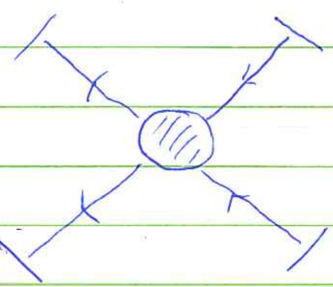
0a

2 particle scattering from finite Volume correlation functions

A. Walker-Loud lecture 4
unproofed notes

In infinite Volume, (Minkowski space) one of the most common ways we learn about the fundamental physics world is to scatter particles of eachother.



All the information about the microscopic interactions can be studied with this scattering - known as the S-matrix

$\uparrow$ if you have multiple channels this is a matrix

$$S_\ell = e^{2i\delta_\ell(p)}$$

Partial wave decomposition, ℓ

$\delta_\ell(p)$ = elastic scattering phase shift

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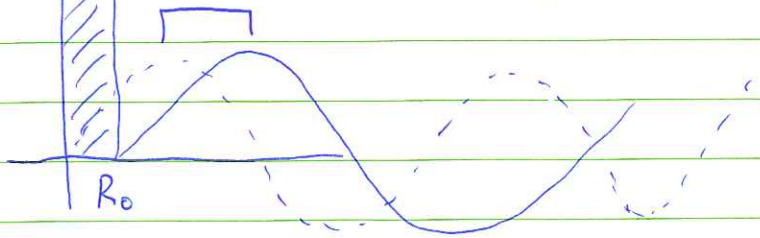
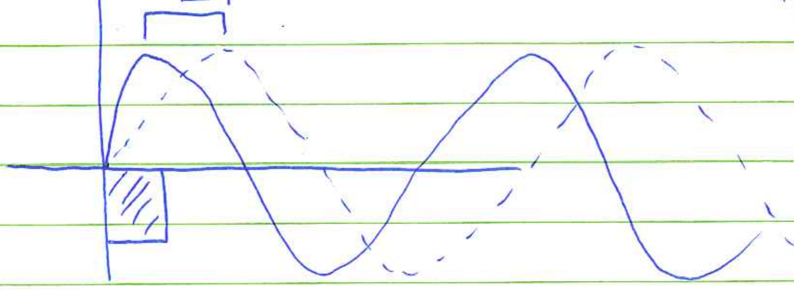
$\Delta\phi(p)$ $\sin(pr - \Delta\phi(p))$



repulsive hard-well

The repulsive hard-well pushes the wave function further out.

$-\Delta\phi$

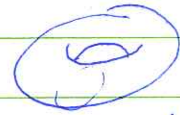


"soft" attractive-well

$\sin(pr + \Delta\phi(p))$ The attractive well pulls the wave function in.

We can measure these phase shifts at assymptotically far distances from the scattering region. These phase shifts allow us to reconstruct information about the microscopic interactions.

In a finite volume w/ periodic boundary conditions



The universe is a torus. We can not take the individual particles assymptotically far away, so we can not directly compute scattering on the lattice.

But we can still compute the phase shifts – Lüscher's Method

 – I will assume continuum limit

~~15~~

0c

Recal from QM scattering

$$T(s) = \frac{32\pi\sqrt{s}}{\sqrt{s-4m^2}} \frac{e^{2i\delta(s)}-1}{2i}$$

$S = e^{2i\delta}$, COM, $s = (P_1+P_2)^2$

$= 4E^2 - (\vec{P}_1+\vec{P}_2)^2$

$= 4E^2 \qquad \vec{P}_2 = -\vec{P}_1$

$= 4(m^2+\mathbf{k}^2)$

$$= \frac{32\pi\sqrt{s}}{2|\mathbf{k}|} \frac{e^{i\delta}\left(e^{i\delta}-e^{-i\delta}\right)}{2i}$$

$k \equiv |\mathbf{k}|$

$$= \frac{16\pi\sqrt{s}}{k} \frac{\sin\delta}{e^{-i\delta}}$$

$$= \frac{16\pi\sqrt{s}}{k} \frac{\sin\delta}{\cos\delta - i\sin\delta}$$

$$= \frac{16\pi\sqrt{s}}{k(\cot\delta - i)}$$

$$= \frac{16\pi\sqrt{s}}{k\cot\delta - ik}$$

$k\cot\delta$ is Real part of inverse scattering amplitude $= \mathbb{K}$ matrix

Also

$$k\cot\delta = -\frac{1}{a} + \frac{1}{2} r k^2 + \cdots$$ effective range expansion

a = scattering length

r = effective range

these are quantities we can measure and we can compute

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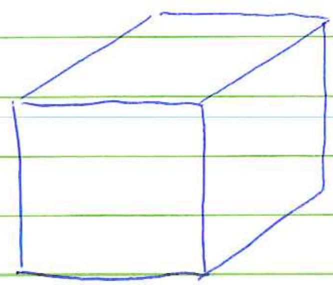
Let's begin with a 2 point correlation function that overlaps with a g.s. of a single particle.

$$C_{\Lambda'\mu',\Lambda\mu}(y_0-x_0,\vec{k}) \equiv \langle 0|\,\varphi_{\Lambda'\mu'}(y_0,\vec{k})\,\phi^{\dagger}_{\Lambda\mu}(x_0,-\vec{k})\,|0\rangle \propto \delta_{\Lambda\Lambda'}\delta_{\mu\mu'}$$

Λ irrep (irreducible representation) of cubic group

μ row of Λ



We don't have spherical symmetry so instead of classifying operators in terms of angular momentum basis, we should use operators which respect cubic symmetry.

Then, we can determine how the ∞-V rotationally invariant states overlap w/ these cubic irreps.

A_1^+ : most cubic symmetric, positive parity group overlaps with $L = 0, 4, 6, 8, \ldots$

eg. $|A_1^+, L=0\rangle = \overset{L\,m}{|0,0\rangle}$

$$|A_1^+, L=4\rangle = \frac{1}{2}\sqrt{\frac{5}{6}}\,|4,4\rangle + \frac{1}{2}\sqrt{\frac{7}{3}}\,|4,0\rangle + \frac{1}{2}\sqrt{\frac{5}{6}}\,|4,-4\rangle$$

$$|A_1^+, L=6\rangle = \frac{\sqrt{7}}{4}\,|6,4\rangle - \frac{1}{2\sqrt{2}}\,|6,0\rangle + \frac{\sqrt{7}}{4}\,|6,-4\rangle$$

How do you map finite volume states onto rotational states?

For now, we will drop the labels

What do we expect the time dependence of this correlation function to be?

$$C(t, \vec{k}) = e^{-E_k t} \left| \langle 0 | \varphi(0, \vec{k}) | E_{\vec{k}}; L \rangle \right|^2 + O\left(L^s \frac{e^{-E_3 t}}{E_3} \right) \qquad t = y_0 - x_0$$

We get this from inserting a complete set of states

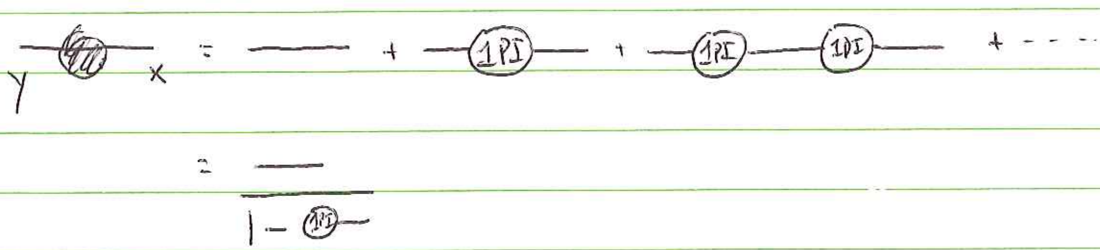
$$1 = \sum_n |n\rangle\langle n|$$

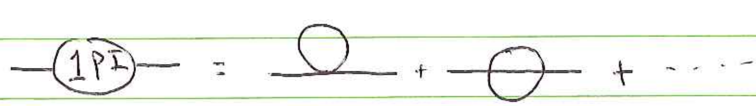
and time evolving the operators to $t=0$.

The first contributions is that from the single particle ground state.

The second term is from a 3-particle state, and so one

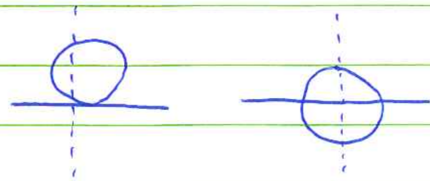
We can depict this graphically





The first term comes from the 1-particle pole

The second term (E_3) comes from the 3-particle pole



When the Energy of the correlator is ~~#~~ $3m_\phi \leq E < 5m_\phi$ so all particles can go on-shell.

There is another way we can understand this correlation function - the dressed single particle propagator

$$C(t,\vec{k}) = L^3 \int \frac{dP_0}{2\pi} \left[\frac{1}{2\omega_k(iP_0+\omega_k)} + \frac{Z_3}{2\omega_k^{(3)}(iP_0+\omega_k^{(3)})} + \cdots \right] e^{iP_0 t}$$

$$\omega_k = \sqrt{m^2+\vec{k}^2}$$

$\omega_k^{(3)}$ = 3 particle energy pole

The 1 particle pole goes on-shell for $P_0 = +i\,\omega_k$

~~who~~ We have chosen the residue of the pole to be $\underline{1}$ (we are using renormalized field operators with on-shell renormalization)

$$C(t, \mathbf{k}) = L^3 \left[\frac{e^{-\omega_k t}}{2\omega_k} \right] + O\left(\frac{L^3 e^{-\omega_k^{(3)} t}}{\omega_k^{(3)}} \right)$$

Our on-shell condition is the ~~same as~~ equivalently expressed as

$$\underbrace{\langle 0 | \varphi(0, \vec{x}=0) | E^{(1)}, k; \infty \rangle}_{\text{Fourier Transform of } \phi(0,\mathbf{k})} = 1$$

$|E^{(1)}k, \infty\rangle$ = infinite Volume, one particle state

$$\langle E^{(1)} \mathbf{k}' ; \infty | E^{(1)}_{\mathbf{k}} ; \infty \rangle = 2\omega_k (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}')$$

From here we can deduce

$$|\langle 0 | \phi(0, \mathbf{k}) | E_k ; L \rangle| = \sqrt{\frac{L^3}{2\omega_k}}$$

These relations hold up to corrections which are exponentially suppressed in e^{-mL}.

these are finite Volume corrections you learned about in χPT lectures

This was a warm up to think about our 2-particle states.

Let's construct interpolating fields that respect cubic symmetry from "single" particle interpolating fields

$$O_{\Lambda\mu}(x_0, \vec{P}, |\vec{P}-\vec{k}|, |\vec{k}|) = \sum_{R \in LG(P)} C(\vec{P}\Lambda\mu; R\vec{k}; R(\vec{P}-\vec{k}))\, \phi(x_0, R\vec{k})\, \phi(x_0, R(\vec{P}-\vec{k}))$$

This operator will create a state with total momentum $\vec{P}$

In our finite Volume calculations, we can not project individual states onto definite momentum, only the total momentum is a good QN (Quantum Number)

The sum $R \in LG(P)$ sums over all rotations of k, $\underline{P}-k$ that preserve $\underline{P}$, so we ~~are~~ don't have access to $\underline{t, u}$ dependence, only $\underline{s}$ (Mandelstam variables)

These Clebsch Gordon coefficients connect the two single particle states, in their own irreps, onto the two particle state $C(\underline{P}\Lambda\mu; R\vec{k}; R(\underline{P}-k)) \equiv \langle \Lambda(\underline{P}), \mu \mid \Lambda_1(\underline{P}-k)\ \Lambda_2(k) \rangle$

$$\sum_{R \in LG(P)} |C(\ )|^2 = \underline{1}$$

So now, we can construct our "2 particle" correlator

$$C^{(2)}(t,\vec{P}) = \langle 0 | \mathcal{O}_{\Lambda'\mu'}(t,\vec{P})\, \mathcal{O}^{\dagger}_{\Lambda\mu}(0,-\vec{P}) | 0 \rangle$$

$$= \delta_{\Lambda\Lambda'}\delta_{\mu\mu'} \sum_n e^{-E_{\Lambda,n} t} \left| \langle 0 | \mathcal{O}_{\Lambda\mu}(0,\vec{P}) | E_{\Lambda,n}, \vec{P} : L \rangle \right|^2 + \mathcal{O}\left(L^6 \frac{e^{-E_{th} t}}{E_{th}} \right)$$

where E_{th} = energy needed to create multi-particle's

in our case $E_{th} = 4m_\phi$

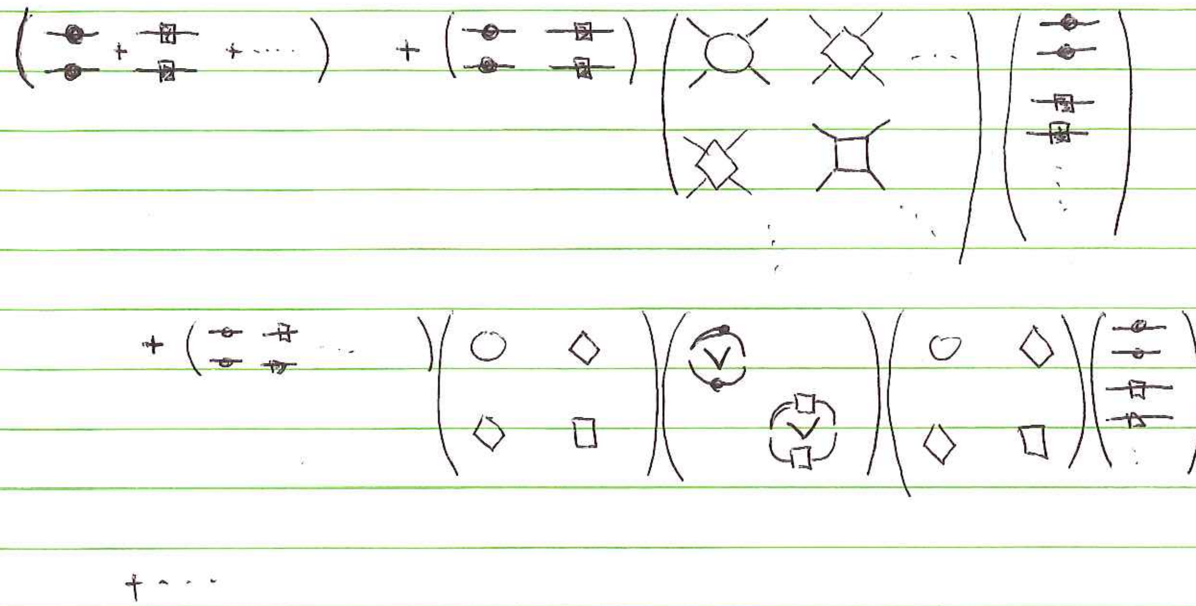
the $\sum_n$ runs over all energy states below this inelastic threshold.

* Important point: at this time, we only know how to compute elastic scattering processes such as these, with ~~Ns~~ $N=2$ particles.

The formalism to understand $N=3$ or more particles is still being developed. It is an active area of research (last week, a paper by Hansen & Sharpe appeared doing the beginning of 3-particles in QFT FV).

Multiple particles have been studied in non-relativistic theories, but the general QFT understanding is not complete. If you are interested in this type of work - it is a good time to get involved

The correlation function can be understood in terms of the interactions of the theory

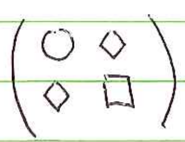


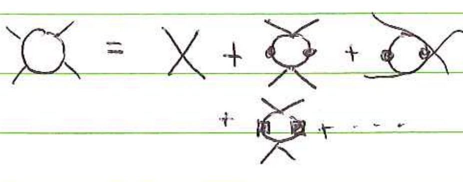
Here, I have allowed for coupled channels.

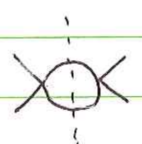
For example: $\pi\pi$, KK $I=0$

 single particle propagator for state ●

 single particle propagator for state ☒

 $= -K^*$ matrix of interactions (Bethe Salpeter kernel)

 all terms which contribute to scattering that do not contribute to s-channel process

s-channel  when all intermediate states can go on-shell

Note: what we are doing is setting up a geometric series for this two particle state, just as we did for single particles, so that we can re-sum and look for poles in the scattering.

Recal from QM, poles in the scattering amplitude correspond to discrete- bound states.

In FV, we are using the same notion. In FV, all we have are discrete states (no continuous momentum) So we are looking for poles of the 2-particle correlation function.

The first term  comes from the limit of no interactions

$$C^{(2,L0)}(t,\underline{P}) = L^6 \int \frac{dP_0}{2\pi}\, e^{iP_0 t}\, \tilde{C}^{(2,L0)}(P_0,\vec{P})$$

$$\tilde{C}^{(2,L0)}(P_0,\underline{\vec{P}}) = \int \frac{dk_0}{2\pi} \sum_{R\in LG(P)} C(P\Lambda\mu, Rk, R(P-k))\, G(k)\, G(P-k)\, C^*(P\Lambda\mu, Rk, R(P-k))$$

$$G(k) = \int d^4x\, e^{-ik\cdot x} \langle 0|\, \phi(x)\, \phi^\dagger(0)\, |0\rangle$$

[perturbation theory
$G(p) = \dfrac{i}{p^2-m^2+i\epsilon}$ M
$= \dfrac{1}{p^2+m^2}$ E]

Here, we are free to use infinite volume propagators, because they differ from FV only by e^{-mL} corrections, which we are currently neglecting

Note: this expression for $\tilde{C}^{(2,L0)}(P_0,P)$ is only valid for a strip, bounded by the inelastic threshold

We can evaluate the k_0 integral, encircling one of the poles

$$\omega_1 = \sqrt{m^2 + (P-k)^2}\ , \quad \omega_2 = \sqrt{m^2 + k^2}$$

let's encircle the 2^{nd} pole

ω_2 = on-shell, non-interacting energy

the 1-particle will have energy $-iP_0 - \omega_2$

$$C^{(2,LO)}(t,P) = L^6 \int \frac{dP_0}{2\pi} \, e^{iP_0 t} \sum_{R \in LG(P)} \frac{|C(P\Lambda\mu, Rk, R(P-k))|^2}{4\omega_1\omega_2\left(iP_0 + (\omega_1+\omega_2)\right)} + O\left(L^6 \frac{e^{-E_{th} t}}{E_{th}}\right)$$

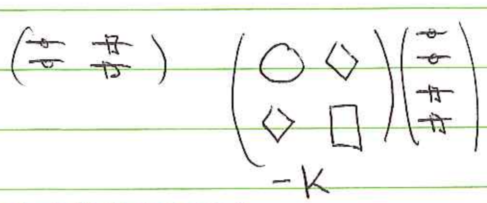
The next contribution comes from a single insertion of the Bethe-Salpeter Kernel, K

$$C^{(2,NLO)}(t,\vec{P}) = L^6 \int \frac{dP_0}{2\pi} \, e^{iP_0 t} \, \tilde{C}^{(2,NLO)}(P_0,\vec{P})$$

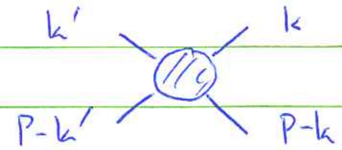
$$\tilde{C}^{(2,NLO)}(P_0,\vec{P}) = -\frac{1}{L^3} \sum_{R,R' \in LG(P)} C(P\Lambda\mu, R'k, R'(P-k))$$

$$\times \int \frac{dk_0}{2\pi} \int \frac{dk'_0}{2\pi} \, G(k') G(P-k') \, K(P,k,k') \, G(k) G(P-k)$$

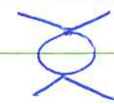
$$\times \, C^*(P\Lambda\mu, Rk, R(P-k))$$



focussing on a single channel

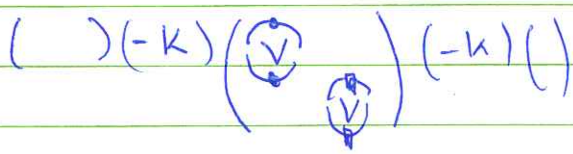


Recall, the Kernel is built from all interactions/diagrams that do-not contribute to the s-channel. Therefore

e.g. 

we are below the energy when all particles can go on shell. This means we can replace K_{FV} by K_{∞},
~~using Poisson resc~~
the corrections are suppressed by e^{-mL}

The finite Volume dependence we can not ignore, which will allow us to determine the ∞-V scattering phase shift, comes from the next diagram



What we are interested in, is how the interactions of the system modify the energy levels in finite Volume.

For non interacting systems $E = 2\sqrt{m^2 + k^2}$, $k \in \frac{2\pi}{L}\vec{n}$

The interactions will distort this relation

$$\Delta E = 2\sqrt{m^2+\vec{k}^2} - 2m$$

$$\vec{k} \neq \frac{2\pi\vec{n}}{L}$$

We will find ΔE depends strongly on the volume.

We will find a relationship between ΔE and the infinite volume scattering phase shift.

The correction we wish to evaluate has a UV divergence. But all we are interested in is the difference between FV & ∞V, so if we take the difference, the UV divergences cancel, and we can determine precisely the FV corrections

$$\left[\frac{1}{L^3}\sum\!\!\!\!\!\!\int_{\vec{k}}\right] \equiv \frac{1}{L^3}\sum_{\vec{k}} - \int\frac{d^3k}{(2\pi)^3}$$

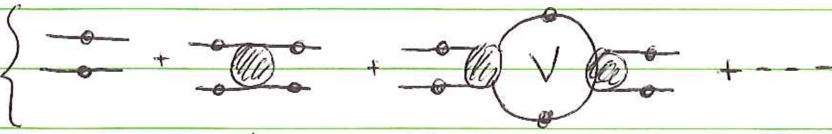
It is convenient to also use the Principle Value prescription to define the integral at the pole

$$\left[\frac{1}{L^3}\sum\!\!\!\!\!\!\int_{\ell}\right] \text{P.V.} \frac{K(P,k,\ell)\; K(P,\ell,k')}{4\,\omega_{1,\vec{P}-\vec{\ell}}\;\omega_{2,\vec{\ell}}\left(\omega_{1,\vec{P}-\vec{\ell}} + \omega_{2,\vec{\ell}} - P_{0,M}\right)} \equiv -K^{*}_{\text{off,on}}\, F_V\, K^{*}_{\text{on,off}} + O(e^{-mL})$$

Minkoswki Energy

So, what do we have?

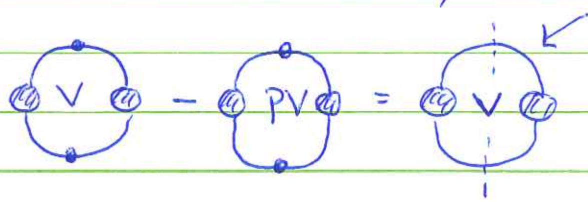
$$C^{(2)}(t,\vec{P}) = \int \frac{dP_0}{2\pi} e^{iP_0 t} \Big\{$$



1 $\qquad$ $-K$

This intermediate state can go on-shell and propagate to the edge of the box, and tell it is trapped in a finite volume. This is where our non-trivial FV dependence comes from

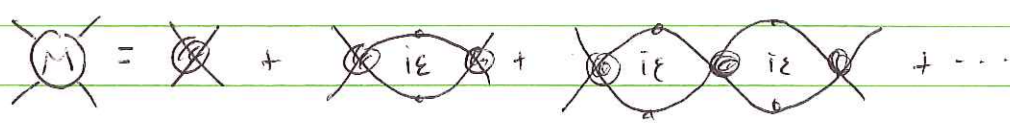
What we are doing is



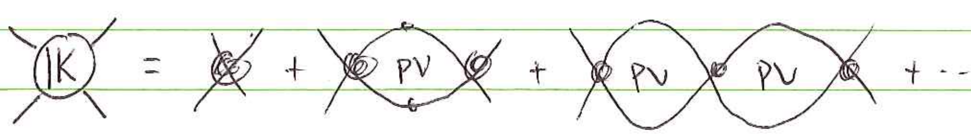
⋮ denotes cut, or on-shell state

Why PV?

In infinite volume, full scattering amplitude



There is a useful quantity, called the $\mathcal{K}$ matrix



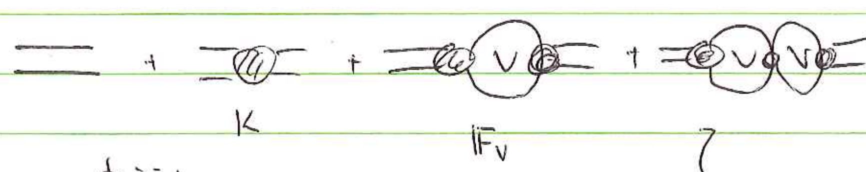
recal $\dfrac{1}{k\cot\delta - ik}$

The principle value prescription picks ~~of~~ up the Real part, and hence is related to the ∞V $\mathbb{K}$ matrix.

In fact, They only differ by corrections which scale as e^{-mL}

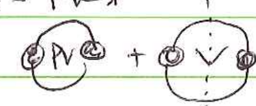
So back to our correlation function:

$$C^{(2)}(t,\vec{P}) = \int \frac{dP_0}{2\pi} e^{iP_0 t} \Big\{ \qquad + \cdots \Big\}$$



The poles of the infinite series are the interacting energy eigenvalues of the system

$$K + K\,\mathbb{F}_V\,K + K\,\mathbb{F}_V\,K\,\mathbb{F}_V\,K + \cdots \quad , \mathbb{F}_V = PV +$$



$$\Downarrow$$

$$\mathbb{K}_{off,off} - \mathbb{K}_{off,on}\,\mathbb{F}_V\,\mathbb{K}_{on,off} - \mathbb{K}_{off,on}\,\mathbb{F}_V\,\mathbb{K}_{on\,on}\,\mathbb{F}_V\,\mathbb{K}_{on,off}$$

FV function has cuts which put intermediate states on shell

$$= \mathbb{K}_{off,off} - K_{off,on}\cdot \mathbb{K}^{-1}_{on,on} \underbrace{\frac{1}{\mathbb{K}^{-1}_{on,on} + \mathbb{F}_V}}_{\text{Poles live here}} \mathbb{F}_V \cdot \mathbb{K}_{on,off}$$