Metric theories of gravity

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- Only one metric theories
- Only Riemann geometry
- Only gravitational sector
- Some cosmological applications

Notations

- $c = \hbar = 16\pi G (\equiv 2\kappa^2) = 1$
- $(+ - \ldots -)$ or $(- + \ldots +)$
- $R_{ik} = R^l{}_{ilk}$ or $R_{ik} = R^l{}_{ikl}$
- $R^l{}_{ikm} = \partial_m \Gamma^l_{ik} - \partial_k \Gamma^l_{im} + ...$ or $R^l{}_{ikm} = \partial_k \Gamma^l_{im} - \partial_m \Gamma^l_{ik} + ...$

Lovelock gravity

D. Lovelock, J. Math. Phys. 12 (1971) 498

$$S = \int d^4x\sqrt{-g}R$$

$$\Downarrow$$

$$G_{ik} \equiv R_{ik} - \frac{1}{2}g_{ik}R = G_{ik}(g_{ik}, g_{ik,l}, g_{ik,lm})$$

what is all possible tensors A_{ik} in space-time with general dimensions d which

- symmetric $A_{ik} = A_{ki}$
- $A_{ik} = A_{ik}(g_{ik}, g_{ik,l}, g_{ik,lm})$
- divergence free $A^{ik}{}_{;k} = 0$

the answer is

$$A^i_k = \sum_{n=0}^{d} \alpha_n \delta^{i\mu_1\nu_1\ldots\mu_n\nu_n}_{k\alpha_1\beta_1\ldots\alpha_n\beta_n} \prod_{r=1}^{n} R^{\alpha_r\beta_r}{}_{\mu_r\nu_r}$$

associated Lagrange density is

$$L = \sqrt{-\mathrm{g}} \sum_{n=0}^{d} \alpha_n \Omega_n, \quad \Omega_n = 2\delta^{\mu_1\nu_1\ldots\mu_n\nu_n}_{\alpha_1\beta_1\ldots\alpha_n\beta_n} \prod_{r=1}^{n} R^{\alpha_r\beta_r}{}_{\mu_r\nu_r}$$

where generalized Kronecker delta

$$\delta^{\mu_1\nu_1\ldots\mu_n\nu_n}_{\alpha_1\beta_1\ldots\alpha_n\beta_n} = det \begin{vmatrix} \delta^{\mu_1}_{\alpha_1} & \ldots & \delta^{\mu_1}_{\beta_n} \\ \vdots & & \vdots \\ \delta^{\nu_n}_{\alpha_1} & \ldots & \delta^{\nu_n}_{\beta_n} \end{vmatrix}$$

where $\sqrt{-\mathrm{g}}\Omega_n$ – is total derivative up to $D = 2n$
it produce non-trivial contribution to the equations of motions only beginning from $D = 2n + 1$

examples:

- Ω_0 – Λ-term, non-trivial contribution in any non-trivial space
- Ω_1 – scalar curvature R, non-trivial contribution beginning from $D = 3$
- Ω_2 – Gauss-Bonnet invariant $G = R_{iklm}R^{iklm} - 4R_{ik}R^{ik} + R^2$, non-trivial contribution beginning from $D = 5$

- Ω_3 – third Euler density

$$\Omega_3 = 8(-R^3 + 12RR_{ik}^2 - 3RR_{iklm}^2 - 16R_i{}^kR_k{}^lR_l{}^i +$$

$$+24R_{ik}R_{lm}R^{likm} + 24R_{ik}R^{ilmj}R^k{}_{lmj} -$$

$$-4R_{ik}{}^{lm}R_{lm}{}^{jn}R_{jn}{}^{ik} - 8R_{ilmk}R^{ljnm}R_j{}^{ik}{}_n),$$

non-trivial contribution beginning from $D = 7$
- $\Omega_{15} = -R^{15}+$ 79536629 additional terms! non-trivial contribution beginning from $D = 31$

Lovelock gravity is multidimensional theory of gravity which is take into account non trivial contribution from all possible Euler densities.

Starobinsky's inflation model

A.A. Starobinsky, Phys. Lett. B 91 **(1980) 99**

$$R_{ik} - \frac{1}{2}g_{ik}R = \langle T_{ik} \rangle$$

$$\langle T_{ik} \rangle = \frac{m_2}{2880\pi^2}(R_i{}^l R_{kl} - \frac{2}{3}RR_{ik} - \frac{1}{2}g_{ik}R_{lm}R^{lm} + \frac{1}{4}g_{ik}R^2)$$
$$+\frac{m_3}{2880\pi^2}\frac{1}{6}(2R_{;i;k} - 2g_{ik}R_{;l}^{;l} - 2RR_{ik} + \frac{1}{2}g_{ik}R^2)$$

$$k_2 = \frac{m_2}{60(4\pi)^2} = \frac{N + 11N_{\frac{1}{2}} + 62N_1 + 1411N_2 - 28N_{HD}}{60(4\pi)^2}$$

$$k_3 = \frac{m_3}{60(4\pi)^2} = -\frac{N + 6N_{\frac{1}{2}} + 12N_1 + 611N_2 - 8N_{HD}}{60(4\pi)^2}$$

$$\rho_q = k_2 H^4 + k_3(2\ddot{H}H + 6\dot{H}H^2 - \dot{H}^2)$$

$$6H^2 = \rho_q$$

- Vacuum stability condition: $k_3 < 0$
- Exist de Sitter solution for $k_2 > 0$
- Singularity problem may be solved for $K = -1$

R^2 example

$$S = \int d^4x\sqrt{-\mathrm{g}}(R + \alpha R^2).$$

$$2\alpha\nabla_i\nabla_k R - (1 + 2\alpha R)R_{ik} + \mathrm{g}_{ik}[\frac{1}{2}\alpha R^2 + \frac{1}{2}R - 2\alpha\Box R] = 0.$$

$$6\alpha\Box R = R \quad \Rightarrow \quad m_{eff}^2 = \frac{1}{6\alpha}.$$

$$\tilde{\mathrm{g}}_{ik} = (1 + 2\alpha\varphi)\mathrm{g}_{ik}.$$

$$\tilde{R}_{ik} - \frac{1}{2}\tilde{g}_{ik}\tilde{R} = \frac{6\alpha^2}{(1+2\alpha\varphi)^2}\left(\nabla_i\varphi\nabla_k\varphi - \frac{1}{2}\tilde{g}_{ik}\left[\nabla_i\varphi\nabla^i\varphi + \frac{\varphi^2}{6\alpha}\right]\right).$$

This correspond to the theory:

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - \frac{6\alpha^2}{(1+2\alpha\varphi)^2}\left(\nabla_i\varphi\nabla^i\varphi + \frac{\varphi^2}{6\alpha}\right)\right].$$

and the field equation for φ:

$$6\alpha(1+2\alpha\varphi)\Box\varphi - 12\alpha^2\nabla_i\varphi\nabla^i\varphi = \varphi.$$

$$6\alpha(1+2\alpha\varphi)(-\ddot{\varphi}-3H\dot{\varphi})-12\alpha^2\dot{\varphi}^2=\varphi.$$

in the limit of large φ

$$\varphi \propto -t$$

and from Einstein equation we find

$$H^2=\frac{1}{24\alpha}$$

– that is quasi de Sitter solution.

Quadratic gravity

$$S = \int d^D x \sqrt{-\mathrm{g}} (R + aR^2 + bR_{ik}R^{ik} + cR_{iklm}R^{iklm}),$$

simplification for $D = 4$

$$S = \int d^4 x \sqrt{-\mathrm{g}} (R + aR^2 + bR_{ik}R^{ik} + cR_{iklm}R^{iklm} - cG) =$$

$$= \int d^4 x \sqrt{-\mathrm{g}} (R + AR^2 + BR_{ik}R^{ik}).$$

it is possible further simplification for conformally flat metric $C_{iklm} = 0$

$$C_{iklm}^2 = R_{iklm}^2 + (D-6)R_{ik}^2 + \left(\frac{7}{3} - \frac{13}{18}D + \frac{1}{18}D^2\right)R^2.$$

$$S = \int d^4x\sqrt{-\mathrm{g}}(R + \tilde{A}R^2 + \tilde{B}C_{iklm}C^{iklm}).$$

so for cosmological applications:

$$S = \int d^4x\sqrt{-\mathrm{g}}(R + \tilde{A}R^2).$$

de Sitter stability in quadratic gravity

non trivial question for any high derivative theory

$$S = \int d^4x \sqrt{-g}(R - \Lambda).$$

$$\Downarrow$$

$$6H^2 = \Lambda$$

$$S = \int d^4x \sqrt{-g}(R + \tilde{A}R^2 - \Lambda).$$

$$\Downarrow$$

$$6H^2 + 12\tilde{A}\left(6H\ddot{H} - 3\dot{H}^2 + 28H^2\dot{H}\right) = \Lambda$$

– de Sitter solution as in previous case, but it may be unstable due to high derivatives to investigate stability of de Sitter solution rewrite equation of motion in the form of dynamical system:

$$\begin{cases} \dot{H} = C, \\ \dot{C} = \frac{1}{6H}\left[\frac{1}{12\tilde{A}}(\Lambda - 6H^2) + 3\dot{H}^2 - 28H^2\dot{H}\right] \equiv f(H, C). \end{cases}$$

stationary point – is de Sitter solution $H_0 = \sqrt{\frac{\Lambda}{6}}$
it stable when $\tilde{A} > 0$

note: there are two different de Sitter solution: due to Λ-term and due to gravitational sector

$f(R)$-gravity

$$S = \int d^4x \sqrt{-\mathrm{g}} f(R) + S_m,$$

$$-\frac{1}{2} f \mathrm{g}_{ik} + f_R R_{ik} - \nabla_i \nabla_k f_R + \mathrm{g}_{ik} \Box f_R = T_{ik},$$

$$T_{ik} = 0,$$

$$3\Box f_R - 2f + f_R R = 0$$

$$f_R(R_0)R_0 - 2f(R_0) = 0,$$

$\Rightarrow R^2$ – is degenerated case

$$m_{eff}^2 = \frac{1}{3}\left(\frac{f_R}{f_{RR}} - R\right).$$

- $f_R > 0$ – graviton is not ghost
- $f_{RR} > 0$ – scalaron is not tachyon
- additional possible condition: $f(0) = 0$ – vanish cosmological constant

Cosmological constant Λ is a good candidate for dark energy (late time accelerating), but not for inflation one.
Cosmological constant can not explain possible fantom regime $w < -1$ for $p = w\rho$.

Several examples

- $f(R) = R + \frac{c_1\left(\frac{R}{\mu^2}\right)^n}{c_2\left(\frac{R}{\mu^2}\right)^n + 1}$

- $f(R) = R - \beta R_s\left(1 - e^{-R/R_s}\right)$

- $S = \int d^4x\sqrt{-\mathrm{g}}\,[R + f_1(R) + f_2(R)L_d]$

  with $L_d = \frac{1}{2}\mathrm{g}^{ik}\nabla_i\varphi\nabla_k\varphi$

$$S = \int d^4x\sqrt{-\mathrm{g}}f(R).$$

$$S = \int d^4x\sqrt{-\mathrm{g}}f(R) = \int d^4x\sqrt{-\mathrm{g}}[f(\lambda) + \mu(R - \lambda)].$$

Variation with respect to μ and λ give us correspondingly

$$\lambda = R, \;\; \mu = \frac{\partial f(\lambda)}{\partial \lambda}$$

We may define potential V as

$$V(\lambda, \mu) = f(\lambda) - \mu\lambda,$$

initial action take the form

$$S = \int d^4x\sqrt{-\mathrm{g}}[\mu R + V(\lambda, \mu)].$$

We define $\chi = \ln \mu$ and rescale to the metric $\bar{\mathrm{g}}_{ik} = \mathrm{e}^{\chi} \mathrm{g}_{ik}$. This give us next result

$$S = \int d^4x \sqrt{-\bar{\mathrm{g}}} \left[\bar{R} - \frac{3}{2} \bar{\mathrm{g}}^{ik} \frac{\partial \chi}{\partial x^i} \frac{\partial \chi}{\partial x^k} + \mathrm{e}^{-2\chi} V(\lambda, \mu) \right].$$

$\Rightarrow f_R > 0$ – graviton is not a ghost on quantum field theory level

$$-\frac{1}{2} f \mathrm{g}_{ik} + f_R R_{ik} - \nabla_i \nabla_k f_R + \mathrm{g}_{ik} \Box f_R = \kappa^2 T_{ik},$$

$$\Downarrow$$

$$f_R \left(R_{ik} - \frac{1}{2} R \mathrm{g}_{ik} \right) = \kappa^2 T_{ik} + \frac{1}{2} f \mathrm{g}_{ik} + \nabla_i \nabla_k f_R - \mathrm{g}_{ik} \Box f_R - \frac{1}{2} f_R R \mathrm{g}_{ik},$$

$\frac{\kappa^2}{f_R}$ – effective gravitational constant, so
$\Rightarrow f_R > 0$ – positivity of effective gravitational constant on classical level

$$-2f + f_R R + 3\Box f_R = 0,$$

$$R = R_{backgr} + \delta R,$$

$$-f_R(0)\delta R + 3f_{RR}(0)\Box\delta R = 0,$$

$$u_k \sim e^{i\mathbf{kx} - i\omega t},$$

where $\omega \equiv (k^2 + \mu^2)^{1/2}$, $k \equiv \mid \mathbf{k} \mid$ and μ is the mass of effective scalar field (scalaron).

$$3f_{RR}(0)\mu^2 - f_R(0) = 0.$$

$f_{RR} > 0$ – scalaron is not tachyon on quantum field theory level

$f_{RR} > 0$ – stability of cosmological perturbations

$f_R > 0$ – graviton is not a ghost

$f_{RR} > 0$ – scalaron is not tachyon

High derivative theories

$$S = \int d^4x\sqrt{-\mathrm{g}}f(R, \Box R, \Box^2 R, ..., \Box^k R)$$

$f(\lambda_1, \lambda_2, ..., \lambda_k, \lambda_{k+1})$, where $\lambda_1 = R$, $\lambda_2 = \Box R$ ...

two different cases:

case 1: $\frac{\partial f}{\partial \lambda_{k+1}} = F(\lambda_1, \lambda_2, ..., \lambda_k, \lambda_{k+1})$

case 2: $\frac{\partial f}{\partial \lambda_{k+1}} = F(\lambda_1, \lambda_2, ..., \lambda_k)$ or

$$f(\lambda_1, \lambda_2, ..., \lambda_k, \lambda_{k+1}) = g(\lambda_1, , \lambda_2, ..., \lambda_k)\lambda_{k+1} + h(\lambda_1, , \lambda_2, ..., \lambda_k)$$

$$S = \int d^4x\sqrt{-\mathrm{g}}[f(\lambda_1, \lambda_2, ..., \lambda_k, \lambda_{k+1}) + \mu(R - \lambda_1)$$

$$+\mu_1(\Box\lambda_1 - \lambda_2) + ... + \mu_k(\Box\lambda_k - \lambda_{k+1})]$$

variation over μ_i: $\Box\lambda_i = \lambda_{i+1}$

variation over λ_{k+1}:

$$\mu_k = \begin{cases} \frac{\partial f(\lambda_1,\lambda_2,...,\lambda_k,\lambda_{k+1})}{\partial\lambda_{k+1}}, \text{ in the case 1} \\ \\ g(\lambda_1, \lambda_2, ..., \lambda_k), \text{ in the case 2} \end{cases}$$

it may be solved in case 1: $\lambda_{k+1} = \tilde{\lambda}_{k+1}(\lambda_1, \lambda_2, ..., \lambda_k, \mu_k)$

$$S = \int d^4x\sqrt{-\mathrm{g}}[f(\lambda_1, \lambda_2, ..., \tilde{\lambda}_{k+1}) + \mu(R - \lambda_1) + \mu_1(\Box\lambda_1 - \lambda_2) + ...$$

$$+\mu_k(\Box\lambda_k - \tilde{\lambda}_{k+1})]$$

introduce new fields:

$$\lambda_i = \chi_i + \psi_i, \quad \mu_i = \chi_i - \psi_i$$

$$S = \int d^4x\sqrt{-\mathrm{g}}\mu_i\nabla^2\lambda_i = \int d^4x\sqrt{-\mathrm{g}}[\chi_i\nabla^2\chi_i - \psi_i\nabla^2\psi_i] =$$

$$\int d^4x\sqrt{-\mathrm{g}}[-(\nabla\chi)^2 + (\nabla\psi)^2]$$

$$V(\chi_1, ..., \chi_k, \psi_1, ..., \psi_k) = \mu\lambda_1 + \mu_1\lambda_2 + ... + \mu_{k-1}\lambda_k$$
$$+\mu_k\tilde{\lambda}_{k+1}(\lambda_1, ..., \lambda_k, \mu_k) - f(\lambda_1, ..., \lambda_k, \mu_k)$$

$$S = \int d^4x\sqrt{-\mathrm{g}}\left[\mu R - \sum_i \{(\nabla\chi_i)^2 - (\nabla\psi_i)^2\} - V(\chi_1, ..., \psi_k)\right]$$

$$\chi = \ln\mu$$

$$\bar{\mathrm{g}}_{ik} = e^{\chi}\mathrm{g}_{ik}$$

$$S = \int d^4x\sqrt{-\bar{g}}\left[\bar{R} - \frac{3}{2}(\bar{\nabla}\chi)^2 - e^{-\chi}\sum_i\{(\bar{\nabla}\chi_i)^2 - (\bar{\nabla}\psi_i)^2\}\right.$$
$$\left. - e^{-2\chi}V(\chi_1, ..., \psi_k)\right]$$

$2k + 1$ scalar field:

- $k + 1$ of which propagate physically (χ and χ_i)
- and k of which are ghost-like (ψ_i)

It mean that for even $k = 1$ there is one ghost-like scalar field

case 2:
$f(\lambda_1, \lambda_2, ..., \lambda_k, \lambda_{k+1}) = g(\lambda_1, , \lambda_2, ..., \lambda_k)\lambda_{k+1} + h(\lambda_1, , \lambda_2, ..., \lambda_k)$

More complicate case!

$$g(\lambda_1, \lambda_2, ..., \lambda_k)\nabla^2\lambda_k$$

Nevertheless, it is possible to introduce a set of new fields $\{\chi_1, ..., \chi_k, \psi_1, ..., \psi_k\}$ which simultaneously diagonalize the kinetic terms for λ_i and μ_i. This form of transformation will depend on the function g.
At least $k - 1$ of the new fields will be ghost-like.

The only possibility not to have ghosts in the theory is liner case $k = 1$.

Concrete example

$$f = \alpha + \beta R + \gamma R^2 + \epsilon R \Box R$$

$$S = \int d^4x \sqrt{-\mathrm{g}} [\alpha + \beta R + \gamma R^2 + \epsilon R \Box R]$$

$$\Downarrow$$

$$S = \int d^4x \sqrt{-\bar{\mathrm{g}}} \left[\bar{R} - \frac{3}{2} (\bar{\nabla}\chi)^2 - \epsilon e^{-\chi} (\bar{\nabla}\lambda_1)^2 - V(\lambda_1, \chi) \right]$$

with potential $V = e^{-2\chi}(e^{\chi}\lambda_1 - \alpha - \beta\lambda_1 - \gamma\lambda_1^2)$

$$S = \int d^4x \sqrt{-\mathrm{g}} f(R,\ R\Box R \equiv A).$$

$$-\frac{1}{2}f\mathrm{g}_{ik}+f_RR_{ik}-\nabla_i\nabla_kf_R+\mathrm{g}_{ik}\Box f_R+f_A\Box RR_{ik}+$$

$$\Box(f_AR)R_{ik}-\nabla_i\nabla_k(f_A\Box R+\Box(f_AR))+\Box(f_A\Box R+\Box(f_AR))\mathrm{g}_{ik}$$

$$+\frac{1}{2}\nabla_l(f_AR)\nabla^lR\mathrm{g}_{ik}-\nabla_i(f_AR)\nabla_kR+\frac{1}{2}f_AR\Box R\mathrm{g}_{ik}=0.$$

$$\begin{aligned}&-2f+f_RR+3\Box f_R+3Rf_A\Box R\\&+R\Box(f_AR)+3\Box(f_A\Box R+\Box(f_AR))+\nabla_l(f_AR)\nabla^lR=0.\end{aligned}$$

$$S = \int d^4x\sqrt{-\mathrm{g}}f(R,\ R\Box R \equiv A).$$

all scalarons may propagate physically (not a ghost-like) only in the case when function f is linear with respect to second argument more over it is need $f_A > 0$.

Ahmed Hindawi, Burt A. Ovrut, Daniel Waldram, Phys.Rev. D53 (1996) 5597-5608

$$-2f + f_R R + 3\Box f_R + 3Rf_A\Box R$$
$$+R\Box(f_A R) + 3\Box(f_A\Box R + \Box(f_A R)) + \nabla_l(f_A R)\nabla^l R = 0.$$

$$R = R_* + \delta R$$

- flat background $R_* = 0$
- dS background $R_* = R_{dS} = \mathrm{const}$
- non-flat background $R_* = R_b \neq \mathrm{const}$

on flat background:

$$-f_R(0,0)\delta R + 3f_{RR}(0,0)\Box\delta R + 6f_A(0,0)\Box^2\delta R = 0.$$

$$u_k \sim e^{i\mathbf{kx}-i\omega t},$$

$$6f_A(0,0)\mu^4 + 3f_{RR}(0,0)\mu^2 - f_R(0,0) = 0.$$

on dS background $R = R_{dS} + \delta R$, where R_{dS} is a constant, equations is similar to the previous one

Now let us study not flat background. In this case we have $R = R_b + \delta R$, where R_b is solution of trace equation and not fixed.

$$3f_{AA}R_b^2\Box^3\delta R + (6f_A + 6f_{AR}R_b + f_{AA}R_b^3)\Box^2\delta R$$

$$+(2f_AR_b + 3f_{RR} + 2f_{AR}R_b^2)\Box\delta R + (f_{RR}R_b - f_R)\delta R = 0,$$

where all derivatives of function f is took at the point (R_b, A_b). This relation may be strongly simplified if we study the limit $\mu^2 \gg R_b$ at WKB regime ($Rf_{RR} \ll f_R$):

$$3f_{AA}R_b^2\mu^6 + 6(f_A + f_{AR}R_b)\mu^4 + 3f_{RR}\mu^2 - f_R = 0.$$

$$\mu^2 = \frac{f_{RR}}{4f_A}\left(-1 \pm \sqrt{1 + 8\frac{f_A f_R}{f_{RR}}}\right).$$

Thus we have two possibility for positivity both μ^2:

- $f_A > 0$, $f_{RR} < 0$,
- $f_A < 0$, $f_{RR} > 0$.

The first of them is consistent with previous result (not ghost-like scalarons), but it has wrong limit ($f_A \to 0$), because in $f(R)$-gravity it need $f_{RR} > 0$. The second one contrary has true limit, but contain a ghost. Thus we have two possibilities: or we have a ghost(tachyon) in the theory or we have a theory which is disconnected with usual $f(R)$-gravity.

$$S = \int d^4x\sqrt{-\mathrm{g}}f(R,\, \nabla_i R\nabla^i R \equiv B).$$

$$-\frac{1}{2}f\,\mathrm{g}_{ik} + f_R R_{ik} - \nabla_i\nabla_k f_R + \mathrm{g}_{ik}\Box f_R + f_B\nabla_i R\nabla_k R$$

$$-2\nabla_l(f_B\nabla^l R)R_{ik} + 2\nabla_i\nabla_k[\nabla_l(f_B\nabla^l R)] - 2\mathrm{g}_{ik}\Box[\nabla_l(f_B\nabla^l R)] = 0,$$

and it's trace

$$\begin{aligned}&-2f + f_R R + 3\Box f_R +\\ &f_B\nabla_l R\nabla^l R - 2R\nabla_l(f_B\nabla^l R) - 6\Box[\nabla_l(f_B\nabla^l R)] = 0.\end{aligned}$$

On the flat and de Sitter background equation equivalent to the previous case.

$$-6f_B\Box^2\delta R + (3f_{RR} + 2f_B R_b)\Box\delta R + (f_{RR}R_b - f_R)\delta R = 0.$$

Since we are interesting in WKB-regime ($f_R \gg f_{RR}R$), we may neglect by the first term in the last bracket.

$$[f_B\mu^2(-6\mu^2 + 2R_b) + 3f_{RR}\mu^2 - f_R]\delta R = 0.$$

Here its need to note that we interested in the limit $R_b \ll \mu^2$, therefor finally we find equation for μ^2:

$$-6f_B\mu^4 + 3f_{RR}\mu^2 - f_R = 0,$$

which is totally identical (in linear case) to the flat background case because $f_A = -f_B$. This quadratic (with respect to μ^2) equation have two solution similar to previous case (where it need to change $f_A \to -f_B$) and for its positivity we have two possibilities: first $f_B < 0$, $f_{RR} < 0$ and second $f_B > 0$, $f_{RR} > 0$, as early one.

Let us introduce Lagrange multipliers

$$S = \int d^4x\sqrt{-\mathrm{g}}f(R,\ B) =$$
$$= \int d^4x\sqrt{-\mathrm{g}}[f(\lambda_1,\lambda_2) + \mu_1(R-\lambda_1) + \mu_2(\nabla_i\lambda_1\nabla^i\lambda_1 - \lambda_2)].$$

Variation with respect to μ_1 and μ_2 give us correspondingly

$$\lambda_1 = R, \quad \lambda_2 = B.$$

Variation with respect to λ_1 and λ_2 reads

$$\mu_1 = \frac{\partial f(\lambda_1,\lambda_2)}{\partial\lambda_1}, \quad \mu_2 = \frac{\partial f(\lambda_1,\lambda_2)}{\partial\lambda_2}.$$

Let us rewrite initial action in the canonical form. If we define potential V as

$$V(\lambda_1,\lambda_2,\mu_1,\mu_2) = f(\lambda_1,\lambda_2) - \mu_1\lambda_1 - \mu_2\lambda_2,$$

initial action take the form

$$S = \int d^4x \sqrt{-\mathrm{g}}[\mu_1 R + \mu_2 \nabla_i \lambda_1 \nabla^i \lambda_1 + V(\lambda_1, \lambda_2, \mu_1, \mu_2)].$$

Now to complete the transformation to canonical form we need to make a conformal re-scaling of the metric to remove the $\mu_1 R$ coupling. We use a standard procedure for this one. We define $\chi = \ln \mu_1$ and rescale to the metric $\bar{\mathrm{g}}_{ik} = \mathrm{e}^{\chi}\mathrm{g}_{ik}$. This give us next result

$$S = \int d^4x \sqrt{-\bar{\mathrm{g}}} \left[\bar{R} - \tfrac{3}{2}\bar{\mathrm{g}}^{ik} \frac{\partial\chi}{\partial x^i}\frac{\partial\chi}{\partial x^k} + \mathrm{e}^{-\chi}\bar{\mathrm{g}}^{ik}\mu_2 \frac{\partial\lambda_1}{\partial x^i}\frac{\partial\lambda_1}{\partial x^k} + \mathrm{e}^{-2\chi} V(\lambda_1, \lambda_2, \mu_1, \mu_2) \right].$$

We can see that second kinetic term contain a factor $\frac{\mu_2}{\mu_1} \equiv \frac{f_B}{f_R}$ and it must be negative to physical propagation of field λ_1. Thus we have a similar to the previous case picture: theory contain a tachyon($f_B < 0,\ f_R > 0$) or there is no a limit to the $f(R)$-gravity ($f_B > 0,\ f_R < 0$).

de Sitter stability

$$f_1 = R + \beta R^N$$

$$g_{ik} = diag(-1, a^2, a^2, a^2)$$

$$6H^2 + \beta[(1-N)R^N + 6H^2 N R^{N-1} + 6HN(N-1)R^{N-2}\dot{R}] = 0.$$

$$R_0^{N-1} = \frac{1}{\beta(N-2)}.$$

$H = H_0 + \delta H,\ R = R_0 + \delta R = R_0 + 6(\delta\dot{H} + 4H_0\delta H)$ and
$R^N = R_0^N + 6NR_0^{N-1}(\delta\dot{H} + 4H_0\delta H)$

$$-\frac{N(N-1)}{N-2}H_0^{-1}\delta\ddot{H}+3\frac{N(N-1)}{N-2}\delta\dot{H}+4H_0(N-1)\delta H=0$$

$$\delta H=e^{\lambda t}$$

stability condition:

$$0<N<2$$

$$f = R + \beta R^N + \alpha R\Box R.$$

$$6H^2 + \beta[(1-N)R^N + 6H^2 N R^{N-1} + 6HN(N-1)R^{N-2}\dot{R}]$$

$$+\alpha[2R\ddot{R} + 36H^3\dot{R} - \dot{R}^2 - 48H^2\ddot{R} - 12H\dddot{R}] = 0.$$

$$R_0^{N-1} = \frac{1}{\beta(N-2)}.$$

$$24H_0\alpha\delta H^{(4)} + 12\alpha R_0\delta\dddot{H} + \left(10H_0R_0\alpha - \frac{N(N-1)}{N-2}H_0^{-1}\right)\delta\ddot{H}$$

$$-\left(2R_0^2\alpha + 3\frac{N(N-1)}{N-2}\right)\delta\dot{H} + 4H_0(N-1)\delta H = 0.$$

$$\delta H = e^{\lambda t}$$

Routh-Hurwitz theorem

$$T_0 = 24 H_0 \alpha,$$

$$T_1 = 144 H_0^2 \alpha,$$

$$T_2 = 72 H_0 \left(12 \cdot 28 H_0^4 \alpha^2 - \alpha \frac{N(N-1)}{N-2} \right),$$

$$T_3 = 9 \cdot 24 H_0 \left(-12^2 \cdot 14 \cdot 16 \alpha^3 H_0^8 - 48 \alpha^2 H_0^4 \tfrac{N-1}{N-2} (13N - 16) \right.$$
$$\left. + \alpha \tfrac{N^2 (N-1)^2}{(N-2)^2} \right),$$

$$T_4 = 4 H_0 (N-1) T_3.$$

For sufficiently small values of H_0 all T_i have the same sign in the range $1 < N < 2$. It mean that in this range dS-solution may be stable for sufficiently big $|\beta|$. Actually even for $\beta \approx 2$.