Heavy Hadron Spectra using NRQCD Action

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Foldy-Wouthuysen Transformation

- For spectrum calculation it is necessary that $aM << 1$. For light quark it is true but for charm quark $aM_c > 0.4$ and for bottom qurak $aM_b > 1.2$ with lattice spacing $a = 0.12fm$.
- The Dirac equation $H\psi = i\frac{\partial\psi}{\partial t}$ where

$$H = \overrightarrow{\alpha}.(\overrightarrow{P} - e\overrightarrow{A}) + e\phi + m\beta$$

- Non-relativistic limit is reached by making the following transformation $\psi' = e^{iS}\psi$ where $S = -\frac{i}{2m}\beta\overrightarrow{\alpha}.(\overrightarrow{P} - e\overrightarrow{A})$.

$$\begin{aligned} H' &= e^{iS}He^{-iS} - ie^{iS}\frac{\partial e^{-iS}}{\partial t} \\ &= H + i[S,H] - \frac{1}{2}[S,[S,H]] - \frac{i}{6}[S,[S,[S,H]]] + ... \\ & \quad -\dot{S} - \frac{i}{2}[S,\dot{S}] + \frac{1}{6}[S,[S,\dot{S}]] + ... \end{aligned}$$

- Defining $\theta = \overrightarrow{\alpha}.(\overrightarrow{P} - e\overrightarrow{A})$ we get (up to O(v^4/c^4))

$$\begin{aligned} H' &= \beta(m + \frac{\theta^2}{2m} - \frac{\theta^4}{2m}) + e\phi - \frac{e}{8m^2}[\theta,[\theta,\phi]] - \frac{i}{8m^2}[\theta,\dot{\theta}] \\ &\quad + \frac{e\beta}{2m}[\theta,\phi] + i\beta\frac{\dot{\theta}}{2m} - \frac{\theta^3}{3m^2} \end{aligned}$$

- The last three term are off diagonal and can be removed by suitable transformation.

$$\begin{aligned} \theta^2 &= \overrightarrow{\alpha}.(\overrightarrow{P} - e\overrightarrow{A})\overrightarrow{\alpha}.(\overrightarrow{P} - e\overrightarrow{A}) \\ &= \begin{pmatrix} [\overrightarrow{\sigma}.(\overrightarrow{P} - e\overrightarrow{A})]^2 & 0 \\ 0 & [\overrightarrow{\sigma}.(\overrightarrow{P} - e\overrightarrow{A})]^2 \end{pmatrix} \end{aligned}$$

- $$[\sigma.(P - eA)]^2 u = [-\sum_i D_i^2 - e\sigma.B]u.$$

-
$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}$$

- $i\frac{\partial \psi'}{\partial t} = H'\psi'$

$$\begin{aligned} i\frac{\partial u}{\partial t} &= [m - \frac{1}{2m}\sum_j D_j^2 - \frac{e}{2m}\sigma.B - \frac{1}{8m^3}(\sum_j D_j^2)^2 \\ &\quad + e\phi - \frac{e}{8m^2}\nabla.E - \frac{ie}{8m^2}\sigma.(\nabla \times E - E \times \nabla)]u \end{aligned}$$

- Lowest order action

$$S_0 = \int d^4xu(x)^\dagger(iD_0 + \frac{\vec{D}^2}{2m})u(x)$$

- For free fermions the propagator becomes

$$G(p) = \left[i\sin(p_4 a) - 2\sin^2\left(\frac{p_4 a}{2}\right) + \frac{1}{2ma}\sum_j 4\sin^2\left(\frac{p_j a}{2}\right)\right]^{-1}$$

No doubling problem.

NRQCD Action

- Similarly for QCD we add correction terms. For order O(v^4)

$$\begin{aligned}\delta\mathcal{L}_{v^4} &= c_1\frac{1}{M^3}u^\dagger D^4 u + c_2\frac{g}{M^2}u^\dagger(D.E - E.D)u \\ &\quad + c_3\frac{ig}{M^2}u^\dagger\sigma.(D\times E - E\times D)u + c_4\frac{g}{M}u^\dagger\sigma.Bu\end{aligned}$$

- Correction term, order $O(v^6)$

$$\begin{aligned}\delta\mathcal{L}_{v^6} &= f_1\frac{g}{M^3}u^\dagger\{D^2,\sigma.B\}u + f_2\frac{ig^2}{M^3}u^\dagger(\sigma.E\times E)u \\ &\quad + f_3\frac{ig}{M^4}u^\dagger\{D^2,\sigma.(D\times E - E\times D)\}u\end{aligned}$$

- c_1 can be computed from relativistic kinetic energy formula

$$\sqrt{p^2+M^2}\approx M+\frac{p^2}{2M}-\frac{p^4}{8M^3}$$

- $c_1 = \frac{1}{8}$

- To compute chromoelectric couplings consider $T_E = \bar{\psi}(q)\gamma^0 g\phi(q-p)\psi(p)$ with

$$\psi(p) = \Big(\frac{E_p + M}{2E_p}\Big)^{\frac{1}{2}} \left[\begin{array}{c} u \\ \frac{\sigma.p}{E_p+M} u \end{array} \right]$$

- $T_E = V_E + S_E$

$$\begin{aligned} V(p,q) &= \Big[\frac{i}{4M^2} - \frac{3i}{32M^4}(p^2+q^2)\Big] u^\dagger \sigma.(q\times p) g\phi(q-p)u \\ S_E(p,q) &= \Big(1 - \frac{(p-q)^2}{8M^3}\Big) u^\dagger \phi(q-p)u \end{aligned}$$

- we conclude $c_3 = \frac{1}{8}$, $f_3 = \frac{3}{64}$ and $c_2 = \frac{1}{8}$.

- For chromomagnetic couplings consider $T_B(p,q) = -\bar{\psi}(q) g\gamma.A(q-p)\psi(p)$
- $c_4 = \frac{1}{2}$ and $f_1 = \frac{1}{8}$.
- f_2 can fixed from the amplitude for double scattering of a quark off an external static electric field
- Four-fermion contact interaction $O(v^4)$

$$\delta\mathcal{L}_{4q} = d_1 \frac{1}{M^2} u^\dagger \chi.\chi^\dagger u + d_2 \frac{1}{M^2} u^\dagger \sigma\chi.\chi^\dagger \sigma u$$

- To incorporate in simulation introduce a complex Gaussian noise field

$$\mathcal{L}_\eta = a_1 u^\dagger \eta u + a_2 \chi^\dagger \eta^\dagger \chi + a_3 Tr(\eta^\dagger \eta)$$

- Equivalent to

$$\mathcal{L} = -\frac{a_1 a_2}{a_3} u^\dagger \chi\chi^\dagger u$$

Lattice NRQCD

- Covariant derivatives for quark fields

$$
\begin{aligned}
a\Delta_\mu^+\psi(x) &= U_\mu(x)\psi(x+a\hat{\mu}) - \psi(x) \\
a\Delta_\mu^-\psi(x) &= \psi(x) - U_\mu^\dagger(x-a\hat{\mu})\psi(x-a\hat{\mu})
\end{aligned}
$$

- For gauge fields

$$
\begin{aligned}
a\Delta_\rho^+F_{\mu\nu}(x) &= U_\rho(x)F_{\mu\nu}(x+-a\hat{\rho})U_\rho^\dagger(x) - F_{\mu\nu}(x) \\
a\Delta_\rho^+F_{\mu\nu}(x) &= F_{\mu\nu}(x) - U_\rho^\dagger(x-a\hat{\rho})F_{\mu\nu}(x)U_\rho(x-a\hat{\rho})
\end{aligned}
$$

- Symmetric derivative

$$
\Delta^\pm = \frac{1}{2}(\Delta^+ + \Delta^-)
$$

- Laplacian

$$
\Delta^2 \equiv \sum_i \Delta_i^+\Delta_i^- = \sum_i \Delta_i^-\Delta_i^+
$$

- For $a = 0.12fm$ it is desirable to correct operators upto order $O(a^4)$.
- Symmetric derivative

$$
\begin{aligned}
\Delta_i^{\pm} f(x) &= \frac{1}{2a}[f(x + a\hat{i}) - f(x - a\hat{i})] \\
&= \partial_i f + \frac{a^2}{6}\partial_i^3 f \\
&= \partial_i f + \frac{a^2}{6}\Delta_i^{\pm}\Delta_i^{+}\Delta_i^{-} f \\
\partial_i f &= \Delta_i^{\pm} f - \frac{a^2}{6}\Delta_i^{+}\Delta_i^{\pm}\Delta_i^{-} f \\
\tilde{\Delta}_i^{\pm} f &= \Delta_i^{\pm} f - \frac{a^2}{6}\Delta_i^{+}\Delta_i^{\pm}\Delta_i^{-} f
\end{aligned}
$$

- Laplacian

$$\tilde{\Delta}^2 = \Delta^2 - \frac{a^2}{12}\sum_i [\Delta_i^{+}\Delta_i^{-}]^2$$

- Gauge fields

$$\begin{aligned} a^2 gF_{\mu\nu}(x) &= -Im\Big(1 - \frac{ig}{4}\oint_{(2\times2)} A.dy + ...\Big) \\ &= \frac{g}{4}\int_{(2\times2)} (\partial_\mu A_\nu - \partial_\nu A_\mu) dy^\mu dy^\nu + O(a^6) \\ &= a^2 gF_{\mu\nu}(x) + \frac{a^4}{6}(\partial_\mu^2 + \partial_\nu^2) gF_{\mu\nu}(x) + O(a^6) \end{aligned}$$

- The a^3 and a^5 terms vanishes

$$g\tilde{F}_{\mu\nu}(x) = gF_{\mu\nu}(x) - \frac{a^4}{6}[\Delta_\mu^+\Delta_\mu^- + \Delta_\nu^+\Delta_\nu^-] gF_{\mu\nu}(x)$$

- Lagrangian

$$\mathcal{L} = \psi^\dagger(x,t) D_4 \psi(x,t) + \psi^\dagger(x,t) H \psi(x,t)$$

- Equation of motion

$$\begin{aligned} D_4\psi(x,t) + H\psi(x,t) &= 0 \text{ after discretization} \\ U_t(x)\psi(x,t+1) - \psi(x,t) + aH\psi(x,t) &= 0 \end{aligned}$$

- Green's function obeys

$$U_t(x,t)G(x,t+1;0,0) - (1-aH)G(x,t;0,0) = \delta_{x,0}\delta_{t,0}$$

whose solution is

$$G(x,t+1;0,0) = U_t^\dagger(x,t)(1-aH)G(x,t;0,0)$$

- From mass and wave function renormalization consideration

$$\begin{aligned} G(x,t+1;0,0) &= (1-\frac{aH_0}{2})(1-\frac{a\delta H}{2})U_t(x,t)^\dagger \\ &\quad (1-\frac{a\delta H}{2})(1-\frac{aH_0}{2})G(x,t;0,0) \end{aligned}$$

- Meson propagator

$$
\begin{aligned}
c(t) &= \sum_{x} \langle 0|\mathcal{O}(x,t)\mathcal{O}^{\dagger}(x,0)|0\rangle \\
&= \sum_{x,n} \langle 0|e^{Ht}\mathcal{O}(x,0)e^{-Ht}|n\rangle\langle n|\mathcal{O}^{\dagger}(x,0)|0\rangle \\
&= \sum_{x,n} |\langle 0|\mathcal{O}(x,0)|n\rangle|^{2} e^{-E_n t} \\
&= \sum_{n} A_n e^{-E_n t}
\end{aligned}
$$

- $\mathcal{O}(x,t) = \bar{\psi}\Gamma\psi$, for pions $\Gamma = \gamma_5$ and for ρ meson $\Gamma = \gamma_i$.
- For large t : $c(t) \simeq A_0 e^{-M_0 t}$.
- M_0 can be obtained as

$$
M_0 = \ln\left[\frac{c(t)}{c(t+1)}\right]
$$