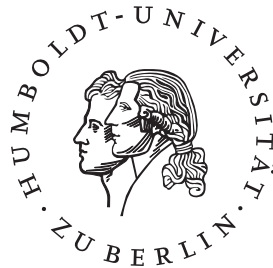


Introduction to Lattice Gauge Theories I

M. Müller-Preussker

Humboldt-University Berlin, Department of Physics



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 - 4. Simulating gauge fields
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(has not been presented during the lectures given)
 - 6.1. Topology of gauge fields
 - 6.2. Measuring topology on the lattice
 - 7. Conclusions, outlook

Literature, text books:

Path integrals:

- R.P. Feynman, A.R. Hibbs, *Quantum mechanics and path integrals*, Mc Graw Hill, 1965
- L.D. Faddeev, in *Methods in field theory*, Les Houches School, 1975
- V. Popov, *Kontinualniye integraly v kvantovoi teorii polya i statisticheskoi fizike*, Moskva Atomizdat, 1976
- H. Kleinert, *Path integrals in quantum mechanics, statistics, polymer physics*, World Scientific
- G. Roepstorff, *Path integral approach to quantum physics*, Springer
- J. Zinn-Justin, *Path integrals in quantum mechanics*, Oxford University Press (2005), translated into russian

Lattice field theory:

- K. G. Wilson, *Confinement of Quarks*, Phys. Rev. D10 (1974) 2445
- J. Kogut, L. Susskind, *Hamiltonian Formulation of Wilson's Lattice Gauge Theories.*, Phys. Rev. D11 (1975) 395
- M. Creutz, *Quarks, gluons and lattices*, Cambridge Univ. Press (1983), translated into russian
- H. Rothe, *Lattice gauge theories - An introduction*, World Scientific (4th ed. 2012)
- I. Montvay, G. Münster, *Quantum fields on a lattice*, Cambridge Univ. Press
- J. Smit, *Introduction to quantum fields on a lattice: A robust mate*, Cambridge Lect. Notes Phys. 15 (2002) 1-271
- C. Davies, *Lattice QCD*, Lecture notes, hep-ph/0205181 (2002)
- A. Kronfeld, *Progress in lattice QCD*, Lecture notes, hep-ph/0209231 (2002)
- T. DeGrand, C. E. DeTar, *Lattice methods for quantum chromodynamics*, World Scientific (2006)
- C. Gattringer, C.B. Lang, *Quantum Chromodynamics on the Lattice - An introductory presentation*, Springer (2010)

More general on non-perturbative methods:

- Yu. Makeenko, *Methods of contemporary gauge theory*, Cambridge Univ. Press (2002)

1st Lecture

1. Introductory remarks: Why lattice field theory?

Allows to understand non-perturbative phenomena and to carry out *ab initio* computations in strong interactions (and beyond):

- **Spontaneous breaking of chiral symmetry:**

QCD Lagrangian for 3 zero-mass flavours - among other symmetries - symmetric with respect to $SU(3)_A$:

$$\psi' = e^{i\xi_a \gamma_5 \lambda_a} \psi, \quad \psi = (u, d, s)$$

but $\langle 0 | \bar{\psi} \psi(x) | 0 \rangle \neq 0$ **quark condensate**
 $\implies$ **octet mesons (π 's, K 's, η) massless (Goldstone bosons).**

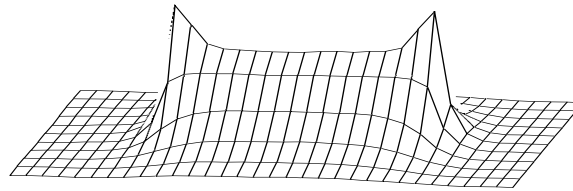
- **Quark confinement:**

Gluon field flux tube between static quarks:

in pure gauge theory $\implies F_{q\bar{q}}(r) = \frac{4}{3} \frac{\alpha_{q\bar{q}}(r)}{r^2} \xrightarrow{r \rightarrow \infty} \sigma$ **string tension**

Scenarios for confinement made ‘visible’ in lattice QCD:

- flux tube structure visualized



From Abelian projection of SU(2) LGT [Bali, Schilling, Schlichter, '97]

- condensation of $U(1)$ monopoles: **dual superconductor**
't Hooft; Mandelstam; Schierholz, . . . ; Bali, Bornyakov, M.-P., Schilling;
Di Giacomo, . . . ; . . .
- center vortices
't Hooft; Mack; Greensite, Faber, Olejnik; Reinhardt, . . . ; Polikarpov, . . . ; . . .
- semiclassical approach via instantons, calorons, dyons . . .
solving e.g. the problem of large η' -mass (“ $U_A(1)$ -problem”)
better understood on the lattice, but responsibility for confinement ???

but cf. lecture by V. Bornyakov

- Hadron masses, hadronic matrix elements, ...:
(cf. lectures by C. Hoelbling, M. Göckeler)

Not calculable within perturbation theory of continuum QCD,
need **non-perturbative (model) assumptions**

▷ condensates $\langle \bar{\psi}\psi \rangle, \langle \text{tr } G_{\mu\nu} G_{\mu\nu} \rangle, \dots$

$\implies$ QCD sum rules

Novikov, Shifman, Vainshtein, Zakharov; ...

▷ vacuum state scenarios: $\implies$ most popular **instanton liquid model**

Callan, Dashen, Gross; Shuryak; Ilgenfritz, M.-P.; Dyakonov, Petrov; ...

▷ Dyson-Schwinger and functional renormalization group equations

or alternatively effective field theories or quark models ...

- **Phase transitions:**

QCD: lattice predicts deconfinement or chiral transition:

hadrons $\iff$ quark-gluon plasma

under extreme conditions $T, \mu, \vec{B}$

cf. lectures by O. Philipsen, G. Endrodi, P. Buividovich

- **Standard model and beyond:**

Non-perturbative lattice approach very useful also for

- strongly coupled QED Kogut,...; Schierholz,...; Mitrjushkin, M.-P.,...; ...
- Higgs-Yukawa model $\implies$ e.g. bounds for Higgs mass
K. Jansen,...; J. Kuti,...; ... ,
- (broken) SUSY
cf. lecture notes HISS school 2011 by D. Kazakov, S. Catteral
- lattice studies (large N_c , N_f) motivated by string theory and AdS/CFT
correspondence M. Teper,...; J. Kuti,...; V. Zakharov,...; ...
- ...

2. Path integrals in quantum (field) theory

2.1. Path integral and Euclidean correlation functions

Quantum physics mostly starts from hermitean, time-independent Hamiltonian

$$\hat{H}|n\rangle = E_n|n\rangle, \quad n = 0, 1, 2, \dots \quad \text{with} \quad \langle m|n\rangle = \delta_{m,n}, \quad \sum_n |n\rangle\langle n| = \hat{1}.$$

Schrödinger Equation for time evolution:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle, \quad |\psi(t)\rangle = e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} |\psi(t_0)\rangle.$$

or

$$\psi(x, t) \equiv \langle x | \psi(t) \rangle = \int dx_0 \langle x | e^{-\frac{i}{\hbar} \hat{H}(t-t_0)} | x_0 \rangle \langle x_0 | \psi(t_0) \rangle.$$

Standard task: find $\{E_n\}$, in particular ground state energy E_0 .

Useful quantity for this: quantum statistical partition function:

$$Z = \text{tr} \left(e^{-\beta \hat{H}} \right) = \int dx \langle x | e^{-\beta \hat{H}} | x \rangle \implies \text{free energy } F(V, T) = -kT \log Z, \quad \beta \equiv \frac{1}{k_B T}$$

Note formal replacement: real time $\frac{i}{\hbar} t \leftrightarrow$ imaginary time $\beta \equiv \frac{1}{k_B T}$.

Extracting the **ground state** E_0 (Feynman-Kac formula):

$$Z = \sum_n \langle n | e^{-\beta \hat{H}} | n \rangle = \sum_n e^{-\beta E_n} \propto e^{-\beta E_0} \left(1 + O(e^{-\beta(E_1 - E_0)}) \right) \quad \text{for } \beta \rightarrow \infty.$$

Extracting the **energy (or mass) gap** $E_1 - E_0$:

two-point correlation function in (Euclidean) Heisenberg picture ($\hbar = 1$):

operators: $\hat{X}(\tau) = e^{\hat{H}\tau} x e^{-\hat{H}\tau}$, states: $|\Psi\rangle = e^{\hat{H}\tau} |\psi(\tau)\rangle$,

quantum statistical mean values:

$$\langle \dots \rangle \equiv Z^{-1} \text{tr} (\dots e^{-\beta \hat{H}}).$$

$$\begin{aligned} \langle \hat{X}(\tau) \rangle &= Z^{-1} \int dx \langle x | e^{-\hat{H}(\beta - \tau)} \hat{x} e^{-\hat{H}(\tau - 0)} | x \rangle \\ &= Z^{-1} \sum_n \langle n | \hat{x} | n \rangle e^{-E_n \beta} \propto Z^{-1} \langle 0 | \hat{x} | 0 \rangle e^{-E_0 \beta} + \dots \\ &\propto \langle 0 | \hat{x} | 0 \rangle + \dots \quad \text{for } \beta \rightarrow \infty. \end{aligned}$$

For two-point function assume $\beta \gg (\tau_2 - \tau_1) \gg 1$:

$$\begin{aligned}
\langle T \left(\hat{X}(\tau_2) \hat{X}(\tau_1) \right) \rangle &= Z^{-1} \int dx \langle x | e^{-\hat{H}(\beta - \tau_2)} \hat{x} e^{-\hat{H}(\tau_2 - \tau_1)} \hat{x} e^{-\hat{H}(\tau_1 - 0)} | x \rangle \\
&\propto Z^{-1} \left\{ \sum_n |\langle 0 | \hat{x} | n \rangle|^2 e^{-(E_n - E_0)(\tau_2 - \tau_1)} \right\} e^{-E_0 \beta} + \dots \\
&\propto Z^{-1} \left\{ |\langle 0 | \hat{x} | 0 \rangle|^2 + |\langle 0 | \hat{x} | 1 \rangle|^2 e^{-(E_1 - E_0)(\tau_2 - \tau_1)} \right\} e^{-E_0 \beta} + \dots
\end{aligned}$$

Then connected two-point correlator:

$$\begin{aligned}
\langle\langle X(\tau_2) X(\tau_1) \rangle\rangle &\equiv \langle T \left(\hat{X}(\tau_2) \hat{X}(\tau_1) \right) \rangle - (\langle \hat{X}(\tau) \rangle)^2 \\
&\propto e^{-(\tau_2 - \tau_1)(E_1 - E_0)} (1 + \dots)
\end{aligned}$$

Higher order correlations can be defined.

Z and $\langle \dots \rangle$ allow path integral representations á la Feynman.

Path integral representation

Subdivide $\beta = N\epsilon = \text{fix}$ with $N \rightarrow \infty$, $\epsilon \rightarrow 0$.

Use $(N-1)$ times completeness relation $\int dx |x\rangle\langle x| = 1$.

$$Z = \int \prod_{i=1}^N dx_i \langle x_N | e^{-\hat{H}\epsilon} | x_{N-1} \rangle \cdot \langle x_{N-1} | e^{-\hat{H}\epsilon} | x_{N-2} \rangle \cdots \langle x_1 | e^{-\hat{H}\epsilon} | x_0 \rangle$$

with $x_0 \equiv x_N$, i.e. “periodic boundary condition”.

For one degree of freedom: $\hat{H} = \hat{T} + \hat{V} = \hat{p}^2/2m + V(\hat{x})$, $[\hat{x}, \hat{p}] = i$

Transfer matrix : $\langle x' | e^{-\hat{H}\epsilon} | x \rangle = \sqrt{\frac{\epsilon}{2\pi}} e^{-\frac{1}{2\epsilon}(x'-x)^2 - \frac{\epsilon}{2}[V(x)+V(x')]} + O(\epsilon^3),$

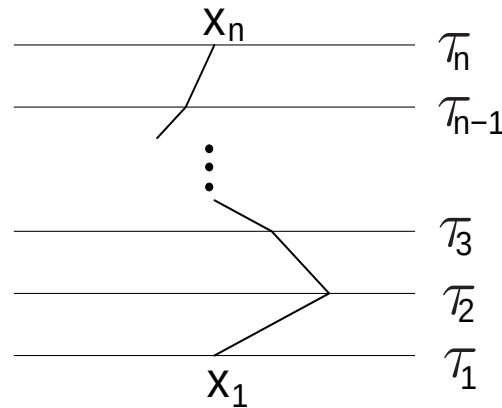
$$\Rightarrow Z \sim \int Dx(\tau) e^{-S_E[x(\tau)]}$$

Proof applies Trotter product formula:

$$\exp(-\beta\hat{H}) \equiv \exp(-(\hat{T} + \hat{V})\epsilon)^N = \lim_{N \rightarrow \infty} \left(\exp(-\epsilon\hat{T}) \cdot \exp(-\epsilon\hat{V}) \right)^N$$

$$Dx(\tau) \equiv \lim_{N \rightarrow \infty} \left(\frac{\epsilon}{2\pi} \right)^{N/2} \prod_{i=1}^N dx_i, \quad S_E[x(\tau)] \sim \int d\tau \left[\frac{m}{2} \left(\frac{dx(\tau)}{d\tau} \right)^2 + V(x(\tau)) \right].$$

Result: summation over fictitious paths in imaginary time



Our main interest: **correlation functions** in Heisenberg picture:

$$\langle \Omega(\tau', \tau'', \dots) \rangle = Z^{-1} \int Dx(\tau) \Omega(\tau', \tau'', \dots) e^{-S_E[x(\tau)]} .$$

(Proof as for partition function Z .)

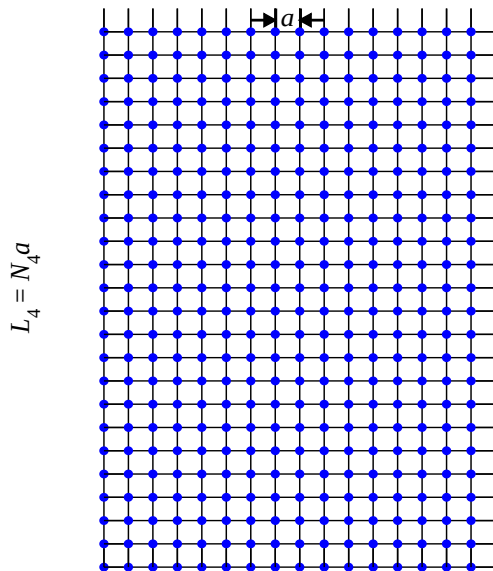
- $\implies$ no operators, but **trajectories** $x(\tau)$ and **Euclidean action** $S_E[x(\tau)]$,
- $\implies$ **correlators resemble classical statistical averages**,
- $\implies$ we use $\beta = N\epsilon$ with N finite, but large, i.e. **“lattice approximation”**.

2.2. Path integral quantization of scalar fields

- Lattice formulation allows non-perturbative computation from first principles.
- Axiomatic field theory starts from lattice formulation.

Illustration: scalar field, Euclidean space-time.

$$S[\phi] = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \right], \quad V(\phi) = \frac{1}{2} m_0^2 \phi^2 + \frac{1}{4!} g_0 \phi^4.$$



$$\begin{aligned}
 x &\longrightarrow x_n \\
 \phi(x) &\longrightarrow \phi_n \equiv \phi(x_n) \\
 \partial_\mu \phi(x) &\longrightarrow \frac{1}{a}(\phi_{n+\hat{\mu}} - \phi_n)
 \end{aligned}$$

$$L = N_s a$$

[“Lattice” taken from A. Kronfeld, hep-ph/0209231]

$$S_{\text{Latt}} = a^4 \sum_n \left\{ \frac{1}{2} \sum_{\mu=1}^4 (\phi_{n+\hat{\mu}} - \phi_n)^2 \frac{1}{a^2} + V(\phi_n) \right\}$$

$\implies$ no unique discretization,

$\implies$ even more - systematic improvement scheme exists (Symanzik).

Boundary condition: $\phi(x + L_\mu \cdot \hat{\mu}) = \phi(x)$

Spectrum: $-\frac{\pi}{a} \leq k_\mu = \frac{2\pi}{L_\mu} n_\mu \leq \frac{\pi}{a}$

Quantization:

Path integral for time ordered correlation functions and vacuum transition amplitude

$$\langle \phi(x_1) \cdots \phi(x_N) \rangle = Z^{-1} \int \prod_x d\phi(x) \{ \phi(x_1) \cdots \phi(x_N) \} e^{-S[\phi]} \quad (1)$$

$$Z = \int \prod_x d\phi(x) e^{-S[\phi]} \quad (2)$$

↕ lattice approximation and rescaling $\phi \longrightarrow \phi \cdot \sqrt{g_0}$.

$$\langle \phi_{n_1} \cdots \phi_{n_N} \rangle = Z_{\text{Latt}}^{-1} \int \prod_n d\phi_n \{ \phi_{n_1} \cdots \phi_{n_N} \} e^{-\frac{1}{g_0} S_{\text{Latt}}} \quad (3)$$

$$Z_{\text{Latt}} = \int \prod_n d\phi_n e^{-\frac{1}{g_0} S_{\text{Latt}}} \quad (4)$$

Compare with: $\mathcal{Z}(T, V) = \int \prod_{i=1}^f dq_i dp_i e^{-\frac{1}{kT} H(q_i, p_i)}$

Scalar Higgs model in some detail:

$$\phi = \sqrt{2\kappa} \frac{\varphi}{a}, \quad g_0 = \frac{6\lambda}{\kappa^2}, \quad m_0^2 = \frac{1 - 2\lambda - 8\kappa}{a^2\kappa}.$$

$$S_{\text{Latt}} = -2\kappa \sum_{n,\mu} \varphi_n \varphi_{n+\hat{\mu}} + \lambda \sum_n (\varphi_n^2 - 1)^2 + \sum_n \varphi_n^2$$

(κ ‘Hopping parameter’)

No lattice spacing visible. How does the continuum limit occur?

In practice need for determining correlation lengths

$$L_4 = N_4 a \gg \zeta \equiv m^{-1} \gg a$$

Continuum limit corresponds to 2^{nd} order phase transition:

$$\zeta/a \rightarrow \infty \quad \text{whereas} \quad m_1/m_2 \rightarrow \text{const.}$$

$\lambda \rightarrow \infty$: $\kappa = \text{fixed} \implies \varphi_n = \pm 1$.

$$Z_{\text{Latt}} \rightarrow \text{const.} \cdot \sum_{\varphi_n = \pm 1} \exp \left(-2\kappa \sum_{n,\mu} \varphi_n \varphi_{n+\hat{\mu}} \right)$$

$\implies$ Ising model in 4D,

$\implies$ 2^{nd} order phase transition at $\kappa_c = 0.0748$.

$\lambda < \infty$: find critical line $\kappa = \kappa_c(\lambda)$.

Comment:

renormalized coupling $\lambda_{ren} \rightarrow 0$ for $\kappa \rightarrow \kappa_c$

$\implies$ non-interacting theory in the continuum limit or

$\implies$ effective theory at finite cutoff.

2nd Lecture

3. Discretizing gauge fields

3.1. QCD: a short introduction

The strong force of **hadrons** realized by **quarks** interacting with **gluons**

- Quarks (= fermionic matter field) occur with
 - $N_c = 3$ “colour” states (gauge group $SU(N_c)$),
 - $N_f = 6$ “flavour” states (up, down, strange, charm, bottom, top),
 - 4 “spinor” states.
- Gluons (= bosonic gauge field) occur with
 - $N_c^2 - 1 = 8$ “colour” states (adjoint repr. of $SU(N_c)$),
 - 4 space-time components.

Colour neutral bound states:

qqq: baryons – $p, n, \dots$,

$q\bar{q}$: mesons – $\pi, \rho, K, \dots$

QCD: quark fields $\psi_f(x) \equiv \{(\psi_f(x))_s^\alpha, \alpha = 1, \dots, N_c, s = 1, \dots, 4\}$
minimally coupled to gluon fields $A_\mu^a(x), \mu = 0, \dots, 3, x = (\vec{x}, t)$.

$$S[\bar{\psi}, \psi, A] = \int d^4x \left(-\frac{1}{2} \text{tr} G^{\mu\nu} G_{\mu\nu} + \sum_{f=1}^{N_f} \bar{\psi}_f (i\gamma_\mu D^\mu - m_f) \psi_f \right)$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_0[A_\mu, A_\nu], \quad D_\mu = \partial_\mu \mathbf{1} + ig_0 A_\mu, \quad A_\mu = A_\mu^a T^a,$$

$$T^a \ (a = 1, \dots, N_c^2 - 1) \text{ -- generators of } SU(N_c), \\ \text{with } [T^a, T^b] = if^{abc} T^c, \quad 2\text{tr}(T^a T^b) = \delta_{ab}, \quad \text{tr} T^a = 0.$$

- **local gauge invariance** – the main principle of the standard model

$$\psi'(x) = g(x)\psi(x), \quad A'_\mu(x) = g(x)A_\mu g^\dagger(x) - \frac{i}{g_0}(\partial_\mu g(x))g^\dagger(x), \quad g(x) \in SU(N_c)$$

leaves action $S[\bar{\psi}, \psi, A]$ invariant for any $g(x)$.

- **quantization with Euclidean path integral analogously to QM:**

$$\langle \Omega(\bar{\psi}, \psi, A) \rangle = Z^{-1} \int DA_\mu D\bar{\psi} D\psi \Omega(\bar{\psi}, \psi, A) e^{-S_E[\bar{\psi}, \psi, A]}$$

$$DA_\mu = \prod_x dA_\mu(x); \quad D\psi = \prod_x d\psi(x) \quad - \quad \text{Grassmannian measure}$$

$\Rightarrow$ weak coupling ($\alpha_s \ll O(1)$): perturbation theory, requires gauge fixing;

$\Rightarrow$ stronger coupling ($\alpha_s \simeq O(1)$): lattice theory, Dyson-Schwinger eqs., ...

3.2. Gauge fields on a lattice

K. Wilson '74:

Local $SU(N_c)$ gauge symmetry kept exact by non-local prescription
on a 4d Euclidean lattice $\implies$ **link variables**

$$\begin{aligned} A_\mu(x_n) \implies U_{n,\mu} &\equiv P \exp i g_0 \int_{x_n}^{x_n + \hat{\mu} a} A_\mu(x) dx_\mu \in SU(N_c) \\ &\simeq e^{i a g_0 A_\mu(x_n)} \simeq \mathbf{1} + i a g_0 A_\mu(x_n) + O(a^2) \end{aligned}$$

Gauge transformation $g_n \equiv g(x_n) \in SU(N_c)$ for $a \rightarrow 0$:

$$\begin{aligned} A'_\mu(x_n) &\simeq g_n A_\mu g_n^\dagger - \frac{i}{a g_0} (g_{n+\hat{\mu}} - g_n) g_n^\dagger \\ U'_{n,\mu} &= \mathbf{1} + i a g_0 A'_\mu(x_n) + O(a^2) \\ &= i a g_0 \textcolor{red}{g}_n A_\mu g_n^\dagger + g_{n+\hat{\mu}} g_n^\dagger + O(a^2), \quad \textcolor{red}{g}_n = g_{n+\hat{\mu}} + O(a) \\ &= g_{n+\hat{\mu}} (\mathbf{1} + i a g_0 A_\mu(x_n) + O(a^2)) g_n^\dagger \\ &= g_{n+\hat{\mu}} U_{n,\mu} g_n^\dagger \end{aligned}$$

Transformation of a product of variables of connected links

$$U'(n + \hat{\mu} + \hat{\nu}, n) \equiv U'_{n+\hat{\mu}, \nu} U'_{n, \mu} = g_{n+\hat{\mu}+\hat{\nu}} U_{n+\hat{\mu}, \nu} U_{n, \mu} g_n^\dagger = g_{n+\hat{\mu}+\hat{\nu}} U(n + \hat{\mu} + \hat{\nu}, n) g_n^\dagger$$

can be generalized to arbitrary continuum path:

“Schwinger line” (parallel transporter)

$$U(x_2, x_1) = P \exp \left\{ ig_0 \int_{x_1}^{x_2} A_\mu dx_\mu \right\},$$

$$\implies U'(x_2, x_1) = g(x_2) U(x_2, x_1) g^\dagger(x_1).$$

$$\implies \text{tr} \left(P \exp \left\{ ig_0 \oint A_\mu dx_\mu \right\} \right) = \text{invariant}.$$

Combination with matter field: $\psi'(x) = g(x)\psi(x), \quad \psi'^\dagger(x) = \psi^\dagger(x)g^\dagger(x)$

$$\implies \psi^\dagger(x_2) U(x_2, x_1) \psi(x_1) = \text{invariant}.$$

Lattice gauge action: from elementary closed (Wilson) loops (“plaquettes”)

$$\begin{aligned}
 U_{n,\mu\nu} &\equiv U_n U_{n+\hat{\mu},\nu} U_{n+\hat{\nu},\mu}^\dagger U_n^\dagger, \\
 S_G^W &= \bar{\beta} \sum_{n,\mu<\nu} \left(1 - \frac{1}{N_c} \operatorname{Re} \operatorname{tr} U_{n,\mu\nu} \right), \quad \bar{\beta} = \frac{2N_c}{g_0^2} \\
 &= \frac{1}{2} \sum_n a^4 \operatorname{tr} G^{\mu\nu} G_{\mu\nu} + O(a^2), \\
 &\rightarrow \frac{1}{2} \int d^4x \operatorname{tr} G^{\mu\nu} G_{\mu\nu}.
 \end{aligned}$$

Homework I:

Prove the continuum limit in the (non-) Abelian case: $\longrightarrow$ tutorial.

Comments:

- **Improvement of S_G^W :** suppression of $O(a)$ -corrections by adding contributions from **larger loops** (e.g. various 6-link loops) or contributions of loops in **higher group representations** (e.g. adjoint representation)
- **Useful test for programmers:** non-trivial vacuum field = pure gauge

$$\begin{aligned}
 U_{n\mu}^{(vac)} &\equiv g_{n+\hat{\mu}} g_n^\dagger, \quad \text{for arbitrary } g_n \in SU(N_c) \\
 \operatorname{Re} \operatorname{tr} U_{n,\mu\nu} &= N_c \implies S_G^W = 0.
 \end{aligned}$$

Path integral quantization:

replacement $\int \prod_{x,\mu} dA_\mu(x) \longrightarrow \int \prod_{n,\mu} [dU]_{n,\mu}$, $[dU]$ “Haar measure”.

Properties: $U, V \in G = SU(N_c)$ in fundamental representation, then

- $\int_G [dU] = 1$ normalization,
- $\int_G f(U) [dU] = \int_G f(VU) [dU] = \int_G f(UV) [dU]$, $[d(UV)] = [d(VU)] = [dU]$,
- $\int_G f(U) [dU] = \int_G f(U^{-1}) [dU]$.

Examples:

$$U(1) : \quad U = e^{i\varphi} \quad \Longrightarrow \quad [dU] = \frac{1}{2\pi} d\varphi, \quad \int f(U) [dU] \equiv \frac{1}{2\pi} \int_0^{2\pi} d\varphi f(U(\varphi));$$

$$SU(2) : \quad U = B^0 + i\vec{\sigma} \cdot \vec{B}, \quad \sum_i (B^i)^2 = 1 \quad \Longrightarrow \quad [dU] = \frac{1}{\pi^2} \delta(B^2 - 1) d^4 B,$$

$\vec{\sigma}$ – Pauli matrices.

Expectation values, correlation functions in pure gauge theory:

$$\langle W \rangle = \frac{1}{Z} \int \left(\prod_{n,\mu} [dU_{n\mu}] \right) W[U] e^{-S_G[U]}, \quad Z = \int \left(\prod_{n,\mu} [dU_{n\mu}] \right) e^{-S_G[U]}.$$

Gauge invariance:

$$U'_{n\mu} = g_{n+\hat{\mu}} U_{n\mu} g_n^\dagger \implies dU'_{n\mu} = dU_{n\mu}, \quad S_G[U'] = S_G[U]$$

Then immediately

$$\begin{aligned} \langle W' \rangle &= \frac{1}{Z'} \int \left(\prod_{n,\mu} [dU'_{n\mu}] \right) W[U'] e^{-S_G[U']} \\ &= \frac{1}{Z} \int \left(\prod_{n,\mu} [dU_{n\mu}] \right) W[U'] e^{-S_G[U]} \end{aligned}$$

$\implies$ invariant, if $W[U]$ itself is gauge invariant.

Fortunately, holds in most of the applications of lattice gauge theory !

Exceptions:

Quark, gluon, ghost propagators or various vertex functions.

Gauge fixing is required, e.g. Landau/Lorenz gauge, as in perturbation theory.

However, gauge condition not uniquely solvable - “Gribov problem”.

Lattice discretized path integral allows to study analytically:

- $\bar{\beta} \rightarrow \infty, g_0 \rightarrow 0$: *(lattice) perturbation theory* by expanding all link variables $U_{n\mu} = e^{ia g_0 A_\mu(x_n)}$ in powers of g_0 .
Inverse lattice spacing a^{-1} acts as UV cutoff.
- $\bar{\beta} \rightarrow 0, g_0 \rightarrow \infty$: *strong coupling expansion* by expanding in powers of $\bar{\beta}$

$$\exp \left\{ -\bar{\beta} \sum_{n, \mu < \nu} \left(1 - \frac{1}{N_c} \operatorname{Re} \operatorname{tr} U_{n, \mu \nu} \right) \right\}.$$

Mostly we have to rely on $g_0 \simeq O(1) \implies$ **Monte Carlo simulation.**

3.3. Wilson and Polyakov loops

Closed *Wilson loops*

$$W[U] \equiv \text{tr} \left(P \exp \left\{ ig_0 \oint A_\mu dx_\mu \right\} \right)$$

play important rôle for explaining quark confinement.

Sketch of the proof [cf. book H. Rothe, '92, new extended edition '12]:

Consider (infinitely) heavy $Q\bar{Q}$ -pair. Gauge-invariant state at fixed t' :

$$\bar{\psi}_\alpha^{(Q)}(\vec{x}, t') U(\vec{x}, t'; \vec{y}, t') \psi_\beta^{(Q)}(\vec{y}, t') |\Omega\rangle$$

with

$$U(\vec{x}, t'; \vec{y}, t') = P \exp \left\{ ig_0 \int_{\vec{x}}^{\vec{y}} A_\mu(\vec{z}, t') dz_\mu \right\} .$$

Compute 2-pt. function for $Q\bar{Q}$ -pair - propagation from 0 to t :

$$G_{\alpha', \beta'; \alpha, \beta}(\vec{x}', \vec{y}', t; \vec{x}, \vec{y}, 0)$$

$$= \langle \Omega | T \{ \bar{\psi}_{\beta'}^{(Q)}(\vec{y}', t) U(\vec{y}', t; \vec{x}', t) \psi_{\alpha'}^{(Q)}(\vec{x}', t) \cdot \bar{\psi}_\alpha^{(Q)}(\vec{x}, 0) U(\vec{x}, 0; \vec{y}, 0) \psi_\beta^{(Q)}(\vec{y}, 0) \} | \Omega \rangle .$$

Euclidean time $t = -i\tau$; for $\tau \rightarrow \infty$, $M_Q \rightarrow \infty$ expect

$$G_{\alpha',\beta';\alpha,\beta} \propto \delta^{(3)}(\vec{x} - \vec{x}') \delta^{(3)}(\vec{y} - \vec{y}') C_{\alpha',\beta';\alpha,\beta}(\vec{x}, \vec{y}) \exp(-E(R)\tau)$$

$E(R)$ = energy of the lowest state (above vacuum), i.e. $Q\bar{Q}$ -potential.

Path integral representation:

$$G_{\alpha',\beta';\alpha,\beta} = \frac{1}{Z} \int DA \int D\bar{\psi}^{(Q)} D\psi^{(Q)} \left(\bar{\psi}_{\beta'}^{(Q)}(\vec{y}', \tau) \dots \psi_{\beta}^{(Q)}(\vec{y}, 0) \right) \exp(-S[\bar{\psi}^{(Q)}, \psi^{(Q)}, A])$$

Carry out (Grassmanian) fermion integration

→ product of Green's functions for Q , $\bar{Q}$ and $\text{Det} \rightarrow \text{const}$ for $M_Q \rightarrow \infty$.

Static limit – neglect spatial derivatives $(i\gamma^\mu D_\mu) \rightarrow (i\gamma^4 D_4)$

→ Green's function:

$$[i\gamma^4 D_4(A) - M_Q] K(z, z'; A) = \delta^{(4)}(z - z').$$

Solution:

$$K(z, z'; A) \sim P \exp \left\{ ig_0 \int_0^\tau A_0(\vec{z}, t') dt' \right\} \delta^{(3)}(\vec{z} - \vec{z}') \cdot \mathbf{K}$$

with $\mathbf{K}$ containing projectors $P_\pm = \frac{1}{2}(1 \pm \gamma^4)$.

Combine time-like phase factors at $\vec{z} = \vec{x}, \vec{y}$ with space-like ones $U(\vec{x}, 0; \vec{y}, 0), \quad U(\vec{y}, \tau; \vec{x}, \tau)$.

Result:

$$G_{\alpha', \beta'; \alpha, \beta} \sim \langle W \rangle \equiv \left\langle \text{tr } P \exp \left\{ i g_0 \oint A_\mu dx_\mu \right\} \right\rangle$$

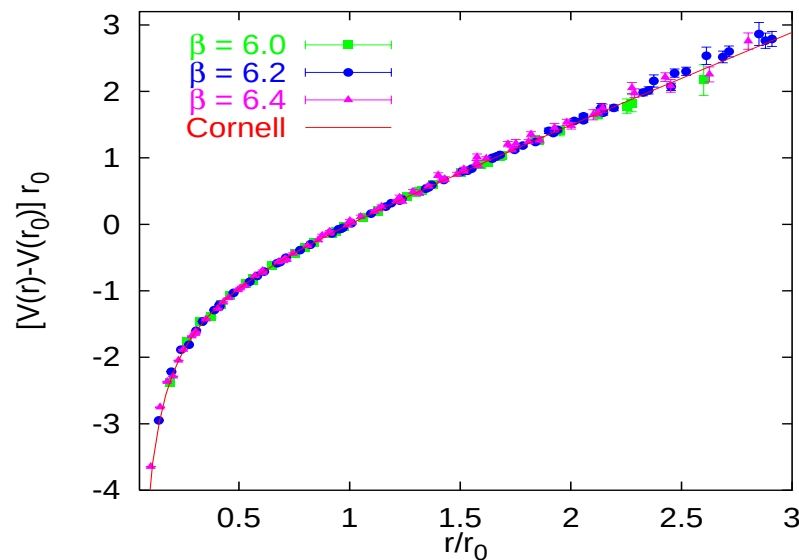
with closed rectangular path of extension $R \times \tau$.

For $\tau \rightarrow \infty$: $\langle W \rangle \propto \exp(-E(R) \cdot \tau)$.

Confinement:

$E(R) \propto \sigma R$ for large $R \iff \langle W \rangle \propto \exp(-\sigma \cdot (R \cdot \tau))$ (area law)

Simulation: $V(R) \equiv E(R)$ (Sommer scale: $r_0 \simeq 0.5$ fm, see below).



[G. Bali, Phys. Reports, '01]

Polyakov loops and non-zero temperature QCD [cf. lecture by O. Philipsen]

[Gross, Pisarski, Yaffe, Rev.Mod.Phys. 53 (1981) 43;

Svetitsky, Yaffe, Nucl.Phys. B210 (1982) 423; Svetitsky, Phys.Rept. 132 (1986) 1]

Partition function:

$$\begin{aligned} Z &= \text{Tr} \exp(-\beta \hat{H}), \quad \beta = 1/k_B T, \\ &= \sum_x \langle x | (\exp(-\epsilon \hat{H}))^N | x \rangle, \quad N\epsilon = \beta, \\ &= C \int Dx \exp(-S_E[x(t)]), \quad S_E = \int_0^\beta dt \left[\frac{m}{2} \dot{x}(t)^2 + V(x(t)) \right] \end{aligned}$$

with $x(0) = x(\beta)$, i.e. **periodicity**.

For Yang-Mills theory analogous proof of path integral representation is non-trivial:

- Hamiltonian approach within gauge $A_0^a = 0$,
 - trace integration produces integration over **auxiliary** field A_4^a ,
- $\implies$ full Euclidean Yang-Mills action recovered.

Result:

$$Z_G = C \int DA e^{-S_G[A]}, \quad S_G[A] = \int_0^\beta dt \int d^3x \frac{1}{2} \text{tr}(G_{\mu\nu} G_{\mu\nu})$$

with $A_\mu^a(\vec{x}, 0) = A_\mu^a(\vec{x}, \beta)$, and $G_{\mu\nu} = G_{\mu\nu}^a T^a$ Euclidean.

Applied to lattice pure gauge theory:

$$Z_G \equiv \int [dU] e^{-S_G[U]}, \quad \text{with } U_\mu(\vec{x}, 0) = U_\mu(\vec{x}, \beta), \quad \beta = aN_4 = \frac{1}{k_B T}.$$

We are interested in the thermodynamic limit: $V^{(3)} \rightarrow \infty$.

In practice, “aspect ratio” $N_s/N_4 \gg 1$ and periodic b.c.’s in spatial directions.

Extension to full QCD: time-antiperiodic boundary conditions for fermionic fields from trace of statist. operator: $\psi(\vec{x}, 0) = -\psi(\vec{x}, \beta)$, $\bar{\psi}(\vec{x}, 0) = -\bar{\psi}(\vec{x}, \beta)$.

Polyakov loop:

$$L(\vec{x}) \equiv \frac{1}{N_c} \text{tr} \prod_{x_4=1}^{N_4} U_4(\vec{x}, x_4), \quad (a=1),$$

invariant w.r. to time-periodic gauge transformations $g(\vec{x}, 1) = g(\vec{x}, N_4 + 1)$.

Physical interpretation:

$$\langle L(\vec{x}) \rangle = \exp(-\beta F_Q),$$

with F_Q free energy of an isolated infinitely heavy quark.

Proof goes analogously as for Wilson loop expectation value.

$\implies F_Q \rightarrow \infty$, i.e. $\langle L(\vec{x}) \rangle \rightarrow 0$ within the confinement phase.

$\implies \langle L(\vec{x}) \rangle$ order parameter for the deconfinement transition.

Spontaneous breaking of $\mathbf{Z}_N$ center symmetry:

$$z_\nu = e^{2\pi i \frac{\nu}{N}} \mathbf{1} \in \mathbf{Z}_N \subset SU(N), \quad \nu = 0, 1, \dots, N-1$$

commute with all elements of $SU(N)$. For $SU(2)$: $z_\nu = \pm \mathbf{1}$.

Global $\mathbf{Z}_N$ -transformation:

$$U_4(\vec{x}, x_4) \rightarrow z \cdot U_4(\vec{x}, x_4) \quad \text{for all } \vec{x} \text{ and fixed } x_4$$

$\implies L(\vec{x}) \rightarrow zL(\vec{x})$ not invariant.

$\implies$ Plaquette values $U_{n,i4} \rightarrow zz^* U_{n,i4} = U_{n,i4}$, i.e. S_G invariant.

$SU(2)$ -case:

$L(\vec{x}) \rightarrow -L(\vec{x})$. Both states have same statistical weight.

$$\implies \langle L(\vec{x}) \rangle = 0 .$$

$\implies$ Order parameter for the deconfinement transition:

$$\langle |\bar{L}| \rangle \equiv \left\langle \left| \frac{1}{V(3)} \sum_{\vec{x}} L(\vec{x}) \right| \right\rangle \sim \begin{cases} 0 & \text{confinement} \\ 1 & \text{deconfinement} \end{cases}$$

analogously to spin magnetization for 3d Ising model.

$SU(3)$ -case: $\bar{L}$ complex-valued.

$$\text{Confinement} \quad \rightarrow \quad \bar{L} \simeq 0,$$

$$\text{Deconfinement} \quad \rightarrow \quad \bar{L} \simeq z_\nu, \quad \nu = 0, 1, 2.$$

Notice: fermions break center symmetry. Then

$\Rightarrow$ Polyakov loop – order parameter for (de)confinement only if $m_q \rightarrow \infty$,

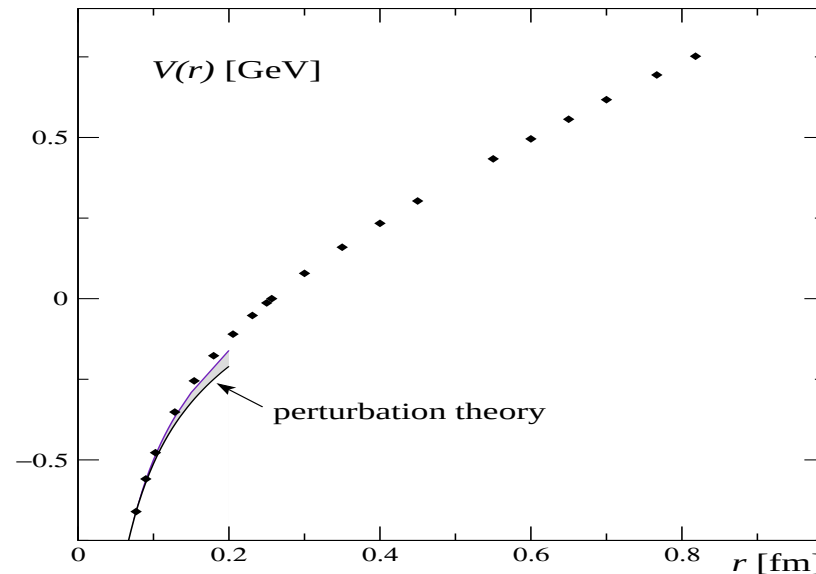
$\Rightarrow$ quark condensate $\langle \bar{\psi}\psi \rangle$ – strict order parameter

for chiral symmetry breaking / restoration only if $m_q \rightarrow 0$.

3.4. Fixing the QCD scale

Expectation values $\langle W \rangle$ (of Wilson loops etc.) in pure gauge theory only depend on parameter $\bar{\beta} \equiv 2N_c/g_0^2$ (and linear lattice extensions N_μ).

View some Monte Carlo results in gluodynamics.

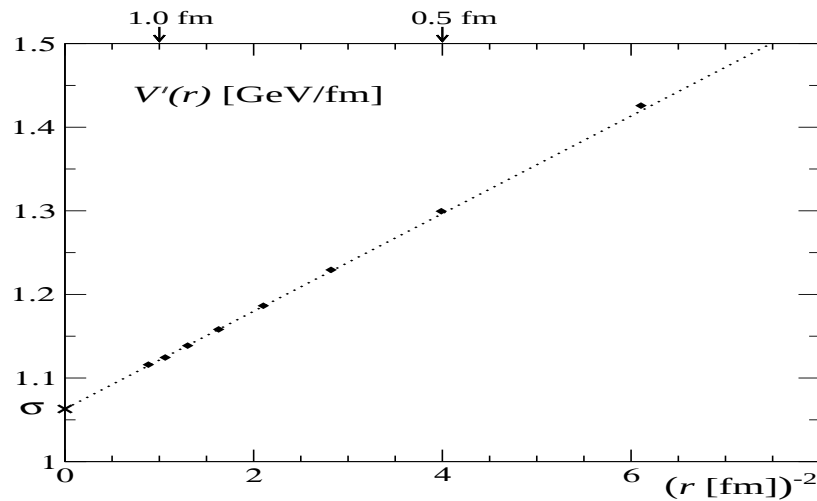


Static $Q\bar{Q}$ potential $V(r)$ ($\equiv E(R)$) at **small** distances from Wilson loops.

Necco, Sommer, '02

Alternative method: Polyakov loop correlator

$$\langle L(\vec{x}_1) L(\vec{x}_2^\dagger) \rangle \propto \exp(-\beta V(r))(1 + \dots), r \equiv |\vec{x}_1 - \vec{x}_2|$$



Static $Q\bar{Q}$ force $F(r) \equiv dV/dr$ from Polyakov loop correlators at $T = 0$.
 $\Rightarrow$ in perfect agreement with [string model prediction](#)

$$V(r) \sim \sigma r + \mu - \frac{\pi}{12r} + O(r^{-2}).$$

Lüscher, Weisz, '02

$F(r)$ allows to fix the scale by comparing with phenomenologically known $\bar{c}c$ - or $\bar{b}b$ -potential [R. Sommer, '94]:

$$F(r_0) r_0^2 = 1.65 \Leftrightarrow r_0 \simeq 0.5 \text{ fm}$$

If *Sommer scale* r_0 is determined in lattice units a for certain $\bar{\beta}$, spacing $a(\bar{\beta})$ can be fixed in physical units.

3.5. Renormalization group and continuum limit

Assume for a physical observable Ω (e.g. string tension σ , critical temperature T_c , glueball mass $M_g, \dots$) in the continuum limit

$$\lim_{a \rightarrow 0} \Omega(g_0(a), a) = \Omega_c$$

Then *renormalization group Eq.*

$$\frac{d \Omega}{d(\ln a)} = 0 \quad \Longrightarrow \quad \left(\frac{\partial}{\partial(\ln a)} - \beta(g_0) \frac{\partial}{\partial g_0} \right) \Omega(g_0, a) = 0$$

with $\beta(g_0) \equiv -\frac{\partial g_0}{\partial(\ln a)}$ known from perturbation theory

$$\beta(g)/g^3 = -\beta_0 - \beta_1 g^2 + O(g^4).$$

For $SU(N_c)$ and N_f massless fermions, independent on renormalization scheme:

$$\begin{aligned} \beta_0 &= \frac{1}{(4\pi)^2} \left(\frac{11}{3} N_c - \frac{2}{3} N_f \right), \\ \beta_1 &= \frac{1}{(4\pi)^4} \left(\frac{34}{3} N_c^2 - \frac{10}{3} N_c N_f - \frac{N_c^2 - 1}{N_c} N_f \right). \end{aligned}$$

For pure gluodynamics: $N_c = 3, N_f = 0 \Rightarrow \beta_0 > 0$.

Solution yields continuum limit

$$a(g_0) = \frac{1}{\Lambda_{Latt}} (\beta_0 g_0^2)^{-\frac{\beta_1}{2\beta_0^2}} \exp\left(-\frac{1}{2\beta_0 g_0^2}\right) (1 + O(g_0^2)).$$

$\Rightarrow 1/a \rightarrow \infty$ for $g_0 \rightarrow 0$ (or $\bar{\beta} \rightarrow \infty$), *asymptotic freedom*.

Corresponds to second order phase transition.

In practice, tune g_0, N_4 for getting correlation lengths ξ , such that:

$$N_4 \gg \xi/a(g_0) \gg 1.$$

At non-zero $T = 1/L_4 = 1/(N_4 a(\bar{\beta}))$:

T can be varied by changing $\bar{\beta}$ at fixed N_4 or N_4 at fixed $\bar{\beta}$.

3rd Lecture

4. Simulating gauge fields

4.1. How does Monte Carlo work?

Realization in quantum mechanics:

M. Creutz, B. Freedman, A stat. approach to quantum mechanics, Annals Phys. 132(1981)427

Here consider n-dim. integral:

$$\langle f \rangle = \int_{\Omega} d^n x f(x) w(x) \quad \text{with} \quad 0 \leq w(x) \leq 1, \quad \int_{\Omega} w(x) d^n x = 1.$$

$$\langle f \rangle = \int_{\Omega} d^n x f(x) \int_0^{w(x)} d\eta = \int_{\Omega} d^n x \int_0^1 d\eta f(x) \Theta(w(x) - \eta)$$

Importance sampling by selecting x in acc. with $w(x)$:

(a) choose randomly: $x \in \Omega$ and $\eta \in [0, 1]$,

(b) acceptance check: accept x , if η satisfies $\eta < w(x)$, otherwise reject.

$\Rightarrow$ From accepted $x^{(i)}$'s estimate $\langle f \rangle \simeq (1/N) \sum_{i=1}^N f(x^{(i)})$.

$\Rightarrow$ However, efficiency small, if acceptance rate in large areas of Ω is low.

More efficient: appropriate Markov chain $x^{(1)}, x^{(2)}, \dots$

generated with transition probability $P(x^{(i)} \rightarrow x^{(i+1)})$

satisfying *detailed balance condition*

$$w(x)P(x \rightarrow x') = w(x')P(x' \rightarrow x)$$

- sufficient for $w(x)$ becoming fix-point of the Markov chain,
- obviously satisfied for $P(x \rightarrow x') \equiv w(x')$.

Markov chains realizable step-by-step by selecting single components x_ν keeping all $x_\mu, \mu \neq \nu$ fixed.

Heat bath method:

- If possible determine x'_ν with probability $\omega(x'_\nu) \sim w(x_1, \dots, x'_\nu, \dots, x_n)$.
- Replace old value x_ν by x'_ν .
- Repeat procedure for other component x_μ .

Metropolis method:

- random shifts $x_\nu \rightarrow x'_\nu = x_\nu + \eta$, $\eta \in (-\epsilon, +\epsilon)$ with ϵ approp. chosen,
- if $\omega(x'_\nu) > \omega(x_\nu)$, then accept x'_ν ,
- if $\omega(x'_\nu) < \omega(x_\nu)$, then accept with probability $\omega(x'_\nu)/\omega(x_\nu)$,
- accepted values x'_ν replace x_ν ,
- repeat procedure at the same x_ν , then go to next x_μ .

Homework II:

Write heat bath code for 4D Ising model $\longrightarrow$ tutorial

4.2. Creutz' heat bath method

Assume statistical weight $w[U] \sim e^{-S_G[U]}$ with plaquette action

$$S_G[U] = \bar{\beta} \sum_{n, \mu < \nu} \left(1 - \frac{1}{N_c} \operatorname{Re} \operatorname{tr} U_{n, \mu\nu} \right).$$

Select a single link variable: $U_{n_0, \mu_0} \equiv U_0$. There are 6 plaquettes containing this link and contributing to S_G :

$$\begin{aligned} S_G[U_0; \{U\} \setminus U_0] &= C[\{U\} \setminus U_0] - \frac{\bar{\beta}}{N_c} \sum_{S=1}^6 \operatorname{Re} \operatorname{tr} (U_0 U_S) \\ &= C - \frac{\bar{\beta}}{N_c} \operatorname{Re} \operatorname{tr} (U_0 A), \quad A = \sum_{S=1}^6 U_S \end{aligned}$$

Call open plaquette $U_S =$ “staple”.

Assume A be fixed (“heat bath”) $\implies$ link variable U_0 to be determined with probability

$$w(U_0) [dU_0] \sim \exp \left(\frac{\bar{\beta}}{N_c} \operatorname{Re} \operatorname{tr} (U_0 A) \right) [dU_0]$$

.

Metropolis:

- apply random shifts $U_0 \rightarrow U'_0 = GU_0$, with $G \in U_\epsilon(\mathbf{1}) \subset SU(N_c)$,
- carry out Metropolis acceptance steps.

Heat bath for $SU(2)$:

Normalize $V = A/\sqrt{\det A} \in SU(2)$, put $VU_0 \equiv U$ and use invariance of Haar measure

$$w(U)[dU] \sim \exp(\rho \operatorname{tr} U) [dU], \quad \rho = \frac{\bar{\beta} \sqrt{\det A}}{2}$$

$$U \equiv B^0 \mathbf{1} + i\vec{\sigma} \vec{B}, \quad [dU] = \frac{1}{\pi^2} \delta(B^2 - 1) d^4 B, \quad \operatorname{tr} U = 2B^0$$

$$\Rightarrow w(U)[dU] \sim \exp(2\rho B^0) \delta(B^2 - 1) d^4 B$$

$$\Rightarrow \text{determine } U \text{ and } U_0 = V^\dagger U.$$

Heat bath for $SU(N_c)$ [Cabibbo, Marinari, '82]

Use $SU(2)$ heat bath algorithm for various subgroup embeddings into $SU(N_c)$.

5. Fermions on the lattice

5.1. Naive discretization and fermion doubling

Path integral for fermions requires anticommuting variables – Grassmann algebra:

$$\eta_i, \bar{\eta}_i, i = 1, 2, \dots, N \quad \text{with} \quad \bar{\eta}_i \text{ adjoint to } \eta_i,$$

$$\begin{aligned} \{\eta_i, \eta_j\} &\equiv \eta_i \eta_j + \eta_j \eta_i \\ &= \{\bar{\eta}_i, \eta_j\} = \{\bar{\eta}_i, \bar{\eta}_j\} = \dots = 0, \\ \eta_i^2 &= 0, \end{aligned}$$

such that any function has representation

$$f(\eta) = f_0 + \sum_i f_i \eta_i + \sum_{i \neq j} f_{ij} \eta_i \eta_j + \dots f_{12\dots N} \eta_1 \eta_2 \dots \eta_N,$$

correspondingly for $f(\eta, \bar{\eta})$.

E.g. $g(\eta, \bar{\eta}) = \exp(-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j) = \prod_{i,j=1}^N (1 - \bar{\eta}_i A_{ij} \eta_j).$

Integration rules (same as differentiation):

$$\begin{aligned} \int d\eta_i &= \int d\bar{\eta}_i = 0, & \int d\eta_i \eta_i &= \int d\bar{\eta}_i \bar{\eta}_i = 1, \\ \{d\eta_i, d\eta_j\} &= \{d\bar{\eta}_i, d\eta_j\} = \{d\bar{\eta}_i, d\bar{\eta}_j\} = \{d\eta_i, \eta_j\} = \dots = 0. \end{aligned}$$

Most important for us:

$$\begin{aligned}
I[A] &= \int \prod_{l=1}^N d\bar{\eta}_l d\eta_l \exp \left(- \sum_{ij=1}^N \bar{\eta}_i A_{ij} \eta_j \right) = \det A, \\
Z[A; \rho, \bar{\rho}] &= \int \prod_{l=1}^N d\bar{\eta}_l d\eta_l \exp \left(- \sum_{ij=1}^N \bar{\eta}_i A_{ij} \eta_j + \sum_i (\bar{\eta}_i \rho_i + \bar{\rho}_i \eta_i) \right), \\
&= \det A \cdot \exp \left(\sum_{ij=1}^N \bar{\rho}_i A_{ij}^{-1} \rho_j \right), \\
\langle \eta_i \bar{\eta}_j \rangle &= \frac{\int D\bar{\eta} D\eta \eta_i \bar{\eta}_j \exp(-\bar{\eta} A \eta)}{\int D\bar{\eta} D\eta \exp(-\bar{\eta} A \eta)} = A_{ij}^{-1}.
\end{aligned}$$

Dirac Propagator (with Euclidean: $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $\gamma_4 \equiv \gamma_M^0$, $\gamma_i \equiv -i\gamma_M^i$)

$$\begin{aligned}
\langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle &= \frac{\int D\bar{\psi} D\psi \psi_\alpha(x) \bar{\psi}_\beta(y) \exp(-S_F[\bar{\psi}\psi])}{\int D\bar{\psi} D\psi \exp(-S_F[\bar{\psi}\psi])}, \\
S_F[\bar{\psi}\psi] &= \int d^4x \bar{\psi}(x) (\gamma_\mu \partial_\mu + m) \psi(x).
\end{aligned}$$

Naive lattice discretization:

$$\text{rescale } m \rightarrow M/a, \quad \psi_\alpha(x_n) \rightarrow \frac{1}{a^{3/2}} \hat{\psi}_\alpha(n), \quad \partial_\mu \psi_\alpha(x_n) \rightarrow \frac{1}{a^{5/2}} \hat{\partial}_\mu \hat{\psi}_\alpha(n)$$

$$\hat{\partial}_\mu \hat{\psi}_\alpha(n) = \frac{1}{2} [\hat{\psi}_\alpha(n + \hat{\mu}) - \hat{\psi}_\alpha(n - \hat{\mu})]$$

$$S_F[\bar{\psi}\psi] \simeq \sum_{n,m;\alpha,\beta} \hat{\bar{\psi}}_\alpha(n) K_{\alpha\beta}(n,m) \hat{\psi}_\beta(m)$$

$$\text{with } K_{\alpha\beta}(n,m) = \frac{1}{2} \sum_\mu (\gamma_\mu)_{\alpha\beta} [\delta_{m,n+\hat{\mu}} - \delta_{m,n-\hat{\mu}}] + M \delta_{mn} \delta_{\alpha\beta}$$

Free lattice Dirac propagator:

$$\langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle = \lim_{a \rightarrow 0} \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} \frac{[-i \sum_\mu \gamma_\mu \tilde{p}_\mu + m]_{\alpha\beta}}{\sum_\mu \tilde{p}_\mu^2 + m^2} e^{ip(x-y)}$$

with $\tilde{p}_\mu = \frac{1}{a} \sin(p_\mu a)$ to be compared with scalar case $\tilde{k}_\mu = \frac{2}{a} \sin(k_\mu a/2)$.

For $M = 0$ in momentum space we get poles at all 2^d corners of the Brillouin zone $[(0000), (\pi/a \ 000), \dots]$. $\Rightarrow$ “**Doubling of fermion degrees of freedom**”.

Homework III:

Determine lattice propagator for free scalar field $\longrightarrow$ tutorial

Theorem by Nielsen, Ninomiya, '81:

Doubling problem can be avoided only by giving up at least one of:

- reflexion positivity,
- cubic symmetry,
- locality,
- chiral invariance in the zero mass case.

5.2. Wilson fermions and improvements

Wilson's choice – break chiral symmetry even at $m = 0$:

$$D_F = \gamma_\mu \partial_\mu \longrightarrow D_{\text{Latt}} \equiv D_F^W = \frac{1}{2} \left[\gamma_\mu (\nabla_\mu^* + \nabla_\mu) - ar \nabla_\mu^* \nabla_\mu \right],$$

where ∇_μ (∇_μ^*) forward (backward) gauge covariant derivatives

$$\begin{aligned} \nabla_\mu \psi(x_n) &= \frac{1}{a} \left[U_{n,\mu} \psi(n + \hat{\mu}) - \psi(n) \right], \\ \nabla_\mu^* \psi(x_n) &= \frac{1}{a} \left[\psi(n) - U_{n,-\mu} \psi(n - \hat{\mu}) \right], \quad U_{n,-\mu} \equiv U_{n-\hat{\mu}}^\dagger \end{aligned}$$

r – arbitrary real parameter, often chosen to be $r = 1$.

Lattice free Wilson fermion action:

$$S_F^W [\bar{\psi}\psi] \simeq \sum_{n,m;\alpha,\beta} \bar{\hat{\psi}}_\alpha(n) K_{\alpha\beta}^W(n,m) \hat{\psi}_\beta(m),$$

$$K_{\alpha\beta}^W(n,m) = (M + 4r)\delta_{mn}\delta_{\alpha\beta} - \frac{1}{2} \sum_{\mu} [(r - \gamma_\mu)_{\alpha\beta} \delta_{m,n+\hat{\mu}} + (r + \gamma_\mu)_{\alpha\beta} \delta_{m,n-\hat{\mu}}]$$

Propagator:

$$\langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle = \lim_{a \rightarrow 0} \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} \frac{[-i \sum_{\mu} \gamma_\mu \tilde{p}_\mu + \textcolor{red}{m}(p)]_{\alpha\beta}}{\sum_{\mu} \tilde{p}_\mu^2 + \textcolor{red}{m}(p)^2} e^{ip(x-y)},$$

$$\tilde{p}_\mu \text{ as before, but } \textcolor{red}{m}(p) = m + \frac{2r}{a} \sum_{\mu} \sin^2(p_\mu a/2)$$

For $a \rightarrow 0$

$$\begin{aligned} m(p) &\rightarrow m \text{ for } p_\mu \neq \pm \frac{\pi}{a}, \\ m(p) &\rightarrow \infty \text{ for } p_\mu = \pm \frac{\pi}{a}. \end{aligned}$$

Problems:

- Chiral $SU(3)_A$ flavor symmetry explicitly broken.
- Eigenvalue value spectrum of D_F^W strongly differs from continuum spectrum.
- Discretization error $\delta S_F^W \sim O(a)$ compared with $\delta S_G^W \sim O(a^2)$
 $\implies$ improvement possible with clover term

$$S_F^{clover} = S_F^W + a^5 \sum_n c_{sw} \psi(x_n) \frac{i}{4} \sigma_{\mu\nu} \hat{F}_{\mu\nu} \psi(x_n) .$$

Sheikholeslami, Wohlert '85

$\implies$ alternative: twisted-mass fermions at maximal twist (cf. K. Jansen).

Chiral improvement: Ginsparg-Wilson fermions

Any lattice Dirac operator satisfying the **Ginsparg-Wilson relation (GWR)**

Ginsparg, Wilson '82

$$\gamma_5 D_{\text{Latt}} + D_{\text{Latt}} \gamma_5 = a D_{\text{Latt}} \gamma_5 D_{\text{Latt}}$$

guaranties approximately local ($\sim O(a)$), but **exact chiral symmetry**.

Lüscher '98

Strategies to solve GWR:

- **Neuberger's operator**: exact solution of GWR

$$D + m \rightarrow D_N = \left\{ 1 + \frac{m}{2} (1 + A (A^\dagger A)^{-\frac{1}{2}}) \right\}, \quad A = 1 + s - D_F^W$$

Neuberger '98

Properties:

- $(\sqrt{A^\dagger A})^{-1}$ numerically involved:
approximation by polynomials (Chebyshev approx.).
- **Det D_N** hard to compute, but required for full fermionic simulation.
- Discretization error still $O(a^2)$.

- Equivalent alternative **domain wall fermions**: extension to 5 dimensions

$$D_{\text{Latt}} = \frac{1}{2} [\gamma_5 (\partial_s^* + \partial_s) - a_s \partial_s^* \partial_s] + D_F^W - \frac{\rho}{a}, \quad 0 < \rho < 2$$

with boundary condition

$$P_+ \psi(o, x) = P_- \psi((N_s + 1) a_s, x) = 0, \quad P_{\pm} = (1 \pm \gamma_5)/2$$

and limit $N_s \rightarrow \infty$ and $a_s \rightarrow 0$

Kaplan '92; Shamir '93

- **Approximative methods**:
 - Renormalization group based perfect action approach
P. Hasenfratz, Niedermayer '94; DeGrand, ... '94
 - generalized (less local) ansatzes for D_{Latt} with parameters fixed from GWR
Gattringer '01

5.3. Staggered fermions

Kogut, Susskind, '75

- Use naive discretization and diagonalize action w.r. to spinor degrees of freedom.
- Neglect three of four degenerate Dirac components.
- Attribute the 16 fermionic degrees of freedom localized around one elementary hypercube to four *tastes* with four Dirac indices each.

Chiral symmetry restored $\iff$ flavor symmetry broken.

Naturally the mass-degenerated four-flavor case is described.

Rooting prescription:

for $N_f = 2 + 1(+1)$ 4th-root of the fermionic determinant is taken.

$\implies$ Locality violated (??)

Improvement possible by 'smearing' link variables.

5.4. How to compute typical QCD observables

Path integral quantization for Euclidean time $\implies$ 'statistical averages'.

Fermions as anticommuting Grassmann variables

$\implies$ analytically integrated $\Rightarrow$ non-local effective action $S^{eff}(U)$.

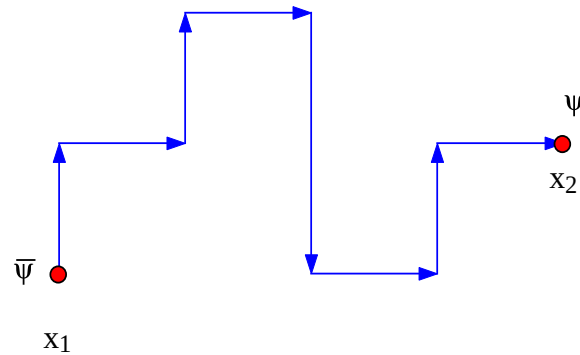
'Partition function' for $f = 1, \dots, N_f$ light quark flavors:

$$\begin{aligned} Z &= \int [dU] \prod_f [d\psi_f] [d\bar{\psi}_f] e^{-S^G(U) + \sum_f \bar{\psi}_f M_f(U) \psi_f} \\ &= \int [dU] e^{-S^G(U)} \prod_f \text{Det} M_f(U) \\ &= \int [dU] e^{-S^{eff}(U)}, \quad S^{eff}(U) = S^G(U) - \sum_f \log(\text{Det} M_f(U)) \end{aligned}$$

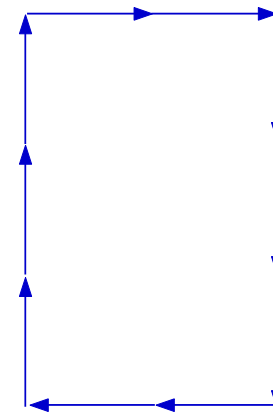
with $M_f(U) \equiv D_{\text{Latt}}(U) + m_f$.

To be simulated on a finite lattice $N_t \times N_s^3$, mostly with periodic boundary conditions for gluons (anti-periodic for quarks).

Observables: mostly gauge invariant.



$\bar{\psi}_u(x_1) U(x_1, x_2) \Gamma \psi_d(x_2)$
Non-local fermionic current



$\Omega(U) = \text{tr} \prod_1 U_1$
Wilson loop

[Taken from C. Davies, hep-ph/0205181]

Pure gauge observables:

$$\begin{aligned} \langle \Omega \rangle &= \frac{1}{Z} \int [dU] \prod_f [d\psi_f] [d\bar{\psi}_f] \Omega(U) e^{-S^G(U) + \sum_f \bar{\psi}_f M(U) \psi_f} \\ &= \frac{1}{Z} \int [dU] \Omega(U) e^{-S^{eff}(U)} \end{aligned}$$

Fermionic observables through correlators,

e.g. for local (u, d) -meson current $H(x) = \bar{\psi}_u^a(x) \Gamma \psi_d^a(x)$

$$\begin{aligned} \langle H^\dagger(x) H(y) \rangle &= \frac{1}{Z} \int [dU] \prod_f [d\psi_f] [d\bar{\psi}_f] H^\dagger(x) H(y) e^{-S^G + \sum_f \bar{\psi}_f M(U) \psi_f} \\ &= \frac{1}{Z} \int [dU] (M(U)^{-1}(x, y))_u^{ab} \Gamma (M(U)^{-1}(y, x))_d^{ba} \Gamma e^{-S^{eff}(U)} \end{aligned}$$

propagator $(M(U)^{-1}(x, y))_f$, $f = u, d$ computed with conj. gradient method.

Quenched approximation: put $\text{Det } M(U) \equiv 1$, i.e. pure gauge field simulation.

$\text{Det } M(U) \neq 1$ can be taken into account, but time consuming

$\Rightarrow$ Hybrid Monte Carlo, multibosonic algorithms,...

$\Rightarrow$ massively parallel supercomputers required.

Typical lattice size for correlation functions in d=4 dimensions:

$$N_s = 32, N_t = 64$$

Gauge field to be simulated:

$$N_c^2 \times d \times N_s^3 \times N_t = 9 \times 4 \times 32^3 \times 64 \simeq 7.5 \cdot 10^7 \text{ degrees of freedom.}$$

Sparse Wilson-Dirac matrix to be inverted:

$$(4N_c N_s^3 N_t) \times (4N_c N_s^3 N_t) \simeq 2.5 \cdot 10^7 \times 2.5 \cdot 10^7$$

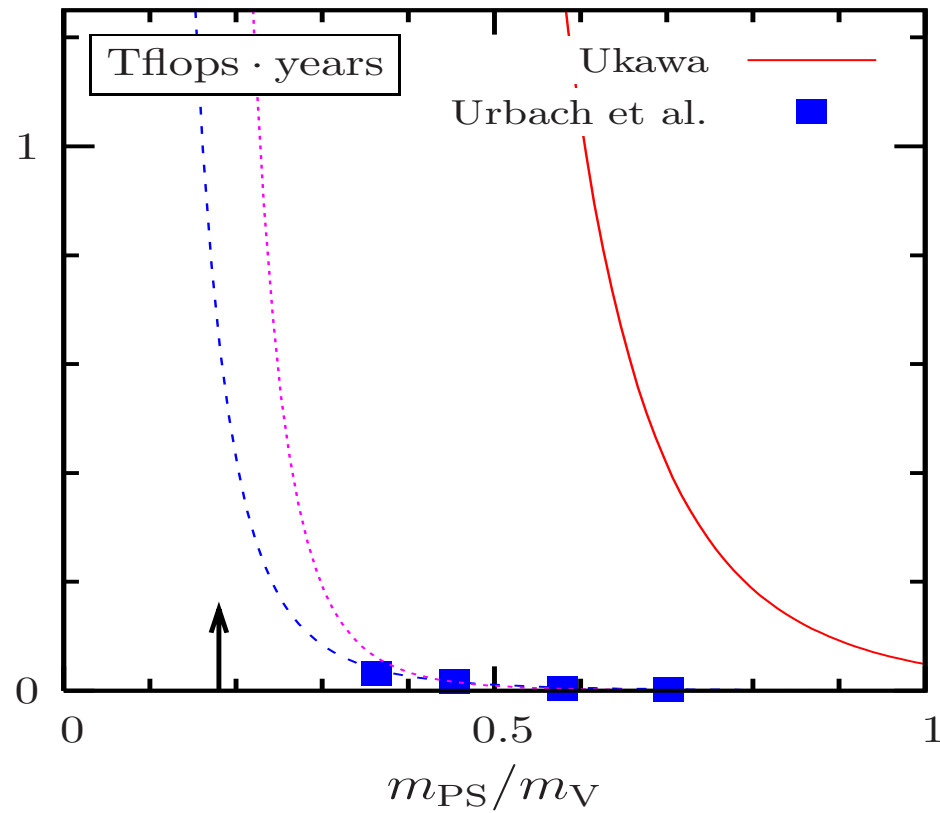
Three critical limites:

- Continuum limit: correlation lengths in lattice units a diverge,
- Box size $L_s^3 \times L_t$ ($L_s \equiv N_s a, L_t \equiv N_t a$) be sufficiently large,
- Realistically small u-, d-quark masses $\Rightarrow \approx$ zero modes of Dirac matrix,

$$CPU \simeq C \left(\frac{20 \text{ MeV}}{m_{\text{quark}}} \right)^{z_m} \left(\frac{L_s}{3 \text{ fm}} \right)^{z_L} \left(\frac{0.1 \text{ fm}}{a} \right)^{z_a}$$

2007: empirical values for HMC algorithm for improved Wilson fermions (ETM collaboration)

$$C \simeq 0.3 \text{ Tflops} \times \text{years}, \quad z_m \simeq 1, \quad z_L \simeq 5, \quad z_a \simeq 6$$



“Berlin Wall” = CPU time versus fixed ratio m_π/m_ρ

A. Ukawa, Lattice '01, HU Berlin;

updated C. Urbach, Lattice '07, U. Regensburg

At present: Simulations at the physical point even with improved Wilson fermions are becoming realistic !!

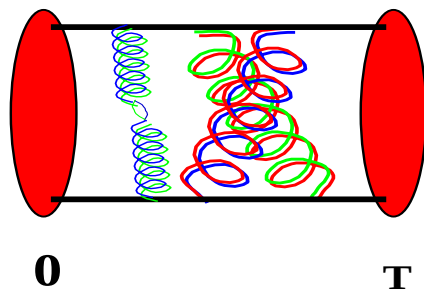
Our aim:

Computation of hadronic masses and matrix elements from various 2-point or 3-point functions

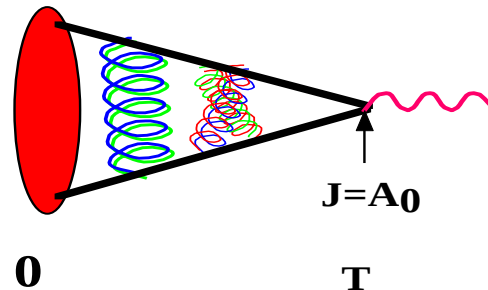
2-point functions:

$$\langle O_f(T) O_i(0) \rangle - \langle O_f \rangle \langle O_i \rangle = \sum_n \frac{\langle \text{vac} | O_f | n \rangle \langle n | O_i | \text{vac} \rangle}{2M_n} e^{-M_n T} \xrightarrow{T \rightarrow \infty} e^{-M_0 T}$$

- $O_f \equiv H^\dagger, O_i \equiv H \implies$ extract masses of hadrons with quantum numbers related to **non-local** current H .
- $O_f \equiv J, O_i \equiv H \implies$ extract vacuum-to-hadron matrix elements (decay constants) with **local** current J .



2pt function for spectrum

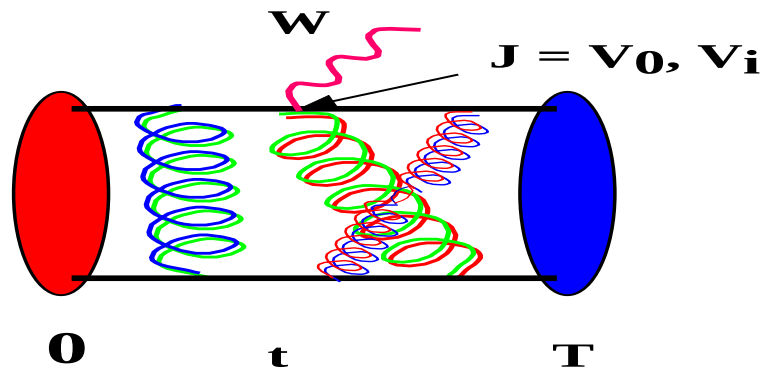


2pt function for decay constant

3-point functions:

$$\langle H'^{\dagger}(T)J(t)H(0) \rangle = \sum_n \sum_m \frac{\langle \text{vac} | H^{\dagger} | m \rangle \langle m | J | n \rangle \langle n | H | \text{vac} \rangle}{2M_n 2M_m} e^{-M_n t} e^{-M_m (T-t)}$$

- allow to extract experimentally relevant hadron-to-hadron matrix elements for decay constants, moments of structure functions and form factors of hadrons.



3pt function for SL decay

[As previous diagrams taken from C. Davies, hep-ph/0205181]

4th Lecture

6. Topology, Cooling and the Wilson (gradient) flow

6.1. Topology of gauge fields [Belavin, Polyakov, Schwarz, Tyupkin, '75; 't Hooft, '76; Callan, Dashen, Gross, '78 -'79]

Euclidean Yang-Mills action: $S[A] = -\frac{1}{2g^2} \int d^4x \operatorname{tr} (G_{\mu\nu} G_{\mu\nu})$

Topological charge:

$$Q_t[A] \equiv \int d^4x \rho_t(x), \quad \rho_t(x) = -\frac{1}{16\pi^2} \operatorname{tr} (G_{\mu\nu} \tilde{G}_{\mu\nu}(x)), \quad \tilde{G}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma}.$$

$$Q_t[A] \equiv \sum_{i=1}^q w_i \in \mathbf{Z},$$

w_i “windings” of continuous mappings $S^{(3)} \rightarrow SU(2)$ (homotopy classes),
invariant w.r. to continuous deformations (**but not on the lattice !!**)

Example for topologically non-trivial field - “instanton”:

[Belavin, Polyakov, Schwarz, Tyupkin, '75]

$$-\int d^4x \operatorname{tr} [(G_{\mu\nu} \pm \tilde{G}_{\mu\nu})^2] \geq 0 \implies S[A] \geq \frac{8\pi^2}{g^2} |Q_t[A]|$$

$$\text{iff } S[A] = \frac{8\pi^2}{g^2} |Q_t[A]|, \quad \text{then } G_{\mu\nu} = \pm \tilde{G}_{\mu\nu} \quad (\text{anti) selfduality}.$$

$|Q_t| = 1$: BPST one-(anti)instanton solution (singular gauge) for $SU(2)$:

$$\mathcal{A}_{a,\mu}^{(\pm)}(x-z, \rho, R) = R^{a\alpha} \eta_{\alpha\mu\nu}^{(\pm)} \frac{2 \rho^2 (x-z)_\nu}{(x-z)^2 ((x-z)^2 + \rho^2)},$$

For $SU(N_c)$ embedding of $SU(2)$ solutions required.

Dilute instanton gas (DIG) $\longrightarrow$ instanton liquid (IL):

path integral “approximated” by superpositions of (anti-) instantons and represented as partition function in the modular space of instanton parameters.

[Callan, Dashen, Gross, '78 -'79; Ilgenfritz, M.-P., '81; Shuryak, '81 - '82; Diakonov, Petrov, '84]

$\implies$ may explain chiral symmetry breaking, but fails to explain confinement.

Case $T > 0$: x_4 -periodic instantons - “calorons”

Semiclassical treatment of the partition function [Gross, Pisarski, Yaffe, '81]

with “caloron” solution $\equiv x_4$ -periodic instanton chain ($1/T = b$)

[Harrington, Shepard, '77]

$$A_{\mu}^{a\text{HS}}(x) = \eta_{a\mu\nu}^{(\pm)} \partial_{\nu} \log(\Phi(x))$$

$$\Phi(x) - 1 = \sum_{k \in \mathbf{Z}} \frac{\rho^2}{(\vec{x} - \vec{z})^2 + (x_4 - z_4 - kb)^2} = \frac{\pi \rho^2}{b|\vec{x} - \vec{z}|} \frac{\sinh\left(\frac{2\pi}{b}|\vec{x} - \vec{z}|\right)}{\cosh\left(\frac{2\pi}{b}|\vec{x} - \vec{z}|\right) - \cos\left(\frac{2\pi}{b}(x_4 - z_4)\right)}$$

$$- \quad Q_t = -\frac{1}{16\pi^2} \int_0^b dx_4 \int d^3x \rho_t(x) = \pm 1.$$

- (as for instantons) it exhibits trivial holonomy, i.e. Polyakov loop behaves as:

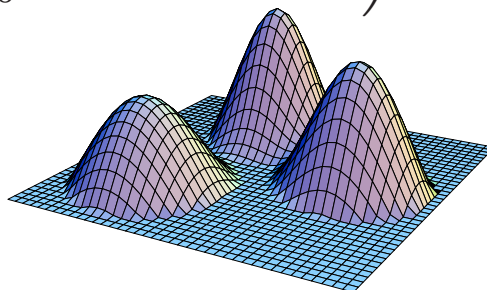
$$\frac{1}{2} \text{tr} \mathbf{P} \exp \left(i \int_0^b A_4(\vec{x}, t) dt \right) \xrightarrow{|\vec{x}| \rightarrow \infty} \pm 1$$

Kraan - van Baal solutions

= (anti-) selfdual caloron solutions with non-trivial holonomy

[K. Lee, Lu, '98, Kraan, van Baal, '98 - '99, Garcia-Perez et al. '99]

$$P(\vec{x}) = \mathbf{P} \exp \left(i \int_0^{b=1/T} A_4(\vec{x}, t) dt \right) \xrightarrow{|\vec{x}| \rightarrow \infty} \mathcal{P}_\infty \notin \mathbf{Z}(N_c)$$



Action density of a single (but dissolved) $SU(3)$ caloron with $Q_t = 1$ (van Baal, '99)
 $\implies$ not a simple $SU(2)$ embedding into $SU(3)$!!

Dissociation into caloron constituents (BPS monopoles or “dyons”) gives hope for modelling confinement for $T < T_c$ as well as the deconfinement transition.

[Gerhold, Ilgenfritz, M.-P., '07; Diakonov, Petrov, et al., '07 - '12; Bruckmann, Dinter, Ilgenfritz, Maier, M.-P., Wagner, '12; Shuryak, Sulejmanpasic, '12-'13; Faccioli, Shuryak, '13]

Systematic development of the semiclassical approach + perturbation theory
“resurgent trans-series expansions” ...

[Dunne, Ünsal and collaborators, '12-'14]

Axial anomaly

[Adler, '69; Bell, Jackiw, '69; Bardeen, '74]

$$\partial_\mu j^{\mu 5}(x) = D(x) + 2N_f \rho_t(x)$$

$$\text{with } j^{\mu 5}(x) = \sum_f^{N_f} \bar{\psi}_f(x) \gamma^\mu \gamma^5 \psi_f(x), \quad D(x) = 2im \sum_{f=1}^{N_f} \bar{\psi}_f(x) \gamma^5 \psi_f(x)$$

$\rho_t \neq 0$ due to non-trivial topology $\implies$ solution of the $U_A(1)$ problem:

η' -meson (pseudoscalar singlet) for $m \rightarrow 0$ not a Goldstone boson, $m_{\eta'} \gg m_\pi$.

Related Ward identity:

$$\begin{aligned} 4N_f^2 \int d^4x \langle \rho_t(x) \rho_t(0) \rangle &= 2iN_f \langle -2m \bar{\psi}_f \psi_f \rangle + \int d^4x \langle D(x) D(0) \rangle \\ &= 2iN_f m_\pi^2 F_\pi^2 + O(m^2) \\ \chi_t \equiv \frac{1}{V} \langle Q_t^2 \rangle \Big|_{N_f} &= \frac{i}{2N_f} m_\pi^2 F_\pi^2 + O(m_\pi^4) \rightarrow 0 \text{ for } m_\pi \rightarrow 0. \end{aligned}$$

$1/N_c$ -expansion, i.e. fermion loops suppressed (“quenched approximation”)

[Witten, '79, Veneziano '79]

$$\chi_t^q = \frac{1}{V} \langle Q_t^2 \rangle \Big|_{N_f=0} = \frac{1}{2N_f} F_\pi^2 [m_{\eta'}^2 + m_\eta^2 - 2m_K^2] \simeq (180 \text{ MeV})^4.$$

Integrating axial anomaly we get **Atiyah-Singer index theorem**

$$Q_t[A] = n_+ - n_- \in \mathbf{Z}$$

$n_{\pm}$ number of zero modes $f_r(x)$ of Dirac operator $i\gamma^\mu \mathcal{D}_\mu[A]$
with chirality $\gamma_5 f_r = \pm f_r$.

$\implies$ For lattice computations employ a chiral operator $i\gamma^\mu \mathcal{D}_\mu$.

$\implies$ Not free of lattice artifacts, use improved gauge action.

Topology becomes unique only for lattice fields smooth enough.

Sufficient (!) bound to plaquette values can be given. [Lüscher, '82].

6.2. Measuring topology on the lattice:

Gauge field approaches:

- Field theoretic with loop discretization of $G_{\mu\nu}$

$$\rho_t^{Latt}(x_n) \equiv -\frac{1}{16\pi^2} \text{tr} (G_{\mu\nu}^{Latt}(x_n) \tilde{G}_{\mu\nu}^{Latt}(x_n))$$

with G^{Latt} in the simplest case from plaquette expansion

$$U_{n,\mu\nu} = \mathbf{1} + i G_{\mu\nu}^{Latt}(x_n) + O(a^4)$$

thus using plaquette loops around site x_n in a symmetrized way

$$\tilde{\epsilon}_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma} \text{ for } \mu, \nu, \rho, \sigma > 0, \quad \tilde{\epsilon}_{\mu\nu\rho\sigma} = \tilde{\epsilon}_{(-\mu)\nu\rho\sigma}, \quad \dots$$

$$Q_t^{Latt} = \sum_n \rho_t^{Latt}(x_n), \quad \rho_t^{Latt}(x_n) = -\frac{1}{2^9 \pi^2} \sum_{\mu\nu\rho\sigma=\pm 1, \dots, \pm 4} \tilde{\epsilon}_{\mu\nu\rho\sigma} \text{tr} U_{n\mu\nu} U_{n\rho\sigma}$$

[Fabricius, Di Vecchia, G.C. Rossi, Veneziano, '81; Makhaldiani, M.-P., '83]

Problems:

- $Q_t^{Latt} \simeq$ integer-valued **only for sufficiently 'smooth'** lattice gauge fields.
- Top. susceptibility $\chi_t^{Latt} = \frac{1}{V} \langle (Q_t^{Latt})^2 \rangle$ has to be renormalized.

- **Geometric definitions** providing always integer Q_t have been invented

[Lüscher, '82; Woit, '83; Phillips, Stone, '86],

(used with and without *smoothing*).

Fermionic approaches:

- **Index of Ginsparg-Wilson fermion operators:** $Q_t = n_+ - n_-$

[Hasenfratz, Laliena, Niedermayer, '98; Neuberger, '01;...]

- From corresponding spectral representation of ρ_t

$$\rho_t(x) = \text{tr} \gamma_5 \left(\frac{1}{2} D_{x,x} - 1 \right) = \sum_{n=1}^N \left(\frac{\lambda_n}{2} - 1 \right) \psi_n^\dagger(x) \gamma_5 \psi_n(x)$$

- Fermionic representation: [Smit, Vink, '87]

$$N_f Q_t = \kappa \text{Tr} \frac{m \gamma_5}{D + m}, \quad \kappa \text{ renorm. factor}$$

- Topological susceptibility from **higher moments and spectral projectors**

[Lüscher, '04; Giusti, Lüscher, '08; Lüscher, Palombi, '10; Cichy, Garcia Ramos, Jansen, '13-'14

-

Stripping off quantum fluctuations:

“Cooling”:

Old days lattice search for multi-instanton solutions

[Berg, '81; Iwasaki, et al., '83; Teper, '85; Ilgenfritz, Laursen, M.-P., Schierholz, '86],

more recently, for non-trivial holonomy KvBLL calorons

[Garcia Perez, Gonzalez-Arroyo, Montero, van Baal, '99; Ilgenfritz, Martemyanov, M.-P., Shcheredin, Veselov, '02; Ilgenfritz, M.-P., Peschka, '05]

- Solve the lattice field equation locally (for a given link variable),
- replace old by new link variable,
- step through the lattice (order not unique),
- find plateau values for the topological charge and action.

Homework IV: Try to formulate and to solve the lattice field equation for a single link variable U_{n_0, μ_0} in the $SU(2)$ case $\longrightarrow$ tutorial

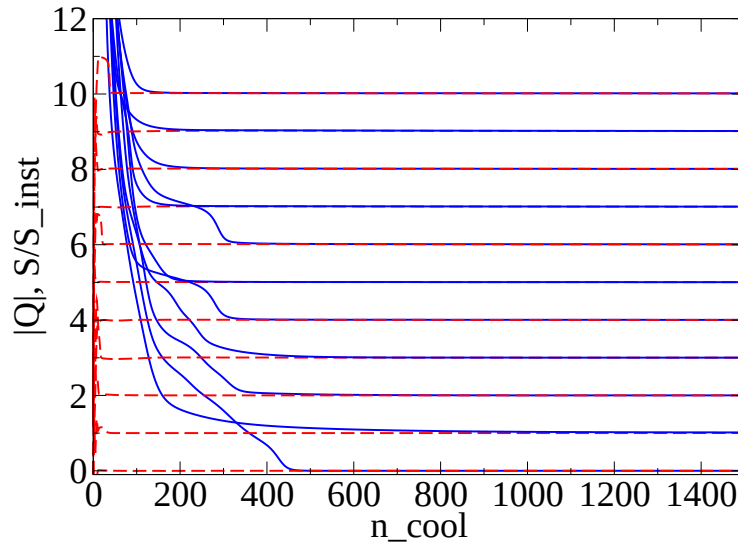
- (Over-) improved cooling and improved $G_{\mu\nu}$

$\implies$ for $T < T_c$ early and extremely stable plateaus at nearly integer Q_t .

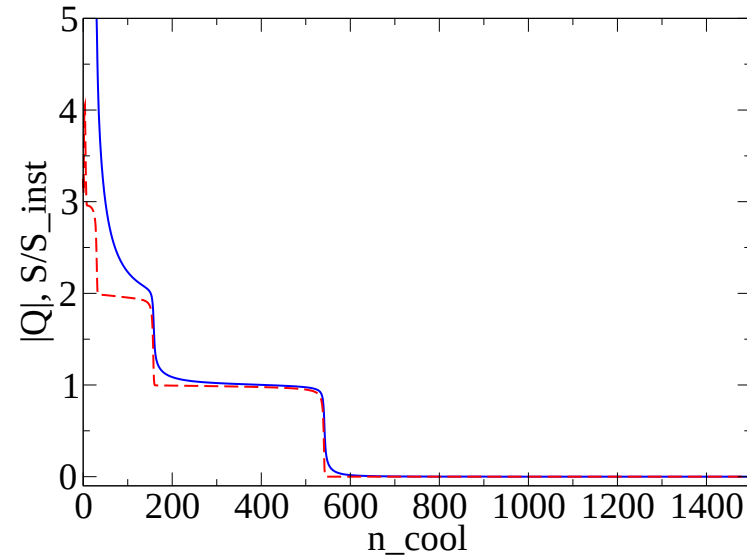
[Garcia Perez, Gonzalez Arroyo, Snippe, van Baal, '94; de Forcrand, Garcia Perez, Stamatescu, '96; Bruckmann, Ilgenfritz, Martemyanov, van Baal, '04]

Typical examples of gluodynamics for $T > 0$ (Q_t , S).

$$T = 0.88 T_c$$



$$T = 1.12 T_c$$



[Bornyakov, Ilgenfritz, Martemyanov, M.-P., Mitrjushkin, '13]

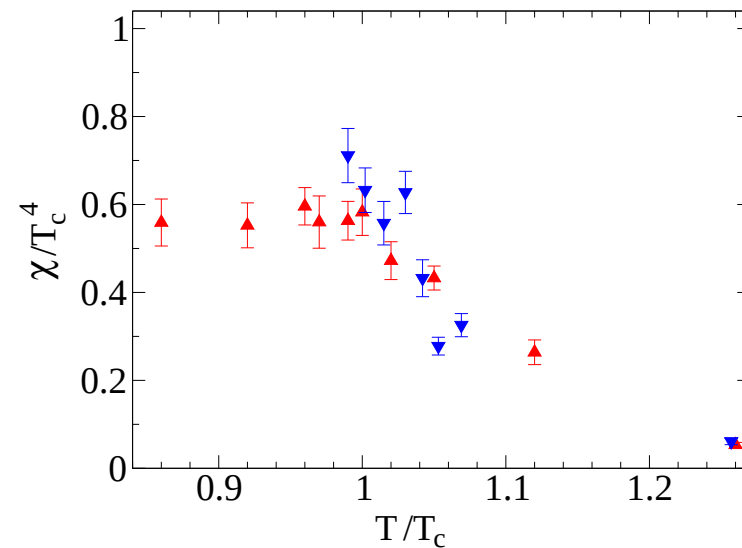
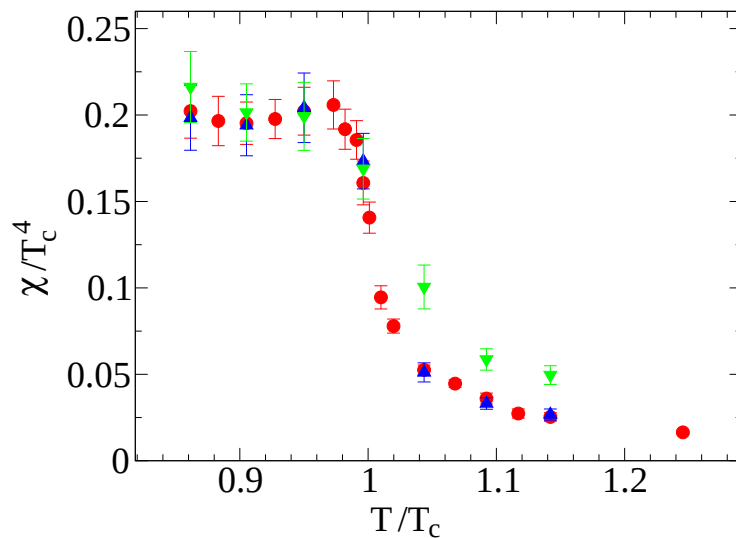
Stability (decay) of plateaus for $T < T_c$ ($T > T_c$) related to KvBLL caloron structure and non-trivial (trivial) holonomy (dyon mass symmetry / asymmetry).

Topological susceptibility χ_t (for two lattice sizes and spacings):

gluodynamics

full QCD

(clover-impr. $N_f = 2$, $m_\pi \simeq 1$ GeV)



- χ_t smoother in full QCD (crossover) than in gluodynamics (1st order).
Should show up also in the $U_A(1)$ restoration.
- What about chiral limit ?

Wilson or gradient flow:

Proposed and thoroughly investigated by M. Lüscher
(perturbatively with P. Weisz) since 2009 (cf. talks at LATTICE 2010 and 2013).

Flow time evolution uniquely defined for arbitrary lattice field $\{U_\mu(x)\}$ by

$$\dot{V}_\mu(x, \tau) = -g_0^2 \left[\partial_{x,\mu} S(V(\tau)) \right] V_\mu(x, \tau), \quad V_\mu(x, 0) = U_\mu(x) .$$

- Diffusion process continuously minimizing action, scale $\lambda_s \simeq \sqrt{8t}$, $t = a^2 \tau$.
- Allows efficient scale-setting (t_0, t_1)
by demanding e.g. $t^2 \langle -\frac{1}{2} \text{tr } G_{\mu\nu} G_{\mu\nu} \rangle|_{t=t_0, t_1} = 0.3, \quad \frac{2}{3}$.
- Emergence of topological sectors at sufficient large length scale becomes clear.
- Renormalization becomes simple (in particular in the fermionic sector).

$\implies$ Easy to handle, theoretically sound prescription !!

How does it work? [Bonati, D'Elia, arXiv:1401.2441 [hep-lat]]

Wilson plaquette action:

$$S = \frac{2N_c}{g_0^2} \sum_{x, \mu < \nu} \left(1 - \frac{1}{2N_c} \text{tr} (U_{\mu\nu}(x) + U_{\mu\nu}^\dagger(x)) \right) ,$$

“Staples” = (in general non unitary) matrices $W_\mu(x)$, defined by

$$W_\mu(x) = \sum_{\nu \neq \mu} \left[U_\nu(x) U_\mu(x + \hat{\nu}) U_\nu^\dagger(x + \hat{\mu}) + U_\nu^\dagger(x - \hat{\nu}) U_\mu(x - \hat{\nu}) U_\mu(x - \hat{\nu} + \hat{\mu}) \right]$$

Part of action involving a given link variable $U_\mu(x)$:

$$-(2/g_0^2) \text{Re tr} [U_\mu(x) W_\mu^\dagger(x)] \equiv -(2/g_0^2) \text{Re tr} [\Omega_\mu(x)] = -(1/g_0^2) \text{tr} [\Omega_\mu(x) + \Omega_\mu^\dagger(x)]$$

Link derivatives (with Hermitean generators T^a):

$$\partial_{x,\mu}^{(a)} f(U_\mu(x)) = \left. \frac{d}{ds} f(e^{isT^a} U_\mu(x)) \right|_{s=0}$$

$$\partial_{x,\mu}^{(a)} S(U) = \frac{2}{g_0^2} \text{Im tr} [T^a \Omega_\mu]$$

$$g_0^2 \partial_{x,\mu} S(U) \equiv 2i \sum_a T^a \text{Im tr} [T^a \Omega_\mu] = \frac{1}{2} \left(\Omega_\mu - \Omega_\mu^\dagger \right) - \frac{1}{2N_c} \text{tr} \left(\Omega_\mu - \Omega_\mu^\dagger \right) .$$

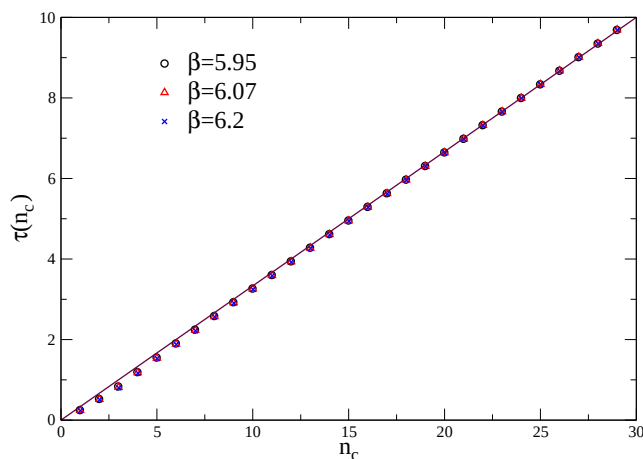
Comparison gradient flow with cooling:

[C. Bonati, M. D'Elia, 1401.2441]

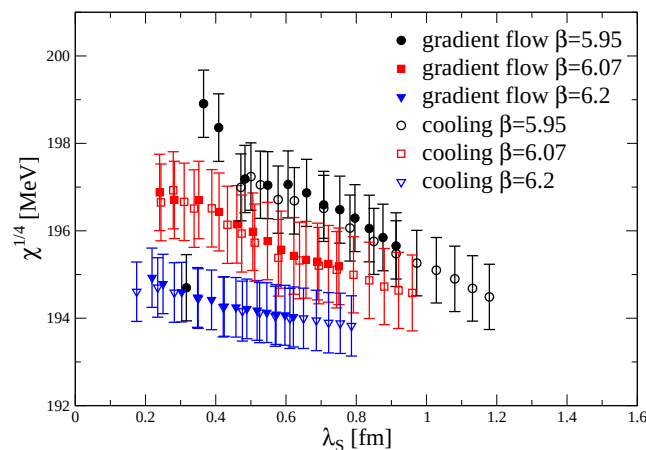
Pure gluodynamics:

- For given number of cooling sweeps n_c find gradient flow time τ yielding same Wilson plaquette action value.
- Perturbation theory: $\tau = n_c/3$

τ/n_c scaling



$\chi_t^{1/4}$ vs. λ_s



- Lattice spacing dependence at fixed λ_s clearly visible.
- Moreover: cooling and gradient flow show same spatial topological structure.

Holds also for $\rho_t(x)$ filtered with adjusted # ferm. (overlap) modes

[Solbrig et al., '07; Ilgenfritz et al., '08].

7. Conclusions, outlook

- Have discussed the basics of lattice gauge theories.
- Main aim was the introduction of lattice QCD.
- As main features were discussed:
 - the path integral method, in particular for fermion matter fields;
 - Wilson's representation of gauge fields on the lattice, such that gauge invariance is maintained;
 - the continuum limit on the basis of the renormalization group and 'asymptotic freedom'.
- Basic Monte Carlo methods for the pure gauge case were introduced.
- The basic approach to compute hadron correlation functions was presented.
- QCD at non-zero temperature was introduced.
- Methods to reveal the topological structure of QCD were presented.