinates $x^\mu \to x^{'\mu}$ characterized by the relation

$$g_{\mu\nu}(x^{'})dx^{'\mu}dx^{'\nu} = \Omega(x)g_{\mu\nu}(x)dx^\mu dx^\nu \ . \tag{292}$$

By considering an infinitesimal transformation $\Omega(x) = 1 + \lambda(x)$ and $x^{'\mu} = x^\mu + \xi^\mu(x)$, (292) implies the relation

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \frac{2}{d}\eta_{\mu\nu}\partial_\sigma\xi^\sigma = 0 \tag{293}$$

Vector fields $\xi_\mu(x)$ that generate conformal transformations are called conformal Killing vectors. In the case in which $d > 2$ the general solution to this equation is

$$\xi_\mu = b_\mu + \omega_{\mu\nu}x^\nu + (c\cdot x)x_\nu - \frac{1}{2}c_\mu x^2 \tag{294}$$

with $\omega_{(\mu\nu)} = 0$. Hence the conformal group in $d > 2$ dimensions is finite-dimensional, since conformal transformations are labeled by a finite number of parameters $(b_\mu, \omega_{\mu\nu}, c_\mu)$. For a space of signature (t, s), the conformal group is $SO(t+1, s+1)$.

The situation is quite different in the two-dimensional case. There we can use light-cone coordinates and (293) becomes

$$\partial_+\xi_- = \partial_-\xi_+ = 0 \tag{295}$$

the general solution being $\xi_+ = \xi(\sigma^+)$ and $\xi_- = \xi(\sigma^-)$. With the metric in light-cone coordinates $ds^2 = d\sigma^+ d\sigma^-$ the conformal transformations take the form

$$\sigma^+ \to f(\sigma^+) \qquad \sigma^- \to g(\sigma^-) \tag{296}$$

If, after Wick rotating, we use complex coordinates z, $\bar{z}$, the conformal transformations correspond to holomorphic and antiholomorphic trasformations for z and $\bar{z}$ respectively

$$z^{'} = f(z) \qquad \bar{z}^{'} = \bar{f}(\bar{z}) \tag{297}$$

In these coordinates the energy-momentum tensor $T_{\mu\nu}(\sigma)$ has only two non-vanishing components, namely, $T_{zz}(z) = T(z)$ and $T_{\bar{z}\bar{z}}(\bar{z}) = \bar{T}(\bar{z})$ while the component $T_{z\bar{z}} = T_{\bar{z}z}$ vanishes because of the tracelessness of $T_{\mu\nu}$. Now it is easy to see that the conformal group in two dimensions is infinite-dimensional, being generated by

$$l_n = -z^{n+1}\frac{d}{dz} \,, \tag{298}$$

and satisfying the Virasoro algebra $[l_m, l_n] = (m-n)l_{m+n}$.

We define primary fields $\Phi_{h\bar{h}}(z,\bar{z})$ with conformal weights $(h,\bar{h})$ as those fields such that $\Phi_{h\bar{h}}(z,\bar{z})(dz)^h(d\bar{z})^{\bar{h}}$ is invariant under conformal transformations

$$\Phi_{h\bar{h}}(z,\bar{z})(dz)^h(d\bar{z})^{\bar{h}} = \Phi_{h\bar{h}}[f(z),\bar{f}(\bar{z})](dz^{'})^h(d\bar{z}^{'})^{\bar{h}} \,. \tag{299}$$

This implies the following transformation property for the fields under (297)

$$\Phi_{h,\bar{h}}(z,\bar{z}) = \Phi_{h,\bar{h}}[f(z),\bar{f}(\bar{z})]|f^{'}(z)|^2 \,, \tag{300}$$

or, if we consider infinitesimal conformal transformations $z^{'} = z + \epsilon(z)$,

$$\delta_\epsilon \Phi_{h,\bar{h}}(z,\bar{z}) = \left(\epsilon\partial_z + h\partial_z\epsilon\right)\Phi_{h\bar{h}}(z,\bar{z}) \,, \tag{301}$$

and similarly for the antiholomorphic part. Since

$$T(\epsilon) = \oint_{|z|=1}\frac{dz}{2\pi i}\epsilon(z)T(z) \tag{302}$$

is the generator of conformal transformations we could equally write

$$\delta_\epsilon\Phi_{h\bar{h}} = [T(\epsilon), \Phi_{h\bar{h}}(z,\bar{z})] \tag{303}$$

For simplicity, we are going to consider from now on only the holomorphic part. Any holomorphic function can be written as a power series in z, so the whole conformal group can be constructed from the generators associated with $\epsilon_n = z^{n+1}$

$$L_n = \oint_{|z|=1}\frac{dz}{2\pi i}z^{n+1}T(z) \tag{304}$$

then we can rewrite $T(z)$ as

$$T(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2} \tag{305}$$

where L_n are the Virasoro generators introduced in the previous section in the context of String Theory. In the quantum theory they will satisfy the central extension of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{306}$$

where c is called the central charge. It is easy to see that $L_{\pm1}$ and L_0 form an $SL(2,\mathbf{C})$ subalgebra. From (303) we can determine the commutator of a Virasoro generator with a primary field

$$[L_n, \Phi_h(z)] = h(n+1)z^n\Phi_h(z) + z^{n+1}\partial_z\Phi_h(z) \,. \tag{307}$$

We learned in the previous section that by means of a conformal transformation we can map the cylinder into the punctured complex plane $\mathbf{C}^*$. Time ordering in the original description is converted into radial ordering in $\mathbf{C}^*$, $\tau = -\infty$ corresponds to $z = 0$ and $\tau = \infty$ to $z = \infty$. Associated with every primary field $\Phi_h(z)$ we have a state in the Fock space defined as the action of this field on the vacuum at $z = 0$ ($\tau = -\infty$)

$$|h\rangle = \Phi_h(0)|0\rangle \; . \tag{308}$$

In the case of the energy-momentum tensor for its action on the vacuum state to be well defined it is neccesary that

$$L_n|0\rangle = 0 \; ; \qquad n \geq -1 \; , \tag{309}$$

so the vacuum state $|0\rangle$ is $SL(2, \mathbf{C})$ invariant. By using (307) the state $|h\rangle$ associated with the primary field $\Phi_h(z)$ satisfies

$$L_0|h\rangle = h|h\rangle \tag{310}$$

$$L_n|h\rangle = 0 \; , \qquad n > 0 \tag{311}$$

due to the second property, states created by primary fields are called highest weight states. On the other hand, by applying L_{-n} with $n > 0$, to a highest weight state we obtain the so-called descendent states which are labeled by the eigenvalue of L_0 (we have from the Virasoro algebra that $[L_0, L_{-n}] = nL_{-n}$)

$$\begin{array}{rl} |h\rangle & \qquad L_0 = h \\ L_{-1}|h\rangle & \qquad L_0 = h+1 \\ L_{-1}^2|h\rangle, L_{-2}|h\rangle & \qquad L_0 = h+2 \\ \vdots & \end{array} \tag{312}$$

The representation of the Virasoro algebra spanned by these states is called a Verma module $V(h, c)$ and depends only on the conformal weight of the primary field and on the central charge of the Virasoro algebra.

Given a highest weight state $|h\rangle$ we associate with it a primary field $\Phi_h(z)$ such that $|h\rangle = \Phi_h(0)|0\rangle$; in the same way, given a descendent state $L_{-n_1} \ldots L_{-n_k}|0\rangle$ we have the corresponding descendant field defined by

$$\Phi^{n_1 + \ldots + n_k}(z) = L_{-n_1}(z) \ldots L_{-n_k}(z)\Phi_h(z) \tag{313}$$

where the operators $L_n(z)$ are defined by

$$L_n(z) = \oint_{C_n} \frac{dw}{2\pi i}(w - z)^{-n+1}T(w) \tag{314}$$

and the contours are taken in such a way that $C_{n_k} \subset \ldots \subset C_{n_1}$. This allows us to make a one-to-one correspondence between fields and states.

Equal time (radius) commutator can be related to the operator product expansion (OPE) of the operators involved in the commutator. Let us consider a complete set of operators $\{\Phi_k(z)\}$.

Figure 3: Integration contours to compute the equal-radius commutator

In general, the product of any two operators in this set $\Phi_i(z)\Phi_j(w)$ will be divergent in the limit $z \to w$. Since the set of operators we are considering is complete, we can write

$$\Phi_i(z)\Phi_j(w) = \sum_k C_{ij}^k(z,w)\Phi_k(w) \tag{315}$$

which is called the operator product expansion (OPE) of $\Phi_i(z)\Phi_j(w)$. Thus, the divergence of the product in the limit in which z and w coincides is given by the behavior of $C_{ij}^k(z,w)$ in that limit. Take now two analytic fields $A(z)$ and $B(z)$ and two arbitary functions $f(z)$ and $g(z)$, and construct the operators

$$\begin{aligned} A(f) &= \oint_{|z|=1} \frac{dz}{2\pi i} f(z)A(z) \\ B(g) &= \oint_{|w|=1} \frac{dw}{2\pi i} g(w)B(w) \end{aligned} \tag{316}$$

We write the equal-time commutator as

$$\begin{aligned} [A(f), B(g)] &= \oint_{C_1} \frac{dz}{2\pi i} f(z)A(z) \oint_{C_2} \frac{dw}{2\pi i} g(w)B(w) \\ &- \oint_{C_2} \frac{dw}{2\pi i} g(w)B(w) \oint_{C_1} \frac{dz}{2\pi i} f(z)A(z) \end{aligned} \tag{317}$$

which is schematically represented in fig. 3a. By deforming the contour as in fig. 3b we obtain the result

$$[A(f), B(g)] = \oint_{|w|=1} \frac{dw}{2\pi i} g(w) \oint_w \frac{dz}{2\pi i} A(z)B(w)f(z) \tag{318}$$

where the second integral is performed along a little circunference around w. This expression tell us that the integral in z is determined by the singularities in the OPE of $A(z)B(w)$. From (307) we get the OPE

$$T(z)\Phi_h(w) = \frac{h}{(z-w)^2}\Phi_h(w) + \frac{1}{z-w}\partial_w \Phi_h(w) + regular\ terms \tag{319}$$

where the regular terms are analytic in the limit $z \to w$. Using (318) we can also represent the Virasoro algebra in terms of an OPE

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) \\ &+ \frac{1}{z-w}\partial_w T(w) + regular\ terms\ , \end{aligned} \tag{320}$$

where c is the central charge. Now we can easily get the vacuum expectation value for the product $T(z)T(w)$:

$$\langle T(z)T(w)\rangle = \frac{c/2}{(z-w)^4} \ , \tag{321}$$

and we recover the gravitational anomaly calculated in the previous chapter

$$\int d^2x \langle T_{++}(x)T_{++}(0)\rangle e^{ipx} \sim \frac{p_+^3}{p_-} \ . \tag{322}$$

The presence of a central charge in the OPE for $T(z)T(w)$ makes the energy-momentum tensor to transform not as a primary field. In fact, $T(z)$ can be written as

$$T(z) = L_{-2}(z)\mathbf{I}(z) \ , \tag{323}$$

where $\mathbf{I}(z)$ is the identity operator (which is trivially a primary field of conformal weight $h=0$). This means that the energy momentum tensor is a descendant field of the identity operator. If instead of considering infinitesimal conformal transformations we deal with a finite one $z \to f(z)$, $T(z)$ transforms according to

$$T(z) \to (\partial_z f(z))^2 T(z) + \frac{c}{12}\{f, z\} \ , \tag{324}$$

where $\{f, z\}$ is the Schwarzian derivative defined by

$$\{f, z\} = \frac{\partial_z f(z)\partial_z^3 f(z) - \frac{3}{2}(\partial_z^2 f(z))^2}{(\partial_z f(z))^2} \tag{325}$$

Recall that we started with a conformal field theory defined over the cylinder with complex coordinate w, and that we have passed from the cylinder to the punctured complex plane by applying the conformal transformation $z = e^w$. If we use the transformation rule for the energy-momentum tensor (324) we get the following relation between the energy-momentum tensor in the cylinder and in the complex plane

$$T_{cyl}(w) = z^2 T(z) - \frac{c}{24} \ , \tag{326}$$

which in terms of the Virasoro generators means that the only generator that changes is L_0, namely

$$(L_0)_{cyl} = (L_0)_{plane} - \frac{c}{24} \ . \tag{327}$$

We can give an interpretation to the trasformations induced by L_0 and $\bar{L}_0$. From their definition in terms of the energy-momentum tensor,

$$\begin{aligned} L_0 &= \oint_{|z|=1} \frac{dz}{2\pi i} z T(z) \ , \\ \bar{L}_0 &= \oint_{|\bar{z}|=1} \frac{d\bar{z}}{2\pi i} \bar{z}\bar{T}(\bar{z}) \ , \end{aligned} \tag{328}$$

Figure 4: Torus with modular parameter τ

we see that $L_0+\bar{L}_0$ is the generator of dilatations $z \to e^{\lambda}z$ in $\mathbf{C}^*$ with λ real. But this dilatations are time traslations $\tau \to \tau+\lambda$ in the cylinder and then $L_0+\bar{L}_0$ plays the role of the hamiltonian in our radial quantization formalism. The interpretation of $L_0-\bar{L}_0$ is also straighforward; this is the generator of the trasformations $z \to e^{i\lambda}z$ for real λ, which are the rotations with center $z=0$. Going back to the cylinder we see that this correspond to traslations in the coordinate σ (i.e., a change in the origin of the coordinate σ) so it can be viewed as a momentum operator.

We are interested now in finding the partition function of our CFT, which counts the number of states per energy level. It is also interesting to formulate the CFT not on the cylinder (or the punctured complex plane) but on a Riemman surface of arbitrary genus, in order to apply our results to String Theory. It is possible to construct a CFT on the torus starting from our formulation in the cylinder (see [108] for the details); to get a torus we identify the two ends of our cylinder. Then we have two periods, one associated to time traslations and the other associated to the spatial ones which, as we have seen, are generated by $(L_0)_{cyl} \pm (\bar{L}_0)_{cyl}$ respectively. First we redefine the complex coordinate w as iw so the spatial period is simply $w \equiv w+2\pi$. The remaining period can be defined by a complex number $\tau = \tau_1 + i\tau_2$ as $w \equiv w + 2\pi\tau$, where τ is called the modular parameter of the torus and parametrizes different tori (see fig. 4). Now we calculate the partition fuction; making use of the relation (327) derived earlier, the result is [108]

$$Z = Tr\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24} \tag{329}$$

where q is defined as $q = e^{2\pi i\tau}$.

The partition function counts the number of states associated with a given conformal family. Let $V(h,c)$ be the Verma module associated with a primary field $\Phi_h(z)$

$$V(h,c) = \{|h\rangle, L_{-1}|h\rangle, L_{-2}|h\rangle, L_{-1}^2|h\rangle, \ldots\} \tag{330}$$

We are interested in the number of states per level in $V(h,c)$. Define the character

$$\chi_h(q) = Tr_{V(h,c)} q^{L_0-c/24} \tag{331}$$

which is the holomorphic part of the partition function defined earlier and restricted to states in $V(h,c)$. By evaluating the trace we have

$$\chi_h = \sum_m d_m q^m \tag{332}$$

where d_m is the number of states in $V(h,c)$ with $L_0 = m + c/24$. For example if $V(h,c)$ contains a single highest weight vector the character can be explicitly evaluated, with the result:

$$\chi_h = q^{h-c/24}\prod_{n=1}^{\infty}\frac{1}{1-q^n} \tag{333}$$

As an example to illustrate our discussion on CFT, we analize a massless free scalar field in two dimensions. Using complex coordinates $(z,\bar{z})$ we can write the action (we set from now on $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$)

$$S = \frac{1}{4\pi}\int d^2z \partial\phi(z,\bar{z})\bar{\partial}\phi(z,\bar{z}) \tag{334}$$

and the field equations are

$$\partial\bar{\partial}\phi(z,\bar{z}) = 0 \tag{335}$$

The general solution to this equation can be written as $\phi(z,\bar{z}) = \phi(z) + \phi(\bar{z})$. Since $\partial\phi$ is a holomorphic field, we can make the Laurent expansion

$$\partial\phi(z) = \sum_{n\in\mathbf{Z}} \alpha_n z^{-n-1} \; , \tag{336}$$

and equivalently for $\bar{\partial}\phi$

$$\bar{\partial}\phi(\bar{z}) = \sum_{n\in\mathbf{Z}} \bar{\alpha}_n \bar{z}^{-n-1} \; . \tag{337}$$

From (334) we can evaluate the propagator:

$$\langle\phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\log|z-w|^2 \; , \tag{338}$$

and the energy-momentum tensor

$$T(z) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) : \tag{339}$$

and similarly for the antiholomorphic component. In order to calculate the central charge we compute the OPE for two energy-momentum tensors. Using Wick's theorem for radial quantization, we obtain

$$T(z)T(w) = \frac{1/2}{(z-w)^4} + \textit{other terms} \tag{340}$$

and the central charge associated to a massless scalar field in two dimensions is $c = 1$. From the OPE

$$T(z)\partial\phi(w) = \frac{1}{(z-w)^2}\partial\phi(w) + \frac{1}{z-w}\partial^2\phi(w) \tag{341}$$

we see that $\partial\phi(z)$ is a primary field with conformal weight $(h,\bar{h}) = (1,0)$. We would like to construct other primary fields with differents weights. This can be achieved by using the vertex operators

$$V_k(z,\bar{z}) =: e^{ik\phi(z,\bar{z})} : \tag{342}$$

which, by a simple OPE computation have conformal dimensions

$$h(V_k) = \bar{h}(V_k) = \frac{k^2}{2} \tag{343}$$

3.3 Quantization of the Bosonic String

The example of the two-dimensional massless scalar field provides a natural connection with the quantization of the closed bosonic string. Recall that we started with the action

$$S_P[X,g] = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \ . \tag{344}$$

After fixing the conformal gauge and performing the mapping of the cylindrical world-sheet into $\mathbf{C}^*$ we obtain a gauge-fixed action

$$S_P = -\frac{1}{4\pi\alpha'}\int dz d\bar{z}\, \partial X^\mu \bar{\partial} X^\nu \eta_{\mu\nu} \ . \tag{345}$$

This action describes a set of d massless scalar fields. Working in the critical dimension $d = 26$ where the Liouville field decouples makes the analysis of the free bosonic string particulary easy. From now on we restrict our considerations to this case. The field X^μ is expanded as

$$X^\mu(z,\bar{z}) = q^\mu - \frac{i}{4}p^\mu \log z\bar{z} + \frac{i}{2}\sum_{n\in\mathbf{Z}}\frac{\alpha_n^\mu}{n}z^{-n} + \frac{i}{2}\sum_{n\in\mathbf{Z}}\frac{\bar{\alpha}_n^\mu}{n}\bar{z}^{-n} \ . \tag{346}$$

The vertex operators [109] can be interpreted geometrically as the Fourier transforms of the operators

$$V(X) \sim \int d^2\sigma\sqrt{g}\delta(X^\mu - X^\mu(\sigma)) \ , \tag{347}$$

which pin a string at the point X. Since only $(1,1)$ fields can be integrated in a conformal invariant way we have that physical states satisfy

$$L_0|phys\rangle = |phys\rangle \ , \qquad \bar{L}_0|phys\rangle = |phys\rangle \ . \tag{348}$$

These conditions fix the value of the normal ordering constant b introduced in sec. 3.1.

Now we proceed to obtain the spectrum of the closed bosonic string (we will work in a system of units in which $\alpha' = \frac{1}{2}$). Canonical quantization produces the following commutation relations

$$\begin{aligned} [q^\mu, p_\nu] &= i\eta^{\mu\nu} \\ [\alpha_m^\mu, \alpha_n^\nu] &= m\eta^{\mu\nu}\delta_{m+n,0} \\ [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] &= m\eta^{\mu\nu}\delta_{m+n,0} \end{aligned} \tag{349}$$

and all other commutators vanish. We split the general solution (346) into the right and left-moving parts (holomorphic and antiholomorphic respectively)

$$\begin{aligned} X_R^\mu(z) &= x_R^\mu - \frac{i}{4}p_R^\mu \log z + \frac{i}{2}\sum_{n\in\mathbf{Z}}\frac{\alpha_n^\mu}{n}z^{-n} \\ X_L^\mu(\bar{z}) &= x_L^\mu - \frac{i}{4}p_L^\mu \log \bar{z} + \frac{i}{2}\sum_{n\in\mathbf{Z}}\frac{\bar{\alpha}_n^\mu}{n}\bar{z}^{-n} \ , \end{aligned} \tag{350}$$

with $x_R^\mu = x_L^\mu = q^\mu/2$ and $p_R^\mu = p_L^\mu = p^\mu$. Using these expressions we can write L_0 in terms of operators α_m^μ as

$$L_0 = \frac{1}{4}p^2 + \sum_{n\geq 1} \alpha_{-n} \cdot \alpha_n \tag{351}$$

and the similarly for $\bar{L}_0$ changing α_n^μ for $\bar{\alpha}_n^\mu$. Setting $L_0 = \bar{L}_0 = 1$ and taking into account that the mass of a physical state is defined by $m^2 = -p_\mu p^\mu = m_R^2 + m_L^2$ we obtain the mass formula

$$\frac{1}{4}m^2 = \sum_{n\geq 1} \alpha_{-n} \cdot \alpha_n + \sum_{n\geq 1} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n - 2\ . \tag{352}$$

The condition $L_0 - \bar{L}_0$, which now states that the theory is invariant under redefinitions of the σ-origin, gives the level matching condition

$$\sum_{n\geq 1} \alpha_{-n} \cdot \alpha_n = \sum_{n\geq 1} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n\ . \tag{353}$$

Looking back to the commutation relations (349) we see that $\sum_{n\geq 1} \alpha_{-n} \cdot \alpha_n$ can be interpreted as the ocupation number associated to the right-moving modes (the same for the left-movers). This allows us to rewrite the mass formula and the level matching condition in a simpler form

$$\begin{aligned} \frac{1}{4}m^2 &= N_R + N_L - 2\ , \\ N_R &= N_L\ . \end{aligned} \tag{354}$$

We are now ready to construct the spectrum of the closed bosonic string. We start with the vacuum state $|0,p\rangle$ defined by

$$\begin{aligned} \alpha_n^\mu |0,p\rangle &= \bar{\alpha}_n^\mu |0,p\rangle = 0\ , \qquad n \geq 1 \\ \hat{p}^\mu |0,p\rangle &= p^\mu |0,p\rangle \end{aligned} \tag{355}$$

where we have written $\hat{p}^\mu$ to distinguish the operator from its eigenvalue. This vacuum state is clearly a tachyon, since $N_R = N_L = 0$ and $m^2 < 0$. This tachyon is created by the action of the tachyon vertex operator defined by

$$V_{tachyon}(p) =: e^{ip\cdot X} : \tag{356}$$

In the first excited level we have only one posibility surviving the level matching condition $N_L = N_R = 1$, so a general state will be of the form $\zeta_{\mu\nu}(p)\alpha_{-1}^\mu \bar{\alpha}_{-1}^\nu |0,p\rangle$ with $\zeta_{\mu\nu}(p)$ the polarization vector with an associated vertex operator given by

$$V(p) = \zeta_{\mu\nu}(p) : \partial X^\mu \bar{\partial} X^\nu e^{ip\cdot X} : \tag{357}$$

The states in the first level are massless as one can see either from the mass formula or from the condition that the previous vertex operator has conformal weights $(1,1)$. This also gives the relation $p^\mu \zeta_{\mu\nu}(p) = 0$ which is the on-shell condition for a massles tensor particle. We can decompose a general state in the first excited level into three states that transform irreducibly under $SO(1,25)$; the first one, with $\xi_{\mu\nu} = \xi_{\nu\mu}$, $\eta^{\mu\nu}\zeta_{\mu\nu}(p) = 0$, is a spin 2 particle that we will

Figure 5: Fundamental vertex in string perturbation theory

identify with the graviton. Secondly we have the antisymmetric rank-2 tensor field for which $\zeta_{(\mu\nu)} = 0$. Finally if we set $\zeta_{\mu\nu} = \eta_{\mu\nu}$ we have a scalar particle that is known as the dilaton. Thus at the massless level of the bosonic string we have three states, the graviton $g_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$ and the dilaton Φ.

Without much effort we have been led to the presence of spin 2 particles. In principle we cannot still identify these states with gravitons since we are dealing with free bosonic strings and we have no interaction between them. However, when we introduce interactions in String Theory following the Polyakov prescription (see below) it is possible to show that the scattering amplitudes for these states in the low energy limit are described by the linearized Einstein-Hilbert action, so they can be consistently identified with the graviton. This is one of the main features of String Theory: every closed string theory (and all of them in the interacting case) contains a massless state of spin 2.

We should remark that the existence of a graviton state is also a consequence of working in the critical dimension $d = 26$. Gravitons seem to be excluded from the spectrum in non-critical dimensions. Thus, if we want to use String Theory to unify gravity with other interactions we should work in the critical dimension. More generally, in String Theory it is now clear that the notion of dimension is encoded in the central extension of the matter Virasoro algebra. We want to have $c = 26$, although the contribution from "geometrical" dimensions may be smaller than 26. By adding up different CFT's without a geometrical interpretation we can reach the critical value for c.

3.4 Interaction in String Theory and the characterization of the moduli space

After studying the theory of free strings it is time to introduce interacting strings. This is done by introducing a perturbative expansion in which the basic vertex describes the splitting of one initial string into two final ones (fig. 5) If we use a particle analog, this vertex would correspond to the vertex of a ϕ^3 scalar field theory. However the situation in String Theory is quite different from that in QFT: in the case of a ϕ^3 theory in order to compute a given amplitude we have to sum over all Feynman graphs with a given number of external states. For example, for the four point amplitude we have to sum the contributions of all graphs with four external legs (fig. 6) In String Theory we have to compute amplitudes with a given number of external string states. To do so we have to sum over all world-sheets that connect *in* with *out* strings (fig. 7). All the surfaces we have to sum over can be constructed from the fundamental vertex of fig. 5. It is easy to check that the number of vertices we need to construct a given

Figure 6: Feynman diagrams contributing to the four-point function in ϕ^3 scalar theory

Figure 7: String graphs contributing to the scattering of four strings

surface depends only on the genus g of the surface and the number n of external strings and it is equal to $2g+n-2$. If g_{st} is the string coupling constant associated with the fundamental vertex, the amplitudes can be schematically written as

$$A(1,\ldots,n) = \sum_{topologies} g_{st}^{2g+n-2} \sum_{metrics} \frac{1}{Vol(\text{Diff}) \times Vol(\text{Weyl})} \sum_{embeddings} e^{-S_P[X,g]} \tag{358}$$

where the first sum indicates that we sum over two-dimensional surfaces of genus g and with n external strings. We also divide out by the volume of the group of diffeomorphisms and of the group of Weyl transformations because these are (in the critical dimension) the gauge groups of the quantum theory. Nevertheless it seems quite difficult to perform the sum over surfaces that have external "legs" extending to infinity. This can be solved if we make use of the Weyl invariance of the theory: we may cut external strings from the surfaces in the perturbative expansion and then we are left with two-dimensional surfaces with n boundaries. Then, since by a conformal mapping we can transform a cylinder into the punctured complex plane, we will attach to every boundary the punctured unit circle $\mathcal{C}^* = \{z \in \mathbf{C} | \, 0 < |z| \leq 1\}$. External string states are mapped to the punctures by means of local operators (vertex operators) that carry the quantum numbers of the external strings. When all this information is put together we can write the amplitude (359) as

$$A(1,\ldots,n) = \sum_{g=1}^{\infty} g_{st}^{-\chi} \int_{\Sigma_{g,n}} \frac{\mathcal{D}X\mathcal{D}g}{Vol(\text{Diff}) \times Vol(\text{Weyl})} \prod_{i=1}^{n} V_{\Lambda_i}(k_i) e^{-S_P[X,g]} \tag{359}$$

where $V_{\Lambda_i}(k_i)$ is the vertex operator representing a external string with momentum k_i and quantum numbers Λ_i, and χ is the Euler characteristic for a genus g manifold with n boundaries

$\chi = 2 - 2g - n$.

This is the Polyakov prescription for the computation of amplitudes in String Theory. We sum over all two-dimensional surfaces with genus g and n punctures non-equivalent under the joined action of the group of diffeomorphisms and Weyl transformations. We can, for every surface $\Sigma_{g,n}$, write the metric in the form (up to diffeomorphisms)

$$g_{ab} = e^{\phi(x)} \hat{g}_{ab} \tag{360}$$

where $\hat{g}_{ab}$ is a fiducial metric. Since we are in the critical dimension the dependence over the conformal factor $\phi(x)$ drops out from (359) and the integration over $\phi(x)$ is cancelled by the $Vol(\text{Weyl})$ in the denominator. Then we are faced with the problem of clasifying fiducial metrics $\hat{g}_{ab}$. We will see that the set $\{\hat{g}_{ab}\}/\text{Diff}(\Sigma_{g,n})$ is finite-dimensional.

We work with compact orientable two-dimensional surfaces of genus g with punctures. If we take complex coordinates in such surfaces the problem of classifying them into conformal classes is equivalent to the classification of complex structures on the manifold: then the sum over conformal classes is transformed into a sum over Riemann surfaces of genus g and n punctures $\Sigma_{g,n}$.

However after performing the sum over conformal classes, we are still left with the integration over Riemann surfaces that are not equivalent under diffeomorphisms. We have to classify Riemann surfaces according to their complex structure; the space of parameters that label differents classes constitutes the moduli space for the surface $\mathcal{M}_{g,n}$. We begin with the case $n = 0$. In order to characterize the moduli space of a Riemann surface of genus g we use the uniformization theorems due to Klein, Poincaré and Koebe. Any given Riemann surface Σ can be constructed by starting with one of the simply connected Riemann surfaces (their universal cover $\hat{\Sigma}$)

- The sphere S^2 with the round metric

$$ds^2 = \frac{dz d\bar{z}}{(1 + z\bar{z})^2} \tag{361}$$

- The complex plane $\mathbf{C}$ with the flat metric $ds^2 = dz d\bar{z}$.

- The upper half plane $\mathbf{H}$ with the metric of constant negative curvature

$$ds^2 = 2\frac{dz d\bar{z}}{(z - \bar{z})^2} \tag{362}$$

We recover the Riemann surface from its universal cover by dividing it by its fundamental group $\pi_1(\Sigma)$ which acts in the cover without fixed points. We also have the Gauss-Bonnet theorem

$$\frac{1}{2\pi} \int_{\hat{\Sigma}} d^2 z R = 2(1 - g) \tag{363}$$

The sphere is the covering space for genus zero Riemann surfaces, the complex plane for $g = 1$ and the upper half plane for any surface with $g > 1$. The conclusion is that we will classify all the Riemann surfaces into diffeomorphism classes by finding out the action of the fundamental group $\pi_1(\Sigma)$ on the universal covering space $\hat{\Sigma}$.

Let us begin with a genus zero Riemann surface. There is a theorem that states that any Riemann surface homeomorphic to the sphere is also isomorphic to it. This means that the moduli space for a genus zero Riemann surface $\mathcal{M}_0$ has only one point. In the case of $g = 1$ the situation is not so simple. Since in the complex plane the only group that acts without fixed points is the group of translations, to get a torus we have to divide $\mathbf{C}$ by the action of a subgroup of discretes translations isomorphic to $\mathbf{Z} + \mathbf{Z}$ which is the fundamental group of the torus. Thus we take two complex numbers ω_1 and ω_2 and construct the lattice generated by them. It is easy to see that without loss of generality we can take one of the complex numbers equal to one ($\omega_1 = 1$) and we are left only with one complex modular parameter $\omega_2 = \tau$; moreover we can take the imaginary part of τ positive since the lattices generated by τ and $-\tau$ are isomorphic.

The tori parametrized by τ are equivalent under transformations that lie in the connected component of the identity. However we have to restrict the values of τ in order to avoid overcounting because different values of τ may correspond to tori equivalent under diffeomorphisms that are not connected with the identity. It can be seen that this is the case for the tori characterized by τ and

$$\tau^{'} = \frac{a\tau + b}{c\tau + d} \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbf{Z} \tag{364}$$

because the lattices defined by τ and $\tau^{'}$ are equivalent, they differ only in the choice of the fundamental cell, hence we have to restrict the values of τ to the fundamental region of the group $SL(2, \mathbf{Z})$, which we take to be

$$\mathcal{F} = \{\tau \in \mathbf{C} | Im\tau > 0, -\frac{1}{2} < Re\tau \leq \frac{1}{2}, |\tau| > 1\} \tag{365}$$

We have found that the modular space for a genus zero Riemann surface $\mathcal{M}_0$ consists of a single point and for a genus one surface $\mathcal{M}_1 = \mathcal{F}$. The case with $g > 1$ is more complicated. To construct the moduli space $\mathcal{M}_g$ with $g > 1$ we are going to use the sewing technique. Let us consider a genus-g Riemann surface with two punctures (fig. 8) located at P_1 and P_2 and let us also consider two annuli around these points with coordinates z_1 and z_2 such that $z_1(P_1) = z_2(P_2) = 0$. We can sew the Riemann surface by identifying the points on the annuli via the transformation

$$z_1 z_2 = q \ , \tag{366}$$

with $q \in \mathbf{C} - \{0\}$ the sewing parameter. After this is done we get a Riemann surface with genus $g + 1$.

This construction allow us to establish a relation between the dimensions of the moduli spaces $\mathcal{M}_g$ and $\mathcal{M}_{g+1}$. In fact, the parameters characterizing the Riemann surface after sewing two of its points are the moduli parameters characterizing the original Riemann surface of genus g plus the three complex parameters (z_1, z_2, q) that label the position of the points which we have sewed and the sewing parameter. Thus we arrive at the relation

$$dim\mathcal{M}_{g+1} = dim\mathcal{M}_g + 3 \ . \tag{367}$$

Figure 8: Sewing of two punctures in a genus g Riemann surface to obtain a genus $g+1$ Riemann surface

Figure 9: Construction of a genus g Riemann surface with n puctures from a sphere with $2g+n$ punctures

If we start with a genus one surface we have that since we can always locate one of the points at $z=0$, due to the isometries of the flat metric on the torus, the number of moduli parameters for a genus two surface is three. Applying the sewing technique again and again we obtain that the dimension of the moduli space for a genus-g surface is given by

$$\begin{aligned} dim\mathcal{M}_0 &= 0 \\ dim\mathcal{M}_1 &= 1 \\ dim\mathcal{M}_g &= 3g-3 \qquad g>1 \end{aligned} \tag{368}$$

As we have seen above, in the computation of string amplitudes we work with Riemann surfaces of genus g and n puctures. To derive the dimension of the moduli space $\mathcal{M}_{g,n}$ for such surfaces we use a slightly different method from the one used for $\mathcal{M}_g = \mathcal{M}_{g,0}$. We start with a sphere with $2g+n$ punctures (we consider first the case $2g+n>3$); by conformal invariance [108] we can take three of the punctures to be located at $z=0$, $z=1$ and $z=\infty$ respectively so we are left with $2g+n-3$ free complex parameters to locate the remaining punctures. Now by sewing $2g$ punctures together with g sewing parameters $q_1,\ldots,q_g$ we have that the final number of independent complex parameters in the moduli space of a genus g Riemann surface with n punctures is $3g+n-3$ for $g>1$ (see fig. 9). In the case with $2g+n\leq 2$ the situation is different because in that cases all the punctures in the sphere are in fixed positions. Then the dimension of the moduli space only depends upon the number of sewing parameters which is equal to g.

We have seen that the space of two-dimensional metrics modulo differmorphisms and Weyl rescalings is finite-dimensional. This means that the path integration over metrics has to be transformed into an integration over the moduli space of the corresponding surface

$$\int_{\Sigma_{g,n}} \mathcal{D}g \longrightarrow \int_{\mathcal{M}_{g,n}} d\mu(g,n) \tag{369}$$

where $d\mu(g,n)$ is the integration measure over the moduli space. In general the problem we face when computing amplitudes is to characterize this measure.

3.5 Bosonic strings with background fields. "Stringy" corrections to the Einstein equations

Up to now we have considered bosonic strings propagating in Minkowski space. We are also insterested in studying the propagation of strings in general manifolds with metric $G_{\mu\nu}(X)$. This means that we have to modify the string action in order to include the effect of the background metric $G_{\mu\nu}(X)$

$$S_1 = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)\ . \tag{370}$$

We have seen that the massless sector of the bosonic string contains, besides the graviton, the antisymmetric tensor and the dilaton. Thus, it seems natural that if we include a background value for the graviton field we should also consider background values for the other massless states, $B_{\mu\nu}$ and Φ. The general form of the action is

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \\ &- \frac{1}{4\pi\alpha'}\int d^2\sigma\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) + \frac{1}{4\pi}\int d^2\sigma\sqrt{g}R^{(2)}(g)\Phi(X) \end{aligned} \tag{371}$$

Where $G_{\mu\nu}(X)$, $B_{\mu\nu}(x)$ and $\Phi(X)$ are the background values of the fields in the massless sector and ϵ^{ab} is the completely antisymmetric tensor with two indices.

However for the string theory described by (371) to be consistent, we have to impose that this action define a CFT; this will constrain the values of the background fields. The best way to study the conformal invariance of (371) is to write the metric $g_{\mu\nu}$ as $e^\phi \hat{g}^{\mu\nu}$ in $2+\epsilon$ dimensions and then to see what are the conditions that have to be fulfilled for the conformal factor to disappear from the effective action in the limit $\epsilon \to 0$. Doing so we obtain the following equations for the background fields to lowest order in α' [110]

$$\begin{aligned} R_{\mu\nu} + \frac{1}{4}H_\mu^{\lambda\rho}H_{\nu\lambda\rho} - 2D_\mu D_\nu \Phi &= 0 \\ D_\lambda H^\lambda_{\mu\nu} - 2D_\lambda\Phi\, H^\lambda_{\mu\nu} &= 0 \\ 4(D_\mu\Phi)^2 - 4D_\mu D^\mu\Phi + R + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} &= 0 \end{aligned} \tag{372}$$

where $H_{\mu\nu\lambda}$ is the field strength associated with the antisymmetric tensor $B_{\mu\nu}$

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\nu} \tag{373}$$

An important remark is that (372) can be derived as the field equations associated to the action functional

$$S_{d=26} = -\frac{1}{2\kappa^2}\int d^{26}x\sqrt{G}e^{-2\Phi}\left(R - 4D_\mu\Phi D^\mu\Phi + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right). \tag{374}$$

When we include higher order effects in $\alpha^{'}$ we obtain corrections to the Einstein-Hilbert action for the background fields. For example, if we consider the action (370) and compute the β-function at one loop order we obtain (for the evaluation of β-functions in non-linear σ-models see [111])

$$\beta_{\mu\nu}(X) = -\frac{1}{2\pi}R_{\mu\nu}(X) \tag{375}$$

this means that, since for conformal invariance we must have $\beta_{\mu\nu}(X) = 0$, we obtain the vacuum Einstein equations for the background metric. However if we include the two-loop correction to the β-function the result is [111]

$$\beta_{\mu\nu}(X) = -\frac{1}{4\pi}\left(R_{\mu\nu}(X) + \frac{\alpha^{'}}{2}R_{\mu\lambda\sigma\tau}R_{\nu}{}^{\lambda\sigma\tau}\right) \tag{376}$$

so we obtain a correction to the Einstein equations for the background metric of order $\alpha^{'}$ coming from String Theory

$$R_{\mu\nu}(X) + \frac{\alpha^{'}}{2}R_{\mu\lambda\sigma\tau}R_{\nu}{}^{\lambda\sigma\tau} = 0\ . \tag{377}$$

We do not know the form of the β-function to all orders in $\alpha^{'}$.

3.6 Toroidal compactifications. R-duality

In this section we would like to discuss a genuine property of strings which has no analogue in QFT. This is the notion of duality. All known critical string theories are formulated in dimension higher that four ($d = 26$ for the bosonic string and $d = 10$ for the super and the heterotic string). This means that, in order to derive phenomenological implications from String Theory we have to pass from 26 or 10 to 4 dimensions. One way in which this can be accomplished is a Kaluza-Klein compactification. Starting with a, say, 26-dimensional manifold we compactify the 22 extra dimensions in a 22-dimensional torus T^{22}. Then we are left with four open dimensions and 22 compactified ones; if we assume that the typical radii of compactification is of the order of the Planck length, the existence of extra dimensions is completely unobservable from the energies available in present experiments. Nevertheless, this does not mean that they have no effect in low energy phenomenology. For example, the isometry group of the internal manifold appears as part of the gauge group of the low energy effective field theory. In last few years a great effort has been devoted to the search of a four-dimensional string theory with acceptable phenomenological properties (by using toroidal compactifications or more sophisticated mechanisms as orbifolds, Calabi-Yau manifolds etc.).

We illustrate the main features of toroidal compactifications in the context of the closed bosonic string [5]. Assume that we want to compactify D bosonic coordinates in a D-dimensional torus T^D. We will denote by X^i the $26 - D$ open dimensions and by X^I the compactified ones.

We define the D-dimensional torus by simply identifying the points in the target space according to the rule

$$X^I \sim X^I + \pi\sqrt{2}\sum_{i=1}^{D} n_i R_i e_i^I = X^I + 2\pi L^I \qquad\qquad n_i \in \mathbf{Z} \tag{378}$$

where $\mathbf{e}_i$ are D linearly independent vectors normalized to $\mathbf{e}_i^2 = 2$ and R_i are the radii of the compact dimensions. The vectors L^I define a D-dimensional lattice Λ and the torus T^D is represented as

$$T^D = \mathbf{R}^D/(2\pi\Lambda) \tag{379}$$

The momenta in the compact dimensions p^I are the generators of translations of the center of mass x^I. From the single valuedness of the wave function $\exp(ix \cdot p)$ we have that momenta p^I have to lie in the lattice dual to Λ^D, $(\Lambda^D)^*$, and they can be expanded as

$$p^I = \sqrt{2}\sum_{i=1}^{D} \frac{m_i}{R_i} e_i^{*I} \ , \tag{380}$$

with $\mathbf{e}_i \cdot \mathbf{e}_j^* = \delta_{ij}$ and $\mathbf{e}_i^{*2} = 1/2$.

Let us now consider a closed bosonic string on the torus T^D. We have the periodicity condition $X^\mu(\sigma+\pi,\tau) = X^\mu(\sigma,\tau)$. When we have compactified dimensions, we have to take into account the twisted sector in which the periodicity condition is fulfilled modulo the identification (378)

$$X^I(\tau,\sigma+\pi) = X^I(\tau,\sigma) + 2\pi L^I = X^I(\tau,\sigma) + \pi\sqrt{2}\sum_{i=1}^{D} n_i R_i e_i^I \ , \tag{381}$$

which means that the string wrap around the compactified dimensions. The integer n_i is the winding number and measures the number of times the string wraps around the ith compactified dimension. Because of this new boundary conditions we have to modify the mode expansions for X^I

$$\begin{aligned} X_L^I(\tau+\sigma) &= \frac{1}{2}x^I + (\frac{1}{2}p^I + L^I)(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\bar{\alpha}_n^I}{n}e^{-2in(\tau+\sigma)} \\ X_R^I(\tau-\sigma) &= \frac{1}{2}x^I + (\frac{1}{2}p^I - L^I)(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\alpha_n^I}{n}e^{-2in(\tau-\sigma)} \end{aligned} \tag{382}$$

Defining $p_R^I = \frac{1}{2}p^I + L^I$ and $p_L^I = \frac{1}{2}p^I - L^I$ we rewrite the mass formulae as

$$\frac{1}{4}m_L^2 = \frac{1}{2}p_L^2 + N_L - 1 \qquad\qquad \frac{1}{4}m_R^2 = \frac{1}{2}p_R^2 + N_R - 1 \ . \tag{383}$$

Then the total mass formula and the level matching condition become

$$\begin{aligned} \frac{1}{4}m^2 = \frac{1}{4}m_R^2 + \frac{1}{4}m_L^2 &= N_R + N_L - 2 + \sum_{I=1}^{D}(\frac{1}{4}p^I p^I + L^I L^I) \\ N_L - N_R &= \sum_{I=1}^{D} p^I L^I \end{aligned} \tag{384}$$

It is worth noticing that the spectrum of the compactified bosonic string contains in the untwisted sector all the states of the uncompactified theory. This means that, in general, tachyons cannot be removed by naïve toroidal compactifications (they can be removed if the compactification is made in the presence of Wilson lines [114]). With respect to the twisted sector, it constains $2D$ massless vectors $\alpha^{\mu}\bar{\alpha}^{I}|0\rangle$ and $\alpha^{I}\bar{\alpha}^{\mu}|0\rangle$ as well as D^2 massless scalars $\alpha^{I}\bar{\alpha}^{J}|0\rangle$. The massless vector bosons are associated with the isometry group of the torus $U(1)^D$ and play the role of gauge bosons.

Let us consider the simpler case in which we have only one compactified dimension with radius R. This means that we have the identification

$$X^{25} \sim X^{25} + 2\pi RL \tag{385}$$

with L the winding number. The mass formula is

$$\frac{1}{4}m^2 = \frac{M^2}{4R^2} + L^2R^2 + N_L + N_R - 2 \tag{386}$$

with $N_L - N_R = ML$, $M, L \in \mathbf{Z}$. We see one important property of the spectrum of the bosonic string with one compactified dimension: the mass formula is invariant under the replacement [71]

$$M \to L \qquad\qquad L \to -M \qquad\qquad R \to \frac{\alpha^{'}\hbar}{R} \tag{387}$$

where we have restored the constants $\alpha^{'}$ and $\hbar$. However, the invariance of the mass formula under (387) does not ensure that the whole (interacting) theory is invariant under such transformations. We have to check that the amplitudes are also invariant.

It was shown in [115] using the low energy field theory that if the theory was invariant under (387) then the string coupling constant (i.e., the vacuum expectation value of the dilaton field) has to transform according to

$$g_{st} \longrightarrow g_{st}\frac{\sqrt{\alpha^{'}\hbar}}{R} \tag{388}$$

where Φ is the vacuum expectation value of the dilaton. This transformation for Φ can also be obtained in a σ-model approach by demanding the transformed σ-model to preserve conformal invariance at zero order in $\alpha^{'}$ [116]. In [117] it was finally proved that the duality transformation

$$R \to \frac{\alpha^{'}\hbar}{R} \qquad\qquad g_{st} \to g_{st}\frac{\sqrt{\alpha^{'}\hbar}}{R} \tag{389}$$

is a symmetry of the whole string perturbation theory either for the bosonic and the heterotic string. This symmetry can also be shown to hold in the $c = 1$ non-critical string for the whole perturbative expansion when the matrix model is truncated to the sector without vortices [118].

The invariance under (389) is called R-duality and means that the theory at radius R is equivalent to the one at radius $\alpha^{'}\hbar/R$ with an appropiate transformation of the couplings (for a general review see [119]). Duality is a typical "stringy" property without any counterpart in field theory, since it depends crucially on the extended nature of the string.

R-duality is not an exclusive property of the bosonic string. It appears also in theories with fermions like the heterotic and closed superstrings. In recent years this property has

Figure 10: Basis of the first homology group $H_1(\Sigma_g)$

been considered as lending support to the idea that string theory has a fundamental minimal length. This and the study of the high energy string scattering has led to the formulation of a generalized uncertainity principle [112] which reads

$$\Delta x \sim \frac{\hbar}{\Delta E} + \alpha^{'} \Delta E \tag{390}$$

It is easy to see that there is a lower bound for the right hand side of (390) corresponding to $\sqrt{\alpha' \hbar}$. This would impose a lower bound to any measurable distance that would be of the order of the self-dual radius since, by duality, we are unable to distinguish between spaces of size R and $(\alpha^{'} \hbar)/R$. In some sense this means that strings cannot probe distances shorter than its proper length (which is of the order of the Planck length).

3.7 Operator formalism

Now we are going to study the operator formalism for the bosonic string in a genus g Riemann surface [13]. This formalism provides a general framework where we can analyze many of the properties of String Theory to arbitrary orders of perturbation theory. In particular we can easily understand the origin of potential infinities in string amplitudes, and also the geometrical interpretation of the physical state conditions. Before doing so, it would be useful to review some features of Riemann surfaces that we will use later [113]. It is well known that every orientable two-dimensional surface is topologically characterized by its genus g (or equivalently by the Euler characteristic $\chi = 2 - 2g$). The homology groups of a genus g surface have dimensions

$$dim\, H_0(\Sigma_g) = 1 \qquad dim\, H_1(\Sigma_g) = 2g \qquad dim\, H_2(\Sigma_g) = 1\ . \tag{391}$$

We choose a basis for the first homology group $H_1(\Sigma_g)$ by considering the $2g$ cycles a_i, b_i $(1 \leq i \leq g)$ as shown in fig. 10. Any closed curve on the surface Σ can be uniquely decomposed in terms of curves belonging to the homology classes of (a_i, b_i). Moreover, once we have chosen a homology basis we can represent the genus g surface by a $4g$-sided polygon with proper identifications on its boundary. In fig. 11 we can see how this works for a genus 2 surface. We have an octagon, and by gluing $a_i a_i^{-1}$ and $b_i b_i^{-1}$ we obtain the original surface. With the basis $\{a_i, b_i\}$ we can associate a set of 1-forms $\{\alpha_i, \beta_i\}$ $(1 \leq i \leq g)$ satisfying

$$\begin{aligned} \int_{a_i} \alpha_j &= \delta_{ij}\ , \qquad & \int_{a_i} \beta_j = 0 \\ \int_{b_i} \alpha_j &= 0\ , \qquad & \int_{b_i} \beta_j = \delta_{ij}\ . \end{aligned} \tag{392}$$

Figure 11: Octagon obtained by cutting a genus 2 Riemann surface along the elements of the basis of $H_1(\Sigma_2)$

When we introduce a complex structure in the two-dimensional surface we have a Riemann surface, and we can decompose the cotangent space $T^*\Sigma$ into two sets

$$T^*\Sigma = T^{*(1,0)}\Sigma \oplus T^{*(0,1)}\Sigma \tag{393}$$

depending of whether the 1-forms are locally of the form $f(z,\bar{z})dz$ or $f(z,\bar{z})d\bar{z}$. In particular there exists a set of g holomorphic $(1,0)$ differentials ω_i (locally of the form $\omega_i = f_i(z)dz$) and which can be normalized according to

$$\int_{a_i} \omega_j = \delta_{ij} \ . \tag{394}$$

This normalization determines completely the differentials ω_i (called Abelian differentials of the first kind). The integration of ω_i along the cycles b_j define the period matrix of the Riemann surface

$$\int_{b_i} \omega_j = \Omega_{ij} \ . \tag{395}$$

It can be shown that the period matrix verifies $\Omega_{ij} = \Omega_{ji}$ and $Im\,\Omega_{ij} > 0$. The set of matrices with these properties define the Siegel upper half plane $\mathbf{H}_g$. In fact, $\mathbf{H}_g$ contains the Teichmüller space for a genus g Riemann surface, i.e., the space

$$\mathcal{T}_g = \frac{\text{Metrics}}{\text{Weyl} \times \text{Diff}_0^+(\Sigma)} \tag{396}$$

with $\text{Diff}_0^+(\Sigma)$ the set of orientation preserving diffeomorphisms on Σ connected with the identity. The moduli space $\mathcal{M}_g$ [5] introduced above is related to the Teichmüller space by

$$\mathcal{M}_g = \frac{\mathcal{T}_g}{\Omega(\Sigma)} \tag{397}$$

with the mapping class group defined by

$$\Omega(\Sigma) = \frac{\text{Diff}^+(\Sigma)}{\text{Diff}_0^+(\Sigma)} \tag{398}$$

Another important subject in the theory of Riemann surfaces is the definition of spin structures. In a genus g Riemann surface there are 2^{2g} spin structures corresponding, roughly

[5] The space of inequivalent complex structures on the surface Σ_g.

Figure 12: Riemann surface containing a long narrow tube. Divergences in the bosonic string amplitudes appear when summing the contributions from this type of surfaces in the limit in which the length of the tube tends to infinity

Figure 13: Riemann surface with a cycle pinched off

speaking, to the assignment of periodic or antiperiodic boundary conditions for the fermions along each generator of the homology group $H_1(\Sigma)$. These spin structures are classified into two groups; a spin structure is even if the number of zero modes of the chiral Dirac operator is even and it is odd otherwise. In the case of the torus, for example, we have four spin structures corresponding to (P,P), (P,A), (A,P) and (A,A) with P (A) indicating periodic (antiperiodic) boundary conditions along the homology basis (a,b). With the flat metric on the torus we can see that (P,P) is the only odd spin structure. The other three are even.

After this brief review about some topics on Riemann surfaces, we analize the origin of the divergences appearing in the partition function of the bosonic string [120]. The analysis in the case of fermionic strings is similar. To do so we have to convince ourselves that, in the sum over world-sheets, the divergences come from those surfaces lying on the boundary of moduli space $\mathcal{M}_g$. The presence of a tachyon in the spectrum of the bosonic string makes the contribution coming from surfaces with long narrow tubes divergent (fig. 12) This can be easily seen from the partition function on the cylinder constructed in sec. 3.2. The tachyon in the spectrum makes the integration over the length of the cylinder divergent since the integrand grows exponentially with the length.

In order to study the divergences of String Theory it is convenient to add to the moduli space $\mathcal{M}_g$ all the points on its boundary to construct the compactified moduli space $\bar{\mathcal{M}}_g$. Points in $\bar{\mathcal{M}}_g - \mathcal{M}_g$ represent Riemann surfaces with infinitely long narrow tubes. Equivalently we can consider the conformally related surfaces in which we have nodes, i.e., in which some curve is pinched off (fig 13). It is easy to see that a neigbourhood of that nodes is not topologically equivalent to a piece of $\mathbf{C}$ but to two disks joined by their centers (fig. 14). We will say that one of this surfaces in the boundary of the moduli space is of type Δ_i if the degenerate surface splits a genus g surface into two joined surfaces of genus i and $g-i$ (fig. 15). By Δ_0 we denote those surfaces in which a homologically non-trivial cycle is pinched off and we obtain a genus

Figure 14: Local aspect of a node

Figure 15: Splitting of a genus g Riemann surface into two Riemann surface by pinching off a homologically trivial cycle

$g-1$ surface with two points identified. This surfaces can be obtained by sewing Riemann surfaces with sewing parameter $q=0$.

We now present the construction of the operator formalism for higher genus Riemann surfaces. In sec. 3.3 we studied the operator formalism in the cylinder by passing to the punctured complex plane where any correlation function can be represented as

$$\langle 0|\Phi_1(z_1)\ldots\Phi_N(z_N)|0\rangle\ . \tag{399}$$

For higher genus surfaces we would like to construct a state $|W\rangle$ characterizing the details about the topology and complex structure of the surface, and such that

$$\frac{\langle 0|\Phi_1(z_1)\ldots\Phi_N(z_N)|W\rangle}{\langle 0|W\rangle} \tag{400}$$

coincides with the path integral computation of the corresponding correlator. To achieve this goal we begin by defining the augmented moduli space $\mathcal{P}(g,n)$ which is the moduli space of genus g surfaces with n parametrized boundaries, i.e., with n distincted points P_i and n local parameters $z_i(P_i)$. This set has the property that the mapping class group of a genus g Riemann surface with n distinguished points $\Sigma_{g,n}$

$$\Omega(\Sigma_{g,n})=\mathrm{Diff}(\Sigma_{g,n})/\mathrm{Diff}_0(\Sigma_{g,n}) \tag{401}$$

equals the fundamental group $\pi_1(\mathcal{P}(g,n))$. Moreover, this set is by far more convenient than $\mathcal{M}_{g,n}$ because in order to define oscillators, creation and anihilation operators, etc., in the neighborhood of P_i we need local parameters z_i to define the Laurent expansion of the various fields. Besides, $\mathcal{P}(g,n)$ resolves the orbifold singularities of $\mathcal{M}_{g,n}$.

The construction of the operator formalism for a general Riemann surface amounts to the construction of a map between the space $\mathcal{P}(g,n)$ and $\mathcal{H}\otimes\ldots\otimes\mathcal{H}$ where $\mathcal{H}$ is the Hilbert space

Figure 16: Construction of a genus $g_1 + g_2$ Riemann surface by sewing two Riemann surfaces with genera g_1 and g_2

Figure 17: Buildings blocks for the construction of any element of $\mathcal{P}(g, n)$

of the theory. This means that to every CFT we associate a ray in $\mathcal{H}^{\otimes n}$ for every point of $\mathcal{P}(g, n)$. By making use of the sewing technique we can define a kind of semigroup structure in $\mathcal{P}(g, n)$ by defining a "composition"

$$\begin{aligned} \mathcal{P}(g_1, n_1) \times \mathcal{P}(g_2, n_2) &\rightarrow \mathcal{P}(g_1 + g_2, n_1 + n_2 - 2) \\ P \in \mathcal{P}(g_1, n_1), Q \in \mathcal{P}(g_2, n_2) &\rightarrow P_i \infty_j Q \in \mathcal{P}(g_1 + g_2, n_1 + n_2 - 2) \end{aligned} \tag{402}$$

which correspond to sewing together the ith puncture of the first surface with the jth puncture of the second one (fig. 16). We can also define the operation

$$\begin{aligned} \mathcal{P}(g, n) &\longrightarrow \mathcal{P}(g + 1, n - 2) \\ P \in \mathcal{P}(g, n) &\longrightarrow P8^i_j \in \mathcal{P}(g + 1, n - 2) \end{aligned} \tag{403}$$

in which we sew the ith and jth punctures of P (see fig. 8). This two operations allows us to construct $\mathcal{P}(g, n)$ starting with $\mathcal{P}(0, 3)$ and $\mathcal{P}(0, 2)$ (fig. 17). These operations can be represented in the space of states as follows: let $\{|n\rangle\}$ be an orthonormal basis in $\mathcal{H}$. We will associate with the Riemann surface in fig 17-b the state

$$|S_{ij}\rangle = \sum_n |n\rangle_i |n\rangle_j \tag{404}$$

If we have two surfaces $P \in \mathcal{P}(g_1, n_1)$, $Q \in \mathcal{P}(g_2, n_2)$ with associated states $|P\rangle \in \mathcal{H}^{\otimes n_1}$, $|Q\rangle \in \mathcal{H}^{\otimes n_2}$, the state associated with $P_i \infty_j Q$ is given by

$$|R\rangle = \langle S_{ij}|P\rangle \otimes |Q\rangle \ . \tag{405}$$

We see that starting with $\mathcal{P}(0, 3)$ and using $|S_{ij}\rangle$ we arrive in principle to any surface, so our procedure is quite similar to a Feynman diagram technique [6].

[6] This procedure has been made explicite in the construction of a closed String Field Theory by B. Zwiebach (see lecture by B. Zwiebach and [6]).

Figure 18: Deformation of the Riemann surface by cutting out a disk and gluing it back after deforming it

Once we have a correspondence between elements in $\mathcal{P}(g,n)$ and the states in $\mathcal{H}^{\otimes n}$ we can reformulate Polyakov's prescription in an operator language. For example if we want to evaluate the amplitude for n external strings in the on-shell states $|\chi_i\rangle \in \mathcal{H}$ we have that the contribution coming from a Riemann surface $W \in \mathcal{P}_{g,n}$ to the amplitude will be given by

$$\langle\chi_1| \dots \langle\chi_n|W\rangle \ , \tag{406}$$

so the total amplitude $A(1,\dots,n)$ can be schematically written as

$$A(1,\dots,n) = \sum_{g=0}^{\infty} g_{st}^{-\chi} \sum_{W} \langle\chi_1| \dots \langle\chi_n|W\rangle \ , \tag{407}$$

where the sum over W is given by an integral over the moduli parameters.

It is worth noticing that only for CFT's we obtain a ray associated with a point of $\mathcal{P}(g,n)$. Furthermore, for $g > 1$ the first homology group $H_1[\mathcal{P}(g,n)]$ vanishes (Harer theorem [121]); this is a very useful result in order to characterize the Polyakov measure since it implies that any flat line bundle is necessarily trivial. From (407) it is clear now that the aim of the operator formalism is to obtain an operator representation of "scattering" measures on $\mathcal{M}_{g,n}$.

The problem that arises is to determine the action of the Virasoro generators on the space of states. Let be $R \in \mathcal{P}(g,n)$ and let P_i be a point in R with local parameter z_i (see fig. 18). We now cut an annulus off the disk around the point P_i and transform it by the action of a meromorphic vector field $v(z_i)$

$$z_i \to z_i + \epsilon v(z_i) \ , \tag{408}$$

($v(z_i)$ may have poles at P_i). When this is done, we fill the inside of the annulus to get a disk and glue it back on the surface. This is a deformation of the original Riemann surface and we may wonder what is the relation of the new surface with the old one. There are three posibilities

- The vector field $v(z_i)$ is holomorphic on the disk and vanishes at P_i. In this case the transformation (408) is equivalent to choosing different coordinates at P_i. In the case in which the vector field is holomorphic on the disk but does not vanish at P_i we have an infinitesimal translation of the point P_i.

- The vector field can be extended to a holomorphic vector field on the rest of the surface (i.e., $\Sigma - P_i$). In this case the transformed surface is identical to the original one, since any transformation (408) can be undone by a transformation generated by v in the rest of the surface.

- We have none of the preceeding posibilities. $v(z)$ is holomorphic in the annulus but does not extend holomorphically either to the disk or to $\Sigma - P_i$. Then (408) is an infinitesimal moduli deformation, i.e., an infinitesimal motion in the space $\mathcal{M}_{g,n}$.

The next problem is to represent moduli deformations, or any of the changes induced by $v(z)$ in $\mathcal{H}^{\otimes n}$. This is acomplished by an operator $O(v)$ acting on the states $|W\rangle$. Because of the fundamental rôle played by the Virasoro algebra in any CFT we will consider the energy-momentum tensor in a neighborhood of the point P_i

$$T(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2} \tag{409}$$

and construct the operator $T(v)$ associated with the deformation induced by $v(z)$ as

$$T(v) = \frac{1}{2\pi i}\oint_{P_i} dz\, T(z)v(z) \tag{410}$$

The change in $|W\rangle$ by the deformation generated by $v(z)$ is given by

$$\delta_v|W\rangle = [T(v) + \bar{T}(\bar{v})]|W\rangle \tag{411}$$

This should be taken as the defining property of the action of the Virasoro algebra on the states of the theory. Since for matter fields the Virasoro algebra has a central extension, equation (411) makes sense only in terms of rays.

A final consistency condition in the construction of $|W\rangle$ for $P \in \mathcal{P}(g,n)$ requires that $L_0 - \bar{L}_0 \in \mathbf{Z}$. This condition has to be imposed since we can always make a Dehn twist around the point P, which in terms of the local holomorphic coordinate consists of a trasformation $z \to \exp{(2\pi i\theta)}z$ with $0 \le \theta \le 1$. This is done by acting with the operator $\exp{[2\pi i\theta(L_0 - \bar{L}_0)]}$ on the state $|W\rangle$. In the case $\theta = 1$ for the state to be invariant we must have $L_0 - \bar{L}_0 \in \mathbf{Z}$, condition which is necessary for modular invariance.

We now leave the general set-up and present a few useful examples which should clarify the machinery explained so far. The first example we will study of the operator formalism is a pair of spin $\frac{1}{2}$ holomorphic spinors

$$\begin{aligned} b(z)dz^{\frac{1}{2}} &= \sum_{n\in\mathbf{Z}+\frac{1}{2}} b_n z^{n-\frac{1}{2}} dz^{\frac{1}{2}} \,, \\ c(z)dz^{\frac{1}{2}} &= \sum_{n\in\mathbf{Z}+\frac{1}{2}} c_n z^{n-\frac{1}{2}} dz^{\frac{1}{2}} \,, \end{aligned} \tag{412}$$

with the anticommutation relations

$$\{b_m, c_n\} = \delta_{m+n,0} \,, \qquad \{b_m, b_n\} = \{c_m, c_n\} = 0 \,, \tag{413}$$

which means that $b(z)$ can be represented as the "translation" operator for $c(z)$

$$b(z) \sim \frac{\delta}{\delta c(z)} \,, \tag{414}$$

Figure 19: Riemann surface divided into two parts by the circle S^1

The action for these fields is

$$S = \frac{1}{\pi} \int d^2 z \, b \bar{\partial} c \; . \tag{415}$$

We will try to associate a state Ψ with each side of the surface in fig. 19, Σ_1 and Σ_2, in such a way that on the Fock space of the circle S^1 the partition function becomes

$$Z = \int \mathcal{D}b \mathcal{D}c \exp\left(- \int d^2 z \, b \bar{\partial} c \right) = \langle \Psi_2 | \Psi_1 \rangle \; . \tag{416}$$

We represent the state Ψ as a functional given by

$$\Psi_1[f] = \int_{c|_{S^1} = f} \mathcal{D}b \mathcal{D}c \exp\left(\int_{\Sigma_1} d^2 z \, c \bar{\partial} b + \oint_{S^1} dz \, cb \right) , \tag{417}$$

where the boundary term in the exponential accounts for the flux of the fermionic current through S^1 and we impose no boundary condition on $b(z)$[7]. If w_n is the boundary value at S^1 of a holomorphic spinor on Σ_1 and we consider $\Psi_1[f + w_n]$, after making a shift in the c-field $c \to c - w_n$ it is easy to verify that

$$\Psi_1[f + w_n] = \Psi_1[f] \tag{418}$$

Since, as we indicated above, $b(z)$ plays the role of the translation operator for $c(z)$, we can define the conserved charge

$$Q(w_n) = \frac{1}{2\pi i} \oint_{S^1} dz b(z) w_n(z) \; . \tag{419}$$

and the invariance (418) becomes

$$Q(w_n) \Psi_1 = 0 \tag{420}$$

Furthermore, $Q(w_n)$ depends only on the homology class of the contour.

Notice also that the path integral is invariant under the translation of $b(z)$ by a spinor $\tilde{w}_n$ holomorphic off P (see fig. 20). This invariance can be implemented by defining the operator

$$\tilde{Q}_m = Q(\tilde{w}_n) = \frac{1}{2\pi i} \oint_{S^1} dz \, c(z) \tilde{w}_n \; . \tag{421}$$

[7]For spinor b and c are canonically conjugate variables, and (417) is a hamiltonian functional integral representation of the wave function.

Figure 20: Riemann surface with a single puncture at P

thus $\Psi_1[f]$ satisfies

$$\tilde{Q}_n \Psi_1 = 0 \ . \tag{422}$$

The two charges $Q_n = Q(w_n)$ and $\tilde{Q}_m = Q(\tilde{w}_m)$ verifies the anticommutation relation

$$\{Q_n, \tilde{Q}_m\} = \frac{1}{2\pi i} \oint_{S^1} dz\, w_n(z) \tilde{w}_m(z) \tag{423}$$

which vanishes by deforming the contour of integration, and by the fact that w_n, $\tilde{w}_n$ are holomorphic off P

$$\{Q_n, \tilde{Q}_m\} = 0 \ . \tag{424}$$

Q_n and $\tilde{Q}_m$ give a maximal set of conditions on the state Ψ_1. To associate a state with a given Riemann surface $W \in \mathcal{P}(g, n)$ we look for all the meromorphic sections w_n, $\tilde{w}_m$ of the spinor bundle which are allowed to have poles only at the punctures, we then construct the conserved charges $Q(w_n)$, $Q(\tilde{w}_n)$ and define the state by (420,422). As a first example let us consider the case of a sphere with a single puncture at $z = 0$. In this case the meromorphic sections with poles only at $z = 0$ are of the form

$$w_n(z) = z^{-n} dz^{\frac{1}{2}} \ , \qquad n > 0 \ . \tag{425}$$

Then, taking into account the mode expansion for $b(z)$ and $c(z)$ we find that the operators Q_n, $\tilde{Q}_m$ are

$$\begin{aligned} Q_n &= \int_P \frac{dz}{2\pi i} b(z) z^{-n} = b_{n-\frac{1}{2}} \ , \\ \tilde{Q}_m &= \int_P \frac{dz}{2\pi i} c(z) z^{-n} = c_{n-\frac{1}{2}} \ , \end{aligned} \tag{426}$$

with $n > 0$. Then the state $|W\rangle$ associated with the sphere with one puncture is defined as

$$b_{n-\frac{1}{2}} |W\rangle = c_{n-\frac{1}{2}} |W\rangle = 0 \ , \qquad n > 0 \ , \tag{427}$$

so $|W\rangle$ is the $SL(2, \mathbf{C})$-invariant vacuum state.

We can explicitly construct the state associated with a genus-g Riemann surface with a single puncture at P. To do so we first have to find meromorphic sections that have poles only at P. The Riemann-Roch theorem implies that these sections exist with poles of arbitrary order at the point P. The starting point is the Szegö kernel for spin $\frac{1}{2}$. Let us consider a odd spin structure over Σ labeled by (α, β); then the Szegö kernel, which is the two-point function for fermions $\langle c(z)b(w)\rangle$ on Σ, is given by (for a constructive derivation see [122])

$$S(z,w) = \frac{\vartheta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (z-w|\Omega)}{\vartheta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (0|\Omega) E(z,w)} \tag{428}$$

where Ω is the period matrix and $E(z,w)$ is the prime form associated to Σ. The prime form is defined as the unique $(-\frac{1}{2}, -\frac{1}{2})$ differential such that $E(z,w) = -E(w,z)$ and vanishes only when $z = w$. In order to construct it we begin with the function defined for an odd non-singular spin structure (with just one zero mode)

$$f(z,w) = \vartheta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (\int_w^z \omega|\Omega) \tag{429}$$

By using the Riemann Vanishing Theorem [113] we find that this function vanishes in $g-1$ points $P_1, \ldots, P_{g-1}$ as a function of z with w fixed. Moreover, since it also vanish when $z \to w$, this function, when considered as a function of w with z fixed also vanish at the same points. It can be seen that when $z \sim w \sim P_i$, $f(z,w) \sim (z-w)(z-P_i)(w-P_i)$ so after differentiation with respect to z we can define the holomorphic form

$$h^2(z) = \sum_{j=1}^{g} \omega_i(z) \frac{\partial}{\partial u_i} \vartheta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (\int_z^w \omega|\Omega) \bigg|_{z=w} , \tag{430}$$

where the u_i stand for the arguments of the theta function. $h^2(z)$ has second order zeros at the points P_i, and therefore we can take the square root without introducing cuts. The prime form is defined as

$$E(z,w) = \frac{\vartheta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (z-w|\Omega)}{h(z)h(w)} . \tag{431}$$

The sections we are looking for are given by differentiation of the Szegö kernel with respect to z and setting $w = 0$. For z, w in a neigborhood of $P \in \Sigma$ we have

$$w_n(z) = \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial w^{n-1}} S(z,w) \bigg|_{w=0} = \frac{1}{z^n} + \sum_{m=1}^{\infty} B_{nm} z^{m-1} , \tag{432}$$

where B_{nm} is given by

$$B_{nm} = \frac{1}{(n-1)!(m-1)!} \frac{\partial^{m-1}}{\partial z^{m-1}} \frac{\partial^{n-1}}{\partial w^{n-1}} \left(S(z,w) - \frac{1}{z-w} \right) . \tag{433}$$

Looking at the anticommutation relations it is easy to see that we can find a representation of the operators c_n, b_n ($n \in \mathbf{Z} + \frac{1}{2}$) in terms of differential operators in which b_{-n} and c_{-n} for $n > 0$ acts as multiplication operators and

$$b_n = \frac{\partial}{\partial c_{-n}} , \qquad c_n = \frac{\partial}{\partial b_{-n}} , \qquad n > 0 . \tag{434}$$

Then, the charges Q_n, $\tilde{Q}_m$ become:

$$
\begin{aligned}
Q_n &= b_{n-\frac{1}{2}} + \sum_{m=1}^{\infty} B_{nm} b_{-m+\frac{1}{2}} \\
\tilde{Q}_n &= c_{n-\frac{1}{2}} + \sum_{m=1}^{\infty} B_{nm} c_{-m+\frac{1}{2}}
\end{aligned}
\tag{435}
$$

and we obtain a system of differential equations for $\Psi(c_{-n}, b_{-n})$. The resulting state is

$$
|\Psi\rangle = C \exp\left(- \sum_{n,m=1}^{\infty} B_{nm} c_{-n+\frac{1}{2}} b_{-m+\frac{1}{2}} \right) |0\rangle \ . \tag{436}
$$

This is just a Bogoliubov transformation on the standard vacuum state $|0\rangle$ defined by (427). The charge operators Q_n, $\tilde{Q}_m$, which are linear combinations of both creation and annihilation operators with respect to $|0\rangle$, are the annihilation operators with respect to the new vacuum state $|\Psi\rangle$. All the geometric properties are neatly summarized by the Bogoliubov transformation.

In the case in which $\Sigma \in \mathcal{P}(g,n)$ we have to find meromorphic sections of the spinor bundle with poles at $P_1, \ldots, P_n$. The simplest example is the sphere with two punctures that we can arbitrarily locate at 0 and ∞. With this surface we have associated the "sewing" state $|S_{12}\rangle$. The meromorphic sections on the sphere with poles at $z=0$ and $z=\infty$ are generated by

$$
w_n = z^n dz^{\frac{1}{2}} \ , \qquad n \in \mathbf{Z} \ . \tag{437}
$$

It is easy to get the conserved charges. Integrating in patches around the two points we have

$$
\begin{aligned}
Q_n &= \oint_{P_1} \frac{dz_1}{2\pi i} b^{(1)}(z_1) w_n(z_1) + \oint_{P_2} \frac{dz_2}{2\pi i} b^{(2)}(z_2) w_n(z_2) \ , \\
\tilde{Q}_n &= \oint_{P_1} \frac{dz_1}{2\pi i} c^{(1)}(z_1) w_n(z_1) + \oint_{P_2} \frac{dz_2}{2\pi i} c^{(2)}(z_2) w_n(z_2) \ ,
\end{aligned}
\tag{438}
$$

and solving the corresponding set of differential equations, we obtain

$$
|S_{12}\rangle = \prod_{m=1}^{\infty} \exp\left(c^{(1)}_{-m+\frac{1}{2}} b^{(2)}_{-m+\frac{1}{2}} + c^{(2)}_{-m+\frac{1}{2}} b^{(1)}_{-m+\frac{1}{2}} \right) |0\rangle_1 \otimes |0\rangle_2 \ , \tag{439}
$$

which satisfies

$$
(Q_1 + Q_2)|S_{12}\rangle = (\tilde{Q}_1 + \tilde{Q}_2)|S_{12}\rangle = 0 \ . \tag{440}
$$

Previously we saw that by sewing two Riemann surfaces $P \in \mathcal{P}(g_1, n_1)$, $Q \in \mathcal{P}(g_2, n_2)$ with sewing parameter $q=1$ we obtained a new Riemann surface $R = P_i \infty_j Q \in \mathcal{P}(g_1{+}g_2, n_1{+}n_2{-}2)$. As an example of the use of the operator formalism we are going to verify that the state associated with R

$$
|R\rangle = \langle S_{ij} | P \rangle \otimes |Q\rangle \tag{441}
$$

verifies the condition

$$
(Q_1 + \ldots + Q_{i-1} + Q_{j+1} + \ldots + Q_{n_1+n_2})|R\rangle = 0 \tag{442}
$$

Figure 21: Remnants of the two disks located around P_i and P_j with local parameters t_i, t_j after sewing them together

and correspondingly for the $\tilde{Q}_j$ operators. Since (fig. 21) in R we will have some remnants of the two discs around the ith and jth punctures with coordinates t_i and t_j respectively, if we expand holomorphic spinors in R away from the points $P_1, \ldots, P_{i-1}, P_{j+1}, \ldots, P_{n_1+n_2}$ we could express them in terms of the coordinates t_i, t_j. These spinors will have at most poles at $t_i = t_j = 0$, therefore

$$\langle S_{ij}|(Q_i + Q_j) = 0 \tag{443}$$

and we can write

$$\begin{aligned} & (Q_1 + \ldots + Q_{i-1} + Q_{j+1} + \ldots + Q_{n_1+n_2})\langle S_{ij}|P\rangle \otimes |Q\rangle \\ = \; & \langle S_{ij}|(Q_1 + \ldots + Q_i) + (Q_j + Q_{j+1} + \ldots + Q_{n_1+n_2})|P\rangle \otimes |Q\rangle = 0 \end{aligned} \tag{444}$$

as was to be shown.

Another case of interest is a single-valued scalar field ϕ. We proceed along the lines of the spin-$\frac{1}{2}$ case and construct the functional associated with the Riemann surface Σ_1 with boundary S^1 (fig. 22) as the path integral

$$\Psi[f] = \int_{\phi|_{S^1}=f} \mathcal{D}\phi \exp\left(-\frac{1}{2}\int_{\Sigma_1} d\phi \wedge *d\phi\right) \tag{445}$$

with f a real function defined on S^1. It is possible to show that there exists a harmonic function defined on Σ_1 and which coincides with f along the boundary S^1. We also denote this function by f. Then we can make a shift in the scalar field ϕ by f

$$\phi = \phi^{'} + f \tag{446}$$

where the new field $\phi^{'}$ satisfies the boundary condition $\phi^{'}|_{S^1} = 0$. Substituying (446) into the definition of $\Psi[f]$ yields

$$\Psi[f] = \int_{\phi|_{S^1}=f} \mathcal{D}\phi \, e^{-S[\phi]} = \int_{\phi^{'}|_{S^1}=0} \mathcal{D}\phi^{'} \, e^{-S[\phi^{'}+f]} \tag{447}$$

We now evaluate $S[\phi^{'} + f]$

$$\begin{aligned} S[\phi^{'} + f] &= \frac{1}{2}\int_{\Sigma_1} d(\phi^{'} + f) \wedge *d(\phi^{'} + f) = S[\phi^{'}] + \frac{1}{2}\int_{\Sigma_1} df \wedge *df \\ &+ \int_{\Sigma_1} d\phi^{'} \wedge *df = S[\phi^{'}] + \frac{1}{2}\oint_{S^1} f * df \;, \end{aligned} \tag{448}$$

Figure 22: Riemann surface with boundary S^1

where we have used the Stokes theorem, the boundary condition on ϕ' and the fact that f is a harmonic fuction $d * df = 0$ on Σ_1. Finally the state functional becomes:

$$\Psi[f] = (det' \, \Delta)^{-\frac{1}{2}}_{\phi'|_{S^1}=0} \exp\left(-\frac{1}{2}\oint_{S^1} f * df\right) . \tag{449}$$

The conserved charges in this case can be written using the Green's theorem

$$Q(h_n) = \oint_{S^1} (h_n * d\phi - \phi * dh_n) \tag{450}$$

where h_n is a harmonic function analytic in Σ_1. It is easy to see that since field equations for ϕ gives $\partial\bar{\partial}\phi = 0$ the operator $Q(h_n)$ does not change under deformations of S^1 and only depends on the homology class of the boundary. As in the fermionic case, the state Ψ is constructed by requiring that it is annihilated by the conserved charges. By splitting the function h_n into holomorphic and antiholomorphic parts $h_n^H + h_n^A$, we find that the charge (450) can be derived from the conserved current

$$j(h_n) = \partial\phi h_n^H - \bar{\partial}\phi h_n^A . \tag{451}$$

The problem is to find an explicit expression for the functions h_n. For example consider a genus g Riemann surface with one puncture at P. The simplest solutions are the real and imaginary parts of a holomorphic function on $\Sigma - P$ with poles at P. Unfortunately the Weierstrass gap theorem states that the order of the pole of a holomorphic function at P must be bigger than g. The problem arises because we need g extra conditions in order to determine completely the state Ψ. This means that we cannot restrict to holomorphic objects and we have to consider functions with holomorphic and antiholomorphic parts. We start by considering a holomorphic function with poles at P of order less than g but which, in order to bypass Weierstrass' theorem, is allowed to be multivalued around the cycles. Then we add some antiholomorphic piece to obtain a single-valued harmonic function. We begin with the multivalued meromorphic differentials:

$$\eta_n(t) = \frac{1}{(n-1)!} \frac{\partial}{\partial t}\frac{\partial^n}{\partial y^n} \log E(t,y)\bigg|_{y=0} . \tag{452}$$

where $E(t, y)$ is the prime form defined earlier. With these differentials we define the functions

$$h_n(t) = \int^t [\eta_n(t) - A_n (Im\,\Omega)^{-1}(\omega - \bar{\omega})] , \tag{453}$$

where ω_i are the Abelian differentials and the coefficients A_{ni} are defined in terms of the local coordinate at P through the expansion

$$\omega_i(t) = \sum_{n=1}^{\infty} A_{ni} t^{n-1} dt \; . \tag{454}$$

The $\eta_n(t)$ differentials satisfy

$$\oint_{b_i} \eta_n(t) = A_{ni} \tag{455}$$

Using the properties of the prime form $E(t,y)$ [123] it is possible to check that the functions h_n defined by (453) are single-valued.

Now we expand the field $\phi(t)$ in oscillators using the familiar expression

$$\phi(t) = q + ip\log t + ip\log\bar{t} + \sum_{n\neq 0}\left(\frac{a_n}{n}t^n + \text{c.c.}\right) \; , \tag{456}$$

with the non-vanishing commutation relations

$$[q,p] = i \; , \qquad [a_n, a_m] = n\delta_{n+m,0} \; , \qquad [\bar{a}_n, \bar{a}_m] = n\delta_{n+m,0} \; . \tag{457}$$

By using these commutation relations and Green's theorem for harmonic functions it is possible to show that the charges $Q(h_n)$ commutes

$$[Q(h_n), Q(h_m)] = 0 \; . \tag{458}$$

We can construct a representation for operators $(a_n, \bar{a}_n)$ by introducing variables $(x_n, \bar{x}_n)$ and defining

$$a_n = \frac{\partial}{\partial x_n} \; , \qquad a_{-n} = n x_n \; , \qquad n \geq 0 \; , \tag{459}$$

and the corresponding expressions for barred quantities. Then we impose the conditions on the state $|\Psi\rangle$

$$Q(h_n)|\Psi\rangle = \bar{Q}(\bar{h}_n)|\Psi\rangle = 0 \; . \tag{460}$$

Substituying the representation of the operators $(a_n, \bar{a}_n)$ in terms of $(x_n, \bar{x}_n)$ we arrive at a set of differential equations on $\langle x, \bar{x}|\Psi\rangle$ that upon integration yield:

$$\langle x, \bar{x}|\Psi\rangle = \exp\left[(x,\bar{x}) M \begin{pmatrix} x \\ \bar{x} \end{pmatrix}\right] , \tag{461}$$

where the matrix M is given by

$$M = \begin{pmatrix} Q_{nm} + \frac{\pi}{2} A_n (Im\,\Omega)^{-1} A_m & -\frac{\pi}{2} A_n (Im\,\Omega)^{-1} \bar{A}_m \\ -\frac{\pi}{2} \bar{A}_n (Im\,\Omega)^{-1} A_m & \bar{Q}_{nm} + \frac{\pi}{2} \bar{A}_n (Im\,\Omega)^{-1} \bar{A}_m \end{pmatrix} , \tag{462}$$

and

$$Q_{nm} = \frac{1}{2(n-1)!(m-1)!} \frac{\partial^n}{\partial t^n} \frac{\partial^m}{\partial y^m} \log\left(\frac{E(t,y)}{t-y}\right) . \tag{463}$$

As a particular case we can compute the state associated with a sphere with two puctures (the sewing state $|S_{12}\rangle$) the result being

$$|S_{12}\rangle = \int dp \prod_{n=1}^{\infty} \exp\left(a_n^{(1)+}a_n^{(2)+} + \bar{a}_n^{(1)+}\bar{a}_n^{(2)+}\right)|p\rangle_1 \otimes |p\rangle_2 \ , \tag{464}$$

where the state $|p\rangle$ carries momentum p. For a genus g surface with n points $\Sigma_{g,n}$ we introduce the function

$$G(z,w) = -2\pi \left(Im \int_z^w \omega\right)(Im\,\Omega)^{-1}\left(Im \int_z^w \omega\right) - \log|E(z,w)|^2 \ , \tag{465}$$

and define

$$g(z_i,w_j) = \left\{ \begin{array}{ll} G(z_i,w_j) + \log|z_i - w_j|^2 & \text{if } z_i, w_j \text{ lie in the same patch} \\ G(z_i,w_j) & \text{otherwise .} \end{array} \right. \tag{466}$$

We can split $g(z_i,w_j)$ into four parts corresponding to the holomorphic or antiholomorphic contribution in each of the two variable. After constructing with these four pieces the matrix $M(z_i,w_j)$ we can write the unique ray associated with $\Sigma_{g,n}$ as

$$\begin{aligned} |\Psi_g^n\rangle = & \int dp_1 \ldots dp_n \delta(\sum_i p_i) \\ & \times \exp\left\{ -\frac{1}{(2\pi i)^2} \sum_{i,j} \oint_{P_i} \oint_{P_j} (\partial\phi(z_i), \bar{\partial}\phi(z_i)) M(z_i,z_j) \left(\begin{array}{c} \partial\phi(z_i) \\ \bar{\partial}\phi(z_j) \end{array} \right) \right\} \\ & \times |p_1\rangle \otimes \ldots \otimes |p_n\rangle \ . \end{aligned} \tag{467}$$

The last case we shall consider is the ghost system containing two anticommutating fields: $b(z)$, with spin 2, and $c(z)$ with spin -1. In a sphere with a single puncture at P with $z(P) = 0$ we can expand the fields as

$$b(z) = \sum_{n\in\mathbf{Z}} b_n z^{n-2} \ , \qquad c(z) = \sum_{n\in\mathbf{Z}} c_n z^{n+1} \tag{468}$$

with the anticommutation relations

$$\{b_n, c_m\} = \delta_{m+n,0} \tag{469}$$

We have to look for vector fields holomorphic off P and quadratic differentials with the same property. Using the Riemann-Roch theorem [124] on the sphere it is possible to show that the only vector fields with the desired properties are

$$\{z^2, z, 1, z^{-1}, \ldots\}\frac{\partial}{\partial z} \tag{470}$$

and for the quadratic differentials

$$\{z^{-4}, z^{-5}, z^{-6}, \ldots\}dz^2 \tag{471}$$

The conserved charges associated with these sets are, in the case of $b(z)$

$$\oint_P \frac{dz}{2\pi i} b(z) z^k = b_{-k+1} \, , \qquad k = 2, 1, 0, -1, \dots \, , \tag{472}$$

and for $c(z)$

$$\oint_P \frac{dz}{2\pi i} c(z) z^{-k} = c_{k-2} \, , \qquad k = 4, 5, \dots \, . \tag{473}$$

Thus the state $|\phi_0\rangle$ associate with the sphere with one puncture and the ghost system is defined by

$$\begin{aligned} b_n|\phi_0\rangle &= 0 \qquad n > -2 \\ c_n|\phi_0\rangle &= 0 \qquad n > 1 \end{aligned} \tag{474}$$

which is an $SL(2, \mathbf{C})$-invariant state, because it is annihilated by $L_0, L_{\pm 1}$, where L_n are the Virasoro generators associated with the ghost system. We define the ghost current by

$$j_{gh}(z) =: c(z) b(z) : \tag{475}$$

and the associated charge

$$Q_{gh} = \oint_P \frac{1}{2\pi i} j(z) \tag{476}$$

Our first task will be to determine the ghost charge of the vacuum $|\phi_0\rangle$. If we evaluate the commutator of L_0 with the ghost fields we get

$$\begin{aligned} [L_0, b_n] &= -n b_n \\ [L_0, c_n] &= -n c_n \end{aligned} \tag{477}$$

which shows that the state $|\phi_0\rangle$ is not a highest weight state of the oscillators algebra, since it is not annihilated by negative energy modes. Moreover, the zero modes b_0, c_0 form a Clifford algebra since $\{b_0, c_0\} = 1$. Then we can define the highest-weight states $\{|+\rangle, |-\rangle\}$ which provide a representation of the Clifford algebra:

$$\begin{aligned} c_n|+\rangle &= b_n|-\rangle = 0 \, , \qquad n \geq 0 \\ c_0|-\rangle &= |+\rangle \, , \qquad b_0|+\rangle = |-\rangle \, . \end{aligned} \tag{478}$$

It can be argued [2] that the natural assignment of ghost number to these states is $Q|\pm\rangle = \pm\frac{1}{2}|\pm\rangle$. We can construct these states from the $SL(2, \mathbf{C})$-invariant vacuum as

$$|-\rangle = c_1|\phi_0\rangle \qquad |+\rangle = c_0 c_1|\phi_0\rangle \tag{479}$$

and it is now straighforward to verify that $L_0|\pm\rangle = -|\pm\rangle$. This is the origin of the tachyon.

We next construct the sewing state $|S_{12}\rangle$. For a sphere with two punctures (one at 0 and the other one at ∞) the quadratic differentials and vector fields holomorphic off those points are respectively

$$\mu = z^n dz^2 \, , \qquad v = z^{n+1} \frac{d}{dz} \, , \qquad n \in \mathbf{Z} \, . \tag{480}$$

We proceed as in the previous cases. We impose the conditions

$$
\begin{aligned}
(b_n^{(1)} - b_{-n}^{(2)})|S_{12}\rangle &= 0 \\
(c_n^{(1)} - c_{-n}^{(2)})|S_{12}\rangle &= 0
\end{aligned}
\tag{481}
$$

and obtain the solution

$$
|S_{12}\rangle = \prod_{m=1}^{\infty} \exp\left(-c_{-m}^{(1)} b_{-m}^{(2)} - c_{-m}^{(2)} b_{-m}^{(1)}\right)(b_0^{(1)} - b_0^{(2)})|+\rangle_1 \otimes |-\rangle_2 \tag{482}
$$

If we evaluate the total ghost charge associated with the sewing state we obtain

$$
Q_{gh}|S_{12}\rangle = (Q_{gh}^{(1)} + Q_{gh}^{(2)})|S_{12}\rangle = 0 \tag{483}
$$

To generalize the construction to genus g Riemann surfaces with n punctures we have to take into account that for $g > 1$ the Riemann-Roch theorem states that there are $3g - 3$ holomorphic quadratic differentials. In the neighborhood of a point P we can write them using the local parameter z as

$$
\psi_{n+1} = z^n + \sum_{m \geq q} C_{nm}^{(2)} z^m \qquad q = 3g - 3 \tag{484}
$$

with $n = 0, \ldots, 3g - 4$. In a similar way we can write quadratic differentials with poles at P as

$$
s_n = z^{-n} + \sum_{m \geq q} B_{nm}^{(2)} z^m \tag{485}
$$

and the corresponding expressions for the vectors. The coefficients $B_{nm}^{(2)}$ could in principle be written in terms of prime forms and ϑ-functions. It is worth noticing that since

$$
\oint_{P_i} \frac{dz}{2\pi i} c(z)\psi_{n+1}(z) = c_{-n-2} + \sum_{m \geq q} C_{nm}^{(2)} c_{-m-2} \tag{486}
$$

and

$$
\oint_{P_i} \frac{dz}{2\pi i} c(z) s_1(z) = c_{-1} + \sum_{m \geq q} B_{1m}^{(2)} c_{-m-2} \tag{487}
$$

contains only creation operators, the only way in which the associated state $|\phi_g^n\rangle$ can be annihilated by the these charges is if they appear explicitly in the state. In the case of quadratic differentials with poles of higher order we also get annihilation operators. As we did before we use a representation for the operators c_n, b_n $(n > 0)$

$$
c_n = \frac{\partial}{\partial b_{-n}} \qquad b_n = \frac{\partial}{\partial c_{-n}} \qquad n > 0 \tag{488}
$$

and c_{-n}, b_{-n} act as multiplication operators. The conserved charges conditions lead to:

$$
\begin{aligned}
|\phi_g^n\rangle &= C_1 \ldots C_{3g-3} A_1^{(1)} \ldots A_1^{(n)} \\
&\times \exp\left(-\sum_{m \geq q, n \geq 2} B_{mn}^{(2)ij} c_{-m-2}^{(i)} b_{-n+2}^{(j)}\right)|+\rangle_1 \otimes \ldots \otimes |+\rangle_n \, ,
\end{aligned}
\tag{489}
$$

where

$$A_1^{(i)} = \sum_j \oint_{P_j} s_{-1}^{(i)}(z_j) c^{(j)}(z_j) \ , \tag{490}$$

$s_{-1}^{(i)}(z)$ being the quadratic differential with a single pole at P_i, and

$$C_{i+1} = \sum_j \oint_{P_j} \psi_i(z_j) c^{(j)}(z_j) \ . \tag{491}$$

The ghost charge associated with the state $|\phi_g^n\rangle$ can be evaluated if we take into account that $Q_{gh}(b) = -Q_{gh}(c) = 1$. We obtain

$$Q_{gh}|\phi_g^n\rangle = (3g - 3 + \frac{3n}{2})|\phi_g^n\rangle \tag{492}$$

We finally check that this ghost charge associated with a genus g Riemann surface and n punctures is consistent with the sewing rules. In fact, if we take two states $P \in \mathcal{P}(g_1, n_1)$ and $Q \in \mathcal{P}(g_2, n_2)$ taking into account that the sewing state does not carry ghost charge at all, the ghost charge associated with $P_i \infty_j Q$ will be equal to

$$(3g_1 - 3 + \frac{3n_1}{2}) + (3g_2 - 3 + \frac{3n_2}{2}) = 3(g_1 + g_2) - 3 + \frac{3(n_1 + n_2 - 2)}{2}, \tag{493}$$

consistent with the sewing prescription. It is also possible to apply the previous formalism to the commuting ghosts appearing in the quantization of superstrings (see [125] and references therein).

Now we apply the operator formalism to the bosonic string. In this case we have to consider 26 bosons together with the ghost system, in such a way that the total central charge (matter+ghost) is equal to zero. This condition guarantees that energy-momentum tensor acts without a central extension. If v_1, v_2 are two vector fields in the neighborhood of a point P we have

$$[T(v_1), T(v_2)] = T([v_1, v_2]) \tag{494}$$

where $[v_1, v_2]$ is the commutator of the two vector fields. In the free bosonic string we work with 26th tensor products of the one boson Hilbert space described above, times the space of the ghost system. In the combined matter+ghosts Hilbert space we are given a ray by the operator formalism for any point in $\mathcal{P}(g, n)$. Next we have to worry about the relative normalization of the ray for different points in $\mathcal{P}(g, n)$. Since infinitesimal motions in $\mathcal{P}(g, n)$ are generated by $T(v)$, and for the total system $c = 0$, there is no normalization ambiguity between the normalization at different points as long as we can make the comparision along contractible paths. We may encounter problems only along non-trivial elements of $H_1[\mathcal{P}(g, n)]$. However, Harer's theorem [121] guarrantees that this group vanishes for $g > 2$. In this case, the state $|\phi\rangle$ associated to the bosonic string is globally well defined and free of normalization ambiguities. For $g \leq 2$ one can reach the same conclusion using sewing.

Since the state $|\phi\rangle$ is globally defined, we want to obtain the string measure in moduli space $\mathcal{M}_g$ using the operator formalism. This measure is useful in computing the partition function for the bosonic string. Starting for simplicity with $\mathcal{P}(g,1)$, we can obtain a projection map

$$\begin{array}{cc} \mathcal{P}(g,1) & \\ \downarrow & \pi \\ \mathcal{M}_g & \end{array}$$

by simply forgetting the puncture and the local parameter. To construct a measure in $\mathcal{M}_g$ we have to associate with a basis of holomorphic tangent vectors at a point $P \in \mathcal{M}_g$ $\{V_1,\ldots,V_{3g-3}\}$ a volume element that we will denote as

$$\mu(P)(V_1,\ldots,V_{3g-3},\bar{V}_1,\ldots,\bar{V}_{3g-3}) \tag{495}$$

Given a point $P \in \mathcal{M}_g$ and a basis of tangent vectors at that point $\{V_i\}$ we can find at some point in $\pi^{-1}(P) \subset \mathcal{P}(g,1)$ a set of vector fields $\{v_i\}$ in the neigborhood of the puncture which get mapped into $\{V_i\}$ under the projection map π. For any such set of vectors $\{v_i\}$ we define the measure

$$\begin{array}{l} \mu(P)(V_1,\ldots,V_{3g-3},\bar{V}_1,\ldots,\bar{V}_{3g-3}) = \\ \langle 0| \prod_{i=1}^{3g-3} b(v_i) \prod_{i=1}^{3g-3} \bar{b}(\bar{v}_i)|\phi\rangle_P \end{array} \tag{496}$$

where the operator $b(v)$ is defined as

$$b(v) = \oint_P \frac{dz}{2\pi i} b(z)v(z) \tag{497}$$

$|0\rangle$ is the $SL(2,\mathbf{C})$ invariant vacuum and $|\phi\rangle_P$ is the state associated with the point in $\mathcal{P}(g,1)$ (these states belongs to $\mathcal{F}_{matter} \otimes \mathcal{F}_{bc} \otimes \mathcal{F}_{\bar{b}\bar{c}}$). Notice first of all that the ghost numbers match in (496).

It can be shown that (496) defined above is a measure on moduli space. First if we change the representative $\{v_i\}$ in $\mathcal{P}(g,1)$ of $\{V_i\}$ by $v_i \to v_i + \epsilon_i$, if ϵ_i extends holomorphically away from P there is no change in the measure since $b(\epsilon_i)|\phi\rangle = 0$. Furthermore, if we change the local parameter, or move P infinitesimally, since $L_n|0\rangle = 0$ for $n \geq -1$ and the total central charge vanishes, there is no change in the measure. There is a last condition that the measure has to verify in order to be identified with the bosonic string measure. $\mu(P)$ has to verify the Belavin-Knizhnik theorem [126]

$$\partial\bar{\partial}\log\mu = -13\,\partial\bar{\partial}\log\left(Im\,\Omega\right)\,, \tag{498}$$

therefore

$$\mu = \frac{\rho\wedge\bar{\rho}}{(det\,Im\,\Omega)^{13}} \tag{499}$$

where ρ is a holomorphic $3g-3$ form in the moduli space. To prove (498) we make use of (411):

$$\begin{aligned} \partial\bar{\partial}\log\mu(\ldots V\ldots) &= \frac{\langle 0|\ldots T(\,)\bar{T}(\,)\ldots|\phi\rangle}{\langle 0|\phi\rangle} \\ &- \frac{\langle 0|\ldots T(\,)|\phi\rangle\langle 0|\ldots\bar{T}(\,)|\phi\rangle}{\langle 0|\phi\rangle^2} \end{aligned} \tag{500}$$

Since the only obstruction to holomorphic factorization comes from the matter sector, we can restrict the calculation to the matter states. Using Rohrlich's formula [135]

$$\delta\Omega_{ij} = -\oint_P \omega_i \omega_j \nu \tag{501}$$

we obtain

$$\partial\bar{\partial} \log \mu(\ldots V \ldots) = -13\, \partial\bar{\partial} \log det\, Im\, \Omega \tag{502}$$

So $\mu(P)$ is a well defined measure over the moduli space.

As an application we are going to show that the measure has second order poles at the boundary of moduli space. Let us consider a surface which is pinched off along a non-contractible loop (see sec. 3.4). This can be obtained by sewing two surfaces with sewing parameter $q = 0$. Near the $q = 0$ region the measure looks like

$$\left(\frac{dq}{q}\frac{d\bar{q}}{\bar{q}}\right) \langle S_{ij}| b_0 \bar{b}_0 q^{L_0} \bar{q}^{\bar{L}_0} |\phi\rangle \tag{503}$$

since the lowest eigenvalue of L_0, $\bar{L}_0$ is -1 (which correspond to the tachyon) as $q \to 0$ we obtain

$$\left|\frac{dq d\bar{q}}{q^2 \bar{q}^2}\right| \tag{504}$$

This means that ρ has a second order pole at the boundary of moduli space associated with the equivalence class of Riemann surfaces with a node. This structure completes the identification of (496) with the Polyakov measure. It also exhibits in a rigorous way the intuitive expectation that string infinities originate in the exchange of tachyons along thin and long tubes between different elements of the surface.

As a final application, we study string scattering amplitudes to obtain a geometrical interpretation of the physical state conditions. If we have a scattering amplitude with n external states, we want to associate with it a measure on $\mathcal{M}_{g,n}$ if we wish to compute the amplitude to order g. The operator formalism provides us with a well-defined state $|\phi\rangle_P$ for any $P \in \mathcal{P}(g,n)$. Once again we have a projection map

$$\begin{array}{cc} \mathcal{P}(g,n) & \\ \downarrow & \pi \\ \mathcal{M}_{g,n} & \end{array}$$

by simply forgetting the punctures and the local parameters. Now we construct a measure similar to (496)

$$\langle \chi_1 | \ldots \langle \chi_n | b(\,) \ldots \bar{b}(\,) |\phi\rangle_P \tag{505}$$

In this case, however, for (505) to be a well defined measure we need a $(3g-3+n, 3g-3+n)$ form so we must have $3g-3+n$ b's. This means that since $|\phi\rangle_P$ has ghost number $3g-3+3n/2$ the ghost number of the external states has to be $-\frac{n}{2}$. Distributing the ghost number equally among the external states, we learn that $\langle\chi_i|$ must have ghost number $-\frac{1}{2}$. Apply the other requirements neccesary to have a well defined measure over $\mathcal{M}_{g,n}$ we arrive to the conditions

$$\begin{aligned} L_n|\chi_i\rangle = \bar{L}_n|\chi_i\rangle &= 0\,, \\ b_n|\chi_i\rangle = \bar{b}_n|\chi_i\rangle &= 0\,, \qquad n \geq 0\,, \end{aligned} \tag{506}$$

and we have no constraint coming from L_{-1} because the measure depends on the position of the punctures. These conditions, together with the condition over the ghost number of $|\chi_i\rangle$ are solved by

$$|\chi\rangle = \sum |\psi\rangle_{matter} \otimes |-\rangle \ , \tag{507}$$

where $|\psi\rangle_{matter}$ satisfy

$$\begin{aligned} L_n|\psi\rangle &= \bar{L}_n|\psi\rangle = 0 \qquad , n > 0 \ , \\ L_0|\psi\rangle &= \bar{L}_0|\psi\rangle = |\psi\rangle \ , \end{aligned} \tag{508}$$

so we recover the standard physical state conditions.

We see that the operator formalism provide a natural framework to analyze many of the qualitative and quantitative questions in String Theory. From a geometrical point of view, the computation of string amplitudes is related to the construction of measure on $\mathcal{M}_{g,n}$ satisfying the Belavin-Knizhnik condition. In the operator formalism we can compute these measures using the same techniques as in CFT on the plane. All the geometrical information is coded into the Bogoliubov transformation relating the standard vacuum $|0\rangle$ to the state $|\phi\rangle_P$ associted to $P \in \mathcal{P}(g, n)$. This formalism can also be extended to the supersymmetric case with similar results. We will not discuss this extension in detail. The interested reader can find the details in [125].

4 String Theory II. Fermionic Strings

We now begin a brief study of fermionic strings. The aim is to show how in a supersymmetric theory, many of the undesirable features of the bosonic string are eliminated. We will also emphasize how some simple consistency conditions (absence of global anomalies on the world-sheet) provide strong constraints on the spectrum of the theory. It is quite remarkable that once the modular anomalies (global diffeomorphism anomalies) on the world-sheet are cancelled, and after we make the theory invariant under the mapping class group (see previous chapter), all space-time anomalies also cancel [127]. At the end of this chapter we will discuss some special properties of superstring theory: finite temperature behavior, and a brief status report on finiteness of string perturbation theory. A brief introduction to string black holes (covered in more detail in Polyakov's lectures) will appear in the next chapter.

4.1 Fermionic String

In spite of all the beautiful features found in the study of the bosonic string, this is not a satisfactory model. The first problem we meet is the presence of a tachyon in the spectrum which is the origin of many difficulties. Furthermore, the resulting theory in the target space only contains bosonic degrees of freedom so there is no hope of phenomenological implications for this model. We can get rid of these two problems at the same time by adding fermions in the two-dimensional field theory on the world-sheet. This lead to the construction of the fermionic, or more appropiately, supersymmetric string. Now, together with the bosonic coordinates $X^\mu(\tau, \sigma)$ ($\mu = 1, \ldots, d$), we introduce d Majorana-Weyl spinors on the world-sheet $\psi^\mu(\tau, \sigma)$

transforming as a d-dimensional vector under Lorentz transformations in the target space. In the bosonic string the negative-norm states were eliminated from the spectrum by using reparametrization invariance, which gave rise to the Virasoro constraints. Now we also have negative-norm states coming from the time component of the spinor field, $\psi^0(\tau,\sigma)$, so we need a new local invariance to get rid of these ghost states. This new local symmetry is two-dimensional supergravity. To implement it we introduce two fields, the *zweibein* $e_\alpha{}^a(\tau,\sigma)$

$$h_{\alpha\beta} = e_\alpha{}^a e_\beta{}^b \eta_{ab} \tag{509}$$

and the Majorana gravitino χ_α ($a,b=1,\ldots,d$; $\alpha,\beta=0,1$). The complete action is [99, 100]

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int d\tau d\sigma (det\ e)\eta_{\mu\nu}\left\{h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu - ie^\alpha{}_a\bar\psi^\mu\rho^a\partial_\alpha\psi^\nu\right. \\ &+ \left. 2e^\alpha{}_a e^\beta{}_b\bar\chi_\alpha\rho^b\rho^a\psi^\mu\partial_\beta X^\nu - \frac{1}{2}\bar\psi^\mu\psi^\nu e^\alpha{}_a e^\beta{}_b\bar\chi_\alpha\rho^b\rho^a\chi_\beta\right\} \end{aligned} \tag{510}$$

where the ρ^α's are the Dirac matrices satisfying the two-dimensional Clifford algebra

$$\{\rho^\alpha,\rho^\beta\} = -2\eta^{\alpha\beta}\ . \tag{511}$$

We can fix the superconformal gauge:

$$e_\alpha{}^a = e^\phi\delta_\alpha{}^a\ , \qquad \chi_\alpha = \rho_a\lambda\ . \tag{512}$$

In the critical dimension the classical Weyl and super-Weyl transformations are also symmetries of the quantum theory so ϕ and λ can be gauged away. At the classical level this is always true and we obtain the gauge fixed action action

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\left[\partial_\alpha X^\mu\partial^\alpha X^\nu - i\bar\psi^\mu\rho^\alpha\partial_\alpha\psi^\nu\right]\eta_{\mu\nu}\ , \tag{513}$$

which is still invariant under the supersymmetry transformations

$$\begin{aligned} \delta_\epsilon X^\mu &= \bar\epsilon\psi^\mu \\ \delta_\epsilon\psi^\mu &= -i\rho^\alpha\partial_\alpha X^\mu\epsilon\ , \end{aligned} \tag{514}$$

with ϵ a constant Majorana spinor. Associated with the supersymmetry transformation we have the supercurrent

$$J_\alpha = \frac{1}{2}\rho^\beta\rho_\alpha\psi^\mu\partial_\beta X_\mu\ , \tag{515}$$

such that $\partial_\alpha J^\alpha = 0$. We also have the energy momentum tensor

$$T_{\alpha\beta} = \partial_\alpha X^\mu\partial_b X_\mu + \frac{i}{2}\bar\psi^\mu\rho_{(\alpha}\partial_{\beta)}\psi_\mu - (\text{Trace})\ , \tag{516}$$

which is also conserved $\partial_\alpha T^{\alpha\beta}=0$. Classical equation of motion are

$$\begin{aligned} \partial_\alpha\partial^\alpha X^\mu &= 0 \\ \rho^\alpha\partial_\alpha\psi^\mu &= 0 \end{aligned} \tag{517}$$

together with the constraints

$$\begin{aligned} T_{\alpha\beta} &= 0 \\ J_\alpha &= 0 \end{aligned} \tag{518}$$

which follow from the equations of motion for the *zweibein* $e_\alpha{}^a$ and the gravitino χ_α respectively. Invariance under Weyl and super-Weyl transformations imply that

$$\begin{aligned} T^\alpha_\alpha &= 0\,, \\ \rho^\alpha J_\alpha &= 0\,. \end{aligned} \tag{519}$$

We are going to study the free closed superstring. As for the bosonic string, the world-sheet is a cylinder parametrized by the coordinates $-\infty < \tau < \infty$ and $0 \leq \sigma < \pi$. After Wick rotating these coordinates and conformally mapping the cylinder into the punctured complex plane $\mathbf{C}^*$, the action becomes

$$S = -\frac{1}{4\pi\alpha'}\int d^2z(\bar{\partial}X^\mu\partial X_\mu - \psi^\mu\bar{\partial}\psi_\mu - \bar{\psi}^\mu\partial\bar{\psi}_\mu) \tag{520}$$

where the components of the two-dimensional spinor ψ^μ are

$$\psi^\mu = \begin{pmatrix} \psi^\mu \\ \bar{\psi}^\mu \end{pmatrix}\,, \tag{521}$$

and we have used the following representation for the two-dimensional Dirac algebra

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\,. \tag{522}$$

The equations of motion are

$$\begin{aligned} \partial\bar{\partial}X^\mu(z,\bar{z}) &= 0\,, \\ \partial\bar{\psi}(z,\bar{z}) = \bar{\partial}\psi(z,\bar{z}) &= 0\,, \end{aligned} \tag{523}$$

so $\psi(z)$, $\bar{\psi}(\bar{z})$ are respectively holomorphic and antiholomorphic fields. In the same way we can rewrite the energy momentum tensor and the supercurrent as

$$\begin{aligned} T(z) &=: T_{zz}(z) = -\frac{1}{2}\partial X^\mu\partial X_\mu - \frac{1}{2}\partial\psi^\mu\psi_\mu\,, \\ T_F(z) &=: J_z(z) = -\frac{1}{2}\psi^\mu\partial X_\mu\,, \end{aligned} \tag{524}$$

and similarly for barred quantities $\bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}(\bar{z})$, $\bar{T}_F(\bar{z}) = J_{\bar{z}}(\bar{z})$.

The superfield language is specially suited to study the supersymmetric string [2, 3]. We introduce together with the commuting coordinates $z, \bar{z} \in \mathbf{C}^*$ a pair of anticommuting variables $\theta, \bar{\theta}$, and the supercoordinates:

$$\mathbf{z} = (z, \theta) \qquad \bar{\mathbf{z}} = (\bar{z}, \bar{\theta}) \tag{525}$$

Taking into account the anticommuting character of θ and $\bar{\theta}$ any function $f(z,\bar{z},\theta,\bar{\theta})$ can be expanded as

$$f(z,\bar{z},\theta,\bar{\theta}) = f_0(z,\bar{z}) + f_1(z,\bar{z})\theta + f_2(z,\bar{z})\bar{\theta} + f_3(z,\bar{z})\theta\bar{\theta} \tag{526}$$

The supersymmetric derivative is defined by

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z} \;, \tag{527}$$

and the integration over the anticommuting variables θ, $\bar{\theta}$ is made according to the Berezin rules [128]

$$\int d\theta = 0 \;, \qquad \int d\theta\,\theta = 1 \;. \tag{528}$$

We can now reformulate the superstring action using the superspace formalism. Defining the superfield

$$Y^\mu(z,\bar{z},\theta,\bar{\theta}) = X^\mu(z,\bar{z}) + \theta\psi^\mu(z,\bar{z}) + \bar{\theta}\bar{\psi}^\mu(z,\bar{z}) \tag{529}$$

we can rewrite (513) as

$$S = -\frac{1}{4\pi\alpha'}\int d^2z\, d\theta\, d\bar{\theta}\bar{D}Y^\mu DY_\mu \;, \tag{530}$$

which is invariant under superconformal transformations (for many details and results in superconformal field theory, and the formulation of superstrings in this language, see [129]) We can also define the components of the super-energy-momentum tensor $\mathcal{T}(z,\theta)$, $\bar{\mathcal{T}}(\bar{z},\bar{\theta})$ as

$$\begin{aligned} \mathcal{T}(z,\theta) &= T_F(z) + \theta T(z) \\ \bar{\mathcal{T}}(\bar{z},\bar{\theta}) &= \bar{T}_F(\bar{z}) + \bar{\theta}\bar{T}(\bar{z}) \end{aligned} \tag{531}$$

After quantization we obtain the following the OPE's for $T(z)$ and $T_F(z)$

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{1}{z-w}\partial T(w) + regular\ terms \\ T(z)T_F(w) &= \frac{3/2}{(z-w)^2}T_F(w) + \frac{1}{z-w}\partial T_F(w) + \ldots \\ T_F(z)T_F(w) &= \frac{c/6}{(z-w)^3} + \frac{1}{2(z-w)}T(w) + \ldots \end{aligned} \tag{532}$$

which imply that the field $T_F(z)$ is a primary field of spin $\frac{3}{2}$.

In the path integral formalism we need to gauge fix all local symmetries. This is done using the Fadeev-Popov procedure. Now, together with the anticommuting reparametrization ghosts $b_{\alpha\beta}$, c^α we have also a pair of commuting superconformal ghosts β_α and γ with spins $\frac{3}{2}$ and $-\frac{1}{2}$ respectively. Their action is:

$$S_{gh} = \frac{1}{2\pi\alpha'}\int d^2z(b_{zz}\bar{\partial}c^z + \beta_z\bar{\partial}\gamma + \text{c.c.}) \tag{533}$$

In the superspace formalism, the ghost fields can be combined to ghosts superfields

$$\begin{aligned} B(z,\theta) &= \beta(z) + \theta b(z) \\ C(z,\theta) &= c(z) + \theta\gamma(z) \end{aligned} \tag{534}$$

and their combined action becomes

$$S_{gh} = \frac{1}{2\pi\alpha'} \int d^2z d\theta d\bar{\theta} (B\bar{D}C + \text{c.c.}) \tag{535}$$

Thus, in the free quantum superstring, we have the following fields

$$X^\mu(z), \quad \psi^\mu(z), \quad b(z), \quad c(z), \quad \beta(z), \quad \gamma(z) \tag{536}$$

and the corresponding antiholomorphic components. The contribution of every field to the central charge c is given by

$$\begin{array}{ccc} X^\mu & \longrightarrow & d \\ \psi^\mu & \longrightarrow & \frac{d}{2} \\ b,\ c & \longrightarrow & -26 \\ \beta,\ \gamma & \longrightarrow & 11 \end{array} \tag{537}$$

so the total central charge is equal to $c_{tot} = \frac{3d}{2} - 15$. This means that the conformal and superconformal anomaly cancel if the dimension of the target space is $d = 10$. As in the discussion of the bosonic string, what distinguishes critical and non-critical strings is the decoupling of the conformal modes. In the case of fermionic strings the critical value of $d = 10$. Only in this case a graviton is found in the spectrum. The attention is thus dedicated to critical superstrings when one wants to understand Quantum Gravity and its unification with other interactions.

A crucial point when dealing with fermionic string is the question of the boundary conditions for the fermions. In the closed bosonic string we had the periodicity condition over the bosonic fields $X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma)$. Now we have the same boundary condition on the X^μ field, but for the fermions we can have either periodic or antiperiodic boundary conditions. Periodic boundary conditions are called Ramond (R) boundary condition whereas antiperiodic ones are called Neveu-Schwarz (NS) boundary conditions

$$\begin{aligned} \psi^\mu(\tau, \sigma + \pi) &= \psi^\mu(\tau, \sigma) \qquad & \text{(R)} \\ \psi^\mu(\tau, \sigma + \pi) &= -\psi^\mu(\tau, \sigma) \qquad & \text{(NS)} \end{aligned} \tag{538}$$

When mapping the cylinder onto the punctured complex plane $\mathbf{C}^*$ we have to take into account the transformation properties of spinors under conformal transformations. Since $\psi^\mu(z)$ is a $(\frac{1}{2}, 0)$ field, we have that if $z = e^w = \exp[2(\tau - i\sigma)]$

$$\psi^\mu(z) dz^{\frac{1}{2}} = \psi^\mu(w) dw^{\frac{1}{2}} \,. \tag{539}$$

This implies that under the transformation $z \to e^{2\pi i} z$ the periodicity conditions of $\psi^\mu(z)$ and $\psi^\mu(w)$ are opposite when $w \to w + 2\pi i$. Then, the boundary conditions over the complex plane for both sectors (R and NS) will be

$$\begin{aligned} \psi^\mu(e^{2\pi i} z) &= -\psi^\mu(z) \qquad & \text{(R)} \\ \psi^\mu(e^{2\pi i} z) &= \psi^\mu(z) \qquad & \text{(NS)} \end{aligned} \tag{540}$$

In the closed superstring we have four possible sectors corresponding to the boundary conditions for left- and right-moving modes (R,R), (R,NS), (NS,R) and (NS,NS). For one handness we have the two boundary conditions R and NS. We can expand the fields $\psi(z)$ in these two sectors in Fourier modes:

$$
\begin{aligned}
\psi(z) &= \sum_{n\in\mathbf{Z}} d_n^\mu z^{-n-\frac{1}{2}} \qquad (\mathrm{R})\ , \\
\psi(z) &= \sum_{r\in\mathbf{Z}+\frac{1}{2}} b_r^\mu z^{-r-\frac{1}{2}} \quad (\mathrm{NS})\ .
\end{aligned}
\tag{541}
$$

The field $X^\mu(z)$ has the same expansion as in the bosonic string

$$
X^\mu(z) = q^\mu - \frac{i}{4} p^\mu \log z + \frac{i}{2} \sum_{n\in\mathbf{Z}} \frac{\alpha_n^\mu}{n} z^{-n}\ . \tag{542}
$$

The energy-momentum tensor, being a bosonic field, can be expanded in the usual way

$$
T(z) = \sum_{n\in\mathbf{Z}} L_n z^{-n-2} \tag{543}
$$

where L_n are the Virasoro generators. However, the boundary conditions for the supercurrent $T_F(z)$ are the same as those for the fermionic fields. Hence

$$
\begin{aligned}
T_F(z) &= \frac{1}{2} \sum_{n\in\mathbf{Z}} G_n z^{-n-\frac{3}{2}} \qquad (\mathrm{R})\ , \\
T_F(z) &= \frac{1}{2} \sum_{r\in\mathbf{Z}+\frac{1}{2}} G_r z^{-r-\frac{3}{2}} \qquad (\mathrm{NS})
\end{aligned}
\tag{544}
$$

where G_r are the generators of the superconformal transformations. The (anti)commutation relations follow from the OPE's (532):

$$
\begin{aligned}
[L_m, L_n] &= (m-n) L_{m+n} + \frac{\hat{c}}{8} m(m^2-1)\delta_{m+n,0}\ ; \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}\ , \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{\hat{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\ ,
\end{aligned}
\tag{545}
$$

where now $r \in \mathbf{Z}$ or $r \in \mathbf{Z} + \frac{1}{2}$ depending on whether we are in (R) or (NS) sector and $\hat{c} = \frac{2}{3}c$.

We now proceed to the determination of the spectrum for the closed supersymmetric string. First of all we have the following non-zero (anti) commutation relations between oscillators,

$$
\begin{aligned}
[\alpha_m, \alpha_n] &= m\eta^{\mu\nu}\delta_{m+n,0} \\
\{d_m^\mu, d_n^\nu\} &= \eta^{\mu\nu}\delta_{m+n,0} \\
\{b_r^\mu, b_r^\nu\} &= \eta^{\mu\nu}\delta_{r+s,0}\ .
\end{aligned}
\tag{546}
$$

As for the bosonic string we have negative norm states. In order to eliminate these states there are several procedures already explained in sec. 3.1. The constraints, as in the bosonic string, cannot be imposed naïvely because of the anomaly; then we have the following conditions over the physical states (we also have the same conditions for the antiholomorphic parts) if we use covariant quantization:

$$\begin{aligned} L_n|phys\rangle &= 0 \qquad n>0 \\ G_r|phys\rangle &= 0 \qquad r>0\,(\mathrm{NS}) \quad r\geq 0\,(\mathrm{R}) \end{aligned} \tag{547}$$

together with $(L_0-a)|phys\rangle = 0$ and $(L_0-\bar{L}_0)|phys\rangle = 0$ where a is a normal ordering constant that is $a=\frac{1}{2}$ for the NS and $a=0$ for the R sector. To obtain the physical spectrum we will use light-cone gauge where the constraints can be explicitly solved. Making use of the residual invariance left after fixing the superconformal gauge we impose the light-cone gauge conditions

$$\begin{aligned} X^+ &= q^+ + \alpha' p^+ \tau \,, \\ \psi^+ &= 0\,. \end{aligned} \tag{548}$$

In this gauge the theory is completely described in terms of transverse modes.

We begin with the NS sector. The mass formula is computed following the same steps as in the previous chapter (again we use $\alpha'=\frac{1}{2}$)

$$\frac{1}{2}m^2 = \sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{r>0} r b^i_{-r} b^i_r - \frac{1}{2}\,. \tag{549}$$

The vacuum state $|0\rangle$ satisfies the conditions

$$\alpha^i_n|0\rangle = b^i_r|0\rangle = 0 \qquad n,r>0 \tag{550}$$

It follows from (549,550) that $|0\rangle$ has $m^2<0$ and it is a tachyon state. This looks like the kind of trouble we were trying to scape from in the bosonic string. We will see presently how the tachyon states are eliminated from the theory. The first excited level is constructed by acting with $b^i_{-\frac{1}{2}}$ on the vacuum state. This is a massless state that transforms as a vector with respect to $SO(8)$, the little group for massless particles in ten dimensions. In the first massive state we have two states $\alpha_{-1}|0\rangle$ and $b^i_{-\frac{1}{2}}b^j_{-\frac{1}{2}}b|0\rangle$ both with $m^2=1$.

The mass formula in the R sector is given by

$$\frac{1}{2}m^2 = \sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{n>0} n d^i_{-n} d^i_n \tag{551}$$

However in this sector we have a supplementary complication. The zero modes d^μ_0 form a closed subalgebra since

$$\{d^\mu_0, d^\nu_0\} = 2\eta^{\mu\nu} \tag{552}$$

This means that, with respect to the oscillator vacuum $|0\rangle$ defined by

$$\alpha_n|0\rangle = d_n|0\rangle = 0 \qquad n>0 \tag{553}$$

all the states $d^\mu_0|0\rangle$ are also massless because $\{d^\mu_0, d^\nu_n\}=0$ $(n\neq 0)$. These states provide us with a representation of the Clifford algebra (552), therefore the ground state for the R sector is a ten-dimensional spinor $|a\rangle$ with a a spinor index. We know that in ten dimensions we can impose both Weyl and Majorana conditions; then we can choose our ground state to have a definite chirality. We denote two possible chiral ground states by $|a\rangle$, $|\bar{a}\rangle$, $a, \bar{a} = 1, \ldots, 8$. Starting with this massless spinor state we construct the spectrum of the R sector of the fermionic string. The first excited level has (apart from the two possible chiralities) two states, namely, $d^i_{-1}|a\rangle$ and $\alpha^i_{-1}|a\rangle$ both with masses $m^2 = 2$. We obtain higher excited levels in a similar fashion.

To construct the spectrum of the closed fermionic string we take the tensor product of the right- and left-moving states, taking into acount the level matching condition which can be written as $m_L^2 = m_R^2$ for each of the four sectors (R,R), (R,NS), (NS,R) and (NS,NS). Note that we have not taken care of the problems posed by the presence of tachyons in the sector with NS boundary conditions. Historically, a projection was made onto a set of definite G-parity states [130] thus eliminating the tachyon and also rendering the spectrum space-time supersymmetric. We begin by presenting first how the projection is achieved, and we will later see how this projection is naturally understood in terms of modular invariance, the absence of global diffeomorphism anomalies on the world-sheet. The G-parity operator in the NS sector is defined to be:

$$G = (-1)^{F+1} = (-1)^{\sum_{r \in \mathbf{Z}+\frac{1}{2}} b^i_{-n} b^i_n + 1} . \tag{554}$$

Hence the vacuum state in the NS sector has $G|0\rangle = -|0\rangle$. In the R sector we construct the G-parity operator using a string generalization of γ_5:

$$\Gamma = \Gamma^0 \ldots \Gamma^8 (-1)^{\sum_{n>0} d^i_{-n} d^i_n} \tag{555}$$

By using these operators we can at the same time project out the unwanted tachyon and make the string spectrum space-time supersymmetric. This can be accomplished by using the GSO projection [130] introduced by Gliozzi, Scherk and Olive. We project the spectrum of the string onto the states with positive G-parity in each handness separately. Since the fundamental state in the NS sector has $G = -1$ this projection eliminate the tachyon from the theory. With respect to the Γ operator in the R sector we have different choices: $\Gamma = \pm 1$. Since the action of Γ on the R ground state $|a\rangle$ gives us the chirality of this state, different choices will lead to chiral and non chiral spectra. By taking $\Gamma = \bar{\Gamma} = 1$ we have the ground states in the left- and right-R sector will be two spinors with the same chirality $|a\rangle$, $|b\rangle$. This kind of string is called type IIB superstring [131] and its massless sector contains the states [8]

$$\begin{array}{rcl} |a\rangle_L & \otimes & |b\rangle_R \\ \bar{b}^i_{-\frac{1}{2}}|0\rangle_L & \otimes & b^j_{-\frac{1}{2}}|0\rangle_R \\ |a\rangle_L & \otimes & b^i_{-\frac{1}{2}}|0\rangle_R \\ \bar{b}^i_{-\frac{1}{2}}|0\rangle_L & \otimes & |b\rangle_R \end{array} \tag{556}$$

Decomposing these states into irreducible representations of the little group $SO(8)$ we have the graviton, two real scalars, two antisymmetric tensors and a rank four antisymmetric tensor. In addition we have two spin $\frac{3}{2}$ and two spin $\frac{1}{2}$ states all with the same helicity. The states in the massless sector are those of the chiral $N = 2$ SUGRA in $d = 10$ [34, 132].

[8] Apart from the GSO projection we have also required the level matching condition. The same remark applies in the derivation of 557.

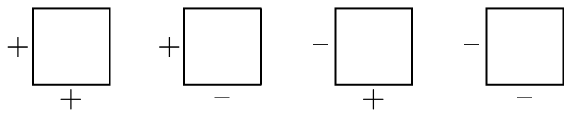
By performing the choice $\Gamma = -\bar{\Gamma} = 1$ we obtain the type IIA superstring [2]. The states in the massless sector are

$$\begin{array}{rcl} |\bar{a}\rangle_L & \otimes & |b\rangle_R \\ \bar{b}^i_{-\frac{1}{2}}|0\rangle_L & \otimes & b^j_{-\frac{1}{2}}|0\rangle_R \\ |\bar{a}\rangle_L & \otimes & b^i_{-\frac{1}{2}}|0\rangle_R \\ \bar{b}^i_{-\frac{1}{2}}|0\rangle_L & \otimes & |b\rangle_R \end{array} \tag{557}$$

Now the theory has the same states than the type IIB superstring but with the difference that fermion states come in both chiralities. Then the particle content for type IIA superstring is that of the non-chiral $N = 2$ SUGRA in $d = 10$ [34] which can also be obtained by dimensional reduction from $N = 1$ SUGRA in $d = 11$.

After the GSO projection, the spectrum of the fermionic string becomes supersymmetric. This is rather surprising because having started with a action that was supersymmetric only on the world-sheet and we obtain in the end supersymmetry in the target space. There is a second formalism developed by Green and Schwarz [133, 134] which starts with a space-time supersymmetric action.

In order to compute amplitudes we will need also to consider world-sheets with non-trivial topology. We have to be specially careful with fermions. Since they are represented by world-sheet spinors, we have to decide how to treat their spin structures. On a genus g surface the first homology group has dimension $2g$. Once a spin structure is fixed, all others differ from the fixed one by some choice of boundary conditions along a basis of the homology cycles. Thus the number of spin structures is 2^{2g}. They can be divided into even and odd depending on whether the number of holomorphic sections of the corresponding line bundle is even or odd. There are $2^{g-1}(2^g + 1)$ even spin structures and $2^{g-1}(2^g - 1)$ odd ones (for a detailed discussion of spin structures on Riemann surfaces see [135]). The physical interpretation of the sum over spin structures is the implementation of the GSO projection. Reciprocally, the geometrical interpretation of the GSO projection lies on the sum over spin structures, the cancellation of modular anomalies. This is best illustrated by studying in some detail the case of genus one, the torus. The spin line bundles on the torus are flat line bundles and we can represent them simply in terms of boundary conditions on the two homology cycles. There are four possibilities schematically drawn below



The horizontal line represents the σ-variable on the world-sheet and the vertical line the τ-variable.

The mapping class group, or modular group on the torus is the group $PSL(2, \mathbf{Z})$ which acts on the modular parameter τ as

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \qquad a, b, c, d \in \mathbf{Z} \, , \qquad ab - cd = \pm 1 \, . \tag{558}$$

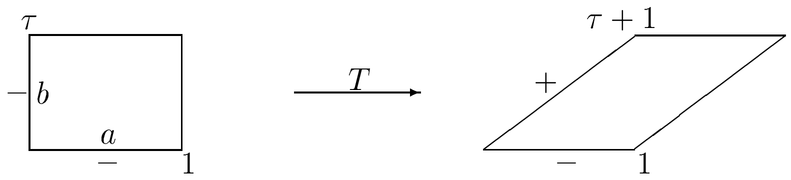
This group is generated by two elements

$$T : \tau \longrightarrow \tau + 1 \, ,$$
$$S : \tau \longrightarrow -\frac{1}{\tau} \, , \tag{559}$$

which satisfy two relations:

$$S^2 = (ST)^3 = 1 \tag{560}$$

The action of this group changes the fermionic boundary conditions. For instance, acting with T on $(-,-)$ the fundamental cell of the torus changes as shown in the figure



Hence, if originally we had the $(-,-)$ spin structure, we end up with $(+,-)$. If we now act with S (which effectively exchange the a and b cycles) $(+,-)$ is transformed into $(-,+)$. Thus, on the torus we have two orbits of $PSL(2,\mathbf{Z})$ on the spin structures: the even orbit $(-,-)$, $(-,+)$, $(+,-)$ and the odd orbit $(+,+)$. The reader can easily check that the operator $\bar{\partial}$ (the Dirac operator in this case) has no zero modes in the even orbit, and one zero mode for $(+,+)$ boundary conditions. If we let Γ_2 [9] be the subgroup of $SL(2,\mathbf{Z})$ leaving the spin structure fixed, the quotient $SL(2,\mathbf{Z})/\Gamma_2$ is a finite group of order six. The issue of global gravitational anomalies on the world-sheet was analized by Witten (see first reference in [12]). After this brief mathematical interlude we can pose the problem of global anomalies on the world-sheet. When we defined scattering measures in the previous chapter, in the measure we had to deal with the integration over metrics modulo diffeomorphisms. Among the latter we can distinguish the normal subgroup of orientation preserving diffeomorphisms in the identity component $\mathrm{Diff}_0^+(\Sigma)$. The mapping class group was defined as $\Omega(\Sigma) = \mathrm{Diff}^+(\Sigma)/\mathrm{Diff}_0^+(\Sigma)$. Since all the quantities appearing in the bosonic string were from the beginning generally covariant, we were rather cavalier in the treatment of $\Omega(\Sigma)$. We need to be more careful now. Since the fields appearing in the fermionic string are acted on non-trivially by $\Omega(\Sigma)$, we have to ensure that the quantum theory is invariant under $\Omega(\Sigma)$. If we fix the spin structures of the fermions the absence of global

---

[9] Γ_2 is the subgroup of elements

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z})$$

such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2$$

anomalies is the requirement that the theory does not change under the action of the elements of Γ_2 (the subgroup of $\Omega(\Sigma)$ preserving the spin structures). Once this is done we can classify all possible ways of summing over spin structures which are invariant under $\Omega(\Sigma)/\Gamma_2$. This should produce all modular invariant string theories with the initial content of two-dimensional fields. The simplest example to analyze the action of Γ_2 is in the case at hand with the GSO projection. The simplest example we are considering has the G-parity projection imposed on left and right movers independently. Let us compute the toral partition functions in the four sectors. Using $q = \exp(2\pi i\tau)$ we obtain:

$$\begin{aligned} Z_{(-,-)} &= tr\, q^{H_{NS}} \\ Z_{(-,+)} &= tr\,(-1)^F q^{H_{NS}} \\ Z_{(+,-)} &= tr\, q^{H_R} \\ Z_{(+,+)} &= tr\,(-1)^F q^{H_R}\,, \end{aligned} \tag{561}$$

where H_{NS} and H_R are respectively the hamiltonians corresponding to the NS and R sectors

$$\begin{aligned} H_{NS} &= (L_0)_{plane} - \frac{1}{2} = \sum_{n>0} \alpha^i_{-n}\alpha^i_n + \sum_{r>0} r b^i_{-r} b^i_r - \frac{1}{2} \\ H_R &= (L_0)_{plane} - \frac{1}{2} = \sum_{n>0} \alpha^i_{-n}\alpha^i_n + \sum_{n>0} n d^i_{-n} d^i_n \end{aligned} \tag{562}$$

since in the light-cone gauge $c = 8 \times 1 + 8 \times \frac{1}{2} = 12$. Note that in order to implement the periodic boundary condition in the τ-direction we need to insert $(-1)^F$ in the traces appearing in (561). This gives the first hint of the geometrical interpretation of the GSO projection. We have dropped the contribution of the zero modes of α^i_0; when computing this contribution coming from both left- and right-moving modes, we get that it is proportional to τ_2^{-4}. We can evaluate the trace for the bosonic modes α^i_n which is a common factor in all the contributions:

$$tr\, q^{\sum_{n>0}\alpha^i_{-n}\alpha^i_n} = q^{\frac{1}{3}}\eta^{-8}(\tau) \tag{563}$$

where $\eta(\tau)$ is the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \tag{564}$$

All the contributions to the partition function can be expressed in terms of Jacobi theta functions [123, 113]

$$\vartheta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z|\tau) = \sum_{n\in\mathbf{Z}} e^{i\pi\tau(n+\alpha)^2 + 2\pi i(n+\alpha)(z+\beta)} \tag{565}$$

with α, $\beta \in (\mathbf{Z}+\frac{1}{2})/\mathbf{Z}$. The computation of the traces in (561) yields:

$$Z_{(-,-)} = \eta^{-8}(\tau)\left(\frac{\vartheta\left[\begin{array}{c}0\\ 0\end{array}\right](0|\tau)}{\eta(\tau)}\right)^4$$

$$Z_{(-,+)} = \eta^{-8}(\tau)\left(\frac{\vartheta\left[\begin{array}{c}0\\ \frac{1}{2}\end{array}\right](0|\tau)}{\eta(\tau)}\right)^4$$

$$Z_{(+,-)} = \eta^{-8}(\tau)\left(\frac{\vartheta\left[\begin{array}{c}\frac{1}{2}\\ 0\end{array}\right](0|\tau)}{\eta(\tau)}\right)^4$$

$$Z_{(+,+)} = \eta^{-8}(\tau)\left(\frac{\vartheta\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right](0|\tau)}{\eta(\tau)}\right)^4 = 0\,. \tag{566}$$

since the factor $q^{\frac{1}{3}}$ cancel out when computing the total trace. We see that the odd spin structure does not contribute as could have been predicted due to the presence of a zero mode. It is not difficult to verify that the fourth power if the ϑ-functions appearing in (566) is the minimal power guaranteeing invariance under Γ_2. For example, using the explicit form (565) with $z = 0$, we see that the ϑ-functions change by a fourth root of unity as $\tau \to \tau + 2$. This modular transformation does not change the boundary conditions. It is an useful exercice to verify that for any element of Γ_2 the change is only a fourth root of unity. The fourth power appearing in (566) is a consequence of the fact that in the light-cone gauge we have eight Weyl-Majorana fermions with the same spin structure. They can be combined into four Weyl fermions, and the determinant of the Weyl operator is proportional to a ϑ-function [113]. Now that the global anomalies have cancelled, we can determine the full modular invariant combination of the partition functions (566). Thus we have to determine the relative phases with which each element in (566) enters in the total partition function

$$\begin{aligned} Z(\tau) &= Z_{(-,-)}(\tau) + w(-,+)Z_{(-,+)}(\tau) \\ &+ w(+,-)Z_{(+,-)}(\tau) + w(+,+)Z_{(+,+)}(\tau) \end{aligned} \tag{567}$$

The phases $w(\pm,\mp)$, $w(+,+)$ are well defined due to the absence of global anomalies. Since the $(-,-)$, $(+,-)$, $(-,+)$ spin structures are determined by the action of T, S, we easily determine the phases to be

$$w(-,+) = w(+,-) = -1 \tag{568}$$

Then we can write the one loop partition function as

$$Z(\tau) = tr\left(1 + (-1)^{F+1}\right) q^{H_{NS}} - tr\, q^{H_R} \tag{569}$$

This means that, in order to maintain modular invariance we have to project onto the sector with $(-1)^{F+1} = 1$. Thus the GSO projection is a consequence of modular invariance of the one loop partition function. Introducing a more convenient notation

$$\vartheta\left[\begin{array}{c}0\\ 0\end{array}\right] = \theta_3 \quad \vartheta\left[\begin{array}{c}\frac{1}{2}\\ 0\end{array}\right] = \theta_2 \quad \vartheta\left[\begin{array}{c}0\\ \frac{1}{2}\end{array}\right] = \theta_4 \quad \vartheta\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right] = \theta_1\,, \tag{570}$$

the partition function becomes

$$Z(\tau) = \eta^{-12}(\tau)[\theta_3^4(0|\tau) - \theta_4^4(0|\tau) - \theta_2^4(0|\tau)] \ . \tag{571}$$

The combination of θ-functions in (571) vanishes identically. An identity called by Jacobi *aequatio identica satis abstrusa*. Since the total partition function for the closed superstring is proportional to $\tau_2^{-4} Z(\tau) Z(\bar{\tau})$ we have that the one loop partition function for the type II superstring vanishes. This is a consequence of supersymmetry since the contribution to $Z(\tau, \bar{\tau})$ from the target space bosons is exactly cancelled by the fermions.

As a final remark we notice that $w(+,+)$ was not determined by the previous arguments. This is not unreasonable. From the space-time point of view, the states in the NS sector transform under single-valued representations of the little group $SO(8)$, and we should quantize them as bosons. In the R sector on the other hand the states are obtained by acting with tensor operators on a spinor ground state, and therefore these states should be treated as space-time fermions. Since there is an ambiguity in how to define space-time chirality, it is not unreasonable that $w(+,+)$ is not determined. To obtain constraints on the relative phases between the NS and R sectors one should consider the factorization properties of two-loop amplitudes. The lesson to be learned from this lengthy discussion is that the spectrum of fermionic string theories is strongly constrained by a simple consistency requirement: absence of global gravitational anomalies, or equivalently, modular invariance. If we consider simultaneously the modular properties of both left and right-movers, more possibilities open up. Each modular invariant combination will correspond to a different spectrum. This way of constructing string theories has been explored in the literature, specially in the context of heterotic strings [136].

As we pointed out earlier, by using Green-Schwarz formalism [133], one can work with an action in which space-time supersymmetry is explicit. The difficulty with this formalism is that it cannot be easily quantized in a covariant way (it requires the use of Batalin-Vilkoviski techniques). Here we use light-cone gauge quantization. In this approach we have, besides the $X^i(\tau, \sigma)$ bosonic coordinates, a field $S^{Aa}(\tau, \sigma)$ where $A = 1, 2$ is a world-sheet spinor index and $A = 1, \ldots, 32$ is a space-time spinor index. We impose on the ten-dimensional spinor both Weyl and Majorana condition, so the 64 complex components of S^{Aa} are reduced to 32 real components. Imposing the light-cone gauge condition on S^{Aa}

$$(\Gamma^+)^{ab} S^{Ab} = 0 \tag{572}$$

with Γ^μ the ten-dimensional Dirac matrices, we are left with 16 real components. Since the Dirac equation reduces further the number of real components to 8, we have that, on-shell, the number of fermionic space-time degrees of freedom is equal to the number of transverse bosonic coordinates, and we can have in principle space-time supersymmetry. The light-cone gauge action is given by

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma (\partial_\alpha X^i \partial^\alpha X^i - i\bar{S}^a \rho^\alpha \partial_\alpha S^a) \ . \tag{573}$$

The equations of motion for the spinors S^{Aa} are simply

$$\partial_+ S^{1a} = \partial_- S^{2a} = 0 \tag{574}$$

This, together with the periodic boundary conditions $S^a(\tau, \sigma + \pi) = S^a(\tau, \sigma)$ for the closed superstring, allow us to write the mode expansion

$$
\begin{aligned}
S^{1a} &= \sum_{n \in \mathbf{Z}} S_n^a e^{2in(\tau - \sigma)} \\
S^{2a} &= \sum_{n \in \mathbf{Z}} \tilde{S}_n^a e^{2in(\tau + \sigma)}
\end{aligned}
\tag{575}
$$

with the anticommutation relations for the modes:

$$
\{S_m^a, \tilde{S}_n^b\} = \delta^{ab} \delta_{m+n,0} \tag{576}
$$

We will now use ˜ to denote left-moving modes in order to avoid confusions with the components of the conjugate spinor. Using these operators the mass formula and the level matching condition for the closed superstring are respectively:

$$
\begin{aligned}
\frac{1}{2}\alpha^{'} m^2 &= \sum_{n>0} (\alpha_{-n}^i \alpha_n^i + \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + n S_{-n}^a S_n^a + n \tilde{S}_{-n}^a \tilde{S}_n^a) \\
\sum_{n>0} (\alpha_{-n}^i \alpha_n^i + n S_{-n}^a S_n^a) &= \sum_{n>0} (\tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + n \tilde{S}_{-n}^a \tilde{S}_n^a)
\end{aligned}
\tag{577}
$$

If the states $|i\rangle$ denote the eight physical degrees of freedom of the massless vector state, we define the supersymmetric partners $|a\rangle$

$$
|a\rangle = \frac{i}{8} (\Gamma_j S_0)^a |j\rangle \ , \tag{578}
$$

which are normalized according to

$$
\langle i|j\rangle = \delta_{ij} \ , \qquad \langle a|b\rangle = \frac{1}{2} (h\Gamma^+)^{ab} \ , \tag{579}
$$

where h is the ten-dimensional helicity operator. The massless spectrum contain 128 bosonic states

$$
|i\rangle_L \otimes |j\rangle_R \qquad |a\rangle_L \otimes |b\rangle_R \tag{580}
$$

and the same number of fermionic states

$$
|i\rangle_L \otimes |a\rangle_R \qquad |a\rangle_L \otimes |i\rangle_R \tag{581}
$$

As indicated above, when $|a\rangle_L$ and $|b\rangle_R$ have the same helicity we have the type IIB superstring (chiral) whereas when their helicities are opposite we have the type IIA superstring (non-chiral).

As a final remark let us indicate the spectrum of the type I (open) superstring. Although now we do not have any periodicity condition over the fields, it is possible to show that the massless spectrum is the same as in one of the sectors of the type II superstring: we have the eight transverse bosonic modes $|i\rangle$ and their fermionic partners $|a\rangle$ defined by (578). Then the states in the massless sector of the open superstring are those of a chiral $N = 1$ SUGRA in $d = 10$. For the open superstring it is possible to introduce gauge symmetries using the Chan-Paton [137] procedure: we attach charges to the endpoints of the string. However by using this method we can only have $USp(N)$, $SO(N)$ and $U(N)$ as gauge groups. By doing so the low energy field theory is a $N=1$ SUGRA coupled to $N=1$ super-Yang-Mills in $d=10$. Green and Schwarz [96] proved that this low energy field theory is anomaly free (from gauge and gravitational anomalies) as long as the gauge groups are $E_8\times E_8$, $SO(32)$ or $E_8\times U(1)^{248}$. Given the restriction imposed over the possible gauge groups introduced by Chan-Paton factors we have that the only consistent open string theory has $SO(32)$ as its gauge group.

4.2 Heterotic String

Up to now we have studied two kinds of string models: the closed bosonic string in which the dimension of the target space is 26, and the type II superstring for which the dimension is ten. In both cases we have to deal with two sectors, the left- and the right-moving modes without mixing. Besides, in these models strings are "neutral" objects, in the sense that they cannot have attached charges by means of Chan-Paton factors.

The independence between the left and right moving modes in the closed string models can be used to construct a hybrid theory, the heterotic string [10] (for a general review see also [2, 5, 138]). We combine the 10 right-moving modes of a type-II superstring with the 26 left-movers of a bosonic string after compactifying 16 of them into an internal manifold. We work in the light cone gauge. The physical degrees of freedom in the right-moving sector are the 8 transverse bosons $X^i(\tau-\sigma)$ and one Majorana-Weyl ten-dimensional fermion $S^a(\tau-\sigma)$ $(a=1,\ldots,8)$. In the left-moving sector we have 24 transverse bosonic coordinates $X^i(\tau+\sigma)$ and $X^I(\tau+\sigma)$ $(I=1,\ldots,16)$. The last 16 bosonic coordinates can be interpreted as parametrizing an internal manifold. Consistency of the resulting theory impose that this internal manifold has to be a 16-dimensional torus $\mathbf{T}$

$$\mathbf{T}=\mathbf{R}^{16}/\Lambda_{16} \tag{582}$$

where Λ_{16} is a 16-dimensional lattice. The action for the free heterotic string can be written in term of the transverse modes as

$$S=-\frac{1}{4\pi\alpha'}\int d\tau d\sigma[\partial_\alpha X^i\partial^\alpha X^i+\partial_\alpha X^I\partial^\alpha X^I+i\bar{S}\Gamma^-(\partial_\tau+\partial_\sigma)S] \tag{583}$$

together with the constraints

$$\Phi\equiv(\partial_\tau-\partial_\sigma)X^I=0 \qquad I=1,\ldots,16 \tag{584}$$

The action (583) is invariant under the supersymmetry transformations

$$\begin{aligned}\delta X^i &= \frac{1}{\sqrt{p^+}}\bar{\epsilon}\Gamma^i S\\ \delta S &= \frac{i}{\sqrt{p^+}}\Gamma_-\Gamma_\mu(\partial_\tau-\partial_\sigma)X^\mu\epsilon\end{aligned} \tag{585}$$

where ϵ is a right-moving Majorana-Weyl light-cone spinor.

It is quite easy to obtain the spectrum for the heterotic string, since what we have to do is to put together the results for the bosonic and supersymmetric strings. Expanding the different fields into modes, we get the set of operators

$$\{\alpha_n^i, S_n^a, \tilde{\alpha}_n^i, \tilde{\alpha}_n^I\} \tag{586}$$

with $n \in \mathbf{Z}$ and $a, i = 1, \ldots, 8$. Besides these operators we also have the center of mass positions x^i and x^I and the momenta p^i and p^I. The commutation relations for the modes of the fields X^I satisfying the constraints (584) are obtained using Dirac brackets (see [10]). The commutation relations for the modes are

$$\begin{aligned}
[\alpha_n^i, \alpha_m^j] &= [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{m+n,0} \\
\{S_n^a, S_m^b\} &= (\Gamma^+ h)^{ab}\delta_{n+m,0} \\
[\tilde{\alpha}_n^I, \tilde{\alpha}_m^J] &= n\delta^{IJ}\delta_{m+n,0} \\
[q^i, p^j] &= i\delta^{ij} \qquad [q^I, p^J] = \frac{i}{2}\delta^{IJ}
\end{aligned} \tag{587}$$

We define the normal-ordered number operators

$$\begin{aligned}
N &= \sum_{n>0}(\alpha_{-n}^i \alpha_n^i + \frac{n}{2}\bar{S}_{-n}\Gamma^- S_n) \; , \\
\tilde{N} &= \sum_{n>0}(\tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \tilde{\alpha}_{-n}^I \tilde{\alpha}_n^I) \; ,
\end{aligned} \tag{588}$$

in terms of which the mass formula becomes

$$\frac{1}{2}\alpha^{'} m^2 = N + \tilde{N} + \frac{1}{2}\sum_{I=1}^{16}(p^I)^2 \tag{589}$$

As in the case of bosonic and supersymmetric strings, we have to implement the level matching condition:

$$N = \tilde{N} + \frac{1}{2}\sum_{I=1}^{16}(p^I)^2 - 1 \; . \tag{590}$$

We still need to determine the internal manifold in which we have compactified the 16 extra bosonic coordinates. The constraints on Λ_{16} come, as might be expected from previous discussions, from the requirement of modular invariance. Once again this requirement determines the spectrum. In the case at hand, it will be shown below that Λ_{16} must be an even integral euclidean self-dual lattice. These lattices are very rare; in fact there are only two such lattices, the weight lattice of $Spin(32)/\mathbf{Z}_2$ (the modding by $\mathbf{Z}_2$ eliminates one of the spin representation, and the root lattice of $SO(32)$ is a sublattice) and the root lattice of $E_8 \times E_8$.

Now we proceed to determine the spectrum of the heterotic string. The ground state $|0\rangle$ for left-moving modes is defined by

$$\tilde{\alpha}_n^i|0\rangle = \tilde{\alpha}_n^I|0\rangle = 0 \; , \qquad n > 0 \; . \tag{591}$$

In the right-moving sector we have the eight bosonic states $|i\rangle$ and the their fermionic partners $|a\rangle$. The mass formula, the level matching condition and the structure of Λ_{16} imply that there are no tachyons in the spectrum. The massless states are

$$\tilde{\alpha}_{-1}^i|0\rangle_L \otimes |i\rangle_R \tag{592}$$
$$\tilde{\alpha}_{-1}^i|0\rangle_L \otimes |a\rangle_R \tag{593}$$
$$\tilde{\alpha}_{-1}^I|0\rangle_L \otimes |i\rangle_R \tag{594}$$
$$\tilde{\alpha}_{-1}^I|0\rangle_L \otimes |a\rangle_R \tag{595}$$
$$|p^I\rangle_L \otimes |i\rangle_R \tag{596}$$
$$|p^I\rangle_L \otimes |a\rangle_R \tag{597}$$

with $(p^I)^2 = 2$. The states (592) and (593) form a $N = 1$, $d = 10$ supergravity multiplet containing the graviton and the gravitino together with the antisymmetric tensor, the dilaton and its supersymmetric partners. States (594)-(597) form the $N = 1$, $d = 10$ super-Yang-Mills multiplet with gauge group $E_8 \times E_8$ or $Spin(32)/\mathbf{Z}_2$. Then the low energy field theory ($\alpha' \to 0$) of the heterotic string is $N = 1$ SUGRA coupled to $N = 1$ super-Yang-Mills in $d = 10$ with gauge group $E_8 \times E_8$ or $SO(32)$. Then the heterotic string provide us with the first example of a string theory with gauge group $E_8 \times E_8$ which could not be obtained in the open string case.

We see, by applying Green-Schwarz cancellation mechanism, that the low energy field theory for the heterotic string is free from both gravitational and gauge anomalies. Moreover, it can be shown that the theory is anomaly-free at the string level if the gauge groups are $E_8 \times E_8$ or $Spin(32)/\mathbf{Z}_2$ [139]. So the cancellation of anomalies with these gauge groups not only works at the level of the low energy effective field theory but it can be verified for the whole string theory.

Let us compute the one loop partition function for the heterotic string. Had we formulated the heterotic string using the old superstring formalism (which is explicitly world-sheet supersymmetric) we would have had to include the GSO-projected NS and R sectors of the right movers. We have also the contribution from the 8 transverse bosonic coordinates of the left-moving sector and the 16 internal degrees of freedom. When all the contributions are included the one loop partition function for the heterotic string (up to constant factors) becomes:

$$Z_{het}(\tau, \bar{\tau}) = \tau_2^{-4} \frac{\bar{\Theta}(\bar{\tau})}{\bar{\eta}^{24}(\bar{\tau})} \eta^{-12}(\tau) [\theta_3^4(0|\tau) - \theta_3^4(0|\tau) - \theta_2^4(0|\tau)] \ , \tag{598}$$

where $\bar{\Theta}(\bar{\tau})$ is the theta function associated with the internal lattice Λ_{16}

$$\bar{\Theta}(\bar{\tau}) = \sum_{\mathbf{p} \in \Lambda_{16}} \bar{q}^{\frac{1}{2}\mathbf{p}^2} \ . \tag{599}$$

In order for the partition function $Z_{het}(\tau, \bar{\tau})$ to be modular invariant, we need $\bar{\Theta}(\bar{\tau})$ to transform under the T and S according to

$$\begin{aligned} T : \bar{\Theta}(\bar{\tau}) &\longrightarrow \bar{\Theta}(\bar{\tau}) \\ S : \bar{\Theta}(\bar{\tau}) &\longrightarrow \bar{\tau}^8 \bar{\Theta}(\bar{\tau}) \end{aligned} \tag{600}$$

It is easy to check that the first condition implies that $\mathbf{p}^2$ is an even integer. This means that Λ_{16} has to be an even lattice. To impose the condition with respect to S we use the Poisson summation formula:

$$\bar{\Theta}\left(-\frac{1}{\bar{\tau}}\right) = \frac{\bar{\tau}^8}{\sqrt{\det g}} \sum_{\mathbf{p} \in \Lambda_{16}^*} \bar{q}^{\frac{1}{2}\mathbf{p}^2} \tag{601}$$

where Λ_{16}^* is the dual lattice of Λ_{16} and g_{ij} is the metric of the lattice. Then the S and T conditions imply that Λ_{16} is even and self-dual. Let us note that, as in the superstring case, the partition function (598) vanishes because of Jacobi's famous theta function identity. This is again a consequence of the space-time supersymmetric character of the heterotic string.

4.3 Strings at finite temperature

In the Standard Cosmological Model (see for example [21]), at very early times the Universe was in a very hot state. Close to the Planck time this temperature could be of the order of the Planck mass. Thus it seems reasonable to think that if String Theory is a theory of Quantum Gravity, it should have a lot to say about the very early Universe [140, 141, 142, 143]. A first step in this direction is to study the thermal properties of strings. This is what we do in this section. As we will see later, modular invariance also plays a rôle in determining the structure of the string free energy.

When studying (critical) strings at finite temperature we meet two properties which are typically "stringy". The first one is the fact that for every string theory there is a temperature at which the canonical partition function diverges. The presence of this temperature, called the Hagedorn temperature [144], is due to the fact that the number of states for a given mass in any critical string theory grows exponentially with the mass. For example, in the bosonic string the asymptotic density of states with mass m is given by

$$\rho(m) \sim m^{-\frac{25}{2}} e^{4\pi\sqrt{\alpha'}m} . \tag{602}$$

The canonical partition function $Z(\beta)$ is defined as the Laplace transform of the density of states ($\beta = 1/T$)

$$Z(\beta) = \int_0^\infty dE \rho(E) e^{-\beta E} . \tag{603}$$

In our case we can write [145]

$$Z(\beta) \sim \int_\mu^\infty dE E^{-\frac{25}{2}} e^{(4\pi\sqrt{\alpha'}-\beta)E} , \tag{604}$$

where μ is a cutoff in order to make (602) applicable. Now it is easy to see that the partition function $Z(\beta)$ is defined as long as

$$\beta > 4\pi\sqrt{\alpha'} . \tag{605}$$

The right-hand side of the inequality defines the Hagedorn temperature for the bosonic string

$$\beta_H = 4\pi\sqrt{\alpha'} \tag{606}$$

On the other hand, the Helmholtz free energy of the bosonic string is, because of the tachyon, undefined (infinite) for every value of β. Nevertheless, the Hagedorn temperature is present in every critical string theory even in those which have a well defined low temperature phase, as the superstring or the heterotic string (cf. for example [146]). The Hagedorn temperature is a consequence of the fact that the density of states grows exponentially with the energy.

The second "stringy" feature mentioned is duality property (sometimes called β-duality). This is expressed by the equality of the canonical partition function of the theory evaluated at β and *constant*/β. For example, for the heterotic string [147]

$$Z(\beta) = Z\left(\frac{\pi^2}{\beta}\right) \tag{607}$$

Note that although formally this duality property is quite similar to the R-duality property that we have treated in sec. 3.6, they are in fact quite different. We will see that the kind of compactification that give raise to both "dualities" are of very different type. β-duality is present formally in the bosonic string (it is a symmetry for the integrand of the free energy), and in the heterotic string (see below); in the type II superstring there is no duality symmetry although it is possible to construct a kind of "generalized" duality relation for the one-loop free energy [148].

As in ordinary Statistical Mechanics, it is convenient to work with the Helmholtz free energy $F(\beta)$:

$$F(\beta) = -\frac{1}{\beta} \log Z(\beta) \tag{608}$$

In QFT, the canonical partition function in $\mathbf{R}^{1,d-1}$ is equal to the vacuum energy of the same theory in $\mathbf{R}^{d-1} \times S^1$ with the length of the compactified circle equal to β. Using this procedure Polchinski computed the Helmholtz free-energy per unit volume for the bosonic string at one-loop [149] with the result

$$F(\beta) = -\pi^{-26} 2^{-14} \int_S \frac{d^2\tau}{\tau_2^2} \tau_2^{-12} |\eta(\tau)|^{-48} \left[\theta_3 \left(0 \left| \frac{i\beta^2}{2\pi^2 \tau_2} \right.\right) - 1 \right] \tag{609}$$

where $\tau = \tau_1 + i\tau_2$ is the modular parameter of the torus, $\eta(\tau)$, $\theta_3(0|\tau)$ are respectively the Dedekind eta function and the Jacobi theta function and the integral is evaluated over the strip $S = \{\tau | \tau_2 > 0, -1/2 < \tau_1 < 1/2\}$.

As we announced earlier, (609) does not converge because of the infrared divergence caused by the tachyon as $\tau_2 \to \infty$. However this result does not correspond to the vacuum energy of a string in $\mathbf{R}^{d-1} \times S^1$ since there are no winding modes corresponding to the would-be space direction in the world-sheet and we know that $\tau \to -1/\tau$ interchanges space and time coordinates on the world-sheet. As a consequence, the integral in (609) is not restricted to the fundamental region of the modular group $PSL(2, \mathbf{Z})$ as we should expect from a path integral calculation, since the integral over the modular parameter comes from the integration over tori not equivalent under diffeomorphisms [150, 147].

It was observed by Polchinski [149] that the path-integral computation of the one-loop free energy (609) coincides with what one would obtain by adding the contributions to the free energy of the different states in the string spectrum. This approach to the computation of the free energy which consists in considering the string as a collection of fields is called the analog model [140, 151, 152, 141, 145, 153] and gives actually the same result than that obtained by Polchinski.

Let us begin [153] with the expression for the free energy per physical degree of freedom and per unit volume of a bosonic (fermionic) quantum field in d dimensions

$$F_{B,F}(\beta) = \pm \frac{1}{\beta} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \ln(1 \mp e^{-\beta\omega_k}) \tag{610}$$

where $\omega_k = \sqrt{\mathbf{k}^2 + m^2}$. After some elementary manipulations (610) can be rewritten as

$$F_B(\beta) = -\pi^{d/2} 2^{d/2-1} \int_0^\infty ds\, s^{-1-d/2} e^{-m^2 s/2} \left[\theta_3 \left(0 \left| \frac{i\beta^2}{2\pi s} \right.\right) - 1 \right] \tag{611}$$

for a bosonic field and as

$$F_F(\beta) = \pi^{d/2} 2^{d/2-1} \int_0^\infty ds\, s^{-1-d/2} e^{-m^2 s/2} \left[\theta_4\left(0\left|\frac{i\beta^2}{2\pi s}\right.\right) - 1\right] \tag{612}$$

for a fermionic field. The general procedure now is to add the contributions of the states in the string given by the preceeding formulae (taking into account the statistic of the corresponding state). As we have seen in previous sections we have to impose the level matching condition. This can be implemented by inserting the following δ-function:

$$\delta_{nm} = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 e^{i2\pi(n-m)\tau_1} \tag{613}$$

One easily convince oneself that the free energy obtained using the analog model corresponds to the free energy of an ensemble of strings, since we sum the contributions of second quantized fields. Using the analog model it is possible to evaluate the one-loop free energy of any string theory. The extension to more loops can be done by substituing the mass in (611), (612) by the renormalized mass calculated to the same loop level minus one [154] (see also [155]).

We concentrate now on the heterotic string at finite temperature, since this is the most interesting string model from many points of view. By using the analog model (or evaluating the path integral in the light-cone gauge for a single string and exponentiating the result) we obtain that the one loop free energy is [153]

$$F(\beta) = -\frac{1}{2^6\pi^{10}} \int_S \frac{d^2\tau}{\tau_2^2} \tau_2^{-4} \frac{\theta_2^4}{\eta^{12}} \frac{\bar{\Theta}_\Gamma}{\bar{\eta}^{24}} \theta_2\left(0\left|\frac{2i\beta^2}{\pi^2\tau_2}\right.\right) \tag{614}$$

As in the case of the bosonic string, our integral is performed over the strip S instead of the fundamental region of the modular group $\mathcal{F}$. It is possible to obtain an explicitly modular invariant form for the one-loop free energy [147, 156]. We have that the integrand of (614) is invariant under a subgroup of the full modular group, namely the subgroup U generated by T which is the subgroup that leaves invariant the spin structure $(+,-)$. We can think of the fundamental region of this subgroup (the strip S) as obtained by acting with transformations belonging to $\Gamma = PSL(2,\mathbf{Z})$ on $\mathcal{F}$. This implies that the translation subgroup U can be written as the union of cosets of the modular group:

$$U = \bigcup_{(c,d)=1} \Gamma\gamma_{cd}\ , \tag{615}$$

where c, d are two co-prime integers and $\gamma_{cd} \in SL(2,\mathbf{Z})$ is given by

$$\gamma_{cd} = \left(\begin{array}{cc} * & * \\ c & d \end{array}\right)\ , \tag{616}$$

the two first entries in the matrix being any integers satisfying the condition on the determinant. Then, if we have an integral over the strip S we can write

$$\int_S \frac{d^2\tau}{\tau_2^2} F(\tau) = \int_{\bigcup \gamma_{cd}\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F(\tau) = \sum_{(c,d)=1} \int_{\gamma_{ab}\mathcal{F}} \frac{d^2\tau}{\tau_2^2} F(\tau)\ . \tag{617}$$

If we perform a change of variables for each term in the sum and commute the sum with the integral we finally obtain

$$\int_S \frac{d^2\tau}{\tau_2^2} F(\tau) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{(c,d)=1} F(\gamma_{ab}\tau) \tag{618}$$

where we have used the modular invariance of the measure $d^2\tau/\tau_2^2$. It can be easily seen that the integrand in (618) is modular invariant, since any modular transformation acting on $\sum_{(c,d)=1} F(\gamma_{ab}\tau)$ amounts to change of representatives in the coset expansion (615). This technique can be applied to any integral whose integrand is invariant under any subgroup of Γ. The equivalence between the original expression and its modular invariant extension depends upon the possibility of performing the change of variables and the commutation of the integral and the sum.

Using the technique just described we can obtain an explicit (and divergent) modular invariant expression for the one-loop free energy of the bosonic string which is also invariant under U. In the case of the heterotic string, in order to obtain the modular invariant extension, we have to work a bit more. The reason is that the extension of the "thermal" θ_2 function from U to Γ directly is quite complicated. Instead we use the congruence subgroup $\Gamma_0(2) \subset \Gamma$ [157] as an intermediary subgroup. Then we make the extension from U to $\Gamma_0(2)$ by restricting γ_{cd} to this subgroup, the result being

$$F(\beta) = -\frac{1}{2^6\pi^{10}} \int_{F(\Gamma_0(2))} \frac{d^2\tau}{\tau_2^2} \tau_2^{-4} \frac{\bar{\Theta}_\Gamma}{\bar{\eta}^{24}} \frac{\theta_2^4}{\eta^{12}} \theta \left[\begin{array}{cc} 0 & \frac{1}{2} \\ 0 & 0 \end{array} \right] (0|4\Omega) \tag{619}$$

where $F(\Gamma_0(2))$ is the fundamental region of the intermediary subgroup and we have introduced Riemann theta functions [123] with the period matrix

$$\Omega = \frac{i\beta^2}{2\pi^2\tau_2} \left(\begin{array}{cc} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{array} \right) . \tag{620}$$

Performing now the extension from $\Gamma_0(2)$ to the full modular group Γ we obtain the modular invariant expression for the Helmholtz free energy of the heterotic string

$$\begin{aligned} F(\beta) \;\; = \;\; & -\frac{1}{2^6\pi^{10}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \tau_2^{-4} \frac{\bar{\Theta}_\Gamma}{\bar{\eta}^{24}} \left\{ \frac{\theta_2^4}{\eta^{12}} \theta \left[\begin{array}{cc} 0 & \frac{1}{2} \\ 0 & 0 \end{array} \right] (0|4\Omega) \right. \\ & + \left. \frac{\theta_4^4}{\eta^{12}} \theta \left[\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & 0 \end{array} \right] (0|4\Omega) - \frac{\theta_3^4}{\eta^{12}} \theta \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{array} \right] (0|4\Omega) \right\} \end{aligned} \tag{621}$$

This second extension is quite simple because now $\Gamma_0(2)$ has index 3 with respect to Γ. If we take the limit $\beta \to \infty$ we obtain that $F(\beta) \to 0$, as it should be, because we are dealing with a supersymmetric string theory.

Using this form of the one-loop Helmholtz free energy and after some calculations it is possible to see that there is a critical value of β at which $F(\beta)$ begins to diverge, the Hagedorn temperature for the heterotic string, which is $1/T_H = \pi(\sqrt{2}+1)$ [10]. By using the Poisson summation formula on the Riemann theta functions it is also straightforward to prove the duality relation for $F(\beta)$ [147]

$$F(\beta) = \frac{\pi^2}{\beta^2} F\left(\frac{\pi^2}{\beta}\right) . \tag{622}$$

The duality property, that we have explicitly seen at the one loop level, is satisfied to all orders in string perturbation theory [117]. Let us consider the perturbation series for the Helmholtz free energy

$$F(g_{st},\beta) = \sum_{g=1}^{\infty} g_{st}^{2(g-1)} F_g(\beta) \; , \tag{623}$$

where g_{st} is the string coupling constant. It is possible to show that the genus g contribution to the free energy has the property

$$F_g(\beta) = \left(\frac{\pi^2}{\beta^2}\right)^g F_g\left(\frac{\pi^2}{\beta}\right) \; , \tag{624}$$

so the whole perturbative expansion satisfies the duality relation:

$$F(g_{st},\beta) = \frac{\pi^2}{\beta^2} F\left(\frac{\pi^2 g_{st}}{\beta}, \frac{\pi^2}{\beta}\right) \tag{625}$$

When the existence of a Hagedorn temperature, and the duality relation are considered together, we obtain a curious three phase structure for the Helmholtz free energy [147]. We know that $F(\beta)$ is finite for all $\beta \geq \beta_H$ and that it diverges for $\beta < \beta_H$. Duality implies that there is another critical temperature $\beta_H^* = \pi^2/\beta_H$ such that $F(\beta)$ is finite for $\beta \leq \beta_H^*$ and diverges when we approximate to this value of β from the right. So the final result is that $F(\beta)$ is finite for all $\beta \leq \beta_H^*$, $\beta \geq \beta_H$ and diverges whenever $\beta_H^* < \beta < \beta_H$.

The high temperature phase ($\beta < \beta_H^*$) implied by duality has singular properties. For example the canonical entropy is negative in this phase. One of the most intriguing features of the high temperature phase is that if we evaluate the limit of the free energy as $T \to \infty$ we obtain

$$\lim_{\beta \to 0} F(\beta) \sim \frac{\pi^2}{\beta^2} \Lambda \tag{626}$$

where Λ is the cosmological constant. Since in our case $\Lambda = 0$, this implies that in the high temperature limit there are no degrees of freedom at all.

Some authors have interpreted these facts as showing that the high temperature phase is not physical and that the Hagedorn temperature must be the maximun temperature of the Universe [142]. Other authors, however, have interpreted the loss of gauge-invariant degrees of freedom at high temperature as evidence that in this regime the theory could be described by a topological phase [158]. In this case the presence of the Hagedorn temperature would indicate the transition between the topological and non-topological phases. There are also a number authors [159] who claim that there is a close resemblance between the possible phase transition at the Hagedorn temperature and a Kosterlitz-Thouless phase transition taking place at the same temperature [160]. Nevertheless, these two phenomena do not seem to be closely related since in the $c = 1$ non-critical string we find a Kosterlitz-Thouless phase transition but there is no Hagedorn temperature [118]. The problem of what happens to the string ensemble at the Hagedorn temperature is still an open problem.

In the canonical ensemble description it can be seen that when approaching the Hagedorn temperature from below the system undergoes violent fluctuations [145]. This may be seen as the breakdown of the canonical ensemble and we could expect that the application of the microcanonical ensemble should be useful to study the system beyond the Hagedorn temperature [151, 152, 145, 161, 162]. The density of states $\Omega(E)$ defined by

$$\Omega(E) = tr\,\delta(E - H) \tag{627}$$

can be written as the inverse Laplace transform of the partition function:

$$\Omega(E, V) = \int_{c-i\infty}^{c+i\infty} \frac{d\beta}{2\pi i} Z(\beta, V) e^{\beta E} \tag{628}$$

with $c \geq \beta_H$. It is expected that by analytically continuing the partition function $Z(\beta)$ to the whole complex β-plane and deforming the contour of integration one could get a density of states valid in regimes different from the Hagedorn one (cf. for example [162, 163]). Looking at the distribution of the energy among the ensemble of strings when the Hagedorn temperature is approached from the low temperature phase, the phase transition would correspond to the fact that, when the number of noncompact dimensions is higher than two, a single string is able to absorb all the energy because of the exponential growth of the number of state per energy level for one string [163]. In the microcanonical description this actually implies that the specific heat is negative, showing an inestability in the system that may be associated to a phase transition. Many issues of finite temperature strings remain to be clarified.

4.4 Is string theory finite?

A very attractive feature of String Theory is that it has a real possibility of being an ultraviolet finite quantum theory of gravity. In the previous chapter we learned that in the bosonic strings the more dangerous infinities came from the boundary of moduli space (surfaces with nodes) and they were due to the presence of tachyons in the spectrum. The integration measure is finite and well behaved except near the boundary of $\mathcal{M}_g$. Since for the superstring and the heterotic string the tachyon is projected out after GSO proyection, one would intuitively expect that no further divergences should arise. This is not completely correct, since the massless states may also contribute some subleading divergences. We now however that for one-loop amplitudes there are no divergences [2, 3]. For higher loops we still lack a complete proof, but many exploratory computations has been carried out. For the bosonic string we have two-loop computations [164] as well for the superstring [165]. For the heterotic string there are several computations of the two-loop cosmological constant [166]. It was finally shown in [167] that there is an explicit modular invariant expression for the two-loop cosmological constant of the heterotic string. The result as one might expect from supersymmetry equals zero, after we sum over all spin structures. There are several general arguments [168] based on space-time supersymmetry which indicate the finiteness of superstring perturbation theory. A detailed analysis of the many technical issues involved in the analysis to all order can be found in [169, 104] and references therein.

Another approach to the proof of finiteness has been pursued by Mandelstam [170]. There are several thorny obstacles encountered in trying to work out a rigorous proof of finiteness. First the moduli space of Riemann surfaces encountered in the previous chapter is now replaced by supermoduli space, the moduli space of super-Riemann surfaces. This space is unfortunately poorly understood. For genus g supersurfaces it is a superalgebraic variety of graded dimension $(3g-3|2g-2)$. If we denote in a given patch a coordinate system by (m_α, τ_α) $\alpha = 1, \ldots, 3g-3$, $a = 1, \ldots, 2g-2$; the transition function between two overlapping coordinate patches in supermoduli will take the form:

$$
\begin{aligned}
m_\alpha^{'} &= f_\alpha(m_\beta, \tau_b) = f_\alpha^{(0)}(m_\beta) + f_\alpha^{ab}(m_\beta)\tau_a\tau_b \ldots && (629)\\
\tau_a^{'} &= g_a(m_\beta, \tau_b) = g_a^{(0)b}(m_\beta)\tau_b + g_a{}^{bcd}(m_\beta)\tau_b\tau_c\tau_d + \ldots && (630)
\end{aligned}
$$

We say that the supervariety is projected if we can find a coordinate cover such that in (629) only $f_\alpha^{(0)}(m_\beta)$ appears; and it is split if we can also make $g_a{}^{bcd}$ and higher order terms vanish. In the split case the supermanifold would be equivalent to a vector bundle. In genus one there is no problem. In genus two it was shown by Deligne [171] that supermoduli space is split, but not much is known for genus $g > 2$. The integration measure for superstring scattering amplitudes is a supermeasure on the supermoduli space, and we can use for example the operator formalism extended to this case [125] in order to compute many properties of the supermeasure. Supermoduli space also has a boundary made of surfaces with nodes, but now these nodes comes in two types depending on whether the punctures identified are NS or R punctures.

The first thing we would like to do is to integrate over the odd moduli τ_a and be left with a measure on spin-moduli space; the moduli space of Riemann surfaces with a fixed spin structure; since the modular group acts transitively on the even and odd spin structures, this space has two components. However if supermoduli space is not split we will encounter ambiguities. All these ambiguities translates into total derivatives which however may give non-trivial contributions on the boundary of moduli space. Remember that the tachyon is eliminated summing over spin structures, and the integration over spin moduli space should include this sum. Related to these integration ambiguities is the existence of certain spurious poles found in the superstring measure in [165]. These brief remarks should give the reader a flavor of the technical obstacles found in trying to actually proof that Superstring Theory is finite. In the references quoted the interested reader can find many more details.

A second issue of interest even accepting the finiteness of superstring perturbation theory is its summability. This would be quite undesirable. The non-summability of perturbation theory is a good indication of the existence of important non-perturbative effects which should dominate many dynamical questions. For critical strings, the compactification of extended dimensions and the breaking of various symmetries is expected to come from non-perturbative effects. It has been shown for instance that in the bosonic string the perturbative expansion is not Borel summable [172] (see also [4]). The genus g contribution for large g grows as $(2g)!$, which is a rate of divergence bigger than in ordinary QFT. One might argue that this behavior is due to the tachyon. This interpretation is probably not correct. In fact, in some recent studies of strings in dimensions $d < 1$, the same growth of $(2g)!$ is found (see [173]) and for these dimensions there is no tachyon. This behavior is the more likely evidence of important non-perturbative effects ([173] and references therein). It is not known whether the same divergent behavior is also characteristic of supersymmetric strings.

5 Other Developments and Conclusions

5.1 String Phenomenology

There are many areas of String Theory not presented in these lectures. Most notably String Phenomenology [18]. Two of the most appealing features of models based on the Heterotic String are: i)The gauge group $E_8 \times E_8$ has plenty of room to contain the SM group. ii)After compactifying it is possible to account naturally for a chiral spectrum of low-energy fermions. Once a classical background of the form $M_4 \times K$ (M_4 is four-dimensional Minkowski space and K is a six-dimensional manifold, or a CFT with the appropiate properties) the low energy theory looks like a supersymmetric extension of the SM. The advantage is that in principle the parameters of the low energy effective action are computable once the background K is known.

As we learned in sec. 3.5 the motion of a string in some space-time background requires a CFT on the world-sheet. In Superstring Theory, this two-dimensional field theory should be superconformal invariant. If we want furthermore the theory to be space-time supersymmetric, there is a general argument [174] showing that this is equivalent to $N = 2$ world-sheet superconformal invariance with an integral lattice for the $U(1)$ generator of the $N = 2$ algebra. For backgrounds like $M_4 \times K$, the effective action describing the interactions between the light states looks like any other $N = 1$ SUGRA lagrangian in four dimensions [175]. Any of them can be characterized in terms of three functions: the Kähler potential $\mathcal{K}$ which determines the kinetic term of the scalars in the theory, the superpotential W responsible among other things for the Yukawa couplings, and the gauge kinetic function f which determines de gauge coupling constants.

The subject of String Phenomenology is about the computation of the three functions $\mathcal{K}$, W, f. In ordinary model building the form of these functions depends very much on how the model is constructed. In String Theory these functions are in principle computable directly in terms of the CFT in the given string vacuum solution. The couplings constant of the low-energy lagrangian are computed in terms of correlation functions of the two-dimensional theory. One of the massless scalars which is ubiquitous in String Theory is the dilaton. Its vacuum expectation value is related to the tree-level gauge coupling constants ($\langle\Phi\rangle = 1/g^2$). One of the specific features of four-dimensional strings is that coupling constants are given in terms of expectation values of fields. They depend on the dilaton field as well as on other scalar fields known as moduli fields. An important problem encountered in the determination of $\langle\Phi\rangle$ is that this expectation value is a flat direction in pertubation theory, and therefore non-perturbative effects should generate a non-trivial dilaton potential. As we will see, there are some interesting recent results in this direction.

We should also mention that the breaking of space-time supersymmetry is required to have special characteristics in order to obtain reasonably realistic models. The basic scale in the problem is the Planck scale M_P ($\sim 10^{19}$GeV), and supersymmetry should be broken at scales of the order of 10^{10}-10^{11} GeV. This value is not so easy to obtain with current model building technology. In the resolution of some of the problems listed, some properties of String Theory have been applied with encouraging success. One of these properties is duality (see sec. 3.6) which has no analoge in conventional Field Theory. The structure of the duality group is only known in few cases, but it proved to be very useful in constraining the couplings of the low-energy theory (for reviews and references, see [176]). The other symmetry with purely "stringy" origin is the mirror symmetry [72]. This symmetry is based on the general properties of $N = 2$ superconformal field theories, and again shows that apparently different geometries correspond to just one string model. Using this symmetry it is possible to compute explicitly some of the Yukawa couplings in the superpotential W. The full physical implications of these symmetries is now being vigorously explored.

In the determination of the gauge coupling constants, we need as mentioned before to determine the non-perturbative form of the dilaton effective potential. In QFT non-perturbative effects display a dependence on the coupling constant of the form $\exp(-1/g^2)$. This dependence can be generated by gaugino condesation (for a review see for example [177]). The effective potential generated has the form

$$V_{eff} \sim \sum_a c_a \exp\left(-\frac{24\pi^2}{b_a g_a^2}\right), \tag{631}$$

where the sum runs over the different factors on the gauge groups responsible for gaugino condensation, b_a are the one-loop coefficient of the β functions, and $\langle\Phi\rangle$ enters in (631) through the relation between g_a and $\langle\Phi\rangle$. This potential as written has a minimum at $\langle\Phi\rangle = \infty$ corresponding to $g_a = 0$. This changes once loop corrections and threshold effects are included [178]

$$\frac{1}{g_a^2(p^2)} = \langle\Phi\rangle + b_a \log\left(\frac{M_P^2}{p^2}\right) + \Delta_a(T) \tag{632}$$

The $\Delta_a(T)$ are the threshold corrections. The argument T denotes collectively the dependence of the masses of the heavy fields integrated out on the light scalar fields expectation values. These threshold corrections are crucial to reconcile string model predictions with the current value of the Weinberg angle. The inclusion of (632) into (631) fixes the dilaton expectation value, and this may break space-time supersymmetry (for more details and references see the second reference in [18]). Explicit studies have been carried out so far only in few models, and hopefully a more general picture will develop in the not very distant future. Many more things could be said about Superstring Phenomenology, but this seems as a good point to stop.

5.2 Black Holes and Related Subjects

We finally turn to some of the exciting recent work on String Black Holes. It is well known that in GR, collapsing matter, under certain conditions generates black holes. The singularities in the space-time metric are supposed to always come with a horizon (Cosmic Censorship Hypothesis). A property of classical black holes is to come with no hair [47]. The only parameters which characterize a static black hole are its mass, angular momentum and charge. Any other attribute of the matter which collapses to form it dissapears. This obviously provides a rather efficient mechanism to violate global symmetries like baryon number. In the mid seventies Hawking made the momentous discovery that black holes radiate with a thermal spectrum whose temperature is inversely proportional to its mass [50]. Thus, unlike better known thermodynamics systems it heats up as it radiates. The thermal nature of the radiation found by Hawking lead him to assert that black holes produce an inherent loss of quantum coherence [179]. This has generated a big controversy in the last fifteen years. The computations carried out by Hawking and others involve quantizing fields in the presence of a classical background gravitational field, and the back-reaction is not taken into account. Among other reasons because to properly understand the back reaction we need a well defined quantum theory of the gravitational field. The last sentence should of course be qualified by String Theory, since it is alledged to provide a consistent quantum theory of gravity. Two approaches has been followed on analyzing this problem. On the one hand one can study some simple toy model of the gravity+dilaton system in two dimensions suggested by the equation of motion of String Theory (for details and references see [56]). These simple systems provide a precise formulation of Hawking's paradoxes in a situation where we have some control on the approximation used. The second approach began after Witten [180] found an exact string theory in two dimensions based on an $SL(2,\mathbf{R})/U(1)$ coset model which describe a true string black hole in two dimensions. We know from QFT that often in the translation of a problem form four to two dimensions it loses most of its flavor. This does not seem to be the case here. Many of the conceptual issues involved in the four-dimensional problem continue to be non-trivial in the two-dimensional case. Since Witten's solution is perfectly sensible as a CFT (and more importantly as a String Theory), an important research effort was initiated after Witten's paper to try to understand what answers it gives to black hole paradoxes. Furthermore, recent revisions of the no-hair theorem (see [181] for details and references) also find a setting in this two-dimensional scenario, in a way possibly connected with the loss of quantum coherence. The authors of [182], on the basis of the properties of Witten's solution have proposed solutions to many of the riddles of black holes and string physics.

The explanation of dilaton black hole solutions and of Witten's black hole string theory is currently going at a fast pace. Details can be found in the literature and in Polyakov's lectures (this volume). The fact that some of the deepest questions in Quantum Gravity are beginning to be addressed by String Theory is both exciting and encouraging. It is however hard to guess at this moment what will be the outcome of these investigations.

We have finally reached the end of these lectures. We hope to have conveyed the impression that String Theory is an important, and active area of research which may ultimately provide us with a consistent picture of Quantum Gravity and of its relations with the other interactions. We may only hope that some of the ideas and proposals put forward in the last few years contain the seeds which will lead to this Grand Synthesis.

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