# Lecture II:

Owe Philipsen

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

- The ideal gas on the lattice
- QCD in the static and chiral limit
- The strong coupling expansion at finite temperature
- The equation of state

# The ideal gas on the lattice

Starting point: propagator of a free scalar field

$$\begin{aligned}\ln Z_0 &= -\frac{1}{2}\ln\det\Delta = \frac{1}{2}\mathrm{Tr}\ln\Delta^{-1} \\ &= V\sum_{n=-N_\tau/2}^{N_\tau/2-1}\int_{\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^3p}{(2\pi)^3}\ln(\hat{p}^2+(am)^2) \\ &= V\sum_{n=-N_\tau/2}^{N_\tau/2-1}\int_{\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{d^3p}{(2\pi)^3}\ln\left(4\sin^2(\frac{a\omega_n}{2})+4\hat{\omega}^2\right)\end{aligned}$$

Lattice momenta:

$$\hat{p}^2 = 4\sin^2(\frac{a\omega_n}{2}) + 4\sum_{j=1}^{3}\sin^2(\frac{ap_j}{2}) \qquad 4\hat{\omega}^2 = 4\sum_{j=1}^{3}\sin^2(\frac{ap_j}{2}) + (am)^2$$

Matsubara sum by analytic continuation, use:

$$\frac{1}{N_\tau}\sum_{n=-N_\tau/2}^{N_\tau/2-1} g(\mathrm{e}^{i\omega_n}) = -\sum_{z_i}\frac{\mathrm{Res}(\frac{g(z_i)}{z_i})}{z_i^{N_\tau}-1}$$

Substitution: $\hat{\omega} = \sinh(aE/2)$

$$\ln Z_0 = -V \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^3p}{(2\pi)^3} \ln(1 - \mathrm{e}^{-N_\tau aE})$$

Expand in small lattice spacing about continuum limit:

$$\xrightarrow{a\to 0} 0$$

$$I(a) = \int_{\frac{\pi}{a}}^{\infty} \frac{d^3p}{(2\pi)^3} \ln(1 - \mathrm{e}^{-N_\tau aE}) = I(0) + \frac{dI}{da}\,a \qquad I(0) \;=\; 0 \qquad I'(a) \;\propto\; \exp -N_\tau\pi$$

So we may put a=0 in the integration limits! Now expand the dispersion relation

$$\sinh^2(\frac{aE}{2}) = \sum_{j=1}^{3} \sin^2(\frac{ap_j}{2}) + \frac{(am)^2}{4}$$

$$E(\mathbf{p}) = E^{(0)}(\mathbf{p}) + aE^{(1)}(\mathbf{p}) + a^2E^{(2)}(\mathbf{p}) + \ldots, \qquad E^{(0)}(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$$

$$\frac{(aE)^2}{4} + \frac{(aE)^4}{48} \;=\; \sum_{j=1}^{3} \left( \frac{(ap_j)^2}{4} - \frac{(ap_j)^4}{48} \right) + \frac{(am)^2}{4} + O(a^6)$$

➡ $$E^{(2)}(\mathbf{p}) = -\frac{1}{24E^{(0)}(\mathbf{p})}\left(\sum_{j=1}^{3} p_j^4 + E^{(0)4}(\mathbf{p})\right)$$

breaks rotation invariance

➡ The bosonic dispersion relation has leading $O(a^2)$ cut-off effects

Improvement: subtracting these, the dispersion relation is “p4-improved”

Expansion of the pressure now simple: expand down $e^{-aEN_\tau}$ , then expand log

Use dimensionless variables: $x = p/T, \varepsilon = E/T$

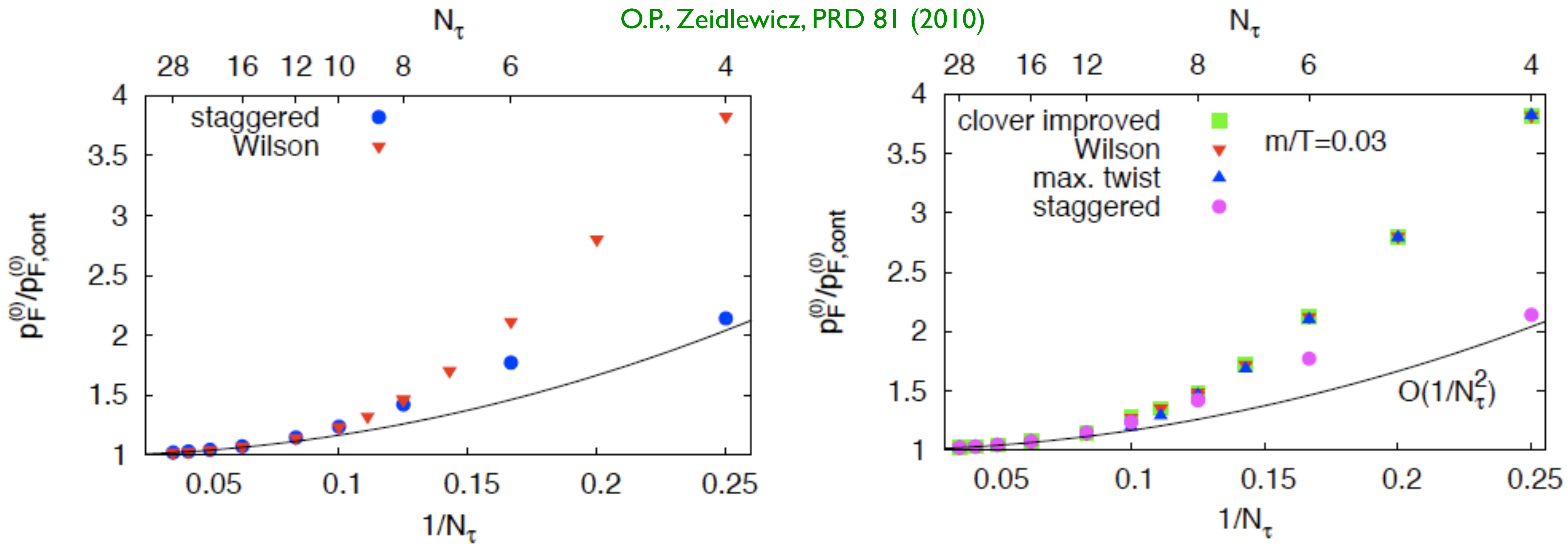
$$\frac{p}{T^4} = \left(\frac{p}{T^4}\right)_{\mathrm{cont}} - a^2 \int \frac{d^3x}{(2\pi)^3} \frac{\varepsilon^{(2)}(x)}{\mathrm{e}^{\varepsilon^{(0)}(x)} - 1} + \dots$$

$$\frac{p}{p_{\mathrm{cont}}} = 1 + \frac{8\pi^2}{21}\frac{1}{N_\tau^2} + O\left(\frac{1}{N_\tau^4}\right)$$

Free boson gas has leading $O(a^2)$ cut-off effects!

# Free fermion gas on the lattice

Analogous calculation, massless case starts also at $O(a^2)$



O.P., Zeidlewicz, PRD 81 (2010)

For Wilson fermions with finite mass, the leading correction is $O(a)$, staggered $O(a^2)$

Note: $N_\tau \geq 10$ required for leading cut-off effects to dominate!

# Quenched limit of QCD and Z(N) symmetry

Infinite quark masses (omitting flavour index) $m \rightarrow \infty$

Static quark propagator: $\langle \psi_\alpha^a(\tau, \mathbf{x}) \bar{\psi}_\beta^b(0, \mathbf{x}) \rangle = \delta_{\alpha\beta}\, e^{-m\tau} \left( T e^{i \int_0^\tau d\tau A_0(\tau, \mathbf{x})} \right)_{ab}$

On the finite T lattice: Polyakov loop $L(\mathbf{x}) = \prod_{x_0}^{N_\tau} U_0(x)$

Static QCD:
(one flavour)

$$S_{\text{static}}[U] = S_g[U] + \sum_{\mathbf{x}} \left( e^{-mN_\tau} \text{Tr} L(\mathbf{x}) + e^{-mN_\tau} \text{Tr} L^\dagger(\mathbf{x}) \right)$$

$$\xrightarrow{m \rightarrow \infty} S_g[U]$$

Gauge transformations: $U_\mu^g(x) = g(x) U_\mu(x) g^{-1}(x + \hat{\mu}), \quad g(x) \in SU(N)$

Periodic b.c.: $U_\mu(\tau, \mathbf{x}) = U_\mu(\tau + N_\tau, \mathbf{x}), \quad g(\tau, \mathbf{x}) = g(\tau + N_\tau, \mathbf{x})$

Action gauge invariant: $S_g[U^g] = S_g[U] \qquad L^g(\mathbf{x}) = g(x) L(\mathbf{x}) g^{-1}(x) \qquad \text{Tr} L^g = \text{Tr} L$

# Topologically non-trivial gauge transformations:

Modified b.c. for trafo matrix: $g'(\tau + N_\tau, \mathbf{x}) = \underset{\substack{\big\uparrow \\ \text{global “twist”}}}{h} g'(\tau, \mathbf{x}) \;, \quad h \in SU(N)$

$U_\mu^{g'}(\tau + N_\tau, \mathbf{x}) = h\, U_\mu^{g'}(N_\tau, \mathbf{x})\, h^{-1}$ needs to be periodic for correct finite T physics!

➡ $h = z\mathbf{1} \in Z(N), \quad z = \exp i\frac{2\pi n}{N}, \quad n \in \{0, 1, 2, \ldots N-1\}$ Centre of SU(N)

$S_g[U^{g'}] = S_g[U]$ invariant: centre symmetry of pure gauge action

**Note: this is not a symmetry of $H$ , but of $H_z$ !**

Requires compact time direction with periodic b.c. ; finite T!

$$L^{g'}(\mathrm{x}) = g'(1,\mathrm{x})L(\mathrm{x}){g'}^{-1}(1+N_\tau,\mathrm{x}) = g'(1,\mathrm{x})L(\mathrm{x}){g'}^{-1}(1,\mathrm{x})h^{-1}$$

$\mathrm{Tr}L^{g'} = z^*\mathrm{Tr}L$ Polyakov loop picks up a phase under centre transformations

Partition function in the presence of one static quark: $Z_Q = \int DU\ \mathrm{Tr}L(\mathrm{x})\ \mathrm{e}^{-S_g[U]}$

$$\langle \mathrm{Tr}L \rangle = \frac{1}{Z}\int DU\ \mathrm{Tr}L\ \mathrm{e}^{-S_g} = \frac{Z_Q}{Z} = \mathrm{e}^{-(F_Q - F_0)/T}$$

gives free energy difference of thermal YM-system with and without a static quark

Small T: $F_Q = \infty$ because of confinement → $\langle \mathrm{Tr}L \rangle = 0$

Large T: $\beta \to \infty$ $U_0 \to \mathbf{1}$ → $\langle \mathrm{Tr}L \rangle \to \mathrm{Tr}\mathbf{1} = N$

Thus Polyakov loop is non-analytic function of T → phase transition!

Deconfinement phase transition in YM: spontaneous breaking of Z(N) symmetry

# Now add dynamical quarks:

$$\psi^g(x) = g(x)\psi(x), \quad \psi(\tau + N_\tau, \mathbf{x}) = -\psi(\tau, \mathbf{x}), \quad \psi^{g'}(\tau + N_\tau, \mathbf{x}) = -h\psi(\tau, \mathbf{x})$$

needs to be anti-periodic for correct finite T physics! $h = 1 \text{ only}$

➡ Centre symmetry explicitly broken by dynamical quarks!

$\langle \mathrm{Tr} L \rangle \neq 0$ for all T!

➡ Confined and deconfined region analytically connected (only one phase!)
No need for a phase transition!

# Massless QCD and chiral symmetry (continuum)

massless quarks:

$S_f$ invariant under global chiral transformations $U_A(1) \times SU(N_f)_L \times SU(N_f)_R$

spontaneous symmetry breaking: $SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V$

➡ $N_f^2 - 1$ massless Goldstone bosons, pions

order parameter: chiral condensate $\langle\bar{\psi}\psi\rangle = \frac{1}{L^3 N_t} \frac{\partial}{\partial m_q} \ln Z$

$$\langle\bar{\psi}\psi\rangle \begin{cases} > 0 \Leftrightarrow \text{symmetry broken phase,} & T < T_c \\ = 0 \Leftrightarrow \text{symmetric phase,} & T > T_c \end{cases}$$

chiral transition: spontaneous restoration of global $SU(N_f)_L \times SU(N_f)_R$ at high T

Chiral symmetry explicitly broken by dynamical quarks, no need for phase transition!

# Physical QCD

.....breaks both chiral and Z(3) symmetry explicitly

.....but displays confinement and very light pions

- no order parameter → no phase transition necessary!
- if there is a p.t.: are there two distinct transitions?
- if there is just one p.t.: is it related to chiral or Z(3) dynamics?
- if there is no phase transition: how do the properties of matter change?

# Strong coupling expansion (pure gauge)

Wilson action: $S_g[U] = \sum_x \sum_{1\leq\mu<\nu\leq 4} \beta\left(1 - \frac{1}{3}\mathrm{Re}\mathrm{Tr}U_p\right) \equiv \sum_p S_p$ Plaquette action

Character of rep. r: $\chi_r(U) = \mathrm{Tr}D_r(U)$

group element

representation matrix of group element

Character expansion: $\exp -S_p = c_0(\beta)[1 + \sum_{r\neq 0} d_r c_r(\beta)\chi_r(U_p)] \ ,$

dimension of rep. matrix

Expansion coefficients: combinations of modified Bessel fcns. for SU(N)

$c_f \equiv u \ = \frac{\beta}{18} + O(\beta^2)$ all others can be expressed by fundamental one

$$f(N_\tau, u) = -\frac{3}{N_\tau} u^{4N_\tau} c^{N_\tau} \left[1 + 12N_\tau u^4 + 42N_\tau u^5 - \frac{115343}{2048} N_\tau u^6 - \frac{597663}{2048} N_\tau u^7\right]$$
$$- \frac{3}{N_\tau} u^{4N_\tau} b^{N_\tau} \left[1 + 12N_\tau u^4 + 30N_\tau u^5 - \frac{17191}{256} N_\tau u^6 - 180N_\tau u^7\right]$$

Remarkable result:

$$f(N_\tau, u) = -\frac{1}{N_\tau} \left[e^{-m(A_1^{++})N_\tau} + 2e^{-m(E^{++})N_\tau} + 3e^{-m(T_1^{+-})N_\tau}\right] (1 + O(u^4))$$

Glueball masses in SCE:
(plaquette correlators)

$$m(A_1^{++}) = -4 \ln u - 3u + 9u^2 \quad ...$$
$$m(E^{++}) = -4 \ln u - 3u + 9u^2 \quad ...$$
$$m(T_1^{+-}) = -4 \ln u + 3u + \frac{9}{2}u^3 \quad ...$$

At strong coupling the QCD partition function is that of a free hadron resonance gas!

# Equation of state: ideal (non-interacting) gases

partition fcn. for one relativistic bosonic/fermionic d.o.f.:

$$\ln Z = V \int \frac{d^3p}{(2\pi)^3} \ln \left(1 \pm \mathrm{e}^{-(E(p)-\mu)/T}\right)^{\pm 1}, \qquad E(p) = \sqrt{\mathbf{p}^2 + m^2}$$

equation of state for g d.o.f., two relevant limits:

Stefan-Boltzmann

| $Relativistic\ Boson,\ m \ll T$ | $\times\ (Fermion)$ | $Non\text{-}relativistic,\ m \gg T$ |
|---|---|---|
| $p_r = g\frac{\pi^2}{90}T^4$ | $\left(\frac{7}{8}\right)$ | $p_{nr} = gT\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} \exp(-m/T)$ |
| $\epsilon_r = g\frac{\pi^2}{30}T^4$ | $\left(\frac{7}{8}\right)$ | $\epsilon_{nr} = \frac{m}{T}p_{nr} \gg p_{nr}$ |

➡ $p_r = \epsilon_r/3, \qquad p_{nr} \simeq 0$

# The QCD equation of state

**Task:** compute free energy density or pressure

$$f = -\frac{T}{V} \ln Z(T,V)$$

➡ all bulk thermodynamic properties follow:

$$p = -f, \qquad \frac{\epsilon - 3p}{T^4} = T \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{p}{T^4}\right), \qquad \frac{s}{T^3} = \frac{\epsilon + p}{T^4}, \qquad c_s^2 = \frac{\mathrm{d}p}{\mathrm{d}\epsilon}$$

**Technical problem:** partition function in Monte Carlo normalized to 1.

➡ $Z, p, f$ not directly calculable, only $\langle O \rangle = Z^{-1}\,\mathrm{Tr}(\rho O)$

➡ **Integral method:** $$\left.\frac{f}{T^4}\right|_{T_o}^{T} = -\frac{1}{V}\int_{T_o}^{T} \mathrm{d}x \, \frac{\partial x^{-3} \ln Z(x,V)}{\partial x}$$

modify for lattice action: Integration along line of constant physics!

$$\left.\frac{f}{T^4}\right|_{(\beta_o,m_{f0})}^{(\beta,m_f)} = -\frac{N_\tau^3}{N_s^3}\int_{\beta_o,m_{f0}}^{\beta,m_f}\Bigg(\mathrm{d}\beta'\left[\left\langle\frac{\partial\ln Z}{\partial\beta'}\right\rangle - \left\langle\frac{\partial\ln Z}{\partial\beta'}\right\rangle_{T=0}\right] + \sum_f \mathrm{d}m_f'\left[\left\langle\frac{\partial\ln Z}{\partial m_f'}\right\rangle - \left\langle\frac{\partial\ln Z}{\partial m_f'}\right\rangle_{T=0}\right]\Bigg)$$

N.B.: lower integration constant not rigorously defined,
but exponentially suppressed

$$\frac{f}{T^4}(\beta_0) \sim \mathrm{e}^{-\mathrm{m}_{\text{Hadron}}/\mathrm{T}} \approx 0$$

cut-off effects in the high temperature, ideal gas limit: momenta $\sim T \sim \frac{1}{a}$

$$\left.\frac{p}{T^4}\right|_{N\tau} = \left.\frac{p}{T^4}\right|_{\infty} + \frac{c}{N_\tau^2} + \mathcal{O}(N_\tau^{-4}) \qquad \text{(staggered)}$$

Quantities to be calculated:

$$\frac{1}{N_\tau N_s^3}\frac{\partial \ln Z}{\partial \beta} = \frac{1}{N_\tau N_s^3}\left\langle \sum_p U_p \right\rangle = \langle -s_g \rangle$$

$$\frac{1}{N_\tau N_s^3}\frac{\partial \ln Z}{\partial m_f} = \frac{1}{N_\tau N_s^3}\left\langle \sum_x \bar{\psi}_f(x)\psi_f(x) \right\rangle$$

For the numerical integration along lines of constant physics, need beta-functions!

Directly accessible before integration: trace anomaly

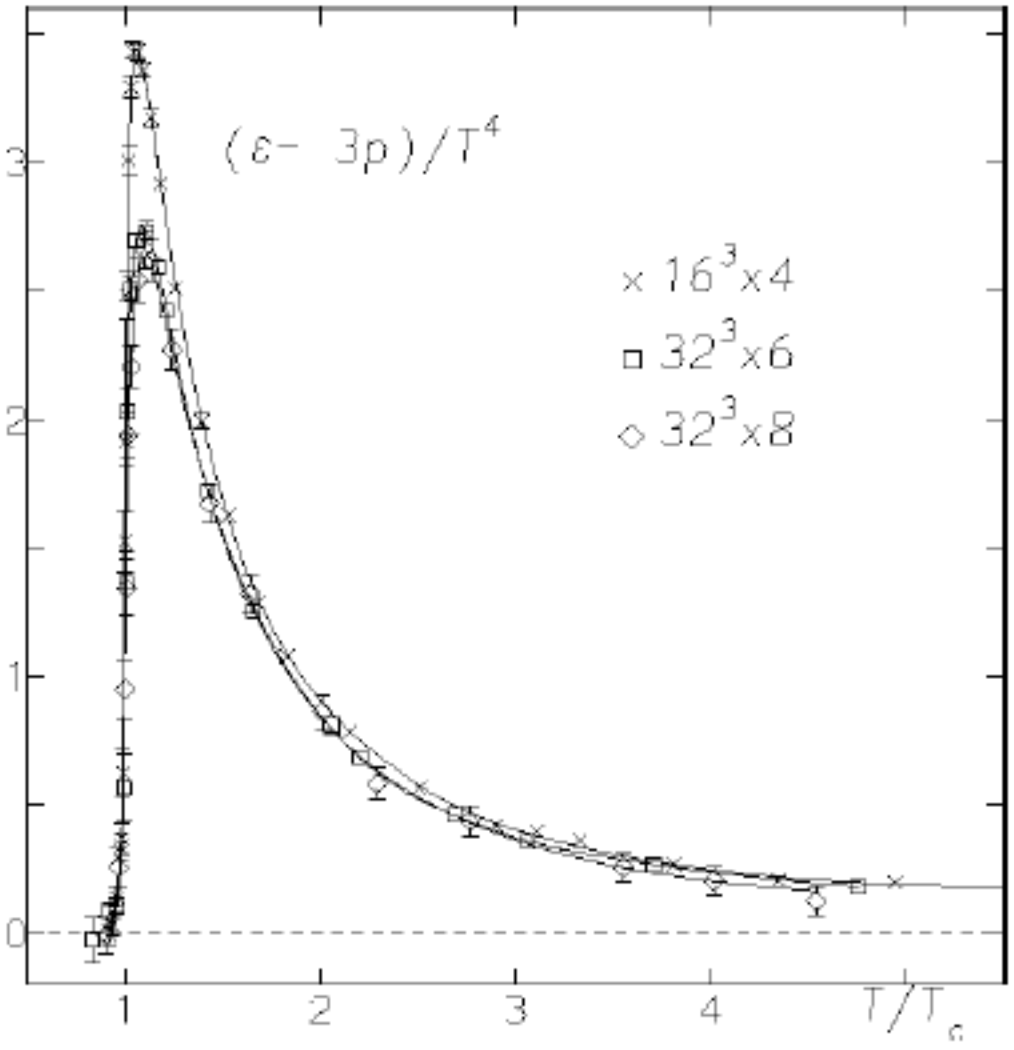
$$I(T) \equiv T^{\mu\mu}(T) = T^5 \frac{\partial}{\partial T}\frac{p(T)}{T^4} = \epsilon - 3p$$

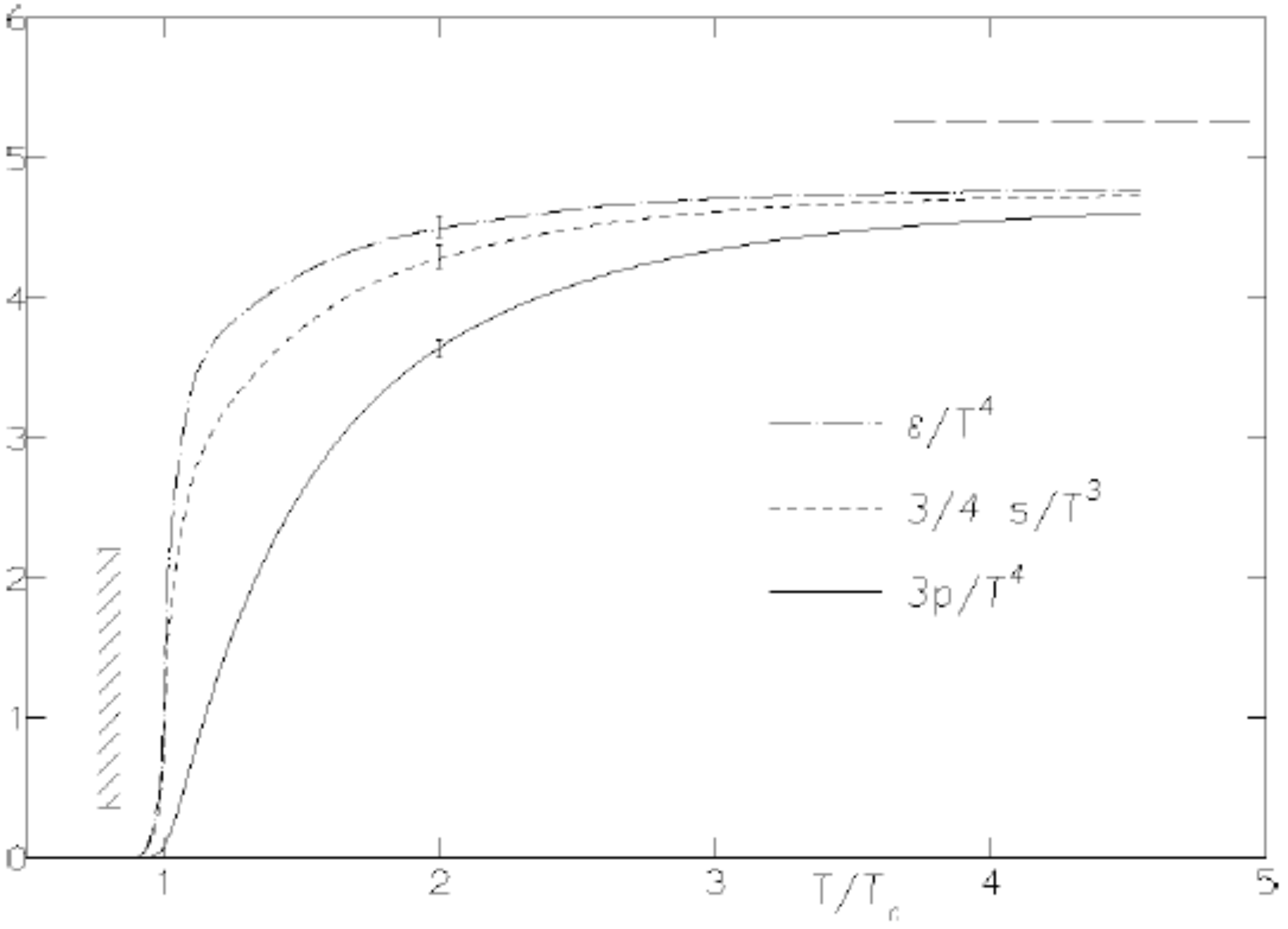
$$\frac{I(T)}{T^4}\frac{dT}{T} = N_\tau^4 \left( d\beta \langle -s_g \rangle^{\text{sub}} + \sum_f dm_f \langle \bar{\psi}_f \psi_f \rangle^{\text{sub}} \right) ,$$

$$\frac{I(T)}{T^4} = -N_\tau^4 \left( a\frac{d\beta}{da} \langle -s_g \rangle^{\text{sub}} + \sum_f a\frac{dm_f}{da} \langle \bar{\psi}_f \psi_f \rangle^{\text{sub}} \right)$$

# Numerical results, pure gauge

Boyd et al., NPB 469 (1996)





Ideal gas behaviour at high and low T

Continuum extrapolation using Nt=6,8

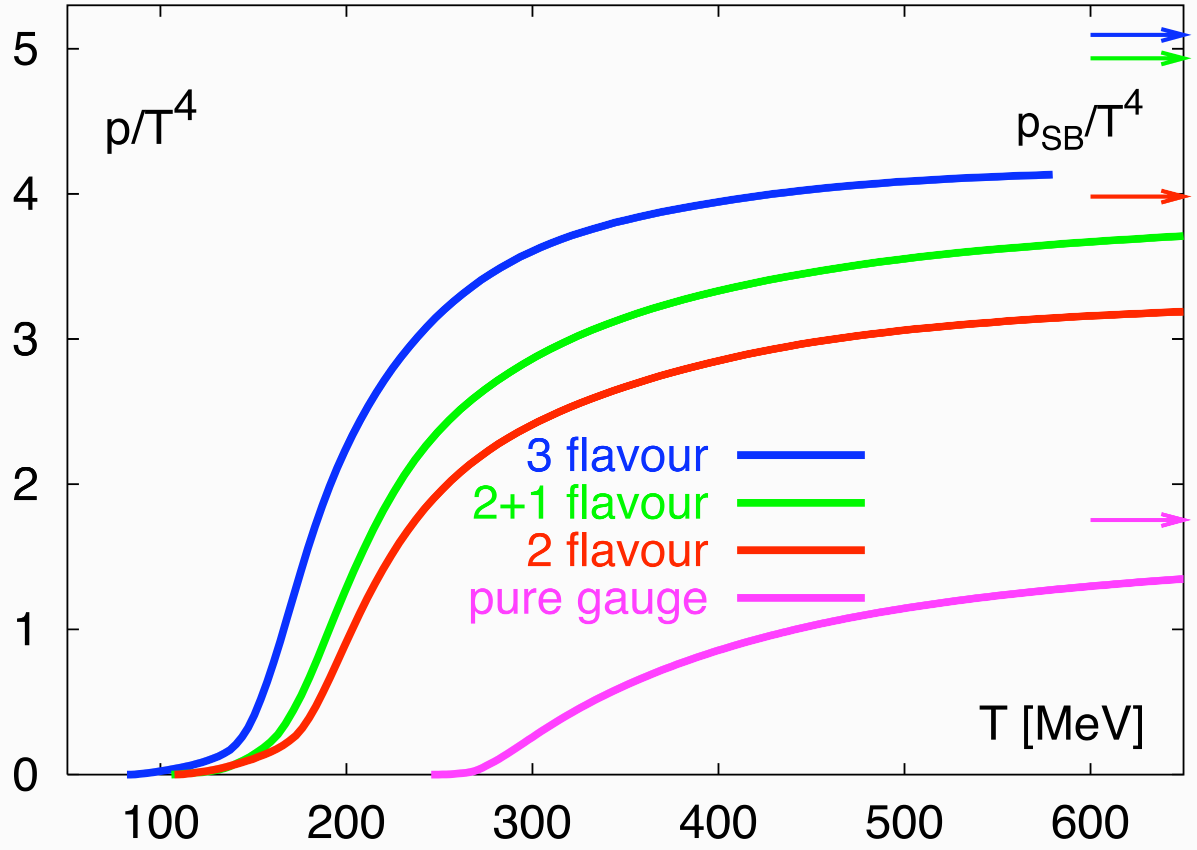
$$\left(\frac{p}{T^4}\right)_a = \left(\frac{p}{T^4}\right)_0 + \frac{c(T)}{N_\tau^2}$$

# Flavour dependence of the equation of state

Bielefeld

staggered p4-improved, $N_\tau = 4$

Karsch et al., PLB 478 (2000)



compare with ideal gas:

Pions

$$\frac{\epsilon_{SB}}{T^4} = \frac{3p_{SB}}{T^4} = \begin{cases} 3\frac{\pi^2}{30} & , T < T_c \\ (16 + \frac{21}{2}N_f)\frac{\pi^2}{30} & , T > T_c \end{cases}$$

Gluons and Quarks

$T > T_c$: more degrees of freedom, but significant interaction!

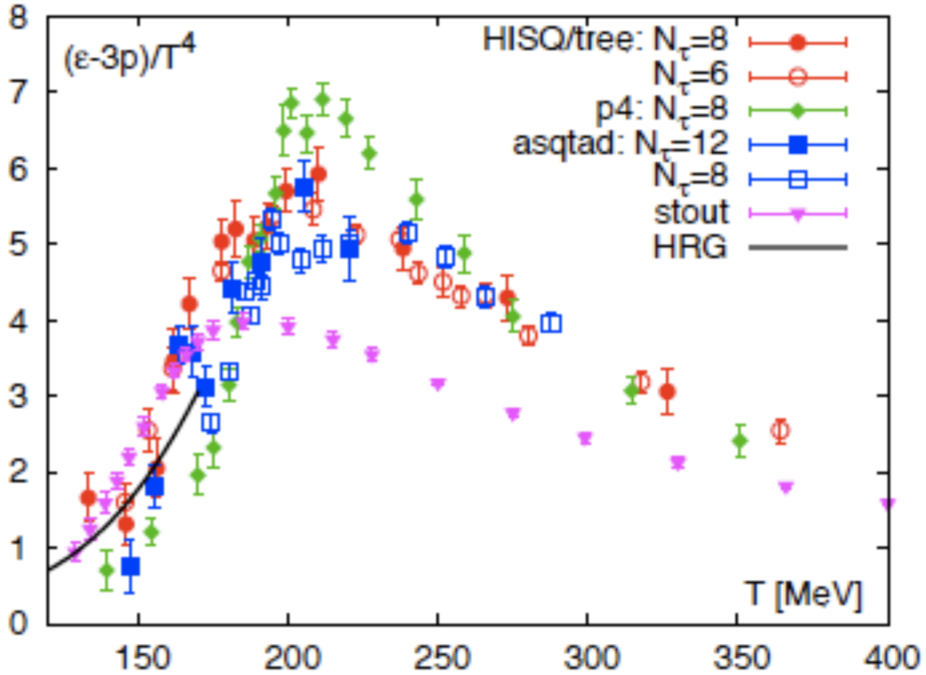
➡ sQGP or `almost ideal' gas....?

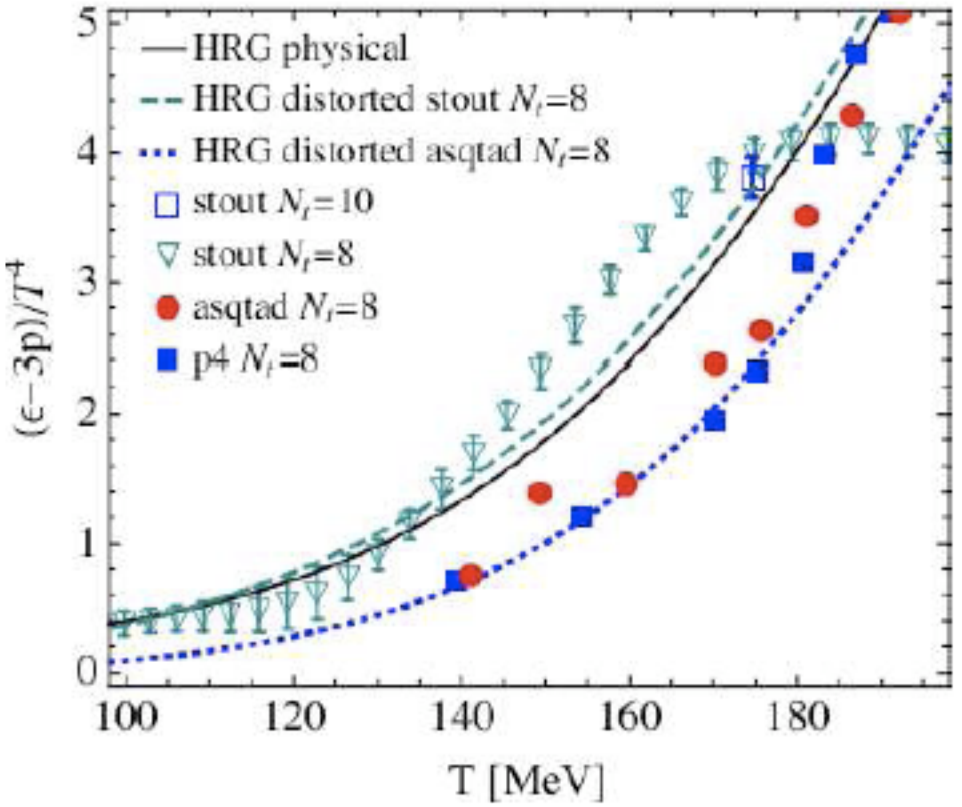
# Deconfinement:

**Free the Quarks!!!**

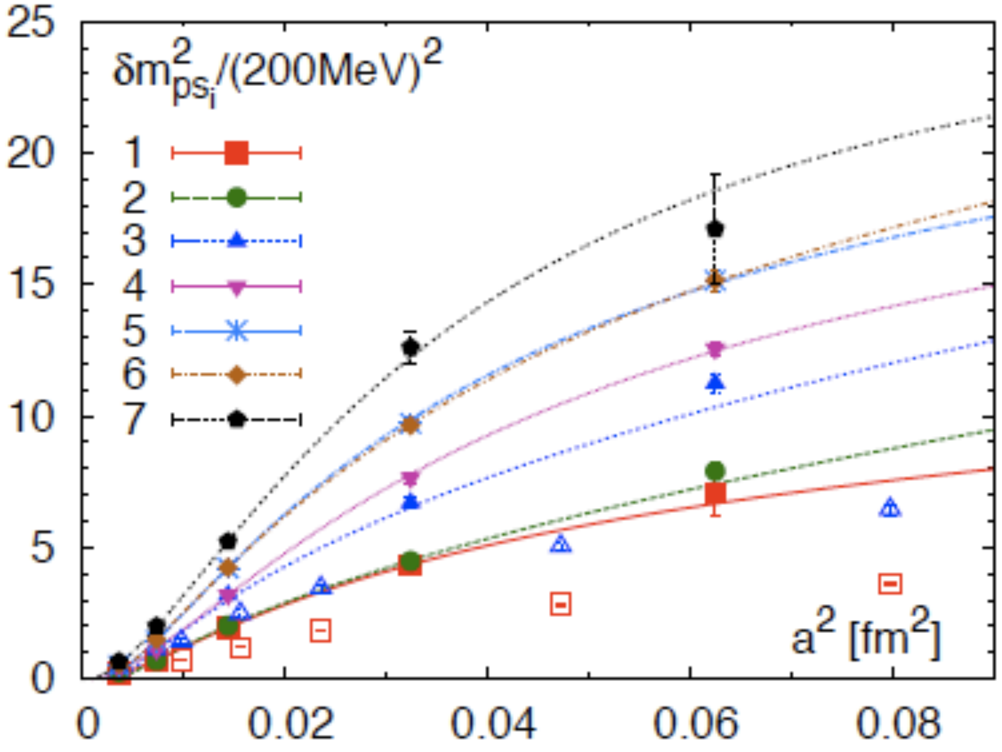
# Beware of cut-off effects!

Different versions of improved staggered actions:





Taste splittings of staggered actions give different contributions to pressure



# Equation of state for physical quark masses, continuum

Karsch et al., PLB 478 (2000)

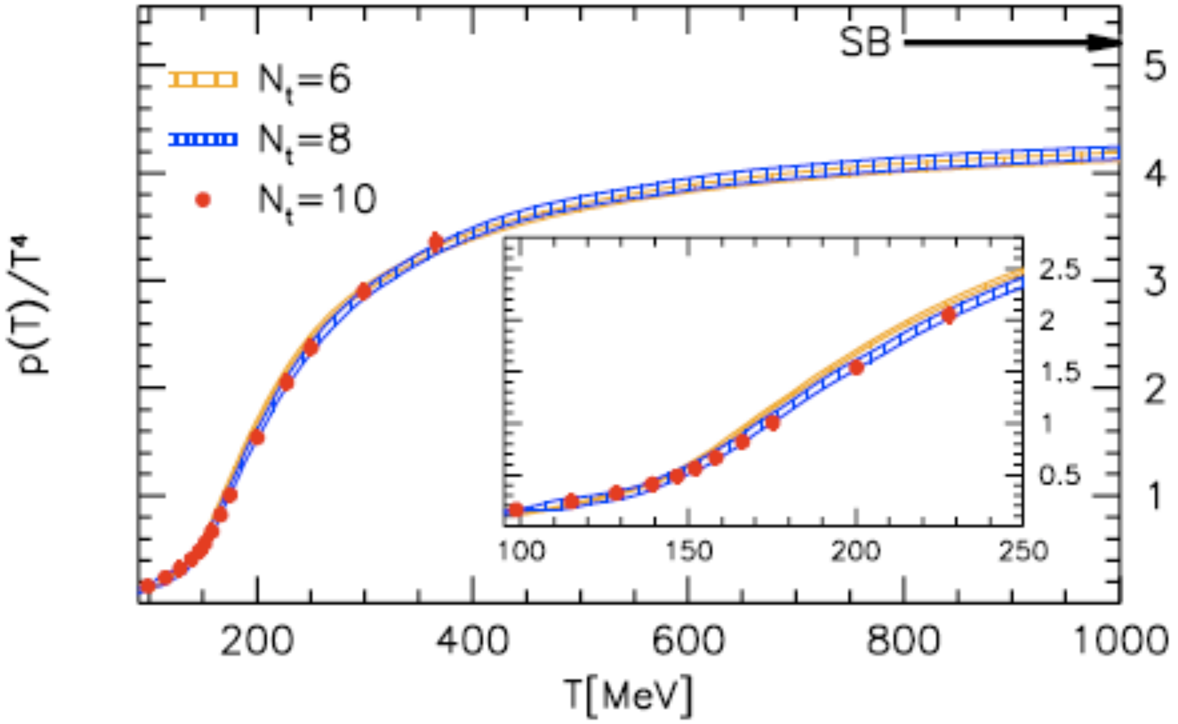


**Figure 10:** The pressure normalized by $T^4$ as a function of the temperature on $N_t = 6, 8$ and 1 lattices. The Stefan-Boltzmann limit $p_{SB}(T) \approx 5.209 \cdot T^4$ is indicated by an arrow. For our highest temperature $T = 1000$ MeV the pressure is almost 20% below this limit.

Hadron resonance gas model

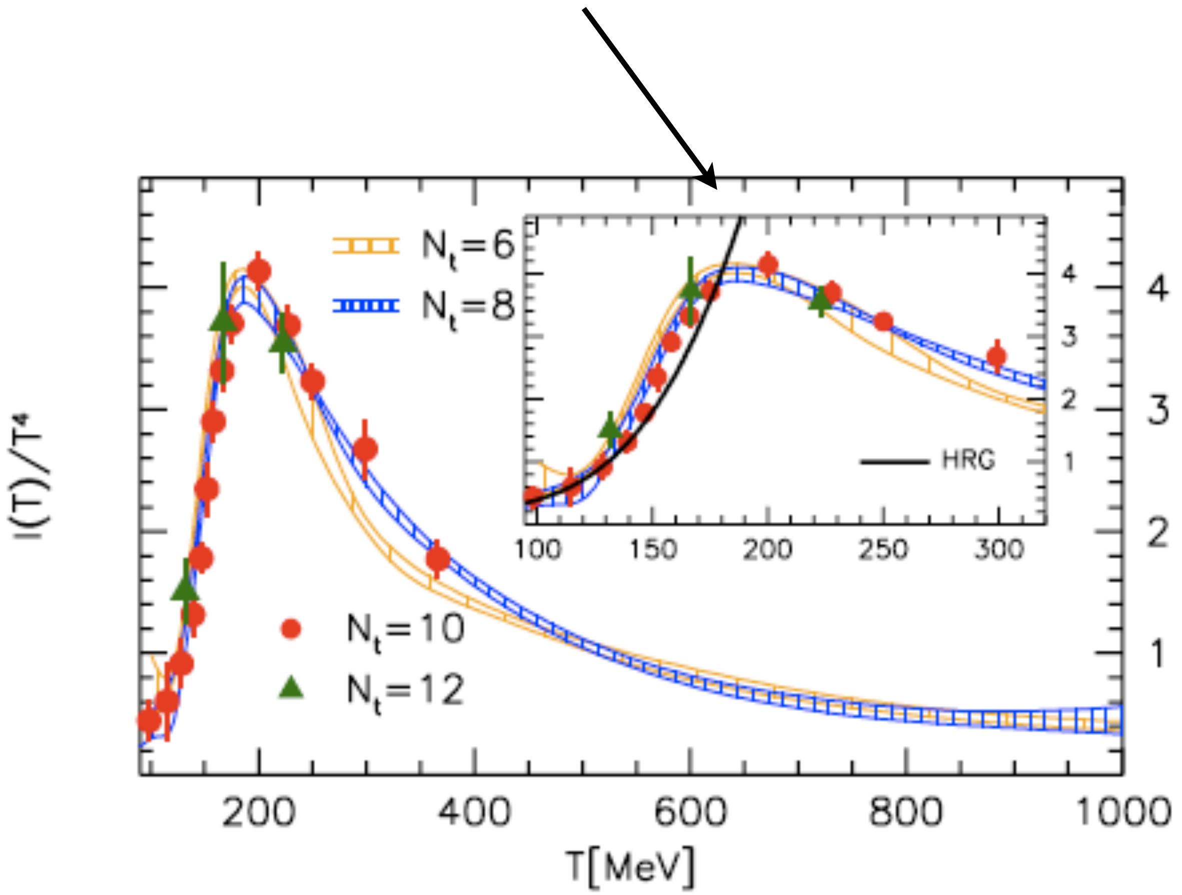


**Figure 9:** The trace anomaly $I = \epsilon - 3p$ normalized by $T^4$ as a function of the temperature $N_t = 6, 8, 10$ and 12 lattices.

Budapest-Marseille-Wuppertal

Symanzik-improved gauge action, staggered quarks with stout links

# Summary Lecture II

- Perturbation theory allows assessment of cut-off effects, but only at high T
- In the strong coupling limit QCD reduces to hadron resonance gas
- Equation of state accessible at physical masses in the continuum limit