# Lattice QCD study for relation between quark confinement and chiral symmetry breaking

Takahiro M. Doi and Hideo Suganuma (Kyoto University) , Takumi Iritani (YITP, Kyoto University)

## Abstract

We derive an analytical gauge-invariant relation between the Polyakov loop $L_P$ and the Dirac eigenvalues $\lambda_n$ in lattice QCD where the temporal lattice size $N_4$ is odd. From this relation, we conclude that low-lying Dirac modes are not essential modes for confinement, which indicates no direct one-to-one correspondence between confinement and chiral symmetry breaking in QCD. In the confinement phase, we find a new "positive/negative symmetry" of the Dirac-mode matrix element of the link-variable operator. In the deconfinement phase, there is no such symmetry.

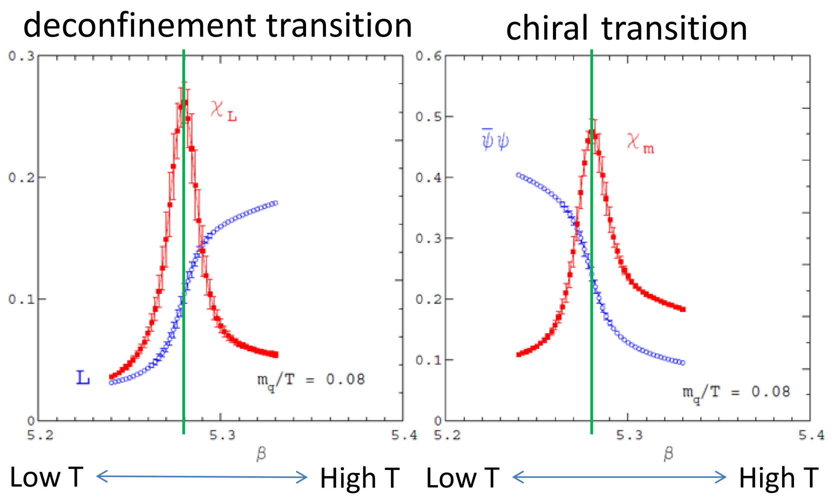
## Introduction

Question: confinement = chiral symmetry breaking in QCD ?

### Transition temperatures are almost same

deconfinement transition    chiral transition

F. Karsch, Lect. Notes Phys. 583, 209 (2002)

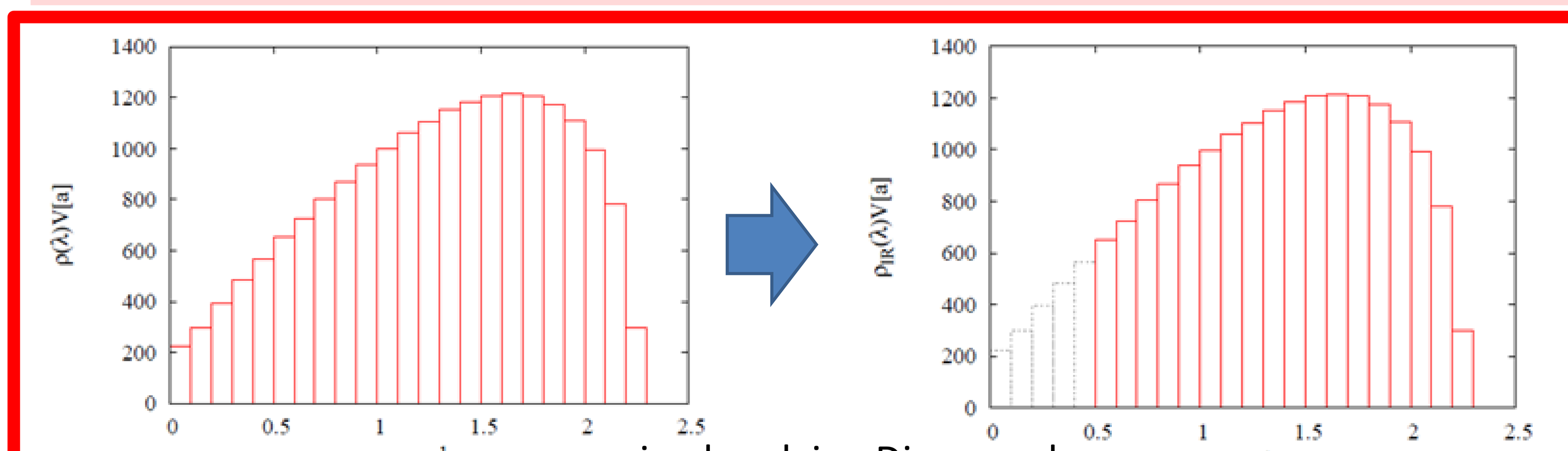



$\langle L\rangle, \chi_L$ : Polyakov loop and its susceptibility
$\langle\bar\psi\psi\rangle, \chi_m$ : chiral condensate and its susceptibility
- zero quark chemical potential
- two flavor QCD with light quarks

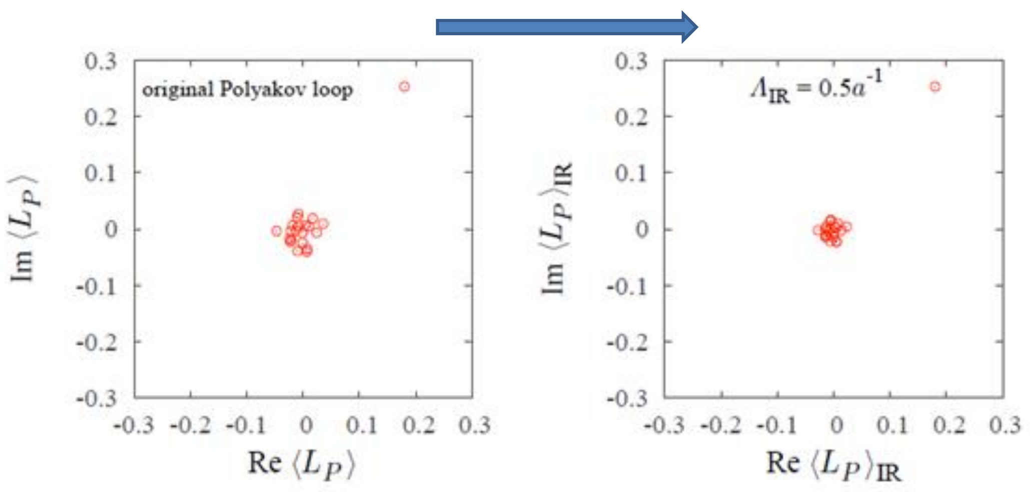
critical temperature is defined as the peak of susceptibility

⇒ Transition temperatures are almost same

### confinement properties are kept after removing low-lying Dirac-modes




removing low-lying Dirac modes




S. Gongyo, T. Iritani, H. Suganuma, PRD86 (2012) 034510
T. Iritani and H. Suganuma, PTEP, 2014 3, 033B03 (2014).

$L_P$: Polyakov loop

$$\langle L_P\rangle \begin{cases} = 0 & (\text{confinement phase}) \\ \neq 0 & (\text{deconfinement phase}) \end{cases}$$

Low-lying Dirac mode: essence of chiral symmetry breaking

c.f.) Banks-Casher relation: $\langle\bar q q\rangle = -\lim_{m\to 0}\lim_{V\to\infty} \pi\langle\rho(0)\rangle$

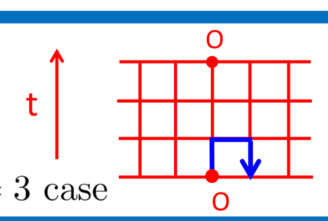
confinement properties are almost unchanged after removing low-lying Dirac-modes

**Notation of Dirac mode**
Dirac operator: $\hat{\not D}$
eigenvalue equation: $\hat{\not D}|n\rangle = i\lambda_n|n\rangle$
Dirac eigenmode: $|n\rangle$
Dirac eigenvalue: $i\lambda_n$
eigenvalue density: $\rho(\lambda) = \frac{1}{V}\sum_n \delta(\lambda-\lambda_n)$

## Analytical relation between Polyakov loop and Dirac mode

### Setup

In this study, we use
- standard square lattice
- with ordinary periodic boundary condition for gluons,
- with the odd temporal length $N_t$ ( temporally odd-number lattice )




### Derivation

Consider the functional trace on the temporally odd-number lattice:

$$I \equiv \mathrm{Tr}_{c,\gamma}(\hat U_4 \hat{\not D}^{N_4-1}) \quad (N_4 : \text{odd}) \quad (\mathrm{Tr}_{c,\gamma} \equiv \Sigma_s \mathrm{tr}_c \mathrm{tr}_\gamma)$$

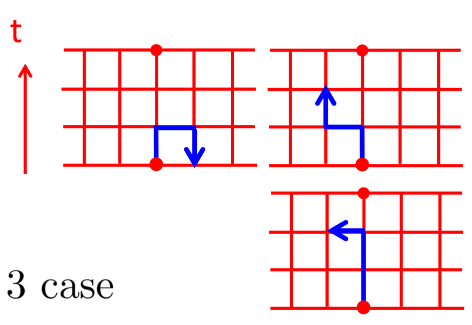
$|s\rangle$ : site

Dirac operator : $\hat{\not D} = \frac{1}{2}\sum_\mu \gamma_\mu(\hat U_\mu - \hat U_{-\mu})$ Link variable operator : $\langle s|\hat U_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat\mu,s'}$

$I \equiv \mathrm{Tr}_{c,\gamma}(\hat U_4 \hat{\not D}^{N_4-1})$ includes many trajectories on the square lattice.
The length of the trajectories is $N_4$ , i.e., odd.

Any closed loop needs even-number link-variables on square lattice.

In this functional trace $I \equiv \mathrm{Tr}_{c,\gamma}(\hat U_4 \hat{\not D}^{N_4-1})$,
it is impossible to form a closed loop on the square lattice,
because the length of the trajectories, $N_4$, is odd.

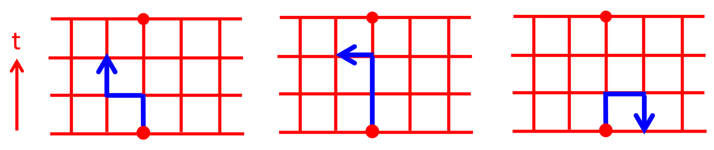



$N_4 = 3$ case

length of trajectories: $N_t = 3$
odd

Almost all trajectories are gauge-variant & give no contribution.

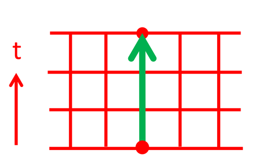
$N_4 = 3$ case




gauge variant
(no contribution)

Only the exception is the Polyakov loop.

$N_4 = 3$ case




gauge invariant !!

$I$ is proportional to the Polyakov loop.

$$I = \mathrm{Tr}_{c,\gamma}(\hat U_4 \hat{\not D}^{N_4-1}) = \frac{12V}{(2a)^{N_4-1}} L_P \quad \cdots ①$$

On the other hand, $I = \mathrm{Tr}_{c,\gamma}(\hat U_4 \hat{\not D}^{N_4-1}) = \sum_n \langle n|\hat U_4 \hat{\not D}^{N_4-1}|n\rangle = i^{N_4-1}\sum_n \lambda_n^{N_4-1}\langle n|\hat U_4|n\rangle \quad \cdots ②$

from ①、②

$$L_P = \frac{(2i)^{N_4-1}}{12V}\sum_n \lambda_n^{N_4-1}\langle n|\hat U_4|n\rangle$$

## relation between confinement and chiral symmetry breaking

The analytical relation connecting the Polyakov loop and Dirac mode

$$L_P = \frac{(2i)^{N_4-1}}{12V}\sum_n \lambda_n^{N_4-1}\langle n|\hat U_4|n\rangle$$ on temporally odd number lattice: $N_4$ is odd (in lattice unit: $a=1$ )

- Low-lying Dirac-modes are important for CSB (Banks-Casher relation) $(\lambda_n \sim 0)$
- Low-lying Dirac-modes have little contribution to Polyakov loop

The relation between Confinement and CSB is not one-to-one correspondence in QCD.

In fact, from similar analysis, we can derivate the similar relation between Wilson loop and Dirac mode.
Therefore, low-lying Dirac-modes have little contribution to the string tension σ, or the confining force.

## Numerical analysis

### Numerical analysis for the relation

$$L_P = \frac{(2i)^{N_4-1}}{12V}\sum_n \lambda_n^{N_4-1}\langle n|\hat U_4|n\rangle \longrightarrow L_P = \frac{(2i)^{N_4-1}}{12V}\sum_n \lambda_n^{N_4-1}\sum_s \psi_n^\dagger(s)U_4(s)\psi_n(s+\hat 4)$$

$L_P,\ U_4(s)$: easily obtained

$\lambda_n,\ \psi_n^\dagger(s),\ \psi_n(s+\hat 4)$ :are determined from $\hat{\not D}|n\rangle = i\lambda_n|n\rangle$

*This formalism is gauge invariant.

explicit form of the Dirac eigenvalue equation

$$\sum_{s',j,\beta} \not D_{ss'}^{ij,\alpha\beta}\psi_n(s')^{j,\beta} = i\lambda_n\psi_n(s)^{i,\alpha}$$

$s, s'$ : site
$i, j$ : color
$\alpha, \beta$ : spinor

where $\not D_{ss'}^{ij,\alpha\beta} = \frac{1}{2}\sum_{\mu=1}^{4}\gamma_\mu^{\alpha\beta}\left[U_\mu(s)^{ij}\delta_{s+\hat\mu,s'} - U_{-\mu}(s)^{ij}\delta_{s-\hat\mu,s'}\right]$   $U_{-\mu}(s) \equiv U_\mu(s-\hat\mu)^\dagger$

notation and coordinate representation

$\langle s|\hat U_4|s'\rangle = U_4(s)\delta_{s+\hat 4,s'}$
$\langle s'|n\rangle = \psi_n(s')$
$\langle n|s\rangle = \psi_n^\dagger(s)$
$\hat D_\mu = \frac{1}{2}(\hat U_\mu - \hat U_{-\mu})$
$1 = \sum_s |s\rangle\langle s|$   $|s\rangle$: site

From the actual numerical calculation,

It is confirmed that the relation is satisfied exactly for each gauge-configuration and low-lying Dirac modes have little contribution to the Polyakov loop in both confinement and deconfinement phases.

### New modified Kogut-Susskind (KS) formalism

We develop the new method (modified KS formalism) for spin-diagonalize the Dirac operator $\hat{\not D}$ applicable to temporally odd number lattice.

$$M(s) \equiv \gamma_1^{s_1}\gamma_2^{s_2}\gamma_3^{s_3}\gamma_4^{s_1+s_2+s_3} \qquad M^\dagger \not D M = \begin{pmatrix} \eta\cdot D & 0 & 0 & 0 \\ 0 & \eta\cdot D & 0 & 0 \\ 0 & 0 & -\eta\cdot D & 0 \\ 0 & 0 & 0 & -\eta\cdot D \end{pmatrix}$$

$$(\eta\cdot D)_{ss'}^{ij} = (\eta_\mu D_\mu)_{ss'}^{ij} = \frac{1}{2}\sum_{\mu=1}^{4}\eta_\mu(s)\left[U_\mu(s)^{ij}\delta_{s+\hat\mu,s'} - U_{-\mu}(s)^{ij}\delta_{s-\hat\mu,s'}\right]$$

$$L = \frac{(2i)^{N_4-1}}{12V}\sum_n \lambda_n^{N_4-1}\langle n|\hat U_4|n\rangle \quad \cdots \text{(A)}$$ Dirac eigenmode $|n\rangle$, $\not D|n\rangle = i\lambda_n|n\rangle$

$$L = \frac{(2i)^{N_4-1}}{3V}\sum_n \lambda_n^{N_4-1}(n|\hat U_4|n) \quad \cdots \text{(A)'}$$ KS Dirac eigenmode $|n)$, $\eta\cdot D|n) = i\lambda_n|n)$

(A)⇔(A)' relation (A)' is equivalent to (A)

case of even lattice ($N_1, N_2, N_3, N_4$ : even)

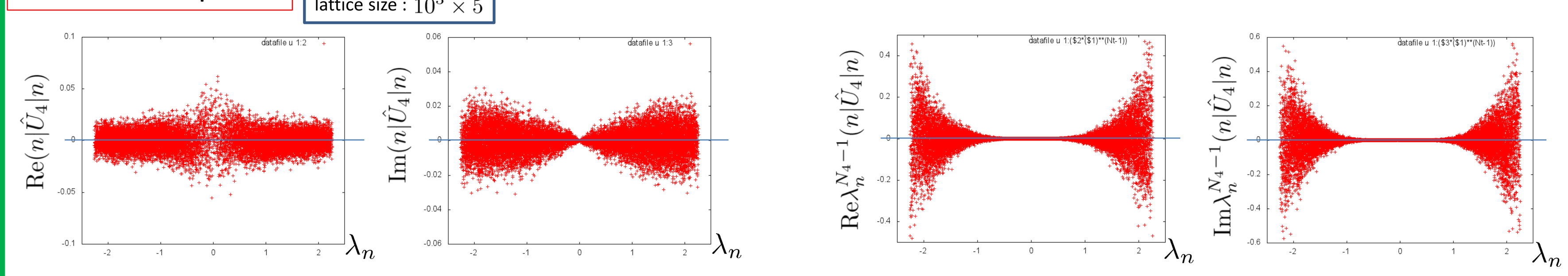
$$T(s) \equiv \gamma_1^{s_1}\gamma_2^{s_2}\gamma_3^{s_3}\gamma_4^{s_4}$$
$$\Rightarrow T^\dagger(s)\gamma_\mu T(s\pm\hat\mu) = \eta_\mu(s)\mathbf{1}_{\text{spinor}}$$
$$\Rightarrow T^\dagger \not D T = \begin{pmatrix} \eta\cdot D & 0 & 0 & 0 \\ 0 & \eta\cdot D & 0 & 0 \\ 0 & 0 & \eta\cdot D & 0 \\ 0 & 0 & 0 & \eta\cdot D \end{pmatrix}$$

staggered phase: $\eta_\mu(s)$

$$\eta_\mu(s) = \begin{cases} 1 & (\mu=1) \\ (-1)^{s_1} & (\mu=2) \\ (-1)^{s_1+s_2} & (\mu=3) \\ (-1)^{s_1+s_2+s_3} & (\mu=4) \end{cases}$$

### Numerical results for the matrix elements $(n|\hat U_4|n)$

confinement phase   $\beta = 5.6$ lattice size : $10^3\times 5$

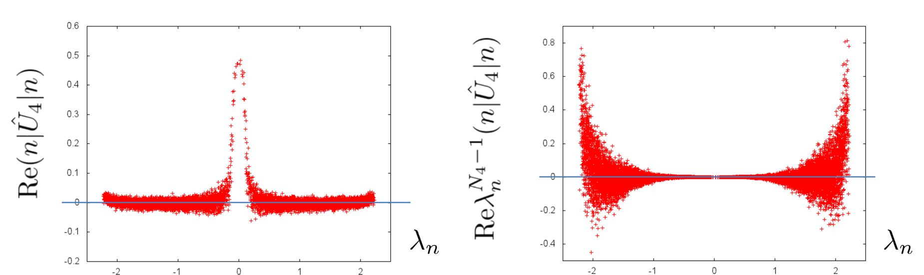



$\langle L\rangle = 0$ is due to the symmetric distribution of positive/negative value of $(n|\hat U_4|n),\ \lambda_n^{N_4-1}(n|\hat U_4|n)$

Low-lying Dirac modes have little contribution to Polyakov loop.

deconfinement phase   $\beta = 6.0$ lattice size : $10^3\times 5$

In our calculation, Polyakov loop is real, so only real part is different from it in confined phase.




In low-lying Dirac modes region, $\mathrm{Re}(n|\hat U_4|n)$ has a large value, but contribution of low-lying Dirac modes to Polyakov loop is very small because of dumping factor $\lambda_n^{N_4-1}$.

There is no positive/negative symmetry, and then the Polyakov loop is nonzero.

## Summary

We derive the relation between the Polyakov loop and the Dirac mode on the temporally odd-number lattice. From this relation, we conclude that low-lying Dirac modes are not essential modes for confinement, which indicates ***no direct one-to-one correspondence between confinement and chiral symmetry breaking in QCD***.

references:

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