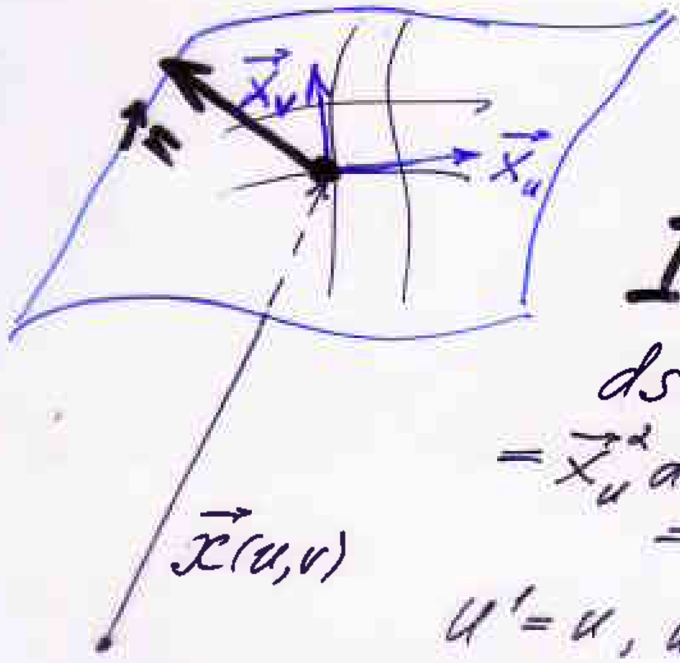
Lecture 5. Rigid string model. Kleinert - Polyakov action.

Lattice simulations with Nambu-Goto string model have shown that string world sheet becomes very crompled. It results in additional divergences. This fact and comparing with QCD have argued that it is reasonable to add an additional term to the Nambu-Goto action.

A short excursition into the classical differential geometry. Surfaces in 3-dimensional Eucledean space

$\vec{x}(u,v)$ position vector



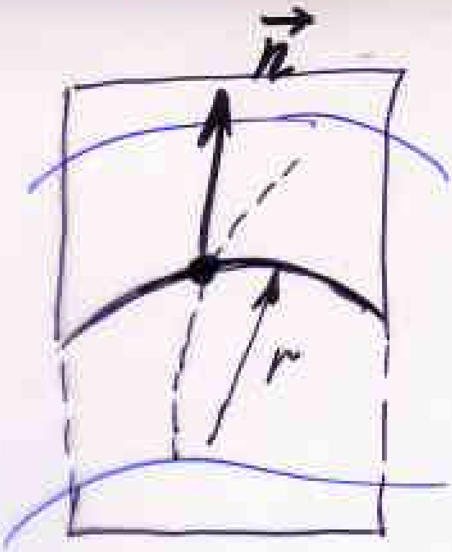
$$d\vec{x} = \vec{x}_u du + \vec{x}_v dv$$

I first fundamental form:

$$ds^2 = d\vec{x}\,d\vec{x} = \vec{x}_u^{\,2} du^2 + 2\vec{x}_u\vec{x}_v du\,dv + \vec{x}_v^{\,2} dv^2 = g_{ij}(u)\,du^i du^j$$

$$u^1 = u, \quad u^2 = v$$

$$g_{ij} = \begin{vmatrix} \vec{x}_u^{\,2} & \vec{x}_u\vec{x}_v \\ \vec{x}_u\vec{x}_v & \vec{x}_v^{\,2} \end{vmatrix}$$



$r_1 \le r \le r_2$

Directions when $r = r_1$ and $r = r_2$ are normal to each other.

Geometrical invariants:

Scalar curvature

$$K = \frac{1}{r_1 \cdot r_2}$$ intrinsic geometry of the surface! $K \equiv K(g_{ij})$!

Mean curvature

$$H = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$ extrinsic geometry!

$$\iint K(u,v)\,dS = \iint K(u,v)\sqrt{g}\,du\,dv = 2\pi\chi$$

χ is the Euler number (topological invariant)

If we would like to controle the shape of the surface then the action should involve the mean curvature.

If $D = 4$ then there are two normals at each point of strong world sheet.

$H \Rightarrow H_\alpha,\ \alpha = 1,2$

$H \Rightarrow \sqrt{H_1^2 + H_2^2}$

II second fundamental form of the surface

$$dl^2 = b_{ij}\,du^i du^j \qquad b_{ij} = b_{ij}(\partial\vec{x}, \partial^2\vec{x})$$

$$\downarrow$$

$$dl_\alpha^2 = b_{\alpha|ij}\,du^i du^j$$

$$H_\alpha = \frac{1}{2} b_{\alpha|ij}\, g^{ij}$$

Weingarten equations

$$\nabla_j x^\mu_{,i} = -\sum_{\alpha=1}^{2} b_{\alpha|ij}\, n^\mu_\alpha \qquad \Big|\, g^{ij}$$

Squaring this, one obtains

$$\sum_{\alpha=1}^{2} H_\alpha^2 = \left(\Delta x^\mu\right)^2,$$ where Δ is the Laplace-Beltrami operator on the string world surface

Polyakov - Kleinert action.

$$S_{rigid} = -\gamma \iint \sqrt{-g}\, ds + \frac{1}{\alpha} \iint d\tau\, d\sigma \sqrt{-g} \sum_{\alpha=1}^{D-2} H_\alpha^2$$

$$+ \frac{1}{\alpha} \iint d\tau\, d\sigma \sqrt{-g}\, \Delta x^\mu \Delta x_\mu$$

Euler-Lagrange equations

$$(-\gamma + \alpha^{-1}\Delta)\Delta x^\mu(\tau, \sigma) + \dots = 0$$

It cannot be linearized by some gauge fixing!

Additional ghost state in the rigid string model are due to the second derivatives in the action.

Example.

$$(\partial^2+m^2)\partial^2\varphi = 0 \quad !$$

$$G(k)=\frac{1}{k^2(k^2-m^2)} = \frac{1}{m^2}\left(\frac{1}{k^2-m^2} - \frac{1}{k^2}\right)$$

$$G \sim [a(\vec{k}), a^+(\vec{k}')] = \delta(\vec{k}-\vec{k}')$$

$$[b(\vec{k}), b^+(\vec{k}')] = -\delta(\vec{k}-\vec{k}')$$

$$|b^+(\vec{k})|0\rangle|^2 < 0 \quad !$$

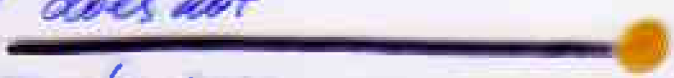
Rigidity term in action (in its Eucledean form) prevents the crampling of the string world sheet (computer simulations).

Membrane theory

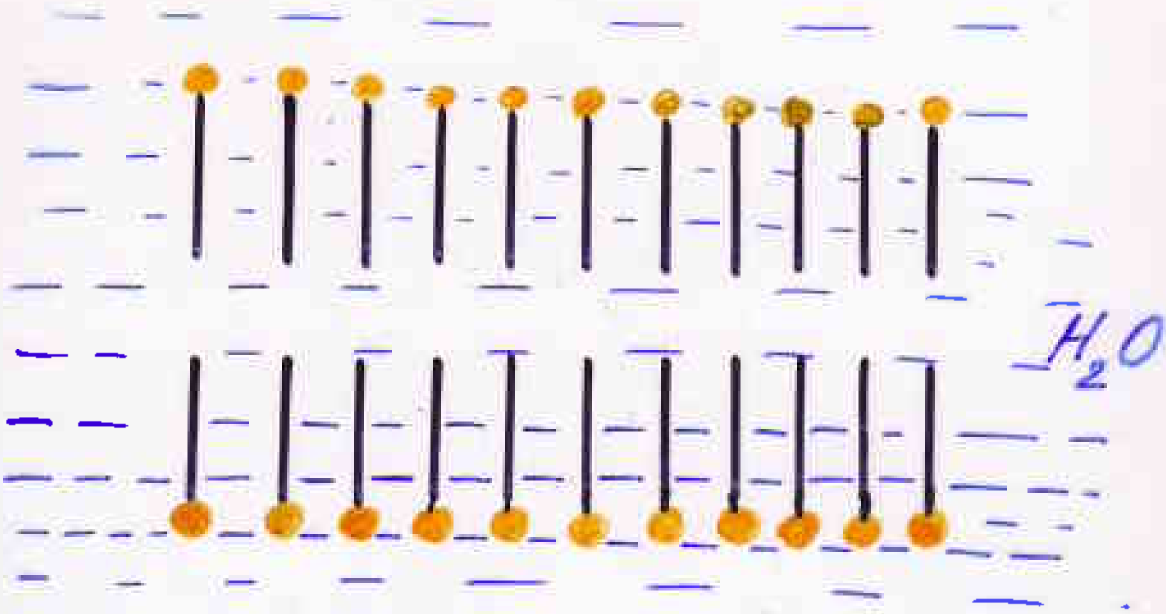
(Helfrich, 1973)

The theory of lipid bilayers

lipid molecules = long chain



Water solution of lipid moleculues



Effective Hamiltonian (closed membranes)

$$\mathcal{H} = \frac{1}{2} k_c \oiint (H - H_0)^2 dS + \Delta p \iiint dV + \lambda \oiint dS$$

H_0 is spontaneous (mean) curvature; (if there is any)

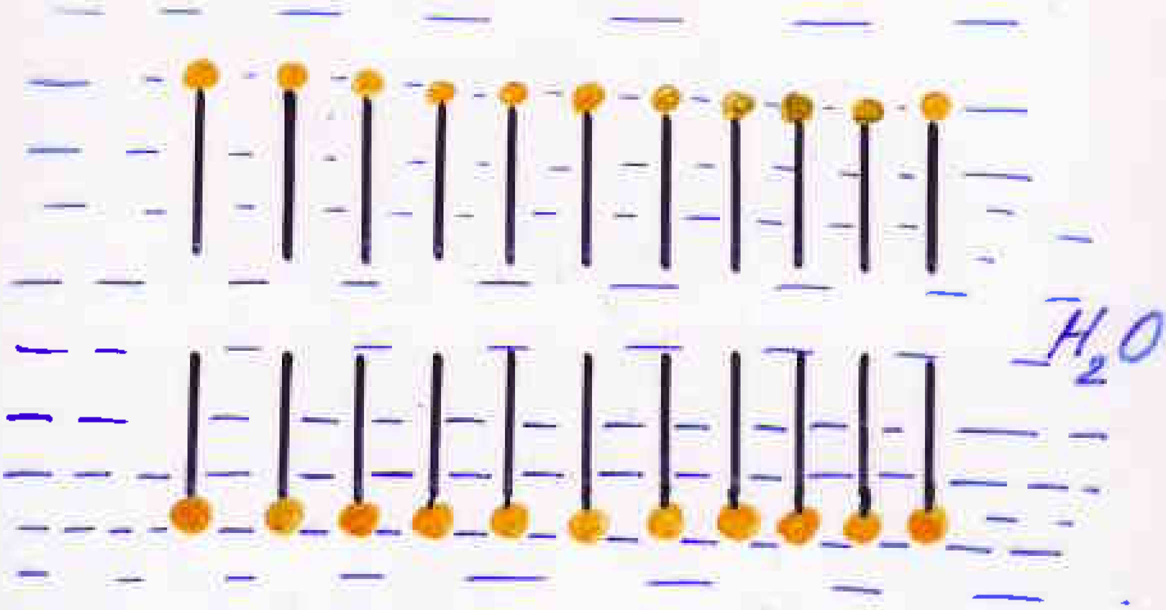
k_c is «the Young modulus» of the membrane;

λ is an induced surface tension due to the condition $S = const$ (λ is simply the Lagrange multiplier).

Δp is the difference of presure (inside and outside the vesicles (Δp is a Lagrange multiplier in order to incorporate the condition $V = const$).

Membrane is liquid, i.e. lipid moleculues can move inside membrane freely!

Water solution of lipid moleculues



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Equations of motion in a geometric form

$$\delta \mathcal{H} = 0$$

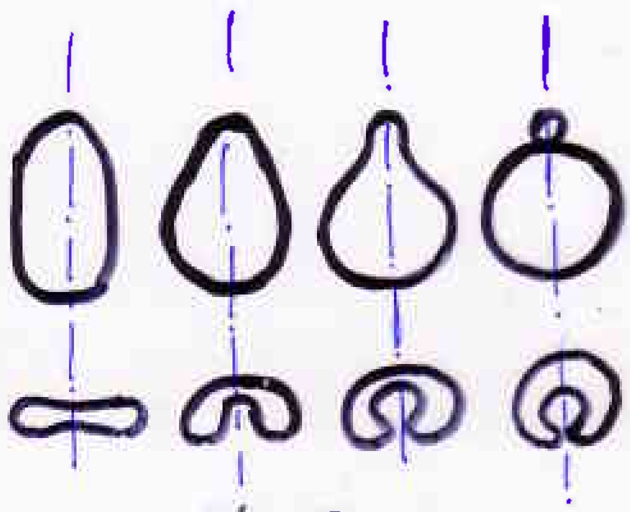
$$2k_c \Delta H + k_c(2H + H_0)(2H^2 - 2K - H_0 H) - 2\lambda H + \Delta p = 0 \quad (*)$$

$H(u^1, u^2)$ is the mean curvature;

$K(u^1, u^2)$ is the Gauss curvature;

$\Delta(u^1, u^2)$ is the Laplace-Beltrami operator on surface to be found.

The observed shapes of vesicles:



These shapes can't be described without H^2 term in Hefrich's Hamiltonian (the Laplace theory)

Willmore surfaces

Willmore functional

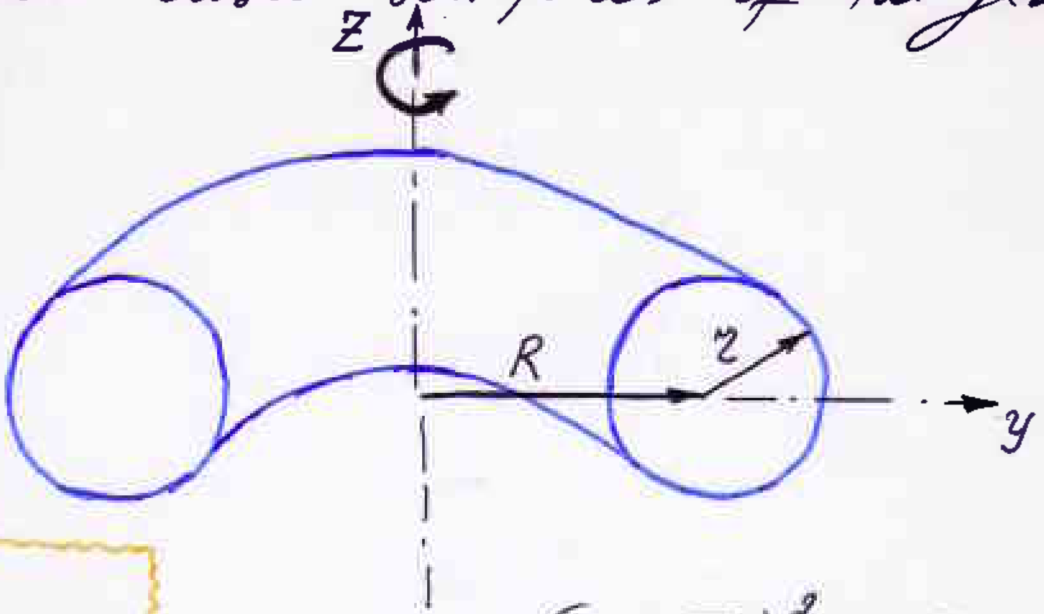
$$W = \oint\!\!\oint ds\, H^2; \qquad \underline{\delta W = 0}$$

$\downarrow$

Willmore surfaces

Willmore conjecture:

The absolute minimum for the functional W is given by the Clifford torus (between closed surfaces of the genus 1)



$$\boxed{\frac{r}{R} = \frac{1}{\sqrt{2}}}$$

$$(y - \sqrt{2})^2 + z^2 = 1$$

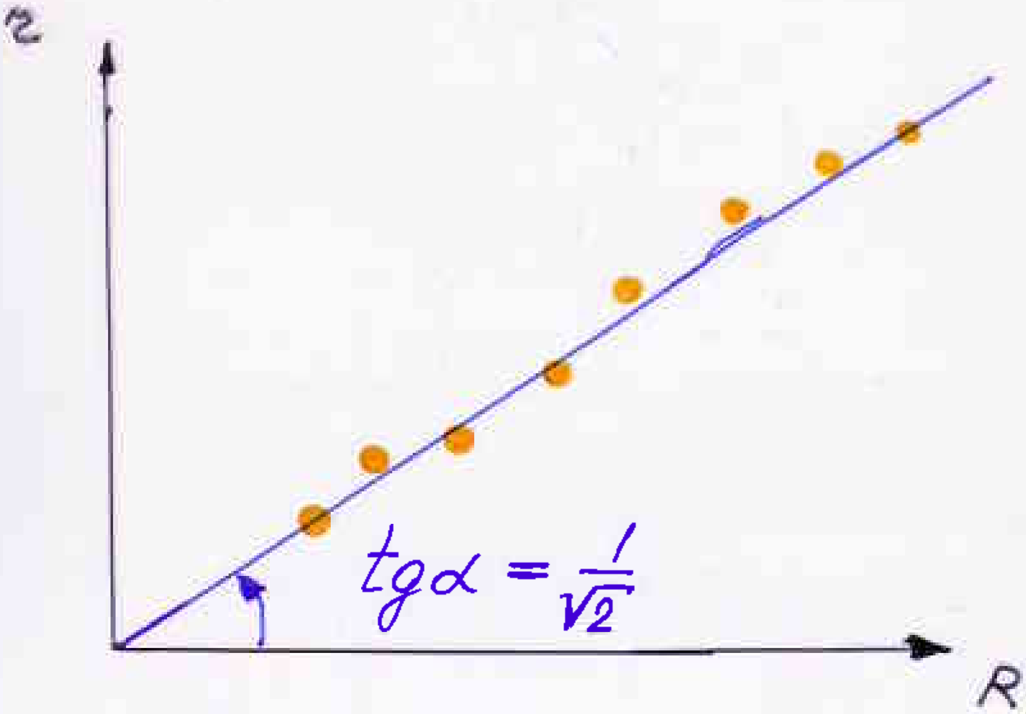
The Clifford torus is a solution to basic equation ⊛ when

$$\Delta p = -2k_c H_0 \quad \text{and}$$

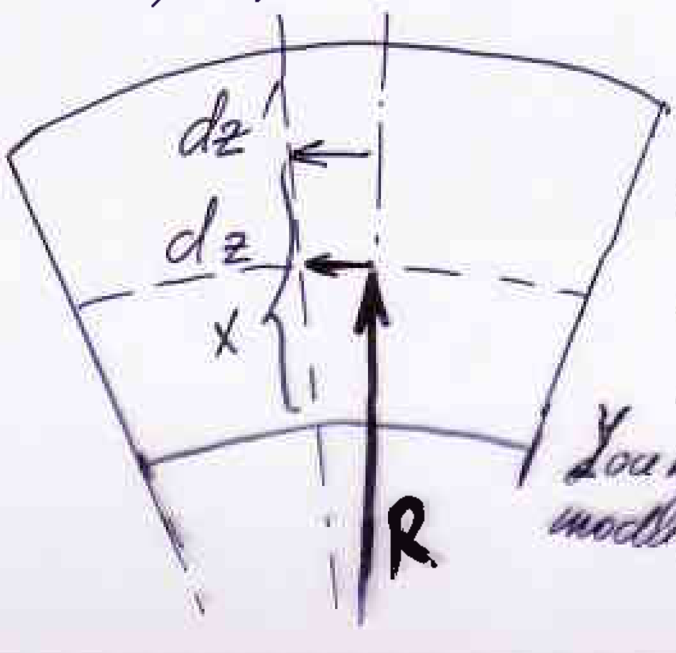
$$\lambda = -2k_c\left(H_0 + \frac{1}{4}H_0^2\right).$$

Experimental «proving» of the Willmore conjacture

«Observation of toroidal vesicles»
M. Mutz and D. Bensimon. Phys. Rev. A43, No.8, (1991) 4525.



Mechanical system: the elastic energy of flexicable rod



$dz' = (1 + \frac{x}{R})dz$

the streng =

$= E_* \frac{x}{R}$ ← Hook's law

Young modulus

$E \sim E_* \frac{x}{R} \cdot \frac{x}{R} = \frac{E_*}{R^2} x^2$

Basic variational problems in classical differential geometry and their applications in theoret. physics

Geometry	Physics
1. Minimal surfaces $A_1 = \alpha \iint dS$; $\delta A_1 = 0$ $H = 0$	Relativistic strings, membranes, p-branes, hadronic strings, cosmic strings
2. Extremal surfaces enveloping fixed volume $A_2 = \alpha \oiint dS + \beta \iiint dV$ $\delta A_2 = 0, \to H = \frac{\beta}{\alpha} = const \neq 0$	W-geometry, W-strings the Laplace theory on membranes
3. Willmore surfaces $A_3 = \alpha \oiint H^2 dS$ $\delta A_3 = 0 \to$ Willmore eq. $\Delta H + 2H^3 + R \cdot H = 0$	Rigid strings; biological (lipid or liquid) membranes, red blood cells

4. Willmore surfaces with fixed enclosed volume and area

Helfrich theory of liquid membranes and vesicles

$$A_4 = \alpha \oiint H^2 dS + \beta \oiint dS + \gamma \iiint dV$$

Lecture 6. Cosmic strings.

The large scale structure of the Universe

charackteristic scale:

i) Stars with planet systems $\sim 10^8$ km

ii) Galaxies containing $\leq 10^{13}$ stars $3 \cdot 10^4 \underbrace{\text{lightyears}}_{ly} \sim$

$M_G \sim 10^{12} M_\odot$

~ 10 kpc

(1pc = 1 parsek = = 3.26 ly)

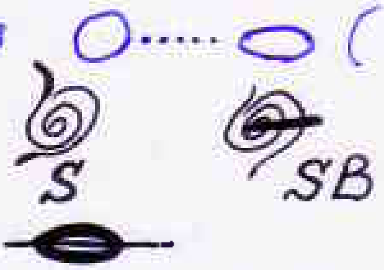
E0 .... E7

25% elliptic ones

50% spiral

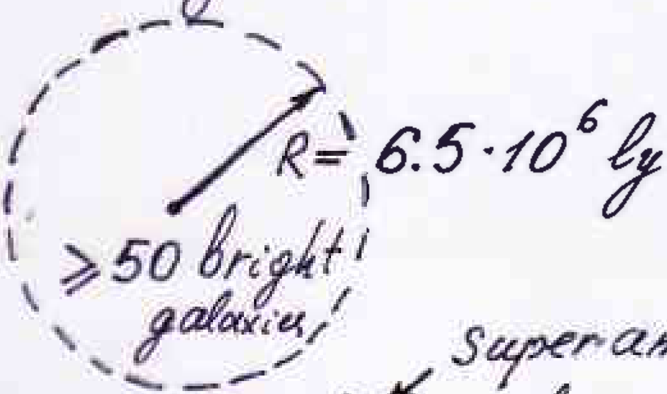
20% linseshape
?

5% irregular



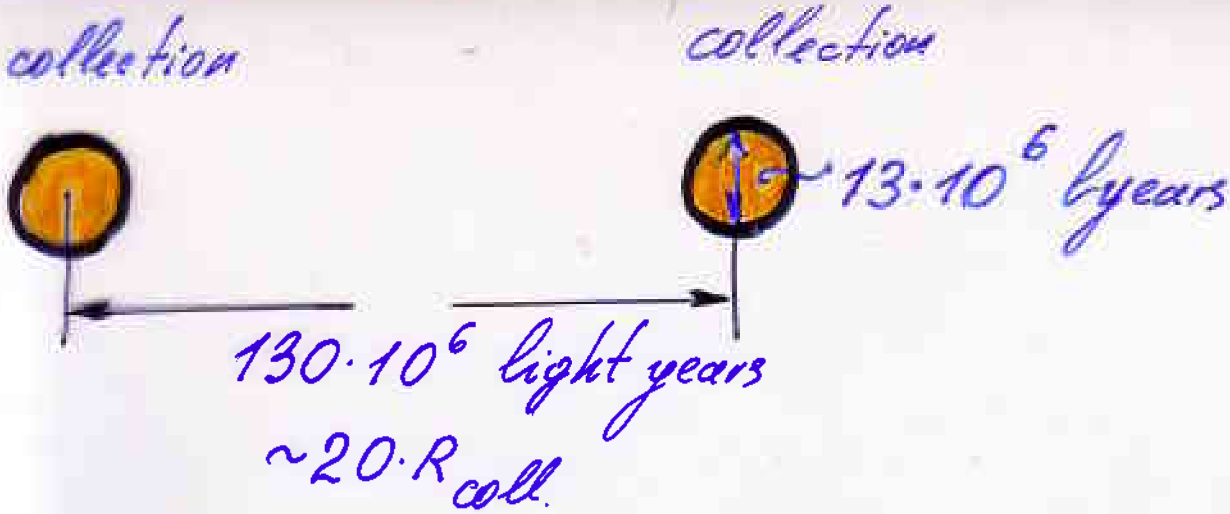
ammassi

iii) Collections of galaxies (J. Abell, 1958) $1 \div 10$ Mpc $\sim 10^7$ ly



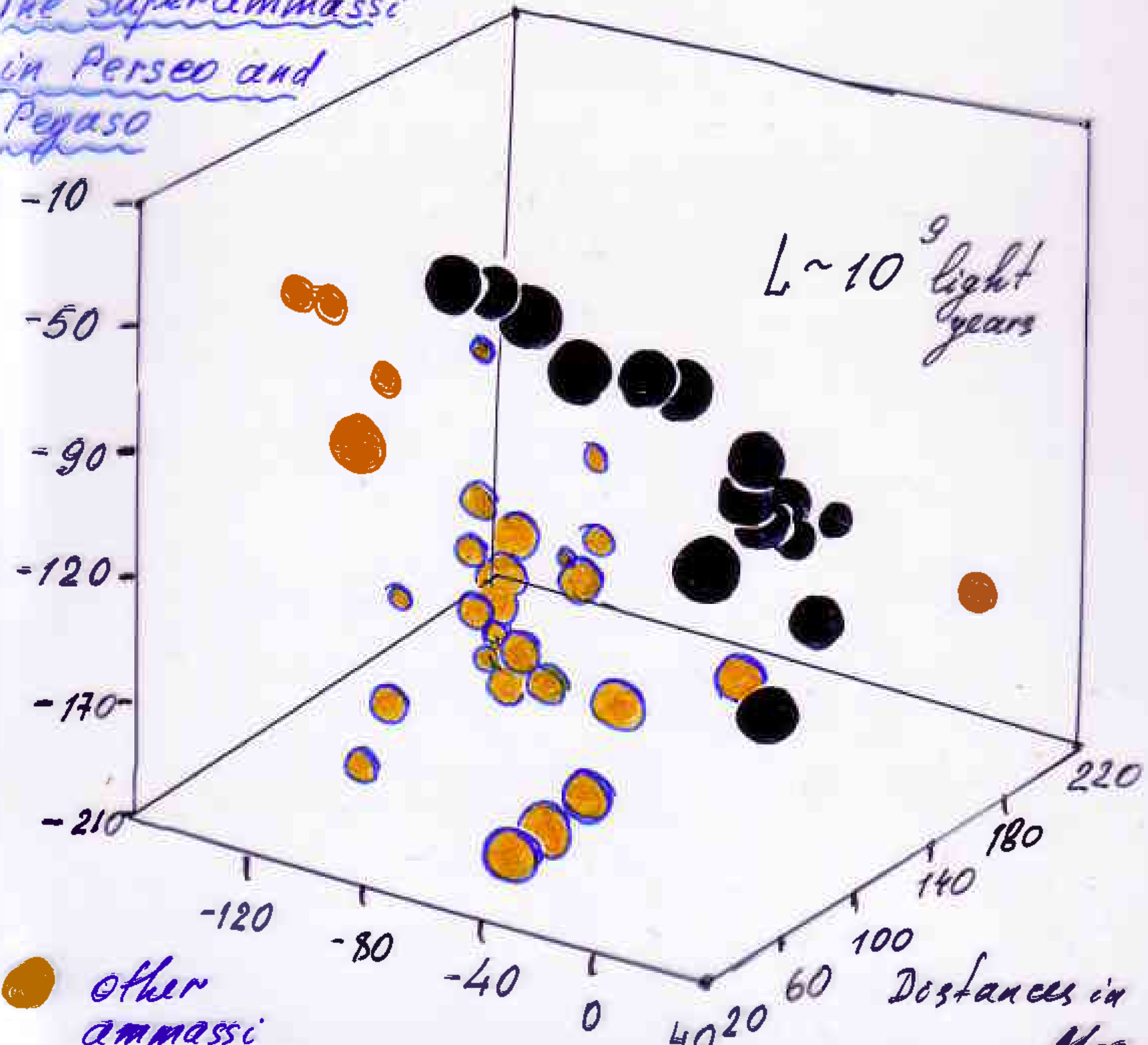
Superammassi

iv) Supercollections of galaxies (collections of collections) ~ 300 Mpc 10^9 lyears



line-like (or fibre-like) structure of supercollections

The Superammassi in Perseo and Pegaso



The idea about endless (infinite) sequence of scales and structures.

(Sharlje, 1908)

This idea does not survived.

At distances > 200 Mpc the matter distribution in the Universe is

homogeneous

The homogeneity of the relict background of γ-quanta

$$\frac{\Delta T}{T} = \frac{1}{3} \cdot 10^{-4}$$ (inside the angle of 6°)

Hubble, 1929 (the red shift)

the running of galaxies

$$V = H \cdot D$$

V is the velocity, D is a distance, and H is the Hubble constant

$$H = 50 \div 100 \frac{km}{сек \cdot Mпс}$$

The standard cosmological model

Input: homogeneous and isotropic Universe

$\oplus$

Einstein eqs.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu};$$

$\oplus$

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p\, g_{\mu\nu};$$

ρ is density of matter, p is a pressure

$\oplus$

The equation of state:

$$F(\rho, p) = 0$$

$\downarrow$

The Robertson-Walker metric of space-time

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta \cdot d\varphi^2)\right]$$

$k = +1$ closed space,

$k = 0$ flat space,

$k = -1$ open space

$$\begin{cases} \ddot R = -\dfrac{4\pi G}{3}(\rho + 3p)R, \\ \ddot R R + 2\dot R^2 + 2k = 4\pi G(\rho - p)R^2 \end{cases} \qquad (*)$$

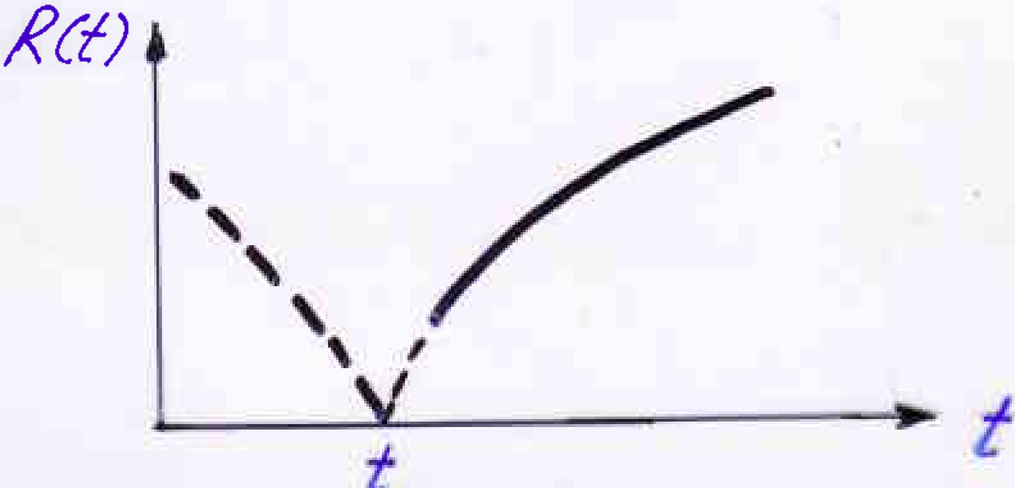
Another form of the Einstein eqs.

$$\begin{cases} \dot R^2(t) + k = \dfrac{8\pi G}{3}\rho(t) R^2(t); & \text{Friedman eq.} \\ \dfrac{d}{dt}(\rho R^3) = -p\dfrac{d}{dt}(R^3); & \text{Energy conservation } dE = -p\,dV \\ F(\rho, p) = 0. & \end{cases}$$

Initial singularity

$\dot R(t) > 0$ (from the red shift of spectrum of galaxies)

$\ddot R < 0$ (from eq. $(*)$).



$R(t_0) = 0, \ \rho \to \infty$

Relation between H and R(t)

$$D = R(t)\int_{r_1}^{r_2}\frac{dr}{1-kr}, \quad V = \dot{D} = \frac{\dot{R}(t)}{R(t)}\cdot D \equiv H\cdot D$$

$= H$

$$H \equiv H(t) = \frac{\dot{R}(t)}{R(t)}$$

Critical density of matter

$$\frac{k}{R^2(t)} = \frac{8\pi G}{3}\left[\rho(t) - \rho_{cr}(t)\right], \quad \rho_{cr} = \frac{3}{8\pi G}\left[\frac{\dot{R}(t)}{R(t)}\right] = \frac{3H^2}{8\pi G}$$

$$\rho_{cr} = 10^{-29}\ g/cm^3$$

$\rho > \rho_{cr}$ $\quad k > 0$ closed Universe

$\rho = \rho_{cr}$ $\quad k = 0$ flat Universe

$\rho < \rho_{cr}$ $\quad k < 0$ open Universe

In current moment:

$$\rho = \rho_{mat} + \rho_{rad} + \rho_{dark}\ ; \ \rho_{mat} = \rho_{galaxies} = 3\cdot 10^{-31}\ g/cm^3$$

$$\rho_{rad} = 4.5\cdot 10^{-34}\ g/cm^3$$

$E_8 \times E_8'$ string

$$\rho_m + \rho_r \sim \frac{1}{30}\rho_{crit}$$

The time behaviour of the scale factor $R(t)$

If $\rho = \rho_{mat}$ $(p = 0)$

$$R(t) \sim t^{2/3}$$

If $\rho = \rho_{rad}$ $(\rho = 3p)$

$$R(t) \sim t^{1/2}$$

Problems in standard cosmological model

i) horizont problem;

ii) curvature problem;

iii) the galaxy formation

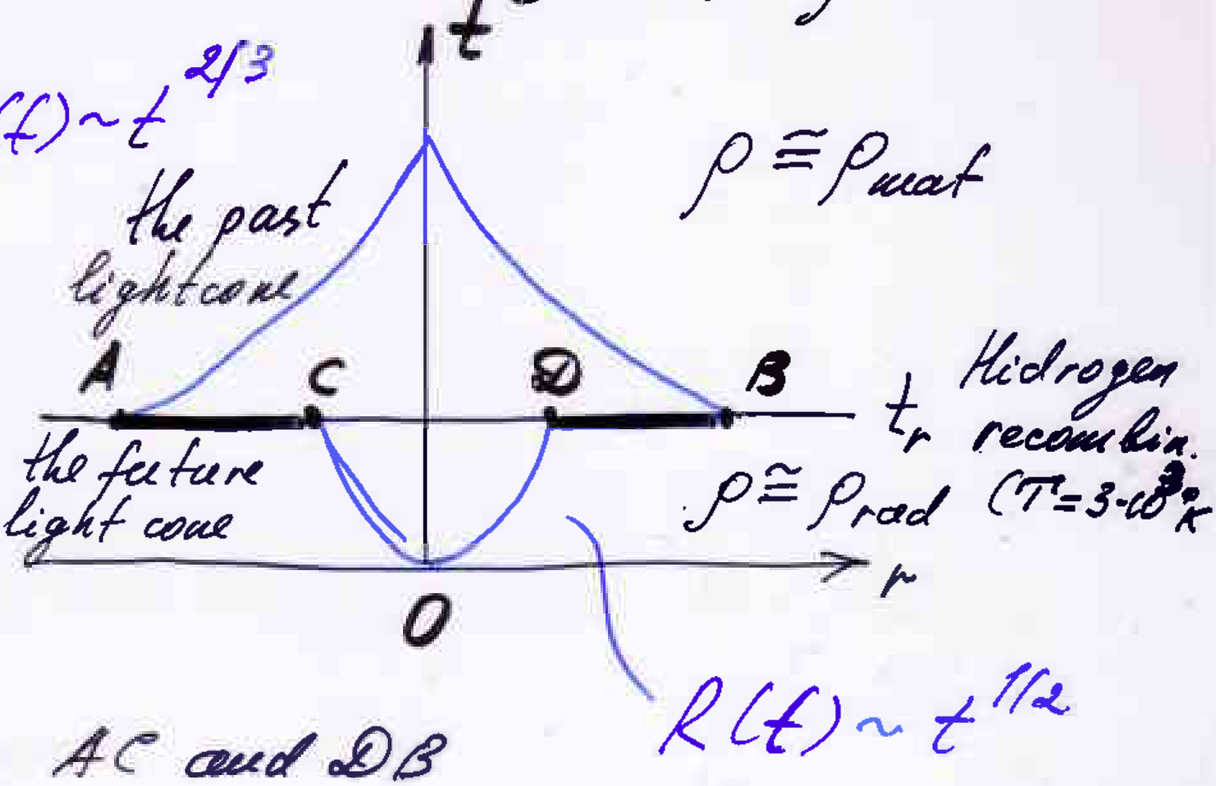
iv) monopole problem

and so on.

If $\rho = -p > 0$
(inflation phase)

$$R(t) = R_0 \exp\left(\frac{8\pi G \rho}{3} t\right) \quad \rho = \text{const}$$

Horizont problem $R(t) \sim t^{2/3}$



Region AC and DB are not causally related!

The curvature problem

Fridmann equation gives:

$$\frac{k}{R^2(t)} \; \frac{8\pi G}{3}(\Omega-1)\cdot\rho_{cr} = (\Omega-1)H^2(t)$$

$$\Omega = \rho/\rho_{cr}$$

$$k = (\Omega(t)-1)\,\dot{R}^2(t)$$

$$R(t) \sim t^n,\; n<1,\quad \dot{R}(t) \cong t^{n-1}$$

$$k = (\Omega(t)-1)\,t^{2(n-1)}$$

$$\Omega(t)-1 \sim t^{2(1-n)}$$

$t \sim 10^{10}$ years now

The age of Universe	Period	Ω
$t = 2\cdot 10^{10}$ years	current	1 ± 0.5
$t = 10^5$ years	H recombin	1 ± 10^{-3}
$t = 1$ sek	~~He~~ Nucl. syntes	1 ± 10^{-16}
$t = 10^{-5}$ sec	quark-gluon plasma	1 ± 10^{-21}
$t = t_{Plank} = 10^{-43}$ cek	quantum gravity	1 ± 10^{-60} !

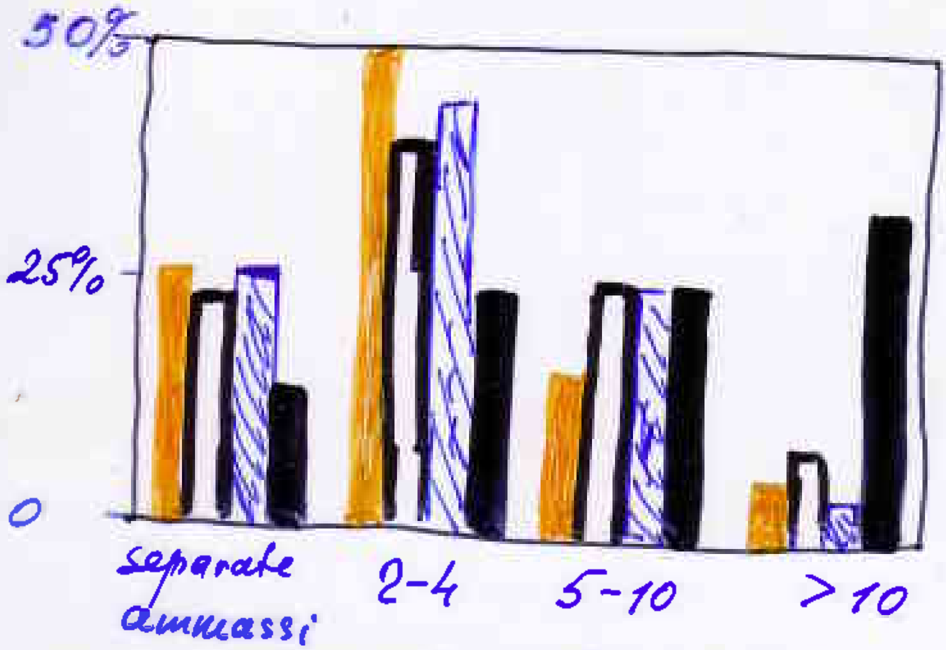
Galaxy formation

Requirements to the perturbation spectrum

$$\frac{\delta\rho}{\rho} = 10^{-4} \div 10^{-5}$$ (there are no any scales)!

This spectrum should ~~be related~~ agree with homogineoty of the relict radiation

$$\frac{\delta\rho}{\rho} \longleftrightarrow \left(\frac{\delta T}{T}\right)_{rel.\,rad.}$$



number of objects in super collection
super ammassi

real destribution of Abell's cell.
Ecolled particles model ⇒
neutrino model
isotermic model

Phase transitions in GUT

G is an initial large gauge group
(simple Lee group)
one ~~int~~ coupling constant

$$G \to H \to \dots \to SU(3) \times SU(2) \times U(1) \to$$
$$\to SU(3) \times U(1)_{em}$$

$$G = SU(5),\ SO(10),\ E_6,\ E_8,\ SO(32)\ \dots$$

$\pi_i(M, x_0)$ are the gomotopic groups ~~of~~
mappings $S^i \to M \quad S_0 \in S^i \to x_0 \in M$

If the homotopic group is nontrivial

$\pi_2(G/H) \to$ monopoles

$\pi_1(G/H) \to$ **strings**

$\pi_0(G/H) \to$ domain walls

and their combinations

The energetic scales of these phase transitions

$$SU(5) \to SU(3)\times SU(2)\times U(1) \to$$

$t = 10^{-37}$ sec

$$\to SU(3)\times U(1)_{em}$$

$m_{GU} \sim 10^{15}$ GeV

$t \sim 10^{-11}$ sec

$\alpha = \frac{g^2}{4\pi} \sim \frac{1}{50}$

$m \sim \sqrt{G_F} \sim 10^2$ GeV

Local cosmic strings

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |(\partial_\mu + ieA_\mu)\phi|^2 +$$
$$+ a\phi^*\phi - b(\phi^*\phi)^2$$

$$a, b > 0$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Static vortex solution

$$A^0_\mu(x_1, x_2) = \varepsilon_{03\mu\nu}\frac{x^\nu}{r}A_\mu(r),$$

$$\phi^0(x_1, x_2) = e^{in\theta}R_n(r), \quad n \in \mathbb{Z}$$

$$R_n(r) = \lambda + C_s e^{-m_s r}, \quad A_\mu(r) \xrightarrow[r\to\infty]{} \frac{n}{er} + \frac{C_v}{\sqrt{er}}e^{-m_v r}$$

$$m_s = \sqrt{2a}, \quad m_v = \sqrt{2}\,e\lambda, \quad \lambda = (a/2b)^{1/2}$$

$H \parallel z$ axis

$$H(r) = \frac{1}{r}\frac{d}{dr}(rA_r) \underset{r\to\infty}{\simeq} \frac{C_V}{\sqrt{er}} e^{-m_v r}$$

Effective action

$$S = \int \mathcal{L}(A,\Phi)\, d^4x = \underbrace{\int d^2x_\perp \mathcal{L}(\bar{A},\bar{\Phi})}_{T} \int d^2S$$

$A = A^0 + \bar{A}$,

$$= T \iint d^2S = S_{N-G}$$

$\Phi = \Phi^0 + \bar{\Phi}$

Density fluctuations generated by strings

If $L_{string} < d_H$ then closed string collapse

$L_{str} > d_H \sim t$

$$\rho_{str} \approx \frac{\mu t}{t^3} = \frac{\mu}{t^2}; \quad \delta\rho \sim \rho_{string}$$

μ is linear density of the string energy (or mass)

$$\rho(t) \sim \rho_{cr}(t) = \frac{3}{8\pi G} H^2(t) = \frac{3}{8\pi G}\left(\frac{\dot{R}(t)}{R(t)}\right)^2 =$$

$$= \begin{cases} \dfrac{3}{32\pi G t^2}, & \text{if } R(t) \sim t^{1/2} \\ \dfrac{1}{6\pi G t^2}, & \text{if } R(t) \sim t^{2/3} \end{cases}$$

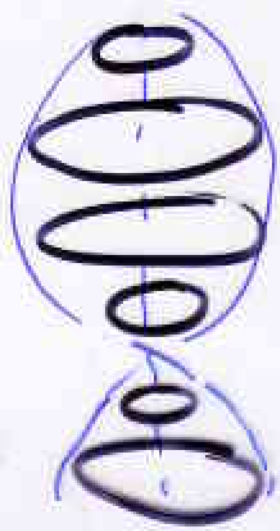
$$\frac{\delta\rho}{\rho} = \begin{cases} \dfrac{32\pi}{3}\mu G \\ 6\pi \mu G \end{cases}$$

In order to obtain $\delta\rho/\rho \sim 10^{-4} \div 10^{-5}$ one should put

$$\mu \sim (10^{15}\,GeV)^2 \div (10^{16}\,GeV)^2$$

Dynamics of cosmic String

Free closed strings oscillate



Gravitation radiation

$$\frac{dE}{dt} \sim \gamma G \mu^2$$

$$\gamma \sim 10^1 \div 10^2$$

Interaction of cosmic string