Lecture 4. Superstring

In spinning string model the supersymmetric sector can be picked up (the Gliozzi-Scherk-Olive projection).

Green and Schwarz (superstring in the light-cone gauge)

$$S^i_A(\tau,\sigma) \to Q^a_A(\tau,\sigma)$$

$$i = 2,3,\dots,9; \quad \underbrace{a = 1,2,\dots,8}_{\text{spinor index for SO(8) group}}, \quad \underbrace{A = 1,2}_{\text{world sheet spinor index}}$$

$$S_{l-c} = \frac{T}{2}\iint d\tau d\sigma \left[\eta^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^i + i\bar{Q}^a \rho^\alpha \partial_\alpha Q^a\right]$$

$\alpha,\beta = 0,1$; ρ^α are the Dirac γ-matrices in two-dim. space $\{\tau,\sigma\}$.

Ramond boundary conditions ($\varepsilon = +1$)

$$Q^a_1(\tau,0) = Q^a_2(\tau,0), \quad Q^a_1(\tau,\pi) = +Q^a_2(\tau,\pi)$$

$$Q^a_1(\tau,\sigma) = \frac{1}{\sqrt{2\pi T}}\sum_k Q^a_k e^{-ik(\tau-\sigma)}$$

$$Q^a_2(\tau,\sigma) = \frac{1}{\sqrt{2\pi T}}\sum_k Q^a_k e^{-ik(\tau+\sigma)}$$

$$[Q^a_n, Q^b_m] = \delta_{n,-m}\delta_{ab}$$

$$D = 10, \quad C_\mp = 0$$

Total superstring action (D=10)

Dynamical variables:

$X^\mu(\tau,\sigma)$

$\left\{\begin{array}{l}\text{two anticom-}\\ \text{muting}\\ \text{space-time}\\ \text{spinors}\end{array}\right\} \rightarrow \theta_A^a$ ← spinor index in 10-dimensional space-time

$A = 1, 2$

$a = 1, 2, \dots, 32 = 2^{[D/2]} = 2^5$

It is not a world sheet index!

θ_A^a should be a Majorana-Weyl spinor in the index a

$$h^{ab}\theta_A^b = 0, \quad \bar{\theta}_A^a = \theta_A^b (\gamma^0)^{ba}, \qquad (*)$$

where h is the Weyl projection operator, $2h = 1 \pm \gamma^{11}$

This condition can be imposed only for $D = 2 \pmod 8$

$$D = 2, 10, 18, \dots$$

Conditions (*) + boundary conditions + Dirac equation for θ_A^a reduce the number of independent fermion degrees of freedom to $2^3 = 8$

$$S=\frac{T}{2}\iint d\tau\, d\sigma\,(\mathcal{L}_1+\mathcal{L}_2),$$

where

$$\mathcal{L}_1=-\frac{1}{2\pi}\sqrt{-g}\,g^{\alpha\beta}\Pi^{\mu}_{\alpha}\Pi_{\mu\beta},\quad \Pi^{\mu}_{\alpha}=\partial_\alpha x^\mu-i\bar{\theta}_A\gamma^\mu\partial_\alpha\theta_A;$$

$$\mathcal{L}_2=-i\varepsilon^{\alpha\beta}\{\partial_\alpha x^\mu[\bar{\theta}_1\gamma_\mu\partial_\beta\theta_1-\bar{\theta}_2\gamma_\mu\partial_\beta\theta_2+$$

$$+\bar{\theta}_1\gamma^\mu\partial_\alpha\theta_1\bar{\theta}_2\gamma_\mu\partial_\beta\theta_2]\},\quad \alpha,\beta=0,1;$$

$$\mu=0,1,\dots,9.$$

Global $N=2$ supersymmetry

$$\delta\theta_A=\frac{1}{4}\omega_{\mu\nu}\gamma^{\mu\nu}\theta_A+\varepsilon_A,\quad \gamma^{\mu\nu}=[\gamma^\mu,\gamma^\nu]_-,$$

$$\delta x^\mu=\omega^\mu_{\ \nu}x^\nu+a^\mu+i\bar{\varepsilon}_A\gamma^\mu\theta_A;\quad \delta g^{\alpha\beta}=0$$

The boundary conditions reduce N to 1.

Light-cone gauge conditions

$$x^-=p^-\tau,\quad \gamma^-\theta_A=0$$

θ_A become the Majorana world-sheet spinor.

БРСТ-формализм в
гамильтоновой формулировке

(*) $[\varphi_a, \varphi_b] = f_{ab}^{\ c}\, \varphi_c$

$\varphi_c(q,p) \overset{=0}{-}$ первого рода связи в теории

$\varphi_c(q,p) \to$ пара духов c_a и $\bar{c}_a$

$$[c_a, \bar{c}_b] = \delta_{ab}, \quad [c_a, c_b]_+ = [\bar{c}_a, \bar{c}_b]_+ = 0$$

$$Q = \sum \varphi_a c_a - \frac{1}{2}\sum f_{ab}^{\ c}\, c_a c_b \bar{c}_c$$

$$\tilde{\varphi}_a \equiv [Q, \bar{c}_a] = \varphi_a + \sum f_{ab}^{\ c}\, \bar{c}_c c_b$$

На класс. уровне связи $\tilde{\varphi}_a$ удовлетворяют той же алгебре (*) и

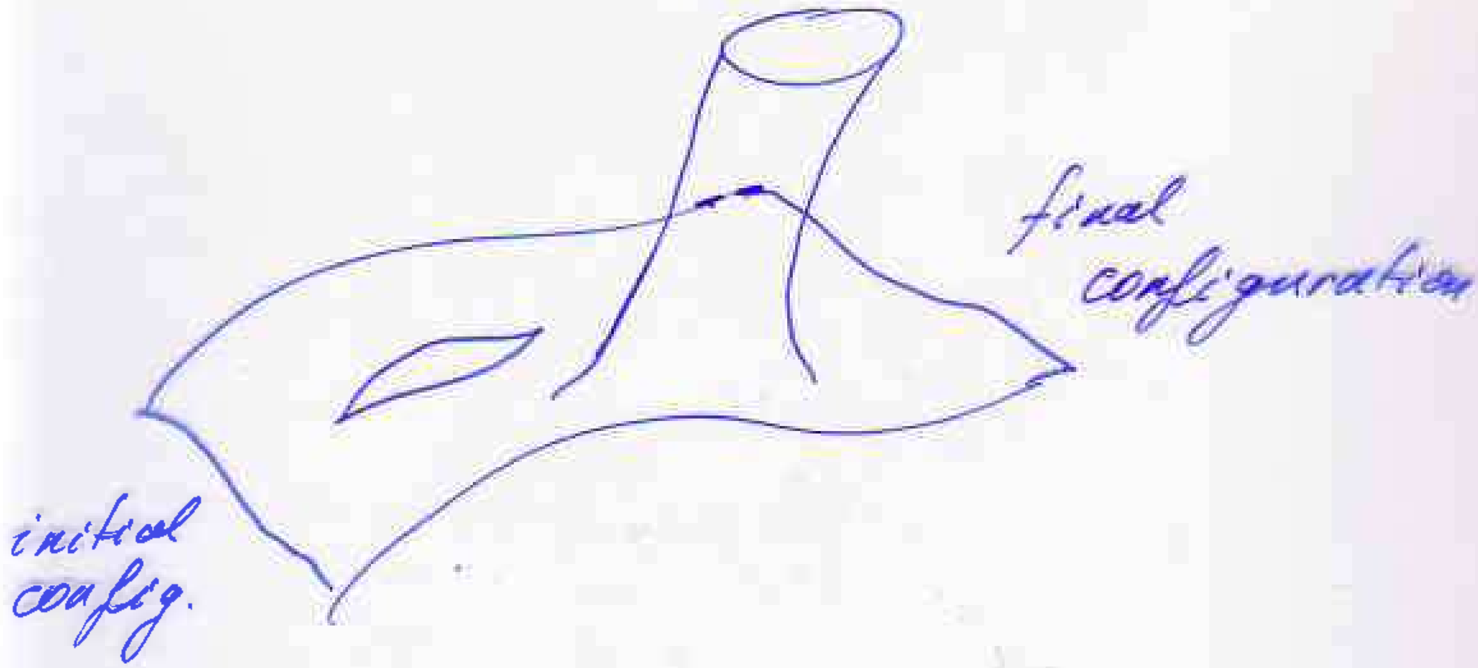
$$[Q, \tilde{\varphi}_a] = 0, \quad Q^2 = 0$$

$$\hat{Q} = \sum_m L_{-m} c_m - \frac{1}{2}\sum_{m,n} (m-n) : c_{-m} c_{-n} \bar{c}_{m+n} : - \alpha(0)\, c_0$$

$$\tilde{L}_m^{\text{кв}} = \{\hat{Q}, \bar{c}_m\}_+ = L_m + \sum_{m,n} (m-n) : \bar{c}_{m+n} c_{-n} : - \alpha(0)\, \delta_{m,0}$$

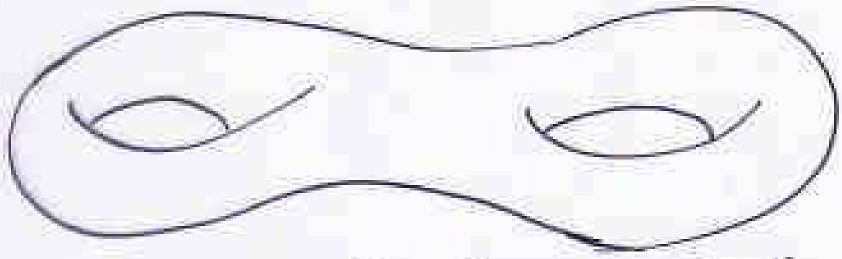
String interaction (path integral approach)

(Mandelstam, Polyakov)

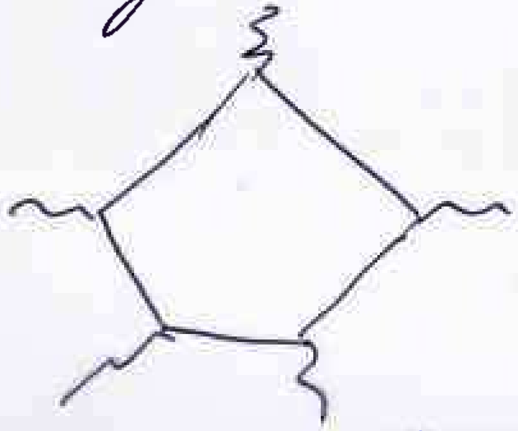


$$A \sim \int \mathcal{D}x_\mu \exp\{S_{str}\}$$

Vacuum amplitudes



String dual amplitudes



For these gauge groups

$SO(32)$

$E_{8\times 8}$

gauge anomaly are cancelled.

Field theory of interecting string

$$\underset{\substack{\uparrow\\ \text{string}\\ \text{field}}}{\Psi}[x(\sigma)] = \{\varphi(x) + A^{(1)}_\mu(x)\alpha^\mu_{-1} + h_{\mu\nu}(x)\alpha^\mu_{-1}\alpha^\nu_{-1} + A^{(2)}_\mu(x)\alpha^\mu_{-2} + \dots\}|0\rangle$$

Expansion in terms of the eigenvectors of the M^2 operator.

$\varphi(x)$ is a scalar field;

$A_\mu(x)$ is an electromagnetic field;

$h_{\mu\nu}(x)$ is ~~gr~~ tensor field with mass 1

and so on.

Gravitation field is described by closed strings.

Virasoro conditions

$$L_n\Psi[x(\sigma)] = \delta_{n,0}\Psi[x(\sigma)],\quad n=0,1,2,\dots$$

$$L_0\varphi(x) = -\alpha' P^2\varphi(x) = \underbrace{\alpha'\partial^2\varphi(x) = \varphi(x)}$$

Klein-Gordon eq. with squared mass $= -(\alpha')^{-1}$

tachyonic field!

$$(\alpha'\partial^2 - 1 + n)A^{(n)}_\mu(x) = 0$$

$$\partial_\mu A^\mu = 0$$

$$(\alpha'\partial^2 + 1)h_{\mu\nu}(x) = 0 \quad \text{and so on}$$

Action for string fields

$$S \sim \Psi * \Psi * \Psi$$

$*$ string product

non commutativ geometry and so on

Complete field theory of interacting string is <u>absent</u>!

Relation with real physics through **<u>compactification</u>** of the space-time.

The solution to the equations of motion in the interacting string theory should have a form

$$V_4 \otimes K_6 ,$$

wher V_4 is the Riemannian space-time and K_6 is a compact manifold (with Plank dimensions).

K_6 may be Kalabi-Yao manifold or something else

(additional Z-boson)!

Strings in external fields

Open bosonic string in a background electromagnetic field

$$S = -T\int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \sqrt{|g|} - \sum_{a=1}^{2} q_a \int_{C_a} dx_\nu A^\nu(x)$$

$$(\dot{x} \pm x')^2 = 0, \quad \text{orthonormal gauge}$$

Eqs. of motion

$$\ddot{x}^\mu - x''^\mu = 0, \quad \mu = 0,1,2,\dots,D-1$$

boundary conditions

$$x'^\mu + f^1_{\mu\nu}\dot{x}^\nu = 0, \quad \sigma = 0; \qquad f^a_{\mu\nu} = \frac{q_a}{T} F_{\mu\nu},$$

$$x'^\mu - f^2_{\mu\nu}\dot{x}^\nu = 0, \quad \sigma = \pi \qquad a = 1,2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \text{const}$$

If $4I_1 = F_{\mu\nu}F^{\mu\nu} = \vec{E}^2 - \vec{H}^2 \neq 0$ and

$$2I_2 = F_{\mu\nu}\tilde{F}^{\mu\nu} \neq 0$$

$$\vec{E}\vec{H}$$

then the matrix $F_\mu{}^\nu$ can be cast by making $O(1,D-1)$ transformations into the block-diagonal form

$$F_\mu{}^\nu = \begin{cases} \operatorname{diag}(\overset{(1)}{F}, \overset{(2)}{F}, \dots, \overset{(d)}{F}), & \text{if } D \text{ is even;} \\ \operatorname{diag}(\overset{(1)}{F}, \overset{(2)}{F}, \dots, \overset{(d)}{F}), & \text{if } D \text{ is odd.} \end{cases} \quad d = [D/2]$$

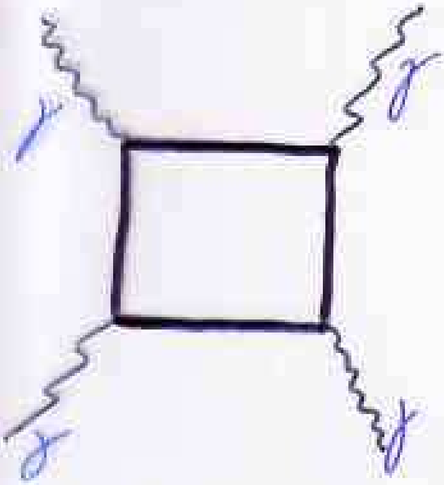
$$\overset{(1)}{F} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}, \quad \overset{(1+\alpha)}{F} = \begin{pmatrix} 0 & H_\alpha \\ -H_\alpha & 0 \end{pmatrix}, \alpha = 1, 2, \ldots, d-1$$

$$E^2 = \sqrt{I_1^2 + I_2^2} - I_1, \quad H^2 = \sqrt{I_1^2 + I_2^2} + I_1$$

From the boundary conditions it follows that

$$\boxed{\left(\frac{q_a}{T} E\right)^2 < 1,} \quad a = 1, 2.$$

This restriction can be treated as indication on <u>the nonlinear Lagrangian</u> of the electromagnetic field that takes into account the vacuum ~~~~ string corrections



$$\mathcal{L}_{N-L} = \mathcal{L}_{B-I} =$$

$$= \sqrt{\det(g_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}$$

(Born–Infeld electrodynamics at Plank scales)

See, for example: E.S. Fradkin, A.A. Tseytlin Phys. Lett. B163 (1985) 123.

$$M^2 = -M_{tr}^2 + 2\pi T(1 - e^2) \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} n \bar{a}_n^i a_n^i$$

$$M_{tr}^2 = \sum_{\alpha=1}^{D/2-1} \frac{e^2 + h_\alpha^2}{1 + h_\alpha^2} \vec{P}_{\alpha\perp}^2 \Leftarrow \text{tachyonic contribution!}$$

Nambu-Goto string in a space-time of a constant curvature

(de Sitter universe)

$$S = -\gamma \iint d^2u \, (-g)^{1/2}, \quad g = \det g_{ij}$$

$$g_{ij} = \frac{\partial x^\mu}{\partial u^i} \frac{\partial x^\nu}{\partial u^j} G_{\mu\nu}(x),$$ $G_{\mu\nu}(x)$ is the metric of the space-time

Equation of motion read

$$\sqrt{-g}\, G_{\mu\nu}(x) \left(\Delta_g x^\nu + \Gamma^\nu_{\rho\sigma}(G) g^{ij} x^\rho_{,i} x^\sigma_{,j} \right) = 0$$

$x_{,i} \equiv \partial x / \partial u^i$ $\quad \Gamma^\nu_{\rho\sigma}$ are the Christoffel symbols for $G_{\mu\nu}(x)$

$$G_{\mu\nu}(x) = \frac{\eta_{\mu\nu}}{1 + \frac{K}{4} x^2}; \quad x^2 = x^\mu x^\nu \eta_{\mu\nu}$$

$\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-,-)$

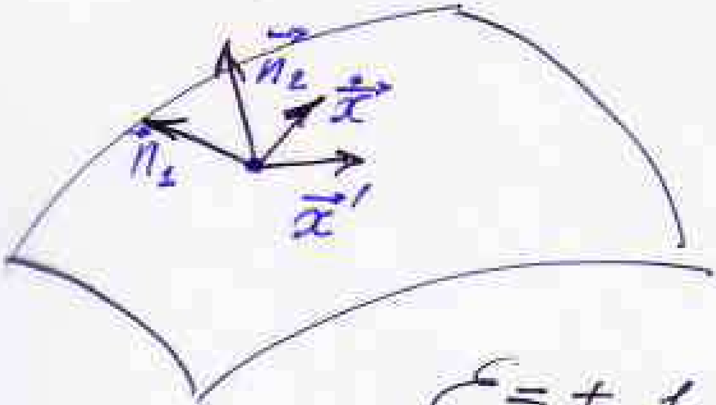
$K = \varepsilon / R^2$ $\quad R$ is radious of the Universe

Geometrical consideration

$$x^\mu(\tau, \sigma) \longrightarrow \{ \underset{\text{(I)}}{g_{ij}(u)}, \; \underset{\text{(II)}}{b_{\sigma | ij}(u)}, \; \nu_{\alpha\beta|i}(u)$$

induced metric — external curvature — torsion vectors

$M_2 \subset V_4 = S^4 \subset E^5$



$(Z^1)^2 - (Z^2)^2 - (Z^3)^2 - (Z^4)^2 + \varepsilon (Z^5)^2 = \varepsilon R^2$

$\varepsilon = \pm 1$ de Sitter space-time of the first or of the second kind

The motion of the Frenet basis is described by

$$\begin{cases} \nabla_j Z^A_{,i} = -\sum_{\alpha=3}^{4} b_{\alpha|ij} n^A_\alpha - K g_{ij} Z^A \\ n^A_{\alpha,i} = -b_{\alpha|ij} g^{jm} Z^A_{,m} - \sum_{\alpha} \nu_{\alpha\beta|i} n^A_\alpha \end{cases}$$

$$i,j = 1,2, \quad \alpha, \beta = 3,4$$

Compitability conditions

$$\frac{\partial^2}{\partial u^1 \partial u^2} = \frac{\partial^2}{\partial u^2 \partial u^1}$$

Gauss equation

$$R_{ijkl} = -\sum_{\alpha=3}^{4} (b_{\alpha|ik} b_{\alpha|jl} - b_{\alpha|il} b_{\alpha|jk}) + K (g_{ik} g_{jl} - g_{il} g_{jk})$$

Codazzi equations

$$\nabla_k b_{\alpha|ij} - \nabla_j b_{\alpha|ik} = \sum_\beta \epsilon_\beta \left(\nu_{\beta\alpha|k} b_{\beta|ij} - \nu_{\beta\alpha|j} b_{\alpha|ik} \right)$$

and Ricci equations

$$\nu_{\beta\alpha|j,k} - \nu_{\beta\alpha|k,j} + \sum \left(\nu_{..}\nu_{...} - \nu_{...}\nu_{..} \right)$$

$$+ \left(b\, b - b\, b \right)$$

Finally we obtain very simple eqs:

$$\begin{cases} \varphi_{,11} - \varphi_{,22} = e^{\varphi} \cos\theta + K e^{-\varphi} \\ \theta_{,11} - \theta_{,22} = e^{\varphi} \sin\theta \end{cases}$$

They can be derived from the Hamiltonian

$H \sim \sin\varphi$ (instability)