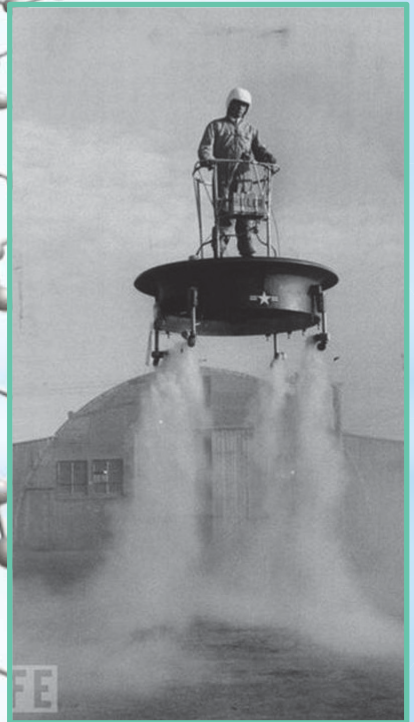
Anomalous transport

on the lattice



Pavel Buividovich

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Unterstützt von / Supported by

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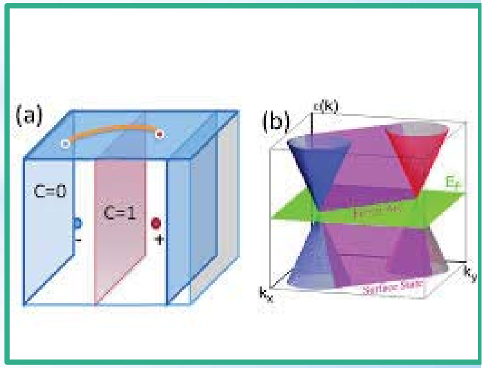
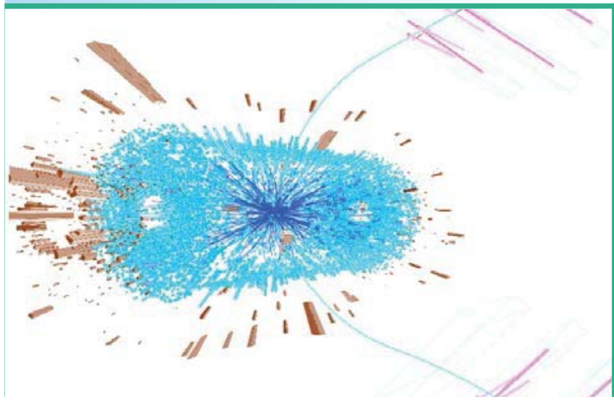
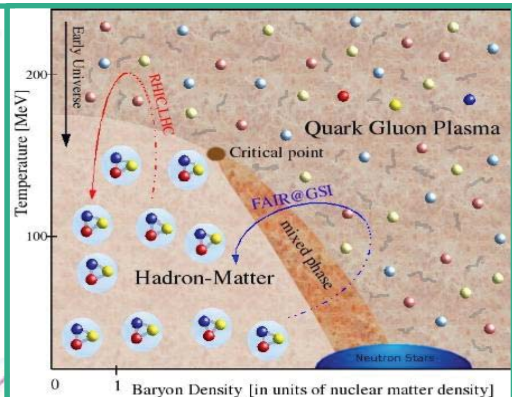
Why anomalous transport?

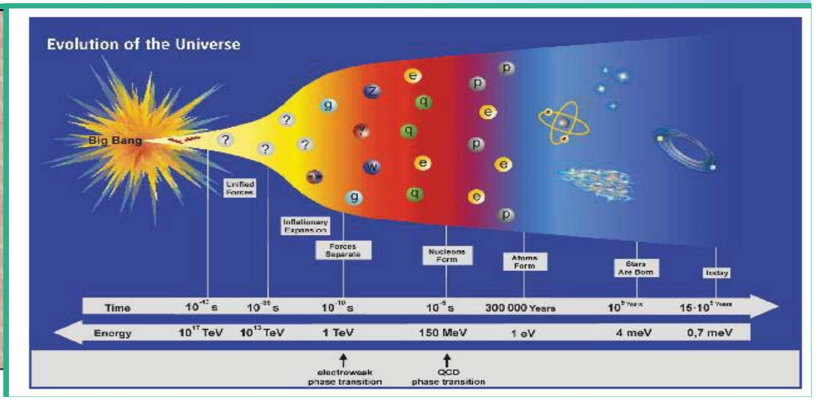
Collective motion of chiral fermions

- **High-energy physics:**
  - ✓ Quark-gluon plasma
  - ✓ Hadronic matter
  - ✓ Leptons/neutrinos in Early Universe
- **Condensed matter physics:**
  - ✓ Weyl semimetals
  - ✓ Topological insulators
  - ✓ Liquid Helium [G. Volovik]









Hydrodynamic approach

Classical conservation laws for chiral fermions

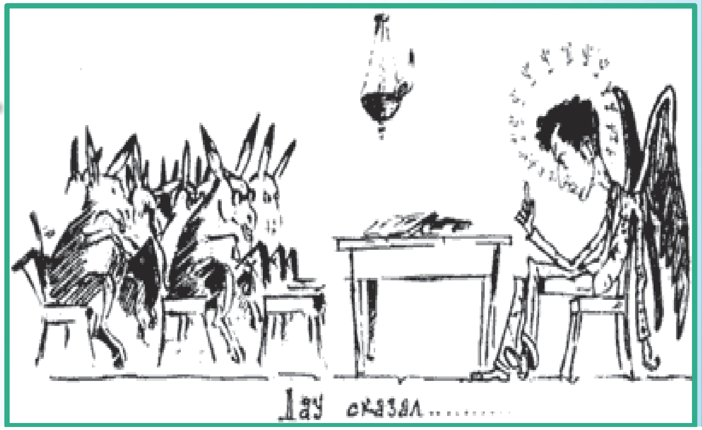
- Energy and momentum
- Angular momentum
- Electric charge ⟷ No. of left-handed
- Axial charge ⟷ No. of right-handed

Hydrodynamics:

- Conservation laws
- Constitutive relations

Axial charge violates parity

New parity-violating transport coefficients



Hydrodynamic approach

Let's try to incorporate

Quantum Anomaly into Classical Hydrodynamics

$$\frac{d}{dt} Q_A = \frac{e^2}{2\pi^2} \int d^3x \, \vec{E} \cdot \vec{B}$$

$$\partial_\mu j_\mu^A = \frac{e^2}{8\pi^2} F_{\mu\nu} \tilde{F}_{\mu\nu}$$

Now require positivity of entropy production…

BUT: anomaly term

can lead to any sign of dS/dt!!!

- Strong constraints on parity-violating transport coefficients [Son, Surowka ' 2009]
- Non-dissipativity of anomalous transport [Banerjee,Jensen,Landsteiner'2012]

Anomalous transport: CME, CSE, CVE

Chiral Magnetic Effect
[Kharzeev, Warringa, Fukushima]

$$j_V^i = \sigma_{VV}^{\mathcal{B}} B^i = \frac{N_c e\, \mu_A}{2\pi^2} B^i$$

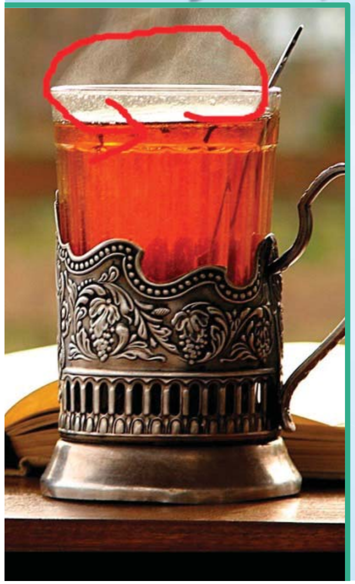
Chiral Separation Effect
[Son, Zhitnitsky]

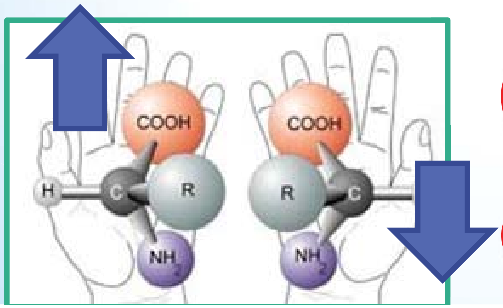
$$j_A^i = \sigma_{AV}^{\mathcal{B}} B^i = \frac{N_c e\, \mu_V}{2\pi^2} B^i$$

Chiral Vortical Effect
[Erdmenger *et al.*, Teryaev, Banerjee *et al.*]

$$j_V = \sigma_V^{\mathcal{V}} w = \frac{N_c e}{2\pi^2} \mu_A \mu_V w$$

$$j_A = \sigma_A^{\mathcal{V}} w = N_c e \left(\frac{\mu_V^2 + \mu_A^2}{4\pi^2} + \frac{T^2}{12} \right) w$$



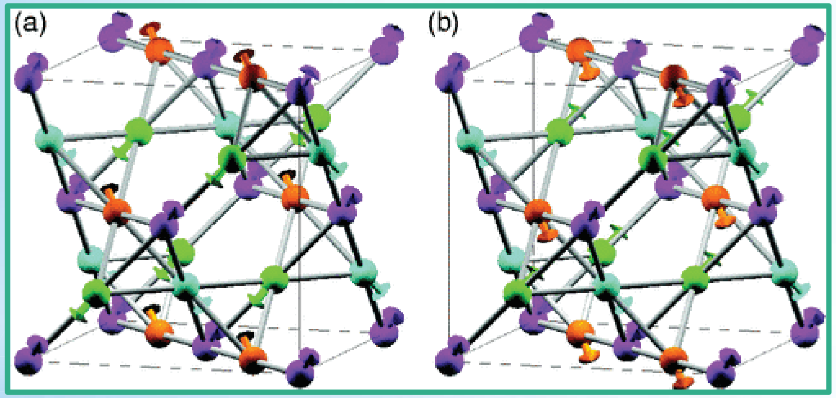


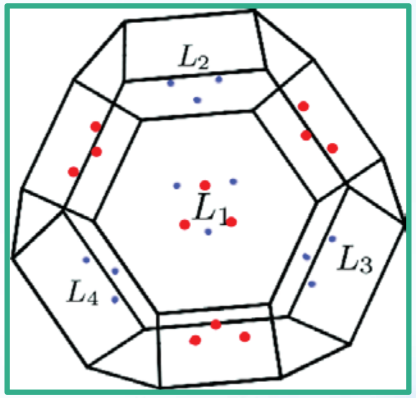
Flow vorticity

Origin in quantum anomaly!!!

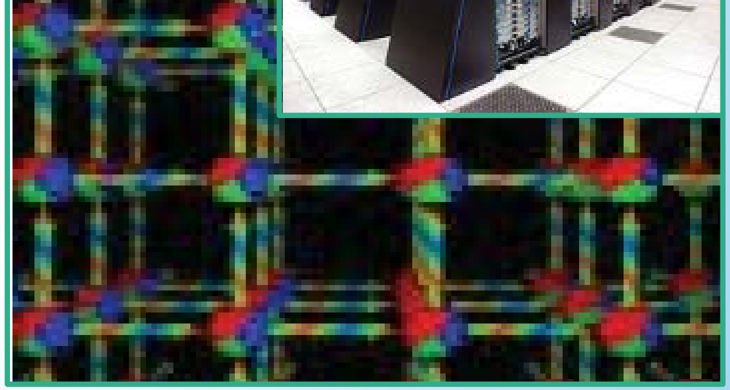
Why anomalous transport on the lattice?

1) Weyl semimetals/Top.insulators are crystals





2) Lattice is the only practical non-perturbative regularization of gauge theories

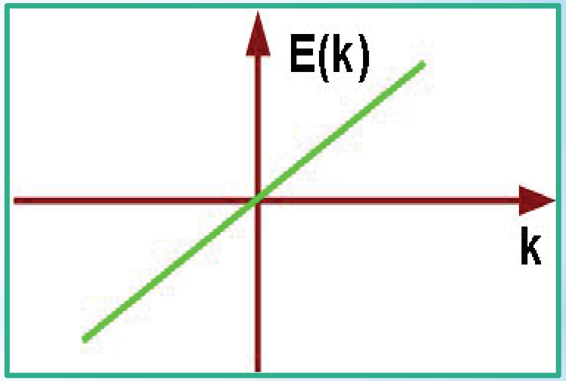


First, let's consider axial anomaly on the lattice

Warm-up: Dirac fermions in D=1+1

- Dimension of Weyl representation: 1
- Dimension of Dirac representation: 2
- Just one “Pauli matrix” = 1

Weyl Hamiltonian in D=1+1



$$\hat{H} = -i \int dx\, \hat{\psi}^{\dagger} (\partial_x - iA_x) \hat{\psi}$$

Three Dirac matrices:

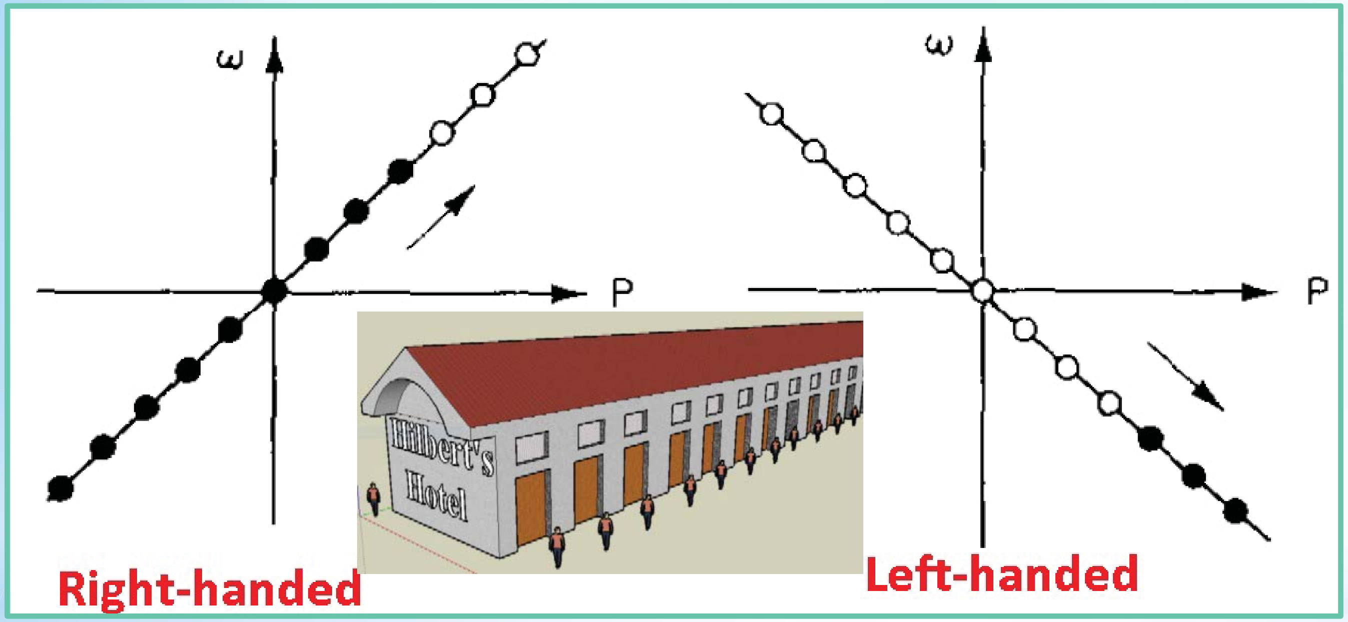
$$\gamma_0 \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_1 \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \gamma_5 \equiv \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\alpha_1 = \sigma_1 \sigma_2 = i\sigma_3$$

Dirac Hamiltonian:

$$\hat{H} = -i \int dx\, \hat{\psi}^{\dagger} \sigma_3 \partial_x \hat{\psi}$$

Warm-up: anomaly in D=1+1



$$\gamma_0 \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma_1 \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\gamma_5 \equiv \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\alpha_1 = \sigma_1 \sigma_2 = i\sigma_3$$

$$\dot{p} = E, \quad w = \pm p \quad \Rightarrow \dot{w} = \pm \frac{e}{2\pi} E$$

$$\frac{d}{dt} n_R = \frac{e}{2\pi} E, \qquad \frac{d}{dt} n_L = -\frac{e}{2\pi} E$$

$$\frac{d}{dt} q_A = \frac{d}{dt}\left(n_R - n_L\right) = \frac{e}{\pi} E$$

$$\partial_\mu j^A_\mu = \frac{e}{2\pi} \varepsilon_{\mu\nu} F_{\mu\nu}$$

Axial anomaly on the lattice

Axial anomaly =

= non-conservation of Weyl fermion number

BUT: number of states is fixed on the lattice???

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THE ADLER–BELL–JACKIW ANOMALY AND WEYL FERMIONS IN A CRYSTAL

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Volume 105B, number 2,3 PHYSICS LETTERS 1 October 1981

A NO-GO THEOREM FOR REGULARIZING CHIRAL FERMIONS

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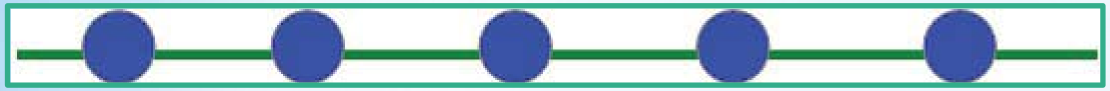
Anomaly on the (1+1)D lattice

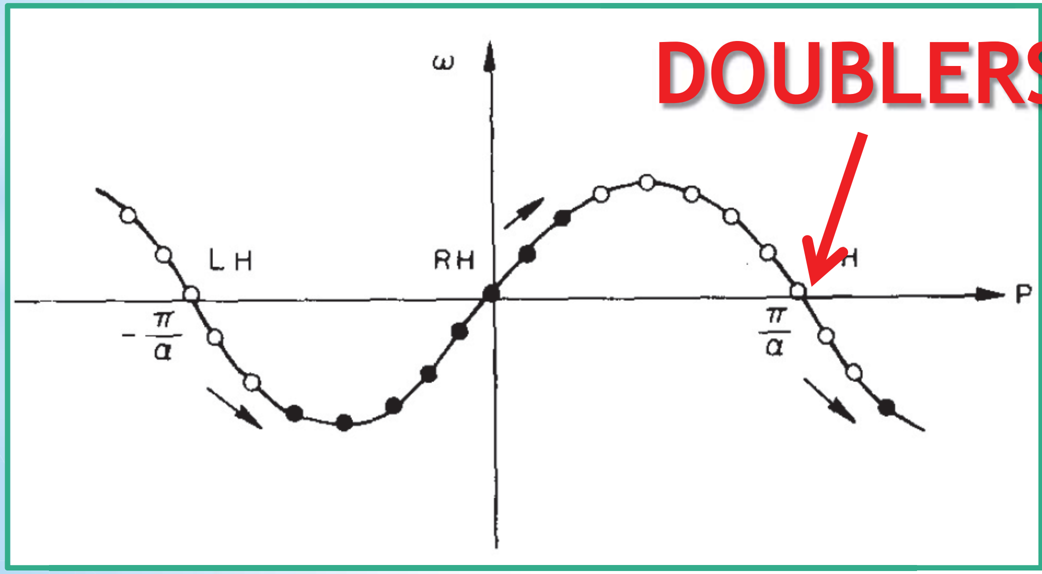


$$\left\{ \hat{\psi}_x^\dagger, \hat{\psi}_y \right\} = \delta_{xy}$$

$$\hat{H} = -i\kappa/2 \sum_x \hat{\psi}_x^\dagger \left(\hat{\psi}_{x+1} - \hat{\psi}_{x-1} \right)$$

$$w\left(k\right) = \kappa \sin\left(k\right), \quad k \in \left[-\pi, \pi\right]$$

1D minimally doubled fermions



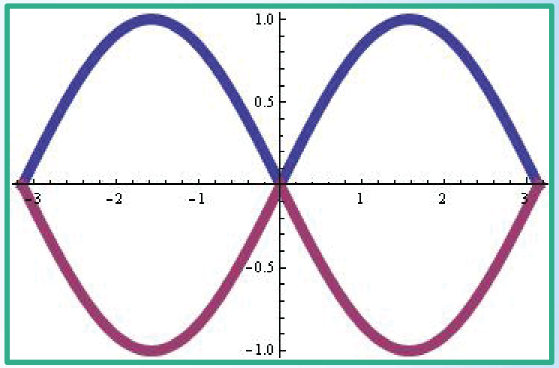
- Even number of Weyl points in the BZ
- Sum of “chiralities” = 0

1D version of Fermion Doubling

Anomaly on the (1+1)D lattice

Let's try "real" two-component fermions

$$\hat{H} = -i\kappa/2 \sum_x \hat{\psi}_x \sigma_3 \left(\hat{\psi}_{x+1} - \hat{\psi}_{x-1} \right)$$

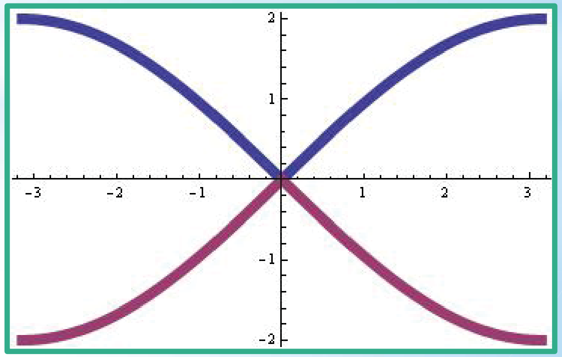


Two chiral "Dirac" fermions

Anomaly cancels between doublers

Try to remove the doublers by additional terms

$$\hat{H} = -i\kappa/2 \sum_x \hat{\psi}_x \sigma_3 \left(\hat{\psi}_{x+1} - \hat{\psi}_{x-1} \right) + \\ + \rho/2 \sum_x \hat{\psi}^\dagger_x \sigma_1 \left(2\hat{\psi}_x - \hat{\psi}_{x+1} - \hat{\psi}_{x-1} \right)$$

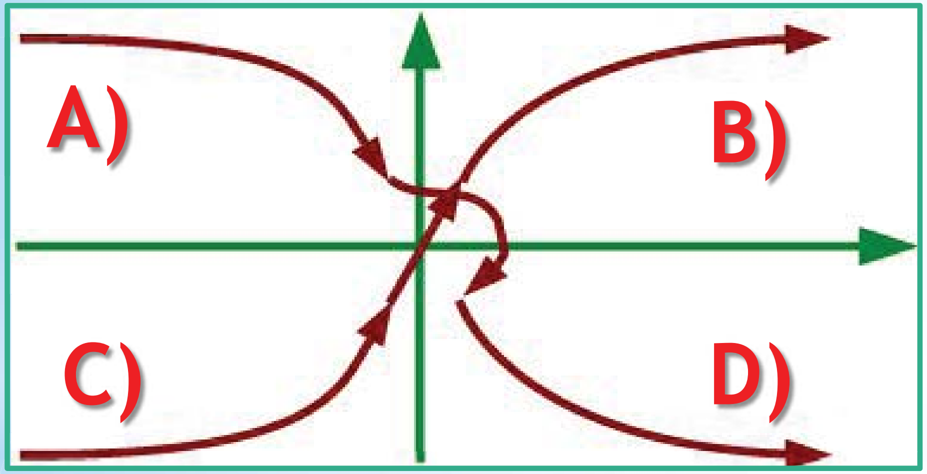


$$w(k) = \pm\sqrt{\kappa^2 \sin^2(k) + 4\rho^2 \sin^4(k/2)}$$

$$H = \begin{pmatrix} \sin(k) & \Delta(k) \\ \Delta(k) & -\sin(k) \end{pmatrix}$$

Anomaly on the (1+1)D lattice

(1+1)D Wilson fermions



In A) and B):

$$\Psi = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

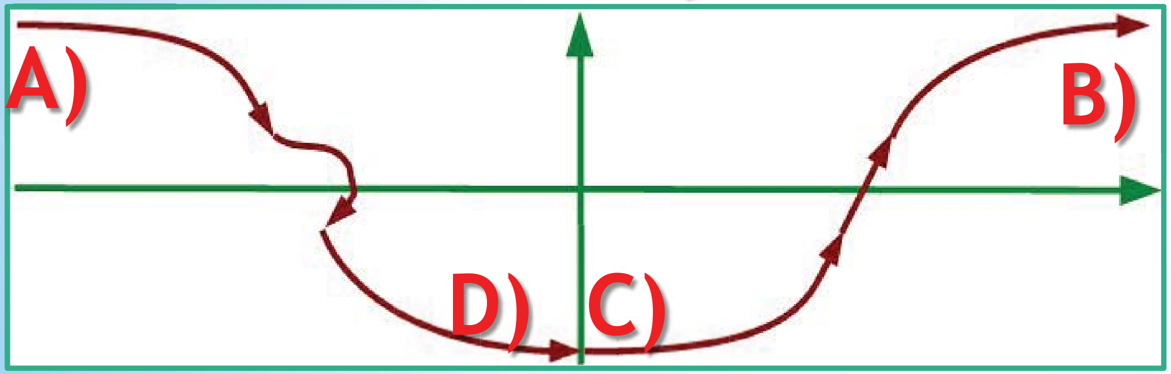
In C) and D):

$$\Psi = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

Maximal mixing of chirality at BZ boundaries!!!

Now anomaly comes from the Wilson term

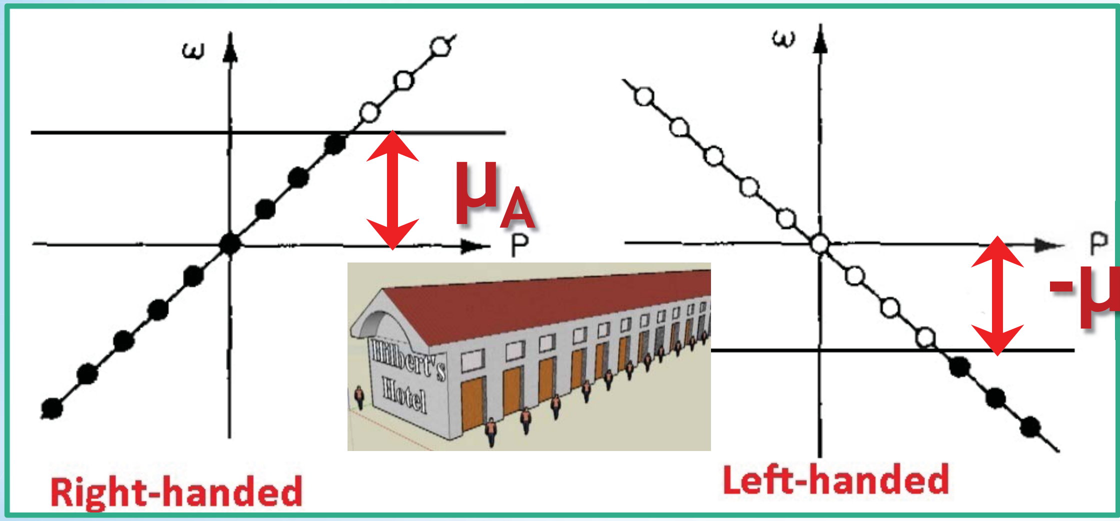
+ All kinds of nasty renormalizations...

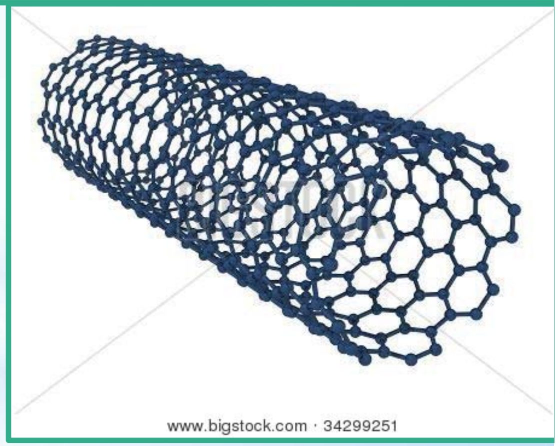


$$\Psi_s = \begin{pmatrix} \sqrt{\frac{\epsilon_s(k)+\sin(k)}{2\epsilon_s(k)}} \\ s\sqrt{\frac{\epsilon_s(k)-\sin(k)}{2\epsilon_s(k)}} \end{pmatrix}$$

$$\epsilon_s(k) = s\sqrt{\sin^2(k) + \Delta^2(k)}$$

Now, finally, transport: "CME" in D=1+1





- Excess of right-moving particles
- Excess of left-moving anti-particles

Directed current

Not surprising - we've broken parity

Effect relevant for nanotubes

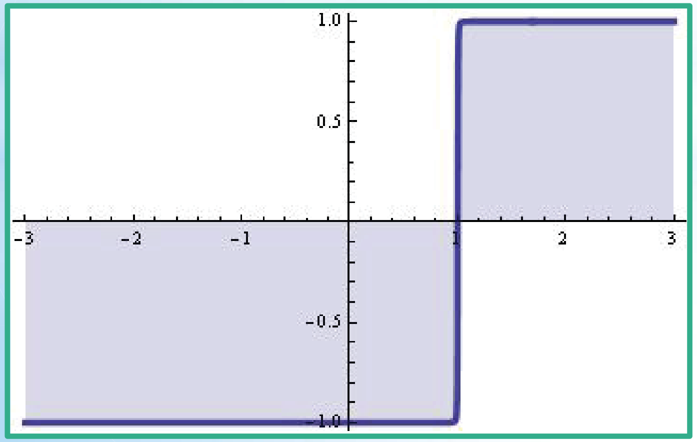
"CME" in D=1+1

$$j_x = -\frac{\partial \mathcal{F}}{\partial A_x} = -T\frac{\partial \log \mathcal{Z}}{\partial A_x}$$

$$\mathcal{F} = \sum_i \ln\left(1 + e^{-\beta\epsilon_i}\right) \rightarrow \sum_{\epsilon_i<0} \epsilon_i$$

$$j_x = -\int \frac{dk_x}{2\pi} \frac{\partial}{\partial A_x} |k_x - A_x - \mu_A| = \\ = \int \frac{dk_x}{2\pi} \,\mathrm{sign}\,(k_x - \mu_A)$$

Fixed cutoff regularization: $j_x = \mu_A/\pi$

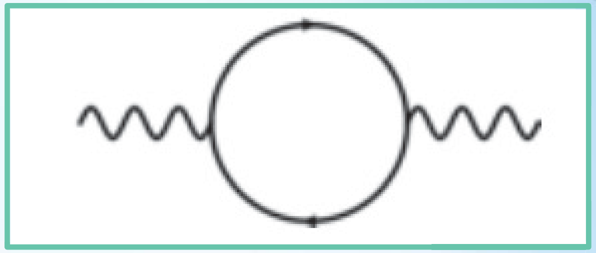


Shift of integration variable: ZERO

UV regularization ambiguity

Dimensional reduction: 2D axial anomaly

Polarization tensor in 2D:



$$j_\mu = \epsilon_{\mu\sigma}\Pi_{\sigma\nu}A_\nu, \qquad A_0 \to i\mu_5$$

Proper regularization (vector current conserved):

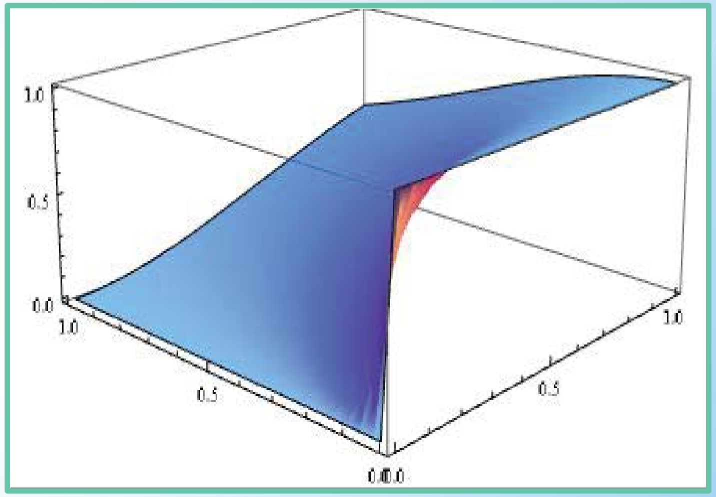
$$\Pi_{\mu\nu} = \frac{1}{\pi}\frac{k^2\delta_{\mu\nu} - k_\mu k_\nu}{k^2}$$

[Chen, hep-th/9902199]

Final answer:

$$j_3(k) = i\Pi_{33}(k)\,\mu_5(k) = \frac{1}{2\pi^2}\,\frac{k_0^2}{k_0^2+k_3^2}\,\mu_5(k)$$

- Value at $k_0=0$, $k_3=0$: <u>NOT DEFINED</u> (without IR regulator)
- First $k_3 \to 0$, then $k_0 \to 0$
- Otherwise zero



Chirality n_5 vs μ_5

μ_5 is not a physical quantity, just Lagrange multiplier

Chirality n_5 is (in principle) observable

Express everything in terms of n_5

To linear order in μ_5 :

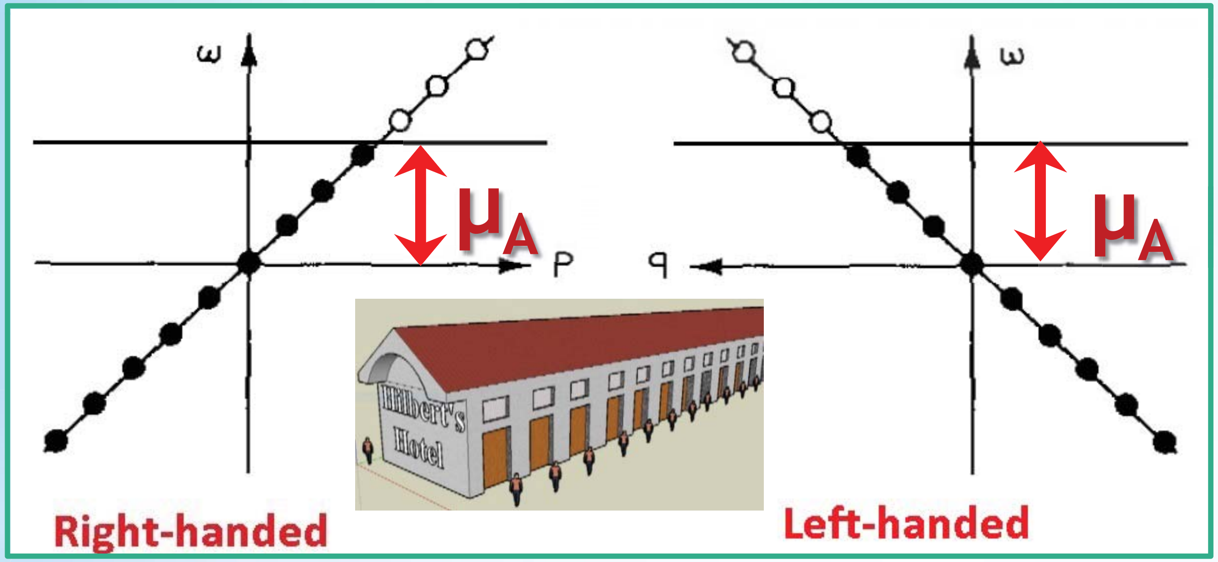
$$\boxed{n_5 = \epsilon_{0\alpha}\Pi_{\alpha\beta}\epsilon_{\beta 0}\mu_5 = \Pi_{33}\mu_5}$$

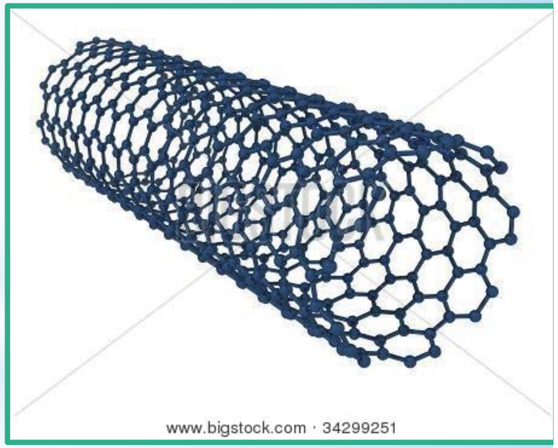
Singularities of Π_{33} cancel !!!

$$\boxed{j_3 = n_5 B}$$

Note: no non-renormalization for two loops or higher and no dimensional reduction due to 4D gluons!!!

"CSE" in D=1+1





- Excess of right-moving particles
- Excess of left-moving particles

Directed axial current, separation of chirality

Effect relevant for nanotubes

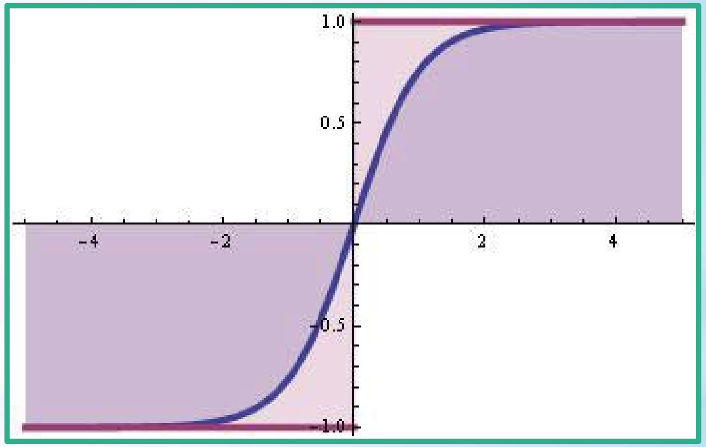
"AME" or "CVE" for D=1+1

Single (1+1)D Weyl fermion at finite temperature T

Energy flux = momentum density

$$\boxed{T_{01} = T_{10} = -i\hat{\psi}^{\dagger}\partial_x\hat{\psi}}$$

$$\boxed{\langle T_{01} \rangle = T \int \frac{dk_x}{2\pi}\, k_x \sum_w \frac{1}{iw - k}}$$



$$\langle T_{01} \rangle = -\int \frac{dk_x}{2\pi}\, k_x\, \frac{1}{2} \tanh\left(\frac{k}{2T}\right) =$$

$$= -\frac{T^2}{4\pi} \int_{-\infty}^{+\infty} dz\, z\, \tanh\left(\frac{z}{2}\right) =$$

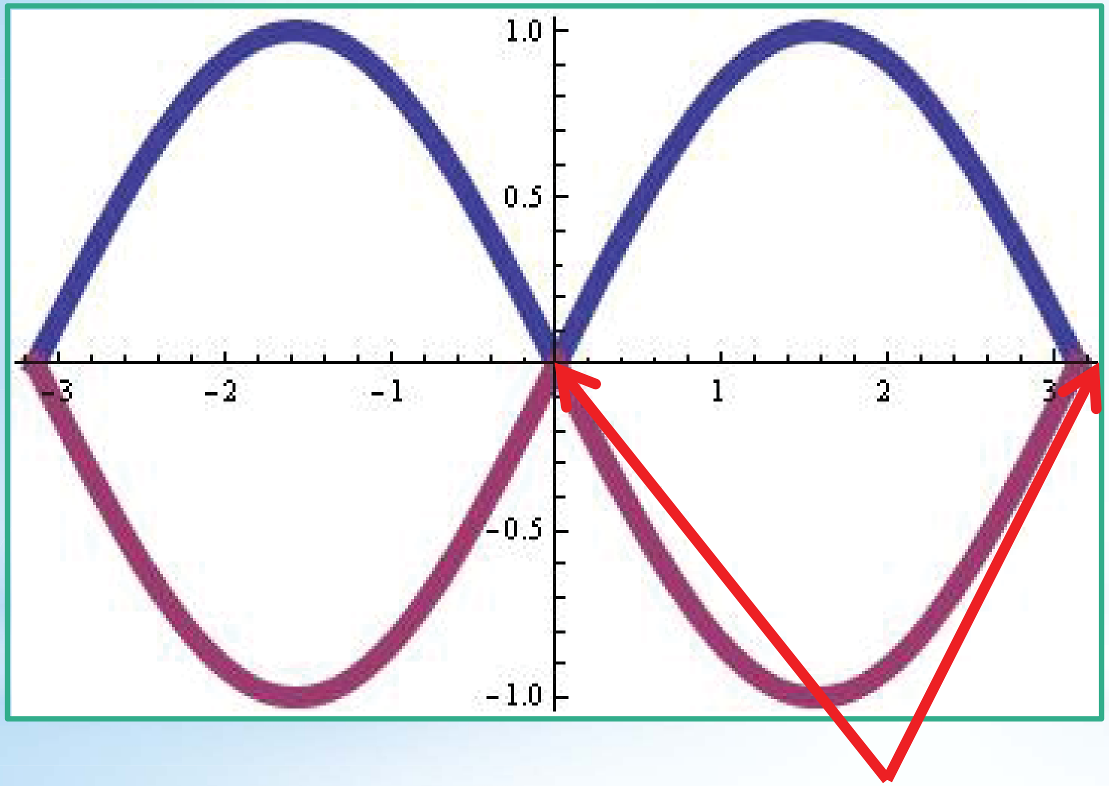
$$= -\frac{T^2}{4\pi} \int_{-\infty}^{+\infty} dz\, z\, \left(\tanh\left(\frac{z}{2}\right) - \mathrm{sign}\,(z) + \mathrm{sign}\,(z)\right) =$$

$$= \frac{T^2}{2\pi} \int_{0}^{+\infty} \frac{dz\, z}{e^z + 1} = \frac{\pi T^2}{12}$$

+ UV divergence

(1+1)D Weyl fermions, thermally excited states: constant energy flux/momentum density

What happens on the lattice (naively)?



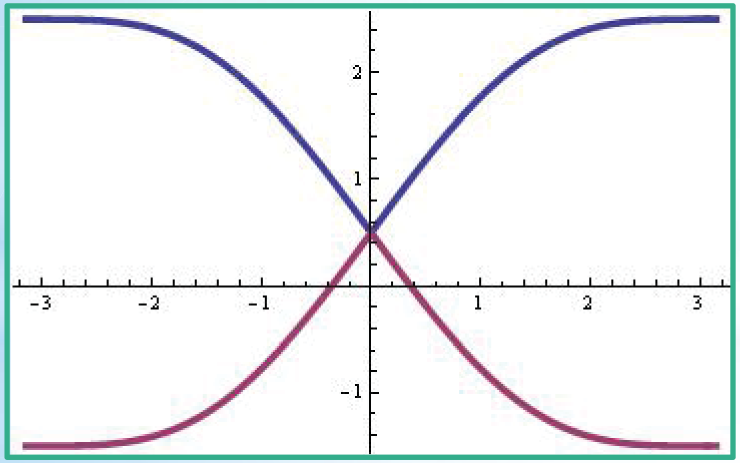
All anomalous effects cancel between doublers!!!

We need to eliminate them somehow…

"CSE" on the D=1+1 lattice

Again, (1+1)D Wilson fermions

$$H = \begin{pmatrix} \sin(k) + \mu & \Delta(k) \\ \Delta(k) & -\sin(k) + \mu \end{pmatrix}$$



$$\hat{j}_x^A = \hat{\psi}^\dagger \sigma_3 \sigma_3 \hat{\psi} = \hat{\psi}^\dagger \hat{\psi}$$

$$j_\mu^A = \bar{\psi} \sigma_\mu \sigma_3 \psi = \epsilon_{\mu\nu} \bar{\psi} \sigma_\nu \psi = \epsilon_{\mu\nu} j_\nu$$

2D magic: Axial current=charge density (for Wilson-Dirac,no unique j_A)

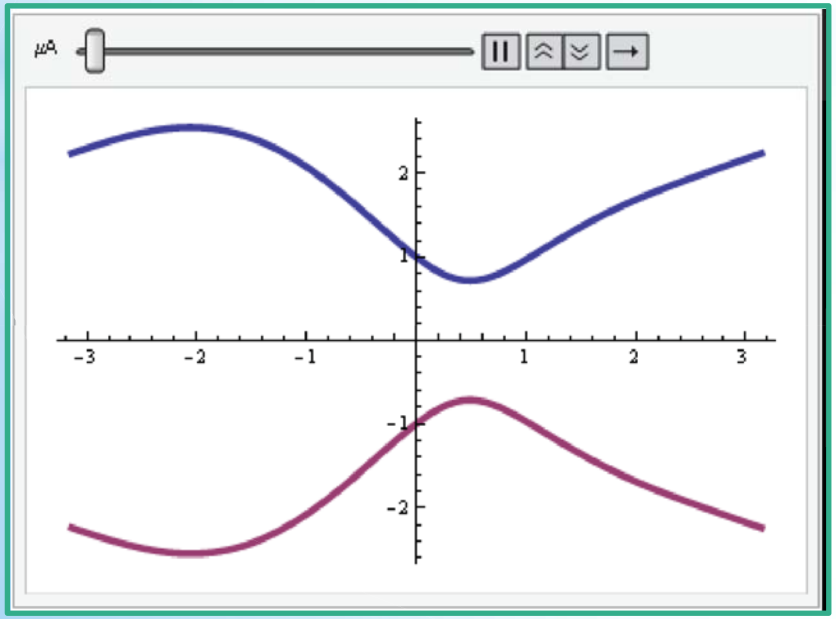
Chemical potential → axial current

"CME" on the D=1+1 lattice

Again, (1+1)D Wilson fermions

$$H = \begin{pmatrix} \sin(k) - \mu_A & \Delta(k) \\ \Delta(k) & -\sin(k) + \mu_A \end{pmatrix}$$

$$\epsilon_s(k) = s\sqrt{(\sin(k) - \mu_A)^2 + \Delta^2(k)}$$



$$j_x = -\frac{\partial \mathcal{F}}{\partial A_x} = -T\frac{\partial \log \mathcal{Z}}{\partial A_x}$$

$$\mathcal{F} = \sum_i \ln\left(1 + e^{-\beta\epsilon_i}\right) \to \sum_{\epsilon_i < 0} \epsilon_i$$

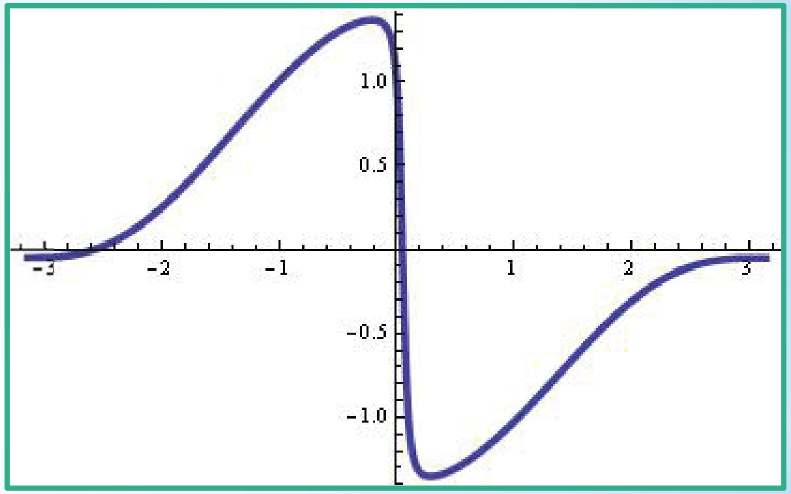
Constant A_x = shift of k_x

$$j_x = \int \frac{dk_x}{2\pi} \frac{\partial \epsilon_{-1}(k_x)}{\partial k_x} = 0$$

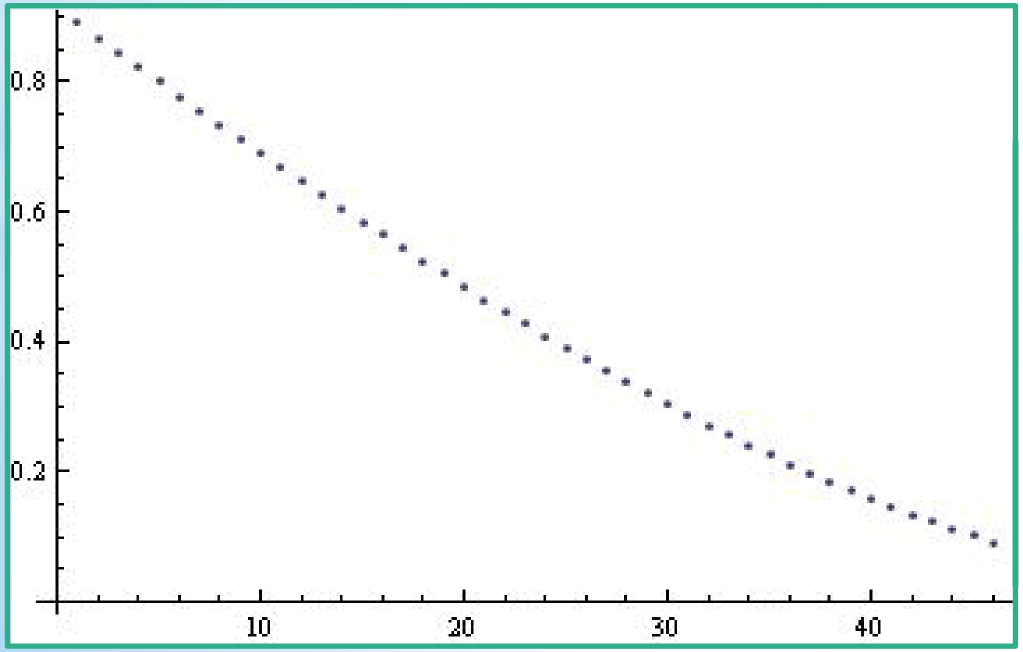
"CME" on the D=1+1 lattice: IR effects

Constant current on a finite lattice:

$$j_x = \sum_{k_x} \frac{\partial \epsilon_{-1}(k_x)}{\partial k_x}$$



CME current vs lattice size



Mixing of IR and UV singularities

Typical for anomaly

Going to higher dimensions: Landau levels for Weyl fermions

$$H = -i\vec{\sigma}\vec{\nabla} =$$
$$= -i\begin{pmatrix} \partial_z & \partial_x - iA_x - i\left(\partial_y - iA_y\right) \\ \partial_x - iA_x + i\left(\partial_y - iA_y\right) & \partial_z \end{pmatrix}$$

$$A_x = By/2, A_y = -Bx/2$$

$$\left[\hat{a}_x, \hat{a}_x^\dagger\right] = 1, \quad \left[\hat{a}_y, \hat{a}_y^\dagger\right] = 1$$

$$H = \begin{pmatrix} \partial_z & (\partial_x + Bx/2) - i\left(\partial_y + By/2\right) \\ (\partial_x - Bx/2) + i\left(\partial_y - By/2\right) & -\partial z \end{pmatrix}$$

$$\hat{a}_x := \frac{1}{\sqrt{B}}\partial_x + \sqrt{B}\,x/2, \quad \hat{a}_y := \frac{1}{\sqrt{B}}\,\partial_y + \sqrt{B}y/2,$$

Going to higher dimensions: Landau levels for Weyl fermions

$$H = \begin{pmatrix} k_z & -\sqrt{B}\,(a_y - i a_x) \\ -\sqrt{B}\,(\hat{a}_y^\dagger + i\hat{a}_x^\dagger) & -k_z \end{pmatrix} =$$

$$= \begin{pmatrix} k_z & \sqrt{2B}\,\hat{A} \\ \sqrt{2B}\,\hat{A}^\dagger & -k_z \end{pmatrix} \qquad \left[\hat{A}, \hat{A}^\dagger\right] = 1$$

$$\Psi_n = \begin{pmatrix} \alpha\,|n-1\rangle \\ \beta\,|n\rangle \end{pmatrix}, \quad n \geq 1 \qquad \Psi_0 = \begin{pmatrix} 0 \\ |0\rangle \end{pmatrix}$$

$$\epsilon_n = \pm\sqrt{k_z^2 + 2|B|n} \qquad \epsilon_0 = -k_z$$

Finite volume:

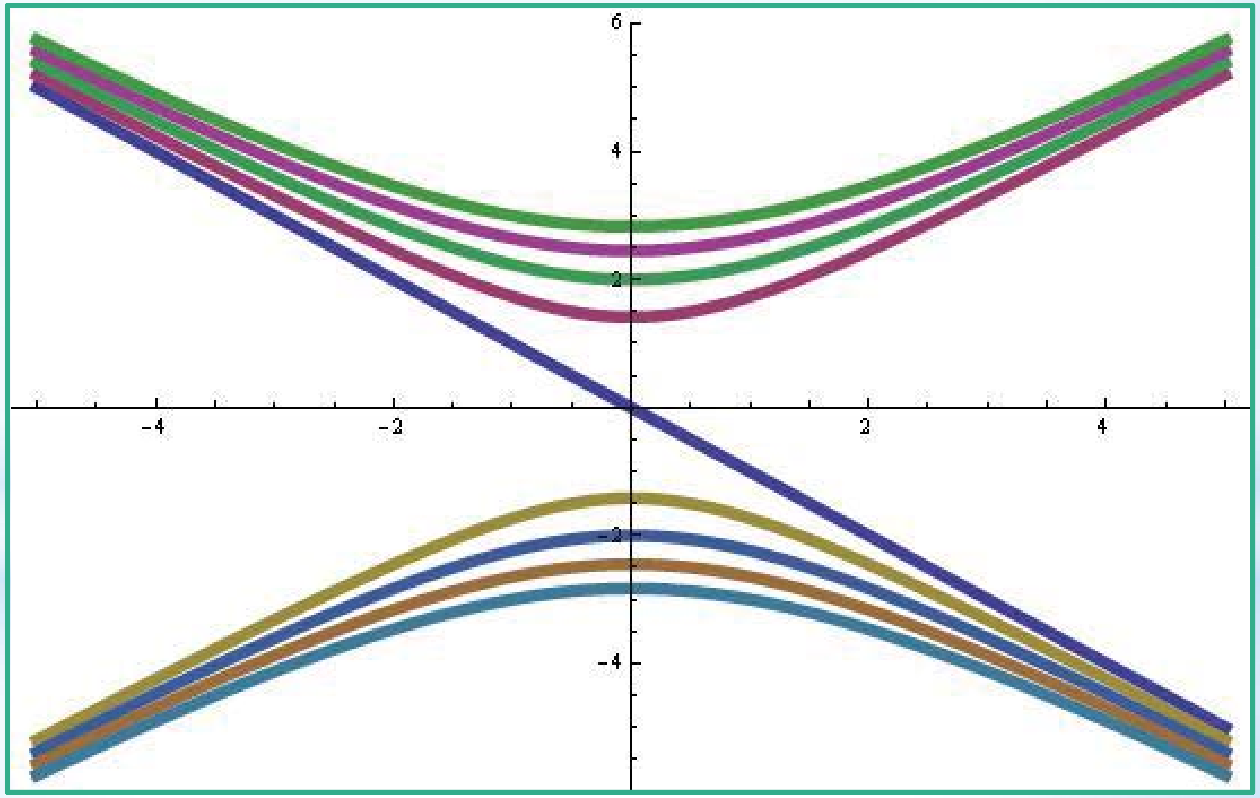
Degeneracy of every level = magnetic flux $\Phi = \frac{BL^2}{2\pi}$

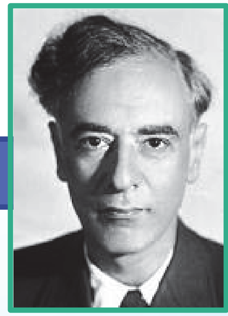
Additional operators [Wiese,Al-Hasimi, 0807.0630]

$$\hat{B} = \hat{a}_y + i\hat{a}_x,\ \hat{B}^\dagger = \hat{a}_y^\dagger - i\hat{a}_x^\dagger \qquad \left[\hat{B}, \hat{B}^\dagger\right] = 1$$

LLL, the Lowest Landau Level

Lowest Landau level = 1D Weyl fermion





Anomaly in (3+1)D from (1+1)D

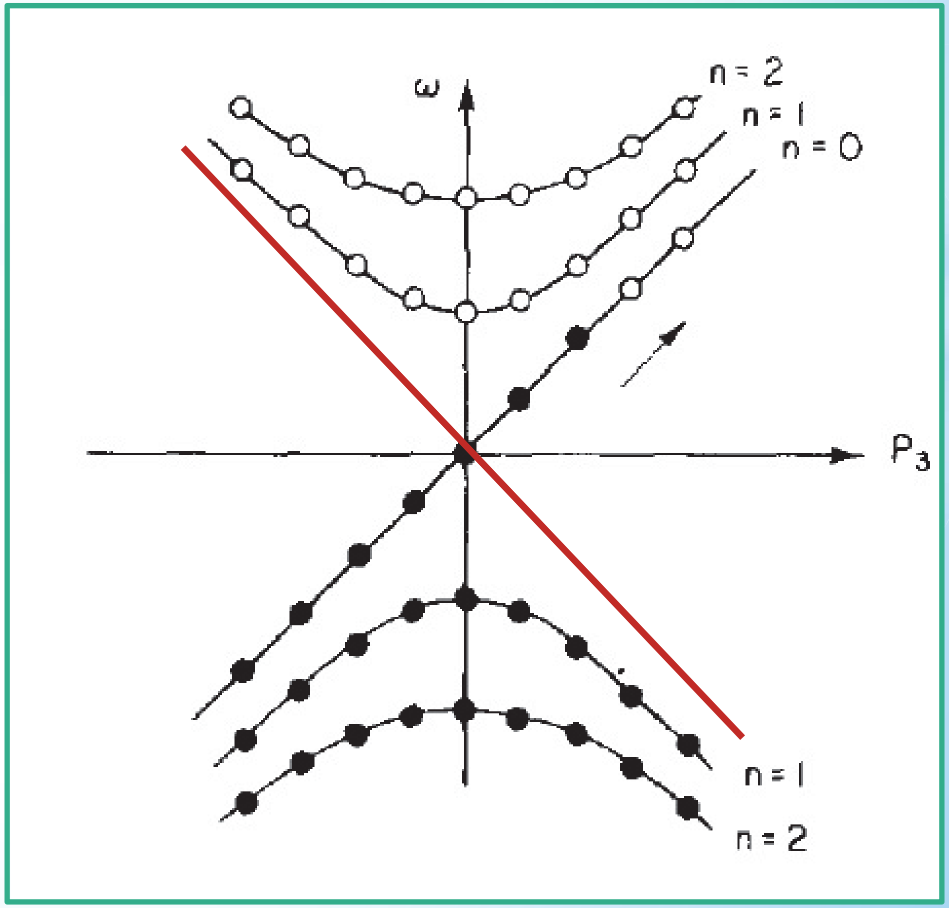
Parallel uniform electric and magnetic fields

The anomaly comes only from LLL

$$\frac{d}{dt}Q_A = (1D\,result) \times \frac{eBL^2}{2\pi} = \frac{e^2}{2\pi^2}EB$$

$$\partial_\mu j^A_\mu = \frac{e^2}{8\pi^2}F_{\mu\nu}\tilde{F}_{\mu\nu}$$

Higher Landau Levels do not contribute

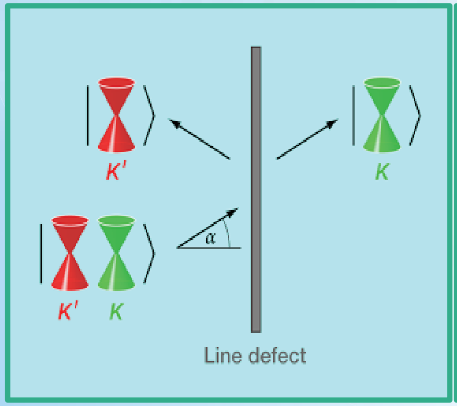


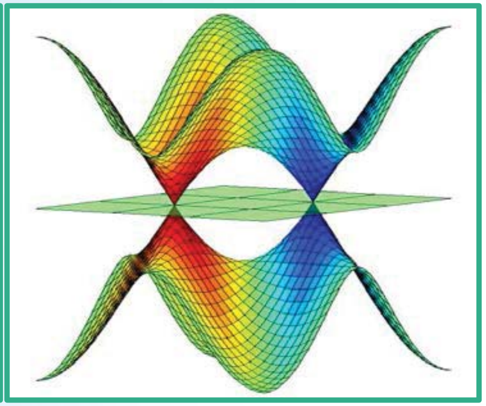
Anomaly on (3+1)D lattice

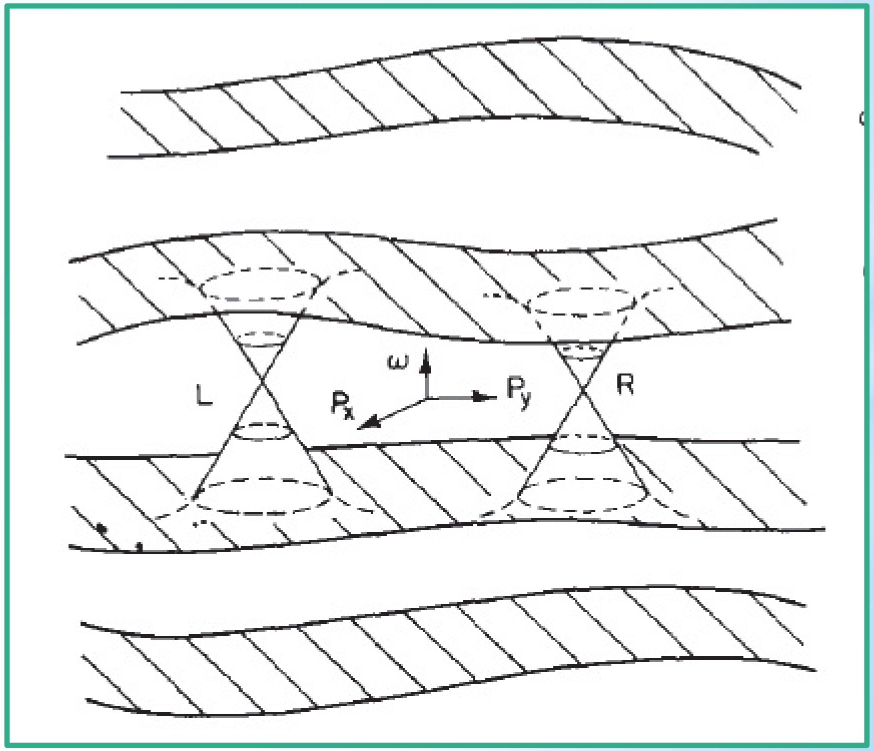
Nielsen-Ninomiya picture:

- Minimally doubled fermions [e.g. Borici-Creutz]
- Two Dirac cones in the Brillouin zone
- For Wilson-Dirac, anomaly again stems from Wilson terms

VALLEYTRONICS







Anomalous transport in (3+1)D from (1+1)D

CME, Dirac fermions - ??? On the lattice

$$\boxed{j_x = \mu_A/\pi} \Rightarrow \boxed{\vec{j} = \frac{\mu_A}{2\pi^2}\vec{B}}$$

CSE, Dirac fermions - OK on the lattice

$$\boxed{j_x^A = \frac{\mu}{\pi}} \Rightarrow \boxed{\vec{j}_A = \frac{\mu}{2\pi^2}\vec{B}}$$

“AME”, Weyl fermions - ??? On the lattice

$$\boxed{\langle T_{01} \rangle = \frac{\pi T^2}{12}} \Rightarrow \boxed{\langle T_{0i} \rangle = \frac{T^2}{24}B_i}$$

Nielsen-Ninomiya and Dirac/Weyl semimetals

5. We assume that we have found a parity non-invariant zero-gap semiconductors which can be simulated by a Weyl fermion theory with a dispersion law $\epsilon^2 = v^2 P^2$. The effect analogous to the ABJ anomaly gives rise to a peculiar behavior of the conductivity of the electric current in the presence of the magnetic field. It is enough to consider one conduction band

Enhancement of electric conductivity along magnetic field

Nielsen-Ninomiya and Dirac/Weyl semimetals

PRL **105,** 132001 (2010) PHYSICAL REVIEW LETTERS week ending 24 SEPTEMBER 2010

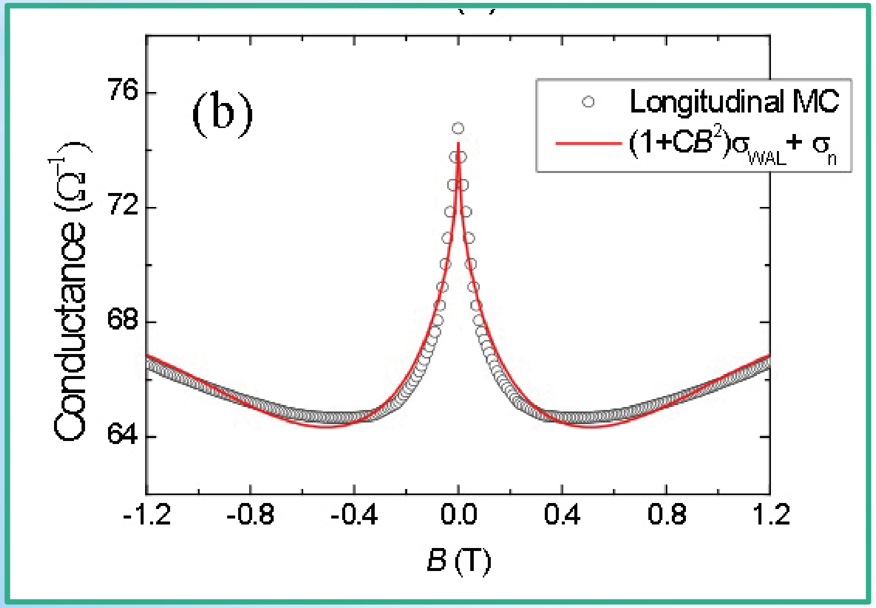
Magnetic-Field-Induced Insulator-Conductor Transition in *SU*(2) Quenched Lattice Gauge Theory

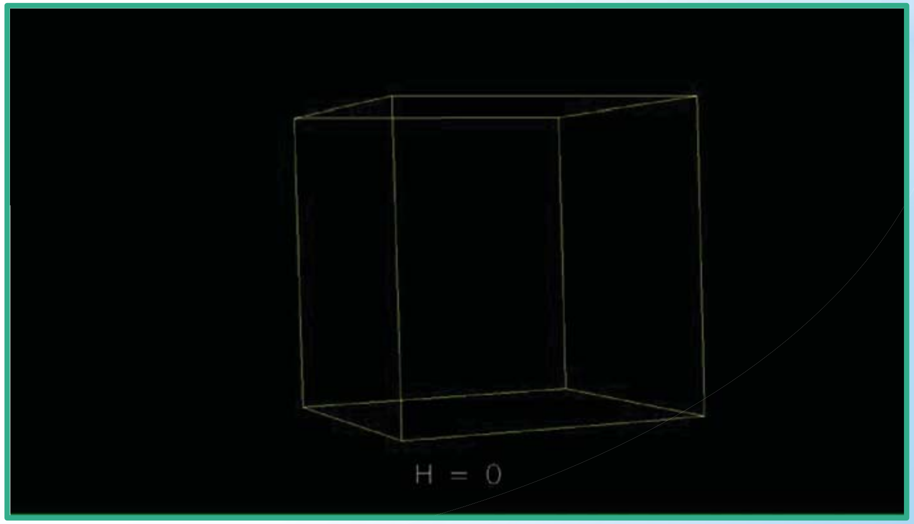
P. V. Buividovich,[1,2] M. N. Chernodub,[3,4,*] D. E. Kharzeev,[5,6] T. Kalaydzhyan,[7,1] E. V. Luschevskaya,[1,2] and M. I. Polikarpov[1]

PRL **111,** 246603 (2013) PHYSICAL REVIEW LETTERS week ending 13 DECEMBER 2013

Dirac versus Weyl Fermions in Topological Insulators: Adler-Bell-Jackiw Anomaly in Transport Phenomena

Heon-Jung Kim,[1,*] Ki-Seok Kim,[2,3,†] J.-F. Wang,[4] M. Sasaki,[5] N. Satoh,[6] A. Ohnishi,[5] M. Kitaura,[5] M. Yang,[4] and L. Li[4]





Topological stability of Weyl points

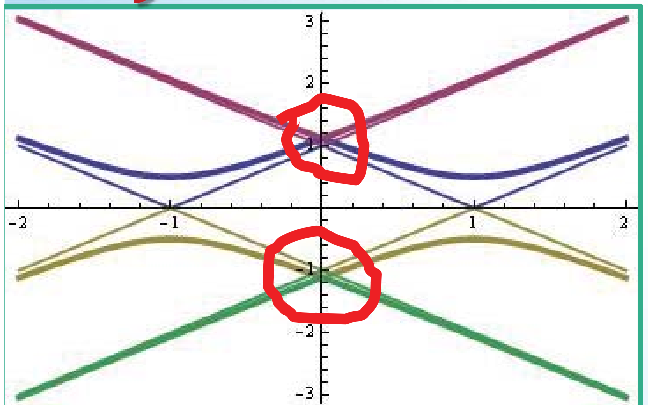
Weyl Hamiltonian in momentum space:

$$\boxed{H = k_i \sigma_i + \mu}$$

Full set of operators for 2x2 hamiltonian
Any perturbation = just shift of the Weyl point

Weyl point are topologically stable
Only "annihilate" with Weyl point of another chirality

E.g. ChSB by mass term:



$$\boxed{\epsilon_{s,\sigma}\left(\vec{k}\right) = s\sqrt{\left(|\vec{k}| - \sigma\mu_A\right)^2 + m^2}}$$

Weyl points as monopoles in momentum space

Free Weyl Hamiltonian: $H = \vec{\sigma} \cdot \vec{p}$

$$\langle f | e^{iHt} | i \rangle = \langle f | \int \mathcal{D}x(\tau)\mathcal{D}p(\tau)\mathcal{P}\exp\left(i\int_0^t d\tau\left(\vec{x}\cdot\dot{\vec{p}} + \vec{\sigma}\cdot\vec{p}\right)\right)|i\rangle =$$

$$= \langle f | \int \mathcal{D}x(\tau)\mathcal{D}p(\tau)\mathcal{P}\exp\left(i\int_0^t d\tau\left(\vec{\sigma}\cdot\vec{p} - \dot{\vec{x}}\cdot\vec{p}\right)\right)|i\rangle$$

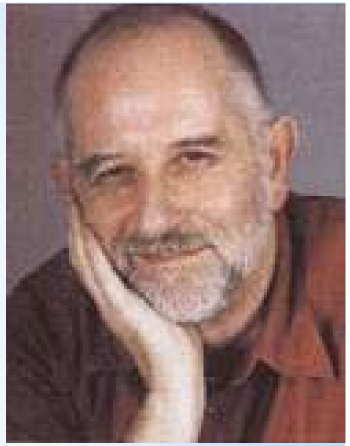
Unitary matrix of eigenstates: $V_p^\dagger \vec{\sigma} \cdot \vec{p} V_p = |\vec{p}|\sigma_3$

Associated non-Abelian gauge field: $\hat{a}_k = i V_p^\dagger \frac{\partial}{\partial p_k} V_p$

$$\langle f | e^{iHt} | i \rangle = \int \mathcal{D}x(\tau)\mathcal{D}p(\tau)$$

$$\mathcal{P}\exp\left(i\int_0^t d\tau\left(\vec{x}\cdot\dot{\vec{p}} + \sigma_3|\vec{p}| + \hat{a}_k\dot{p}_k\right)\right)$$

Weyl points as monopoles in momentum space



Classical regime: neglect spin flips = off-diagonal terms in a_k

Classical action

$$S = \int d\tau \left(\vec{p} \cdot \dot{\vec{x}} - |\vec{p}| - (\vec{a}_p)_{11} \cdot \dot{\vec{p}} \right)$$

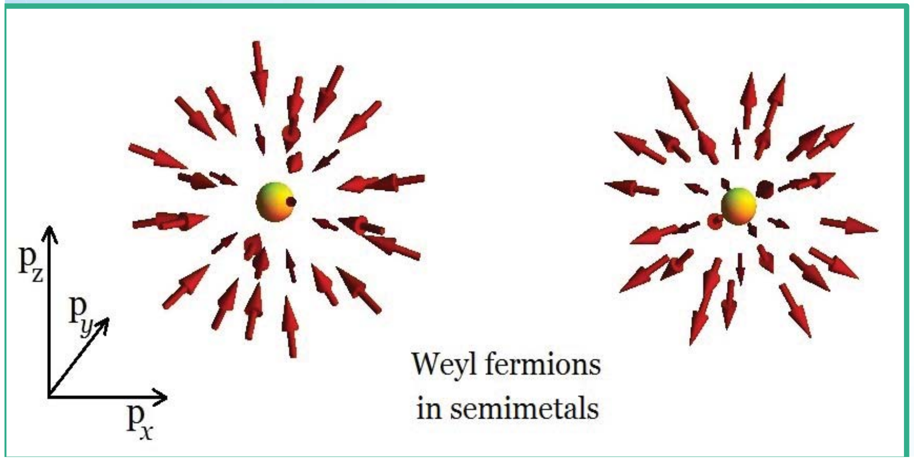
$(a_p)_{11}$ looks like a field of Abelian monopole in momentum space

Berry flux

$$\Phi_B = \int d\vec{S} \cdot \vec{b}, \quad \vec{b} = \vec{\partial}_p \times (\vec{a}_p)_{11}$$

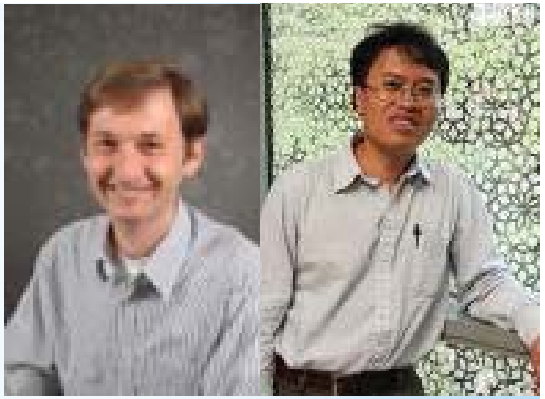
Topological invariant!!!

$$\vec{b} = \frac{\vec{p}}{2p^3}$$



Fermion doubling theorem: In compact Brillouin zone only pairs of monopole/anti-monopole

Chiral kinetic theory
[Stephanov, Son]



Classical action and equations of motion with gauge fields

$$I = \int_{t_i}^{t_f} (\boldsymbol{p}\cdot\dot{\boldsymbol{x}} + \boldsymbol{A}\cdot\dot{\boldsymbol{x}} - \Phi - |\boldsymbol{p}| - \boldsymbol{a}_p\cdot\dot{\boldsymbol{p}})dt$$

$$\dot{\boldsymbol{x}} = \hat{\boldsymbol{p}} + \dot{\boldsymbol{p}}\times\boldsymbol{b};$$
$$\dot{\boldsymbol{p}} = \boldsymbol{E} + \dot{\boldsymbol{x}}\times\boldsymbol{B}.$$

More consistent is the Wigner formalism

Streaming equations in phase space

$$\sqrt{G}\,\dot{\boldsymbol{x}} = \hat{\boldsymbol{p}} + \boldsymbol{E}\times\boldsymbol{b} + \boldsymbol{B}(\hat{\boldsymbol{p}}\cdot\boldsymbol{b})\,;$$
$$\sqrt{G}\dot{\boldsymbol{p}} = \boldsymbol{E} + \hat{\boldsymbol{p}}\times\boldsymbol{B} + \boldsymbol{b}(\boldsymbol{E}\cdot\boldsymbol{B})\,.$$

$$G = (1+\boldsymbol{b}\cdot\boldsymbol{B})^2$$

$$\rho = \sqrt{G}f$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\dot{\boldsymbol{x}})}{\partial\boldsymbol{x}} + \frac{\partial(\rho\dot{\boldsymbol{p}})}{\partial\boldsymbol{p}} = 2\pi\boldsymbol{E}\cdot\boldsymbol{B}\,f\,\delta^3(\boldsymbol{p})\,,$$

Anomaly = injection of particles at zero momentum (level crossing)

CME and CSE in linear response theory

Anomalous current-current correlators:

$$\Pi_{ij}^{AV}(k) = \int d^4x \, e^{ik_\mu x_\mu} \langle j_i^A(x) \, j_j^V(0) \rangle_{\mu_V}$$

$$\Pi_{ij}^{VV}(k) = \int d^4x e^{ik_\mu x_\mu} \langle j_i^V(x) \, j_j^V(0) \rangle_{\mu_A}$$

Chiral Separation and Chiral Magnetic Conductivities:

$$\sigma_{CSE} = \lim_{k_3 \to 0} \lim_{k_0 \to 0} \frac{i}{k_3} \Pi_{12}^{AV}(k_3),$$

$$\sigma_{CME} = \lim_{k_3 \to 0} \lim_{k_0 \to 0} \frac{i}{k_3} \Pi_{12}^{VV}(k_3).$$

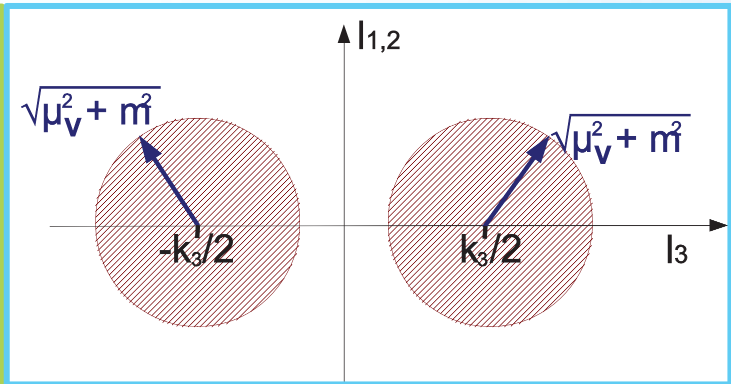
Chiral Separation: finite-temperature regularization

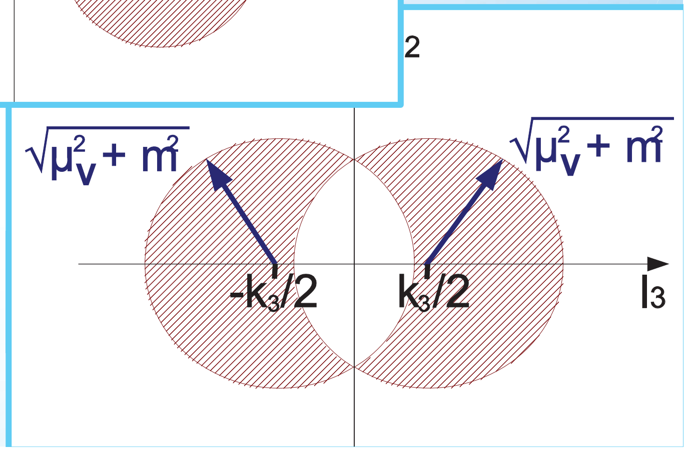
$$\Pi^{AV}_{\mu\nu}(k) = \int \frac{d^4 l}{(2\pi)^4} \mathrm{tr}\left(\gamma_\mu \gamma_5 \mathcal{D}^{-1}(l + k/2, \mu_V)\, \gamma_\nu \mathcal{D}^{-1}(l - k/2, \mu_V)\right)$$

Integrate out time-like loop momentum

Relation to canonical formalism

$$\Pi^{AV}_{12}(k_3) = i \int\limits_{-\infty}^{+\infty} \frac{dl_3}{2\pi l_3} \int\limits_{0}^{+\infty} \frac{d\left(\pi l_\perp^2\right)}{(2\pi)^2}$$
$$\left(\theta\left(\mu_V - \sqrt{(l_3 + k_3/2)^2 + l_\perp^2 + m^2}\right) - \right.$$
$$\left. -\theta\left(\mu_V - \sqrt{(l_3 - k_3/2)^2 + l_\perp^2 + m^2}\right)\right),$$



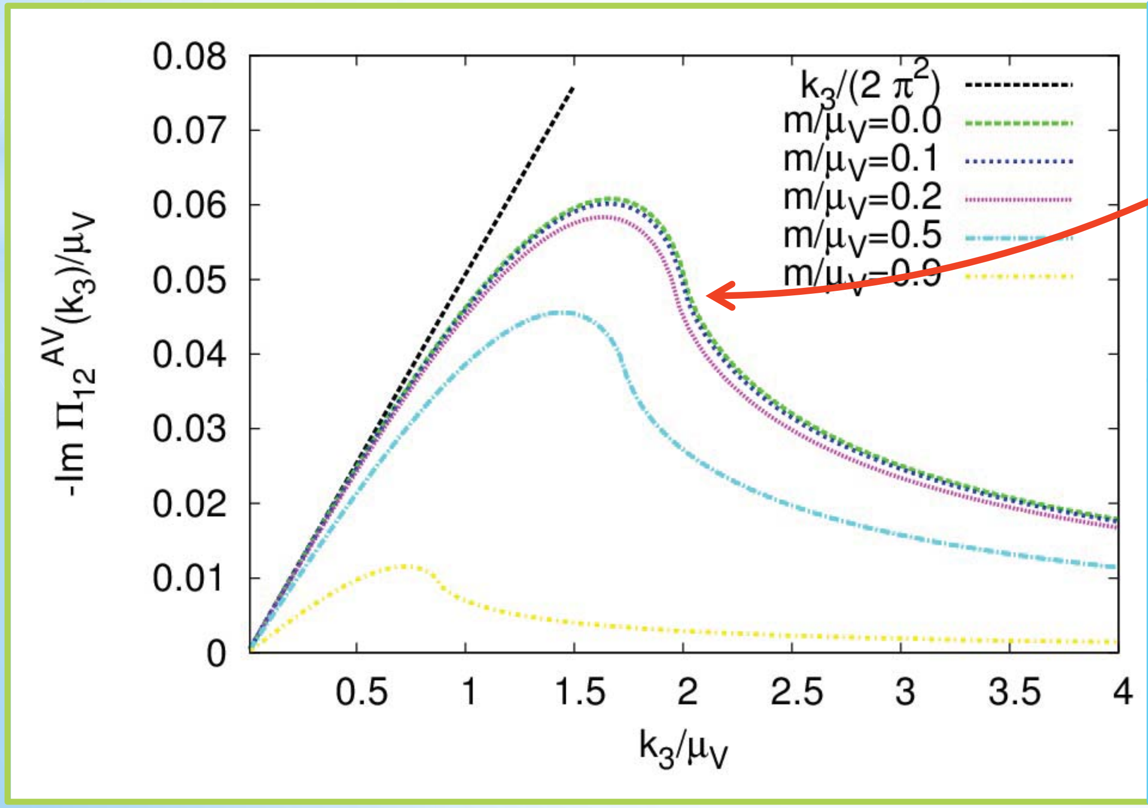


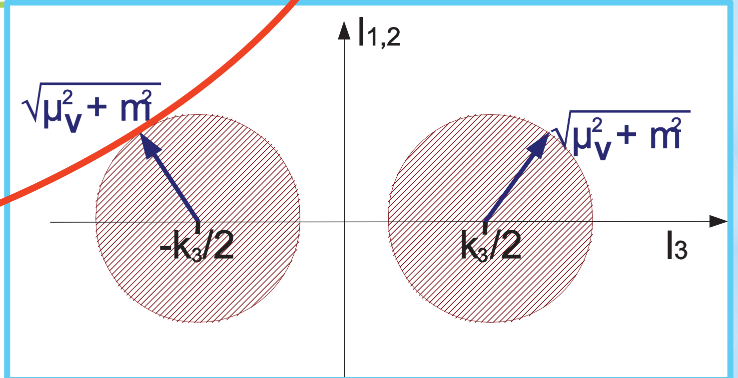
Virtual photon hits out fermions out of Fermi sphere…

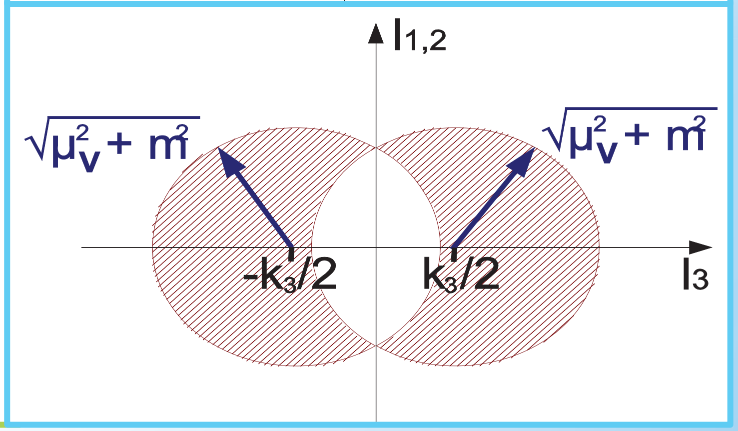
Chiral Separation Conductivity: analytic result

$$\Pi_{12}^{AV}(k_3) = -\frac{i}{(2\pi)^2}\left(\sqrt{\mu_V^2 - m^2}\,k_3 - \right.$$
$$\left. -\left(\mu_V^2 - m^2 - k_3^2/4\right)\log\left|\frac{2\sqrt{\mu_V^2-m^2}-k_3}{2\sqrt{\mu_V^2-m^2}+k_3}\right|\right)$$

Singularity







Chiral Magnetic Conductivity: finite-temperature regularization

$$\Pi^{VV}_{\mu\nu}(k) = \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left(\gamma_\mu \mathcal{D}^{-1}(l + k/2, \mu_A)\, \gamma_\nu \mathcal{D}^{-1}(l - k/2, \mu_A)\right)$$

$$\mathcal{D}^{-1}(p, \mu_A) = \sum_{s=\pm} G_s(p, \mu_A)\, \mathcal{P}_s \times \\ \times \begin{pmatrix} m & -ip_0 + \mu_A - s|\vec{p}| \\ -ip_0 - \mu_A + s|\vec{p}| & m \end{pmatrix}$$

Decomposition of propagators and polarization tensor in terms of chiral states

$$G_s(p, \mu_A) = p_0^2 + (|\vec{p}| - s\mu_A)^2 + m^2$$

$$\Pi^{VV}_{12}(k_3) = -i \int \frac{d^3 l}{(2\pi)^3} \sum_{s,s'} \frac{sqp_3 - s'pq_3}{2pq\,(p - s'sq)} \times \\ \times \left(\frac{p - s\mu_A}{\sqrt{(p - s\mu_A)^2 + m^2}} - \frac{ss'(q - s'\mu_A)}{\sqrt{(q - s'\mu_A)^2 + m^2}} \right)$$

s,s' - chiralities

Chiral Magnetic Conductivity: finite-temperature regularization

$$\Pi_{12}^{VV}(k_3) = -i \int \frac{d^3 l}{(2\pi)^3} \frac{1}{2 l_3} \times$$

$$\times \left(\frac{p-\mu_A}{\sqrt{(p-\mu_A)^2+m^2}} - \frac{q-\mu_A}{\sqrt{(q-\mu_A)^2+m^2}} + \right.$$

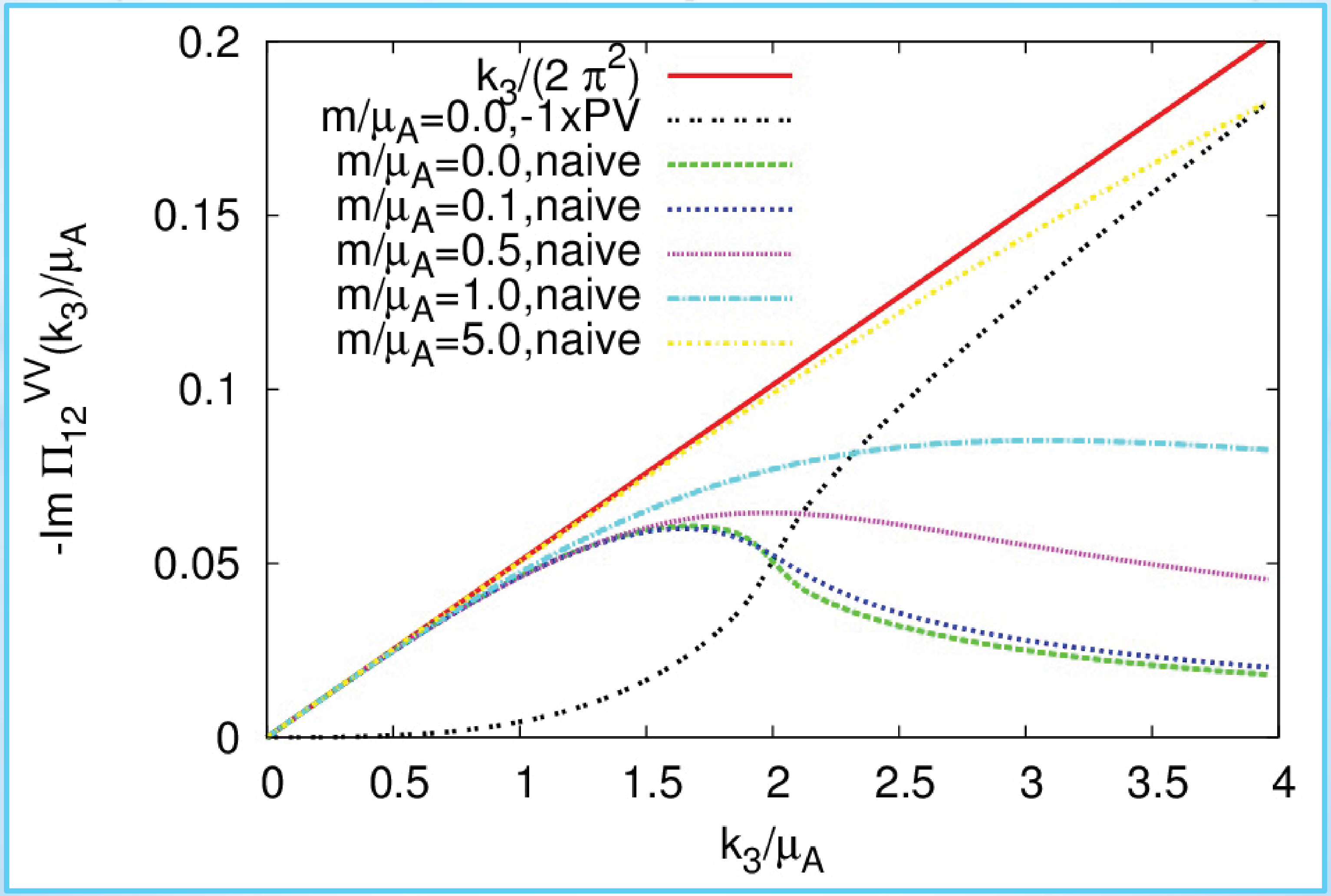
$$\left. + \frac{q+\mu_A}{\sqrt{(q+\mu_A)^2+m^2}} - \frac{p+\mu_A}{\sqrt{(p+\mu_A)^2+m^2}} \right)$$

Careful regularization required!!!

- Individual contributions of chiral states are divergent
- The total is finite and coincides with CSE (upon $\mu_V \to \mu_A$)
- Unusual role of the Dirac mass → non-Fermi-liquid behavior?

Chiral Magnetic Conductivity: still regularization

Pauli-Villars regularization (not chiral, but simple and works well)



CME, CSE and axial anomaly

Expand anomalous correlators in μ_V or μ_A:

$$\frac{\partial}{\partial \mu_V} \Pi^{AV}_{12}(k_3)\,|_{\mu_V=0} \equiv -\Gamma^{VVA}_{201}(0, k_3) =$$
$$= -\int d^4y \int d^4x \, e^{ik_3 x_3} \langle\, j^A_1(x)\, j^V_2(0)\, j^V_0(y)\,\rangle,$$

$$\frac{\partial}{\partial \mu_A} \Pi^{VV}_{12}(k_3)\,|_{\mu_A=0} \equiv -\Gamma^{VVA}_{120}(k_3, -k_3) =$$
$$= -\int d^4y \int d^4x \, e^{ik_3 x_3} \langle\, j^V_1(x)\, j^V_2(0)\, j^A_0(y)\,\rangle.$$

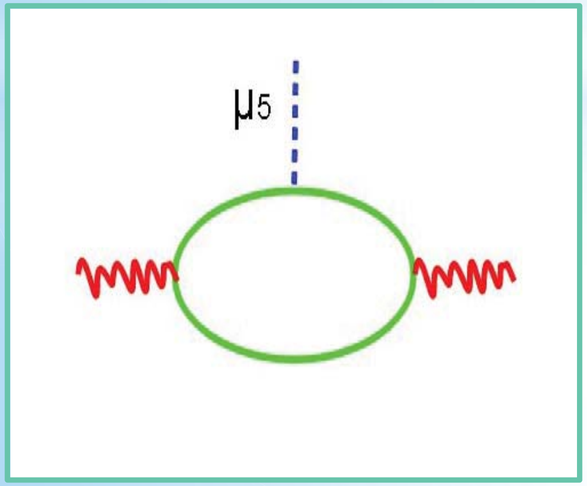
VVA correlator in some special kinematics!!!

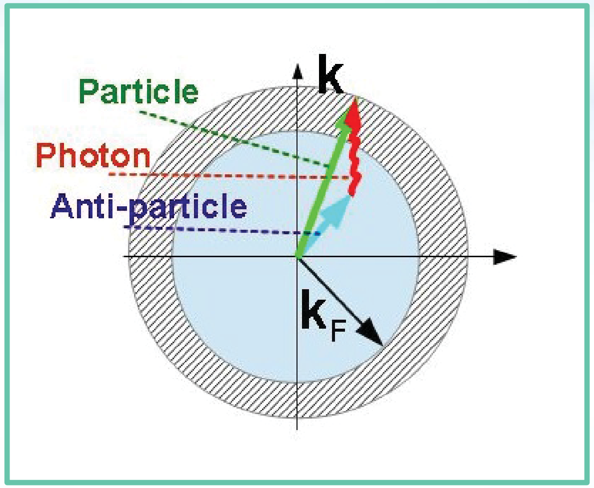
$$\Gamma^{VVA}_{\mu\nu\rho}(p, q) = \int d^4x \int d^4y \, e^{ipx+iqy} \langle\, j^V_\mu(x)\, j^V_\nu(y)\, j^A_\rho(0)\,\rangle.$$

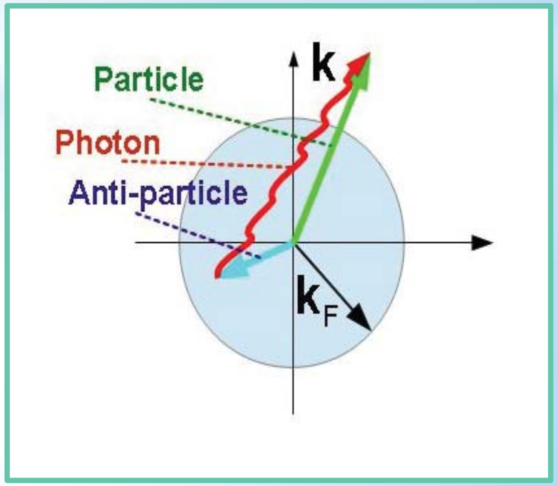
CME, CSE and axial anomaly: IR vs UV

$$\frac{\partial}{\partial \mu} \Pi_{12}(k_3) \big|_{\mu=0} = \lim_{\mu \to 0} \frac{\Pi_{12}(k_3, \mu)}{\mu}$$

At fixed k3, the only scale is μ
Use asymptotic expressions for k3 >> μ !!!
Ward Identities fix large-momentum behavior







- **CME: -1 x classical result, CSE: zero!!!**

General decomposition of VVA correlator

$$\Gamma^{VVA}_{\mu\nu\rho}(p,q) = \frac{i}{4\pi^2}\left(w_L\left(p^2,q^2,(p+q)^2\right)t^L_{\mu\nu\rho}(p,q) + \tilde{w}_T\left(p^2,q^2,(p+q)^2\right)\tilde{t}_{\mu\nu\rho}(p,q) + \right.$$
$$\left. + w_T^{(+)}\left(p^2,q^2,(p+q)^2\right)t^{(+)}_{\mu\nu\rho}(p,q) + w_T^{(-)}\left(p^2,q^2,(p+q)^2\right)t^{(-)}_{\mu\nu\rho}(p,q)\right),$$

$$t^L_{\mu\nu\rho}(p,q) = -(p+q)_\rho\,\epsilon_{\mu\nu\alpha\beta}p_\alpha q_\beta$$
$$t^{(+)}_{\mu\nu\rho}(p,q) = p_\nu\epsilon_{\mu\rho\alpha\beta}p_\alpha q_\beta - q_\mu\epsilon_{\nu\rho\alpha\beta}p_\alpha q_\beta - (p\cdot q)\,\epsilon_{\mu\nu\rho\alpha}(p-q)_\alpha - \frac{2p\cdot q}{(p+q)^2}\epsilon_{\mu\nu\alpha\beta}p_\alpha q_\beta\,(p+q)_\rho$$
$$t^{(-)}_{\mu\nu\rho}(p,q) = \left((p-q)_\rho - \frac{p^2-q^2}{(p+q)^2}(p+q)_\rho\right)\epsilon_{\mu\nu\alpha\beta}p_\alpha q_\beta$$
$$\tilde{t}_{\mu\nu\rho}(p,q) = p_\nu\epsilon_{\mu\rho\alpha\beta}p_\alpha q_\beta + q_\mu\epsilon_{\nu\rho\alpha\beta}p_\alpha q_\beta - (p\cdot q)\,\epsilon_{\mu\nu\rho\alpha}(p+q)_\alpha\,.$$

$$w_L\left(p^2,q^2,(p+q)^2\right) = -\frac{2}{(p+q)^2}$$

- 4 independent form-factors
- Only wL is constrained by axial WIs

[M. Knecht *et al.*, hep-ph/0311100]

Anomalous correlators vs VVA correlator

CSE: p = (0,0,0,k3), q=0, μ=2, ν=0, ρ=1 → ZERO

CME: p = (0,0,0,k3), q=(0,0,0,-k3), μ=1, ν=2, ρ=0

↓

IR SINGULARITY

Regularization: p = k + ε/2, q = -k+ε/2

ε - "momentum" of chiral chemical potential

Time-dependent chemical potential:

$$t^L_{120} = k_3 \epsilon_0^2$$

$$w_L\left(\left(k+\epsilon/2\right)^2, \left(k-\epsilon/2\right)^2, \epsilon^2\right) = -\frac{2}{\epsilon_0^2}$$

$$\frac{\partial}{\partial \mu_A} \Pi^{VV}_{12}\left(k_3\right)|_{\mu_A=0} = -\frac{ik_3}{2\pi^2}$$

Anomalous correlators vs VVA correlator

Spatially modulated chiral chemical potential

$$t^{(+)}_{120}\left(k+\frac{\epsilon}{2},k-\frac{\epsilon}{2}\right)=2k_3^3$$

$$\tilde{t}_{120}\left(k+\frac{\epsilon}{2},k-\frac{\epsilon}{2}\right)=k_3^2\epsilon_3$$

By virtue of Bose symmetry, only $w^{(+)}(k^2,k^2,0)$

⇓

Transverse form-factor

Not fixed by the anomaly

CME and axial anomaly (continued)

In addition to anomaly non-renormalization,
new (<u>perturbative!!!</u>) non-renormalization theorems
[M. Knecht *et al.*, hep-ph/0311100]
[A. Vainstein, hep-ph/0212231]:

$$\left\{\left[w_T^{(+)} + w_T^{(-)}\right]\left(q_1^2, q_2^2, (q_1+q_2)^2\right) - \left[w_T^{(+)} + w_T^{(-)}\right]\left((q_1+q_2)^2, q_2^2, q_1^2\right)\right\}_{\mathrm{pQCD}} = 0$$

$$\left\{\left[\tilde{w}_T^{(-)} + w_T^{(-)}\right]\left(q_1^2, q_2^2, (q_1+q_2)^2\right) + \left[\tilde{w}_T^{(-)} + w_T^{(-)}\right]\left((q_1+q_2)^2, q_2^2, q_1^2\right)\right\}_{\mathrm{pQCD}} = 0$$

$$\begin{aligned}&\left\{\left[w_T^{(+)} + \tilde{w}_T^{(-)}\right]\left(q_1^2, q_2^2, (q_1+q_2)^2\right) + \left[w_T^{(+)} + \tilde{w}_T^{(-)}\right]\left((q_1+q_2)^2, q_2^2, q_1^2\right)\right\}_{\mathrm{pQCD}} - w_L\left((q_1+q_2)^2, q_2^2, q_1^2\right)\\ &= -\left\{\frac{2\,(q_2^2 + q_1\cdot q_2)}{q_1^2}\, w_T^{(+)}\left((q_1+q_2)^2, q_2^2, q_1^2\right) - 2\,\frac{q_1\cdot q_2}{q_1^2}\, w_T^{(-)}\left((q_1+q_2)^2, q_2^2, q_1^2\right)\right\}_{\mathrm{pQCD}}\end{aligned}$$

Valid only for massless QCD!!!

CME and axial anomaly (continued)

Special limit: $p^2 = q^2$

$$
\begin{aligned}
w_T^{(+)}\left(k^2, k^2, \epsilon^2\right) - w_T^{(+)}\left(\epsilon^2, k^2, k^2\right) &= 0, \\
\tilde{w}_T\left(\epsilon^2, k^2, k^2\right) + w_T^{(-)}\left(\epsilon^2, k^2, k^2\right) &= 0, \\
w_T^{(+)}\left(k^2, k^2, \epsilon^2\right) + w_T^{(+)}\left(\epsilon^2, k^2, k^2\right) + \tilde{w}_T\left(\epsilon^2, k^2, k^2\right) & \\
+ \frac{\epsilon^2}{k^2} w_T^{(+)}\left(\epsilon^2, k^2, k^2\right) + \frac{2k^2 - \epsilon^2}{k^2} w_T^{(-)}\left(\epsilon^2, k^2, k^2\right) &= -\frac{2}{k^2}.
\end{aligned}
$$

$$
\begin{aligned}
w_T^{(+)}\left(\epsilon^2, k^2, k^2\right) + w_T^{(-)}\left(\epsilon^2, k^2, k^2\right) - w_T^{(+)}\left(k^2, k^2, \epsilon^2\right) &= 0, \\
\tilde{w}_T\left(\epsilon^2, k^2, k^2\right) + w_T^{(-)}\left(\epsilon^2, k^2, k^2\right) &= 0, \\
w_T^{(+)}\left(\epsilon^2, k^2, k^2\right) + \tilde{w}_T\left(\epsilon^2, k^2, k^2\right) + w_T^{(+)}\left(k^2, k^2, \epsilon^2\right) & \\
+ \frac{2k^2 - \epsilon^2}{\epsilon^2} w_T^{(+)}\left(k^2, k^2, \epsilon^2\right) &= -\frac{2}{\epsilon^2}.
\end{aligned}
$$

Six equations for four unknowns… Solution:

$$
\begin{aligned}
w_T^{(+)}\left(k^2, k^2, \epsilon^2\right) = w_T^{(+)}\left(\epsilon^2, k^2, k^2\right) &= -\frac{2}{2k^2 + \epsilon^2} \\
w_T^{(-)}\left(\epsilon^2, k^2, k^2\right) = \tilde{w}_T\left(\epsilon^2, k^2, k^2\right) &= 0
\end{aligned}
$$

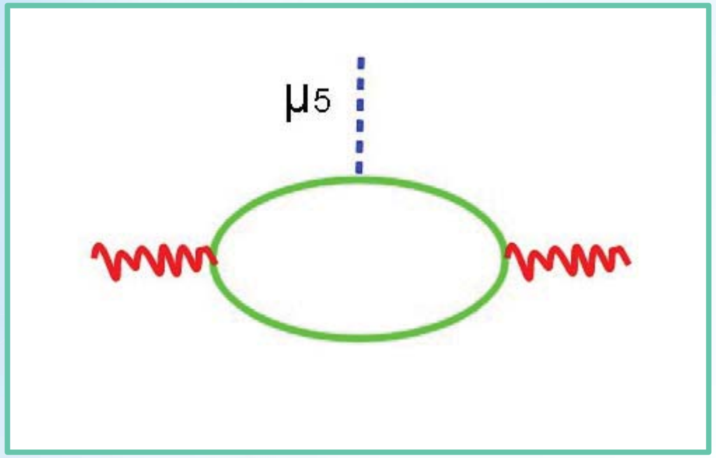
$$
w_T^{(+)}\left(k^2, k^2, 0\right) = -\frac{1}{k^2}
$$

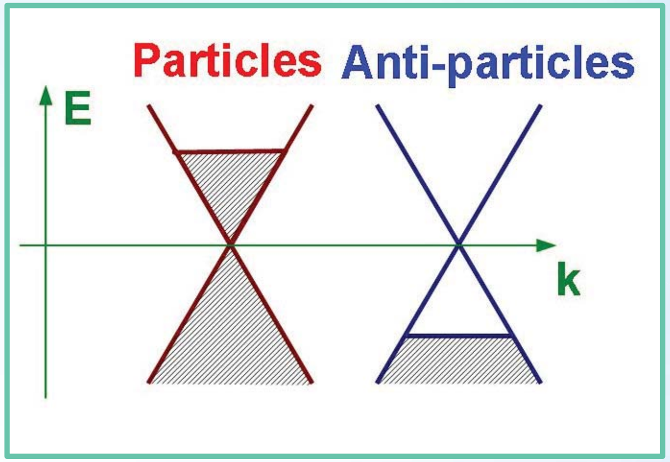
$$
\frac{\partial}{\partial \mu_A} \Pi_{12}^{VV}\left(k_3\right) \Big|_{\mu_A = 0} = -\frac{i k_3}{2\pi^2}
$$

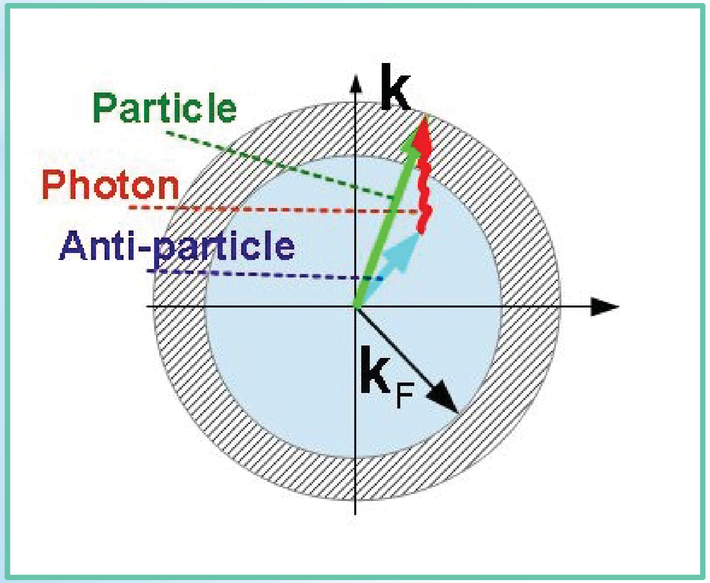
Might be subject to NP corrections due to ChSB!!!

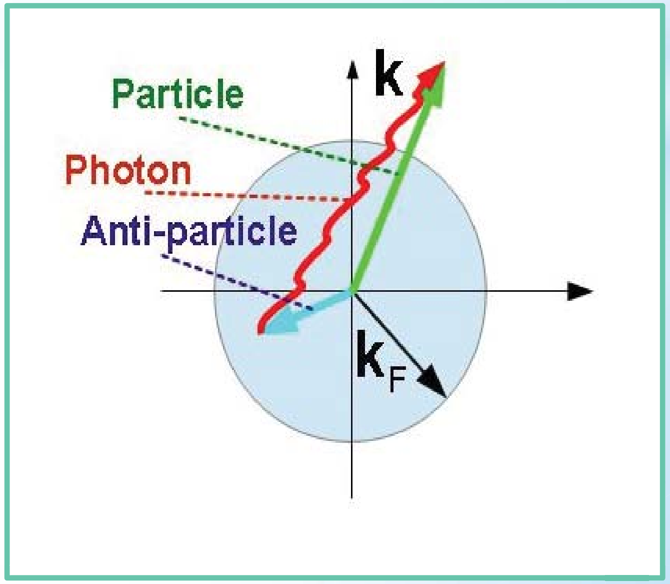
Fermi surface singularity

Almost correct, but what is at small p_3???



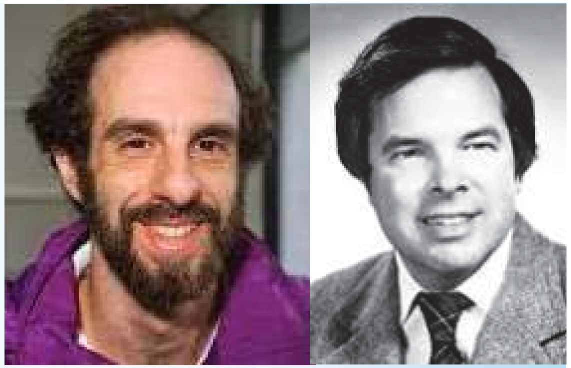






Full phase space is available only at $|p|>2|k_F|$

Chiral fermions on the lattice: Ginsparg-Wilson relation



- Assume chirally invariant action $S\left(\bar{\phi}, \phi\right) = S\left(\bar{\phi} e^{i\gamma_5 \theta}, e^{i\gamma_5 \theta}\phi\right)$

at some very fine physical scale

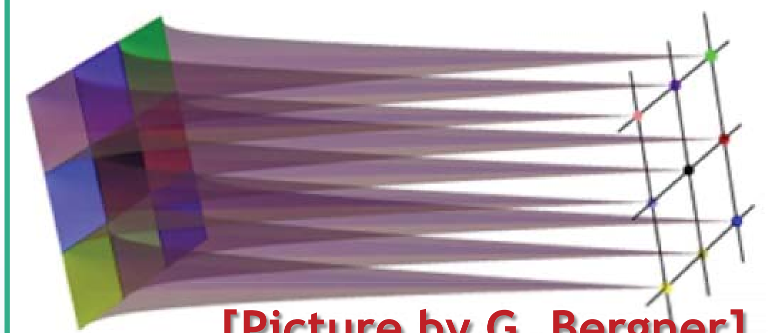
- Construct "blocked variables" ψ

$$\exp\left(-\bar{S}\left(\bar{\psi}_n, \psi_n\right)\right) =$$
$$= \int \mathcal{D}\bar{\phi}\mathcal{D}\phi \exp\left(-\alpha \sum_n \left(\bar{\psi}_n - \int_n d^D x \bar{\phi}\right)\left(\psi_n - \int_n d^D x \phi\right) - S\left(\bar{\phi}, \phi\right)\right)$$

- Change variables to (θ is infinitely small)

$$\bar{\psi}_n \to \bar{\psi}_n e^{i\gamma_5 \theta}, \psi_n \to e^{i\gamma_5 \theta}\psi_n$$

- Assume $\bar{S}\left(\bar{\psi}, \psi\right) = \bar{\psi}\mathcal{D}\psi$



[Picture by G. Bergner]

$$e^{-A(\psi,\bar\psi)}[1+i\epsilon\bar\psi\{\gamma^5,h\}\psi]=\int_{\phi\bar\phi}[1+i\epsilon(\bar\psi-\bar\phi)\{\gamma^5,\alpha\}(\psi-\phi)]\exp[-(\bar\psi-\bar\phi)\alpha(\psi-\phi)-A_I(\phi,\bar\phi)]$$

$$=\left[1-i\epsilon\frac{\partial}{\partial\psi}\alpha^{-1}\{\gamma^5,\alpha\}\alpha^{-1}\frac{\partial}{\partial\bar\psi}\right]\int_{\phi,\bar\phi}\exp[-(\bar\psi-\bar\phi)\alpha(\psi-\phi)-A_I(\phi,\bar\phi)]$$

$$=\left[1-i\epsilon\frac{\partial}{\partial\psi}\{\gamma^5,\alpha^{-1}\}\frac{\partial}{\partial\bar\psi}\right]e^{-A(\psi,\bar\psi)}.$$

$$h\psi e^{-\bar\psi h\psi}=-\frac{\partial}{\partial\bar\psi}e^{-\bar\psi h\psi},$$

$$\bar\psi h e^{-\bar\psi h\psi}=\frac{\partial}{\partial\psi}e^{-\bar\psi h\psi},$$

$$i\epsilon\bar\psi\{\gamma^5,h\}\psi e^{-\bar\psi h\psi}=i\epsilon\bar\psi h\{\gamma^5,\alpha^{-1}\}h\psi e^{-\bar\psi h\psi}$$

Ginsparg-Wilson relation [1982]

$$\{\gamma_5,D\}=2D\gamma_5\alpha^{-1}\gamma_5$$

Usually α = 1 in lattice units

Overlap Dirac operator [Neuberger'1998]

$$\mathcal{D}_{ov}\left(\mu_V\right) = 1 + \gamma_5 \mathrm{sign}\left(\gamma_5 \mathcal{D}_w\left(\mu_V\right)\right)$$

$$\mathrm{sign}\left(\mathcal{H}\right) = \sum_i \mathrm{sign}\left(\mathrm{Re}\,\lambda_i\right)\; |R_i\rangle\langle L_i|,$$

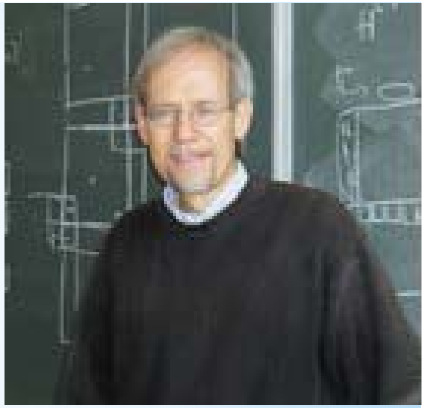
$$\mathcal{H}\,|R_i\rangle = \lambda_i\,|R_i\rangle, \qquad \langle L_i|\,\mathcal{H} = \lambda_i\langle L_i| \qquad \langle L_i|\;|R_j\rangle = \delta_{ij}.$$

Lattice chiral symmetry: Lüscher transformations

$$\left[\delta\mathcal{D}_{ov}\right]_{xy} = \sum_z \left[1 - \mathcal{D}_{ov}/2\right]_{xz} \gamma_5 \delta\theta_z \left[\mathcal{D}_{ov}\right]_{zy} + \\ + \sum_z \left[\mathcal{D}_{ov}\right]_{xz} \delta\theta_z \gamma_5 \left[1 - \mathcal{D}_{ov}/2\right]_{zy}$$

These factors encode the anomaly (nontrivial Jacobian)

Weyl fermions on the lattice?



Continuum index theorem

$$n_R - n_L = \mathrm{Tr}\,\gamma_5 = \frac{1}{8\pi^2}\int d^4x F_{\mu\nu}\tilde{F}_{\mu\nu}$$

Lattice index theorem

$$n_R - n_L = \mathrm{Tr}\,\gamma_5\left(1 - \mathcal{D}_{ov}\right) = \mathrm{Tr}\,\mathrm{sign}\,\left(\gamma_5 \mathcal{D}_w\right)$$

Define the new gamma5 and "chiral states"

$$\bar{\gamma}_5 = \mathrm{sign}\,\left(\gamma_5 \mathcal{D}_w\right), \quad \bar{\gamma}_5^2 = 1 \qquad \bar{\gamma}_5 \left|\psi_{n,\pm}\right\rangle = \pm\left|\psi_{n,\pm}\right\rangle$$

"Weyl operator" on the lattice [Lüscher]

$$\mathcal{D}_{weyl} = \left\langle \psi_{n,-}\right| \mathcal{D}_{ov} \left|\phi_{m,+}\right\rangle$$

→ Eigenstates of gamma5

Path integral with Weyl fermions

$$\mathcal{Z} = \int dg_{x,\mu}\, \det\left(\mathcal{D}_{weyl}\right) e^{-S_{YM}}$$

$$\left|\tilde{\psi}_{n,+}\right\rangle = \sum_m U_{nm}\left|\psi_{n,+}\right\rangle, \quad U \in U\left(V/2\right)$$

← Phase ambiguity!!!

Dirac operator with axial gauge fields

First consider coupling to axial gauge field:

$$j_{5\mu} \sim \frac{\partial \mathcal{D}_{ov}[V_\mu, A_\mu]}{\partial A_\mu}$$

Assume local invariance under modified chiral transformations [Kikukawa, Yamada, hep-lat/9808026]:

$$e^{i\gamma_5\theta}\mathcal{D}\left[V_\mu, A_\mu\right]e^{i\gamma_5\theta} = \\ = \mathcal{D}\left[V_\mu, A_\mu + \partial_\mu\theta\right]$$

$$\delta\psi_x = \sum_y \alpha_x \gamma_5 \left(1 - \frac{\mathcal{D}_{ov}}{2}\right)_{xy} \psi_y \quad \delta\bar{\psi}_x = \sum_y \bar{\psi}_y \left(1 - \frac{\mathcal{D}_{ov}}{2}\right)_{xy} \gamma_5 \alpha_y$$

Require $$\delta S = \delta\left(\bar{\psi}\mathcal{D}_{ov}\psi\right) = \sum_x \alpha_x \partial_{x,\mu} j_{5x,\mu}$$

$$\frac{\partial \mathcal{D}_{ov}[V_\mu, A_\mu]}{\partial A_{x,\mu}} = \frac{\partial \mathcal{D}_{ov}[V_\mu, A_\mu]}{\partial V_{x,\mu}} \gamma_5 \left(1 - \mathcal{D}_{ov}\right)$$

(Integrable) equation for D_{ov} !!!

Overlap fermions with axial gauge fields and chiral chemical potential

We require that Lüscher transformations generate gauge transformations of $A_\mu(x)$

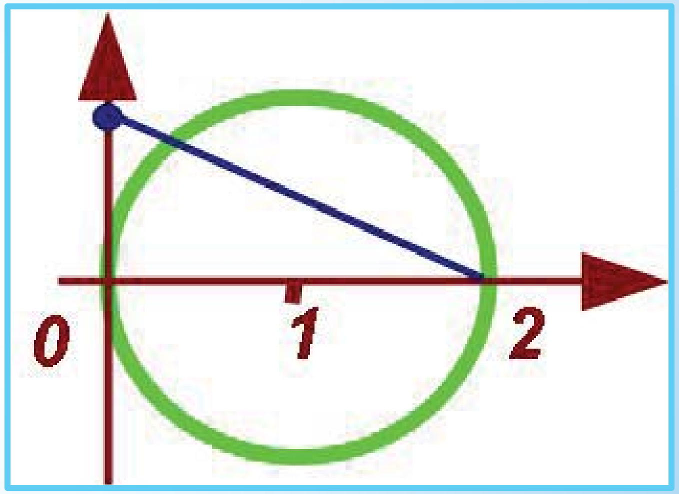
$$\frac{\partial}{\partial A_\mu(x)} \mathcal{D}_{ov}\left[V_\mu(x), A_\mu(x)\right] =$$
$$= \frac{\partial}{\partial V_\mu(x)} \mathcal{D}_{ov}\left[V_\mu(x), A_\mu(x)\right] \gamma_5 \left(1 - \mathcal{D}_{ov}\left[V_\mu(x), A_\mu(x)\right]\right)$$

Linear equations for "projected" overlap or propagator

$$\frac{\partial}{\partial A_\mu(x)} \mathcal{D}_{ov}^{-1} = -\frac{\partial}{\partial V_\mu(x)} \mathcal{D}_{ov}^{-1} \gamma_5$$

$$\frac{\partial}{\partial A_\mu(x)} \tilde{\mathcal{D}}_{ov} = \frac{\partial}{\partial V_\mu(x)} \tilde{\mathcal{D}}_{ov} \gamma_5$$

$$\mathcal{D}_{ov} = \frac{2\tilde{\mathcal{D}}_{ov}}{2+\tilde{\mathcal{D}}_{ov}} \qquad \tilde{\mathcal{D}}_{ov} = \frac{2\mathcal{D}_{ov}}{2-\mathcal{D}_{ov}}$$



Overlap fermions with axial gauge fields and chiral chemical potential: solutions

Dirac operator with axial gauge field

$$\mathcal{D}_{ov}^{-1}\left[V_\mu, A_\mu\right] = \mathcal{P}_+ \mathcal{D}_{ov}^{-1}\left[V_\mu + A_\mu\right] \mathcal{P}_- + \\ + \mathcal{P}_- \mathcal{D}_{ov}^{-1}\left[V_\mu - A_\mu\right] \mathcal{P}_+ + 1/2$$

$$\tilde{\mathcal{D}}_{ov}\left[V_\mu, A_\mu\right] = \mathcal{P}_- \tilde{\mathcal{D}}_{ov}\left[V_\mu + A_\mu\right] \mathcal{P}_+ + \\ + \mathcal{P}_+ \tilde{\mathcal{D}}_{ov}\left[V_\mu - A_\mu\right] \mathcal{P}_-$$

Dirac operator with chiral chemical potential

$$\tilde{\mathcal{D}}_{ov}\left(\mu_V, \mu_A\right) = \mathcal{P}_- \tilde{\mathcal{D}}_{ov}\left(\mu_V + \mu_A\right) \mathcal{P}_+ + \\ + \mathcal{P}_+ \tilde{\mathcal{D}}_{ov}\left(\mu_V - \mu_A\right) \mathcal{P}_-$$

- Only implicit definition (so far)
- Hermitean kernel (at zero μV) = Sign() of what???
- Potentially, no sign problem in simulations

Current-current correlators on the lattice

$$\mathcal{Z} = \int dg_{x,\mu} \det\left(\mathcal{D}\left[g_{x,\mu}, A_{x,\mu}, V_{x,\mu}\right]\right)^{N_f} e^{-S_{YM}[g_{x,\mu}]}$$

$$\langle j^V_{x,\mu} j^V_{y,\nu} \rangle = T^2 \mathcal{Z}^{-1} \partial^V_{y,\nu} \partial^V_{x,\mu} \mathcal{Z}\Big|_{V_{x,\mu}=0},$$

$$\langle j^A_{x,\mu} j^V_{y,\nu} \rangle = T^2 \mathcal{Z}^{-1} \partial^V_{y,\nu} \partial^A_{x,\mu} \mathcal{Z}\Big|_{V_{x,\mu}=0, A_{x,\mu}=0}.$$

$$\langle j^V_{x,\mu} j^V_{y,\nu} \rangle = -T^2 N_f \langle \operatorname{Tr}\left(\mathcal{D}^{-1} \partial^V_{y,\nu} \mathcal{D}\, \mathcal{D}^{-1} \partial^V_{x,\mu} \mathcal{D}\right) \rangle + \\ + T^2 N_f \langle \operatorname{Tr}\left(\mathcal{D}^{-1} \partial^V_{y,\nu} \partial^V_{x,\mu} \mathcal{D}\right) \rangle$$

Consistent currents!Natural definition:charge transfer

Axial vertex:

$$\partial^A_{x,\mu} \mathcal{D}_{ov} = \partial^V_{x,\mu} \mathcal{D}_{ov} \gamma_5 (1 - \mathcal{D}_{ov})$$

Derivatives of overlap

In terms of kernel spectrum + derivatives of kernel:

$$\partial^V_{x,\mu} \mathcal{D}_{ov} = $$
$$= \sum_{i \neq j} \frac{\gamma_5 \, |R_i\rangle\langle L_i| \, \partial^V_{x,\mu} \mathcal{H} \, |R_j\rangle\langle L_j| \;\; (s_i - s_j)}{\lambda_i - \lambda_j}$$

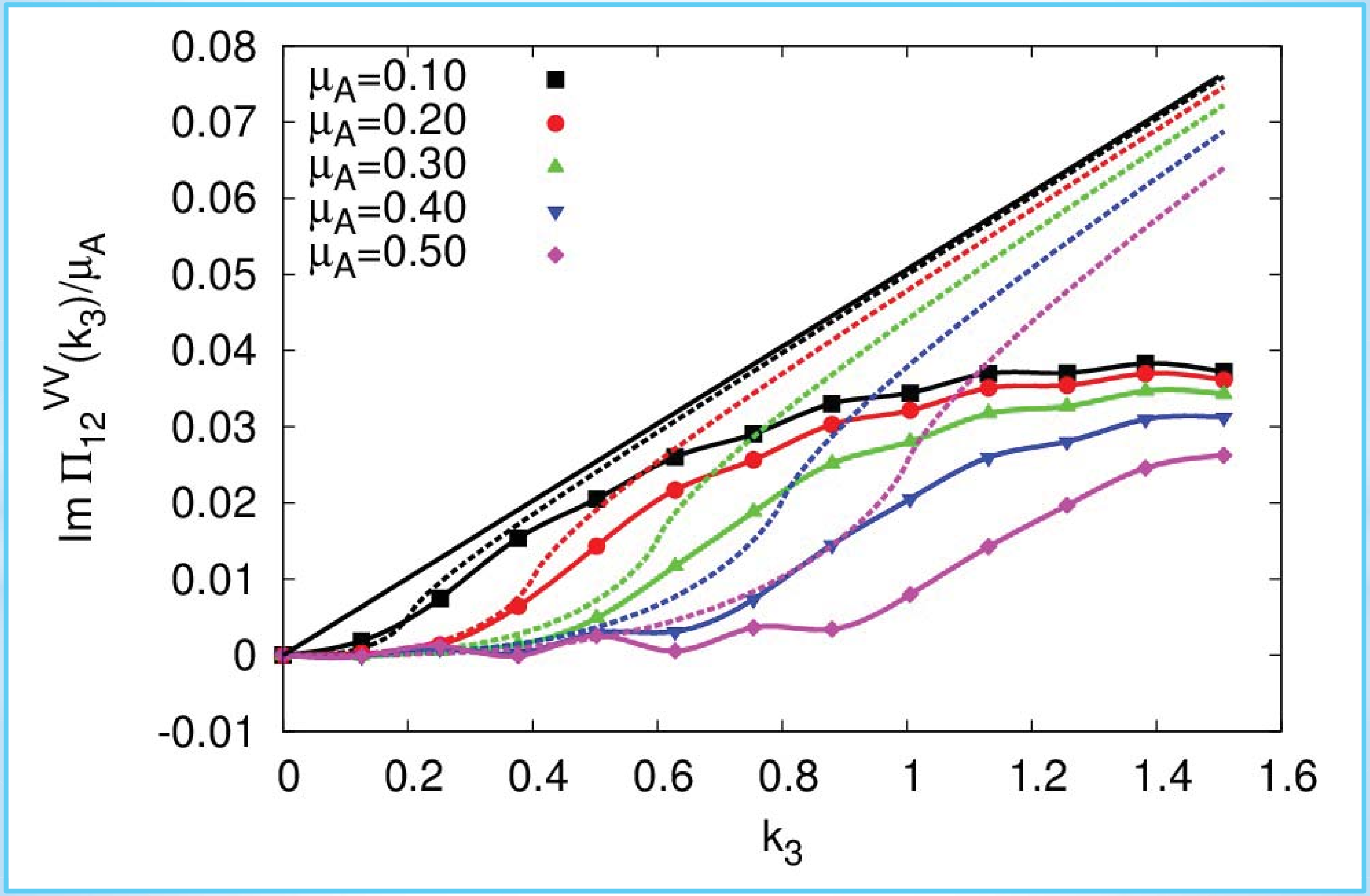
Numerically impossible for arbitrary background…
Krylov subspace methods? Work in progress…

Derivatives of GW/inverse GW projection:

$$\partial^V_{x,\mu} \mathcal{D}_{ov} = \frac{2}{2 + \tilde{\mathcal{D}}_{ov}} \partial^V_{x,\mu} \tilde{\mathcal{D}}_{ov} \frac{2}{2 + \tilde{\mathcal{D}}_{ov}}$$

$$\partial^V_{x,\mu} \tilde{\mathcal{D}}_{ov} = \frac{2}{2 - \mathcal{D}_{ov}} \partial^V_{x,\mu} \mathcal{D}_{ov} \frac{2}{2 - \mathcal{D}_{ov}}$$

Chiral Magnetic Conductivity with overlap



At small momenta, agreement with PV regularization

Chiral Vortical Effect

Linear response of currents to "slow" rotation:

$$[g_{\alpha\beta}] = \begin{pmatrix} -\sqrt{1-\frac{r^2\omega^2}{c^2}} & 0 & r^2\omega & 0 \\ 0 & 1 & 0 & 0 \\ r^2\omega & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_V^{\mathcal{V}} = \lim_{k_z \to 0} \frac{i}{k_z} \langle J_V^x T^{0y} \rangle$$

$$\sigma_A^{\mathcal{V}} = \lim_{k_z \to 0} \frac{i}{k_z} \langle J_A^x T^{0y} \rangle$$

In terms of correlators

$$j_V = \sigma_V^{\mathcal{V}} w = \frac{N_c e}{2\pi^2} \mu_A \mu_V w$$

Subject to PT corrections!!!

$$j_A = \sigma_A^{\mathcal{V}} w = N_c e \left(\frac{\mu_V^2 + \mu_A^2}{4\pi^2} + \frac{T^2}{12} \right) w$$

Coriolis force = axial gauge field

Lattice studies of CVE

A naïve method [Yamamoto, 1303.6292]:

- Analytic continuation of rotating frame metric
- Lattice simulations with distorted lattice
- Physical interpretation is unclear!!!
- By virtue of Hopf theorem:
  only vortex-anti-vortex pairs allowed on torus!!!

More advanced method
[Landsteiner, Chernodub & ITEP Lattice, 1303.6266]:

- Axial magnetic field = source for axial current
- T_{0y} = Energy flow along axial m.f.

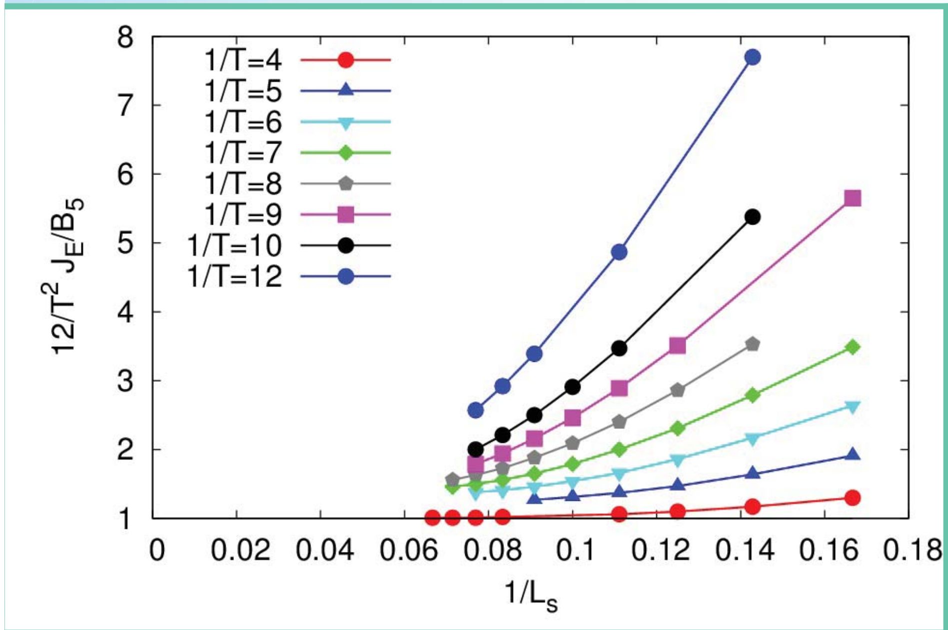
Measure energy flow in the background axial magnetic field

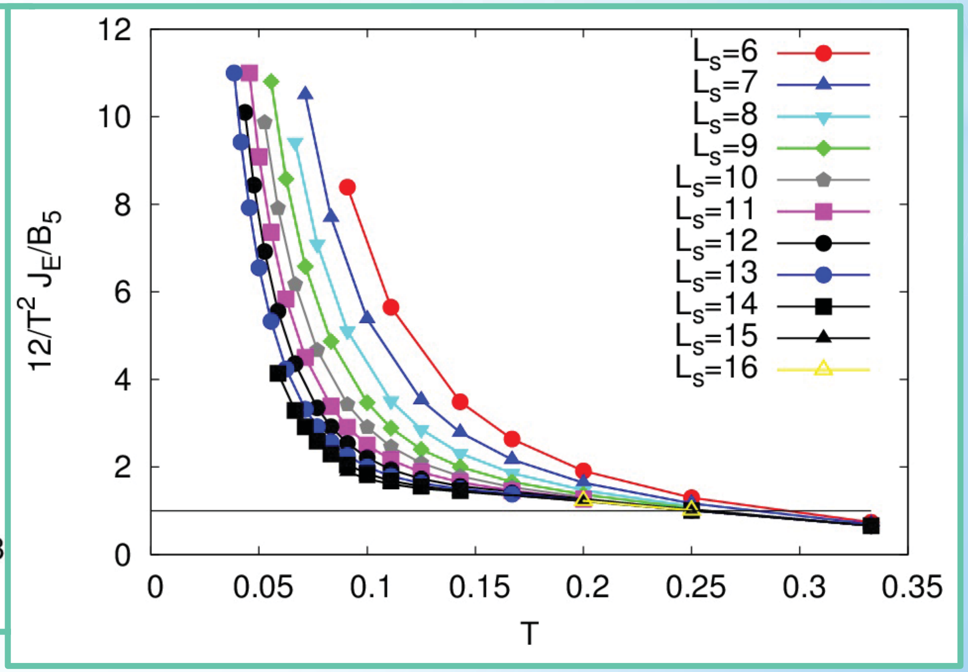
Lattice studies of CVE

First lattice calculations: ~ 0.05 x theory prediction

- **Non-conserved energy-momentum tensor**
- **Constant axial field ➡ not quite a linear response**

For CME: completely wrong results!!!...
Is this also the case for CVE?
Check by a direct calculation, use free overlap





Constant axial magnetic field

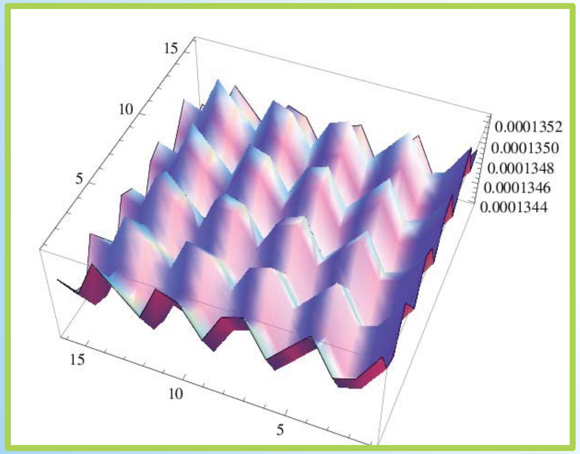
$$\mathcal{D} = \begin{pmatrix} m & \alpha_\mu \nabla_\mu [+B] \\ \alpha_\mu^\dagger \nabla_\mu [-B] & m \end{pmatrix} = \begin{pmatrix} m & ik_0 + \sigma^3 k_3 - i\sigma^a \nabla_a [+B] \\ ik_0 - \sigma^3 k_3 + i\sigma^a \nabla_a [-B] & m \end{pmatrix}.$$

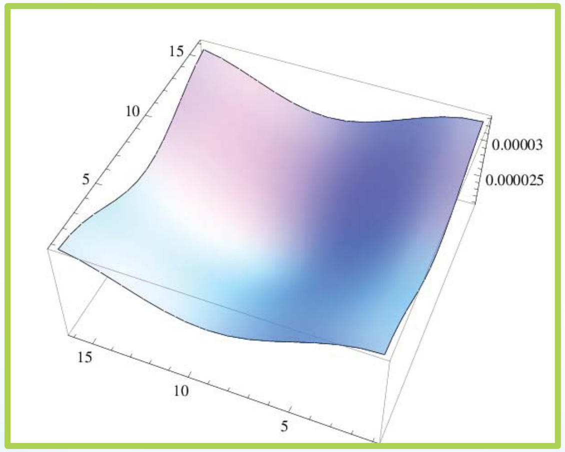
$$\nabla_L = \begin{pmatrix} ik_0 + k_3 & -i\sqrt{2B}A^\dagger \\ i\sqrt{2B}A & ik_0 - k_3 \end{pmatrix}$$

$$\nabla_R^\dagger = \begin{pmatrix} -ik_0 + k_3 & -i\sqrt{2B}C \\ i\sqrt{2B}C^\dagger & -ik_0 - k_3 \end{pmatrix}$$

- No holomorphic structure
- No Landau Levels
- Only the Lowest LL exists
- Solution for finite volume?

$$\nabla_L \nabla_R^\dagger = \begin{pmatrix} k_0^2 + k_3^2 + 2B\, A^\dagger C^\dagger & i\sqrt{2B}\,(ik_0 + k_3)\,(A^\dagger - C) \\ i\sqrt{2B}\,(ik_0 - k_3)\,(C^\dagger - A) & k_0^2 + k_3^2 + 2B\, AC \end{pmatrix}$$





Chiral Vortical Effect from shifted boundary conditions

Conserved lattice energy-momentum tensor:
not known

How the situation can be improved, probably?

Momentum from shifted BC [H.Meyer, 1011.2727]

$$\Phi\left(\vec{x}, \tau + \beta\right) = \pm\Phi\left(\vec{x} + \vec{\xi}, \tau\right)$$

$$\mathcal{Z}_{\xi} = \mathcal{Z}\langle \exp\left(i\vec{P}\cdot\vec{\xi}\right) \rangle$$

We can get total conserved momentum

=

Momentum density

=(?)

Energy flow

Overlap and other chiral Hamiltonians in (1+1)D

Overlap Hamiltonian [Creutz, Horvath, Neuberger]

- Continuous time
- Space-like lattice
- Space-like Overlap Dirac operator and the Hamiltonian

$$h^{(0)} = \gamma_0 \left(1 + \frac{\mathcal{D}_w^{(3D)}}{\sqrt{\mathcal{D}_w^{(3D)} \mathcal{D}_w^{(3D)\dagger}}}\right)$$

$$\mathcal{D}_w^{(3D)} = -\rho + \sum_{i=1}^{3} \left(2 \sin^2\left(\frac{k_i}{2}\right) + i\gamma_i \sin\left(k_i\right)\right)$$

- Left and right projectors from γ_5 and *sign*$(D_w{}^{3D})$
- *1D Weyl Hamiltonian*

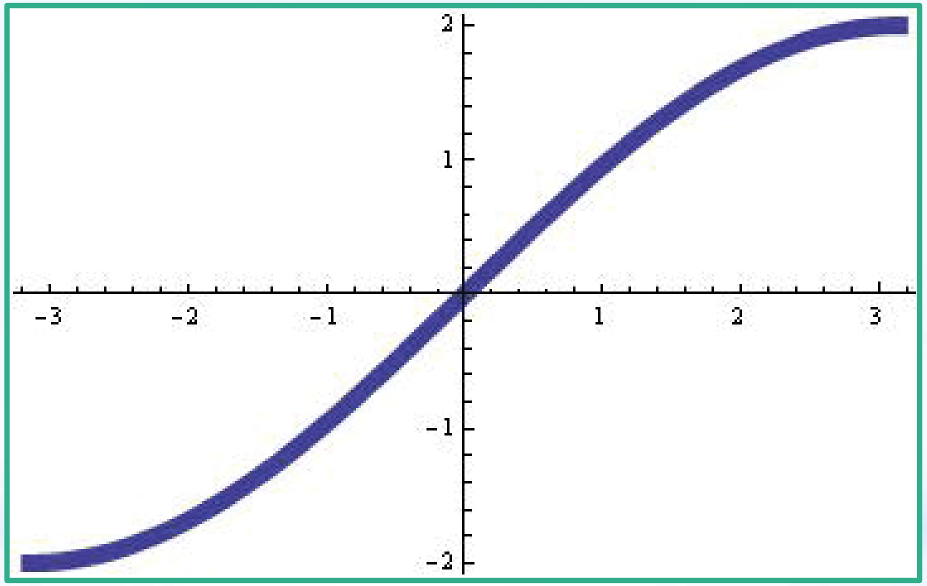
$$h_{weyl} = \langle\psi|\ h\ |\phi\rangle,$$

$$\hat{P}_L = \frac{1 - \mathrm{sign}\left(\mathcal{D}_w^{(3D)}\right)}{2},$$

$$P_L = \frac{1-\gamma_5}{2}$$

Overlap and other chiral Hamiltonians in (1+1)D

Dispersion relation



Periodicity violated!!!

Lattice CVE in (1+1)D from Shifted BC

Spatial shift = total momentum

$$\psi(x, \tau+\beta) = -\psi(x+\Delta, \tau)$$
$$\psi(x+L, \tau) = \psi(x, \tau)$$

$$k = \frac{2\pi m}{L}, \qquad w = 2\pi T(n+1/2) + k\Delta T, \qquad \frac{\partial w}{\partial \Delta} = kT$$

Temporal shift = energy flux

$$\psi(x, \tau+\beta) = -\psi(x, \tau)$$
$$\psi(x+L, \tau) = \psi(x, \tau+\Delta)$$

$$k = \frac{2\pi m}{L} + \frac{w\Delta}{L}, \qquad w = 2\pi T(n+1/2), \qquad \frac{\partial k}{\partial \Delta} = \frac{\Delta}{L}$$

Lattice CVE in (1+1)D from Shifted BC

(1+1)D partition function

$$\mathcal{Z} = \sum_w \sum_k \log\left(w - i\epsilon\left(k\right)\right)$$

Momentum density from spatial shift

$$\frac{\partial \mathcal{Z}}{\partial \Delta} = \sum_w \sum_k \frac{1}{w - i\epsilon(k)} \frac{\partial w}{\partial \Delta} = \sum_w \sum_k \frac{kT}{w - i\epsilon(k)} =$$

$$= \sum_k \frac{ik}{2} \tanh\left(\frac{\epsilon(k)}{2}\right)$$

$$\sum_w \frac{1}{w - i\epsilon} = \frac{i}{2T} \tanh\left(\frac{\epsilon}{2T}\right)$$

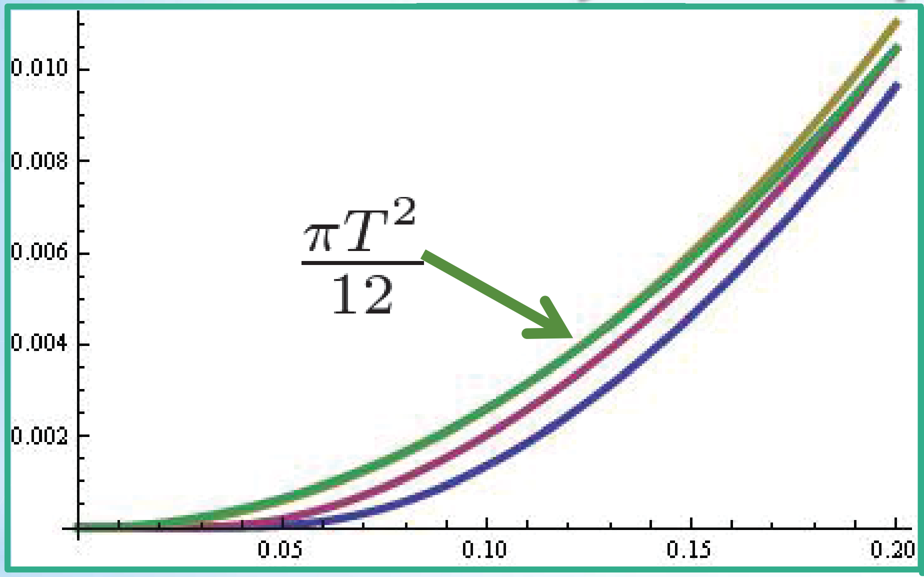
Energy flux from temporal shift

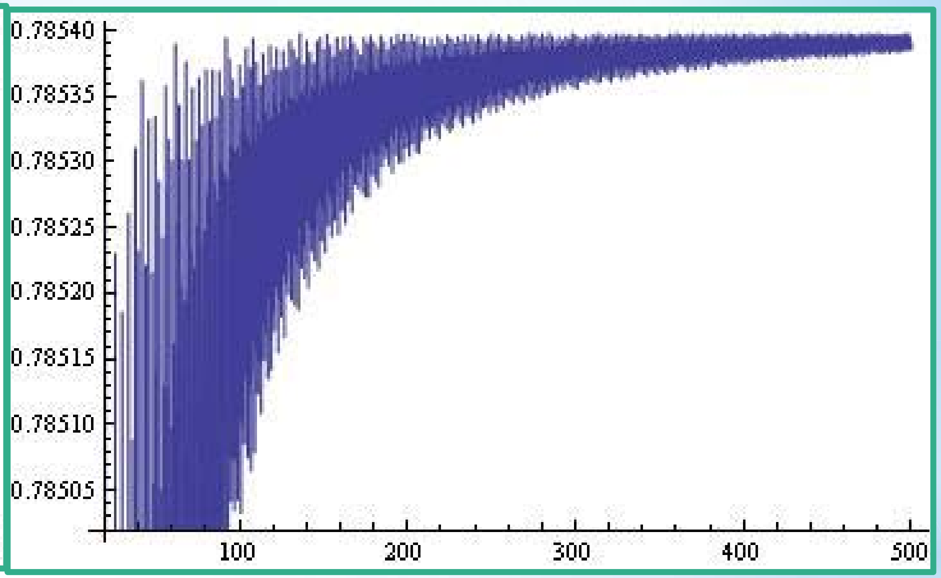
$$\frac{\partial \mathcal{Z}}{\partial \Delta} = \sum_w \sum_k \frac{-i}{w - i\epsilon(k)} \frac{\partial \epsilon(k)}{\partial k} \frac{\partial k}{\partial \Delta} = \sum_w \sum_k \frac{-iw/L}{w - i\epsilon(k)} \frac{\partial \epsilon(k)}{\partial k} =$$

$$= \sum_k \frac{i}{2LT} \epsilon\left(k\right) \frac{\partial \epsilon(k)}{\partial k} \tanh\left(\frac{\epsilon(k)}{2}\right)$$

Lattice CVE in (1+1)D from Shifted BC - momentum

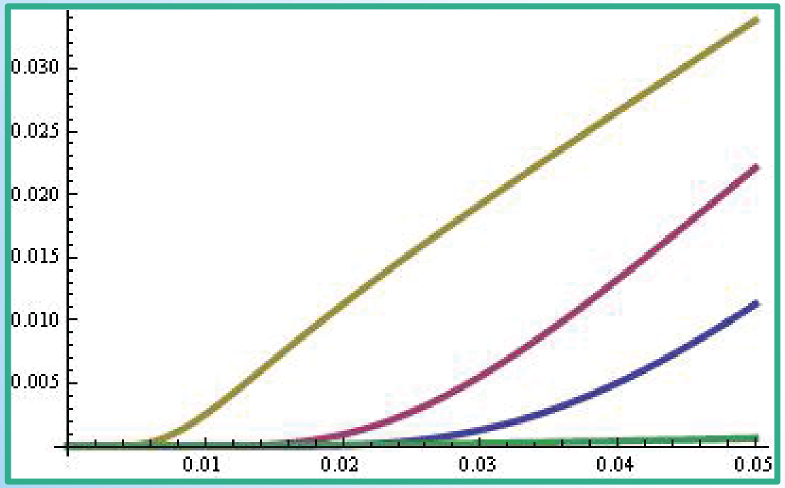
Weyl-Overlap Hamiltonian

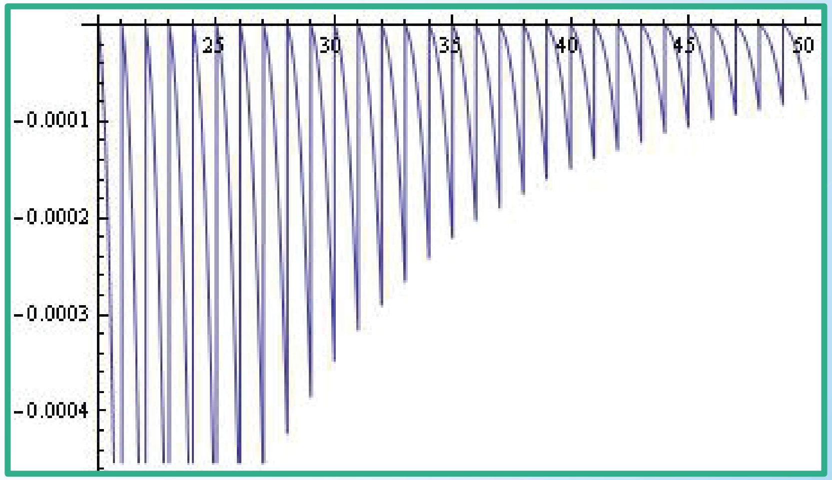




Naïve Dirac Hamiltonian

$$\epsilon(k) \sim \sin(k)$$





Conclusions

Lattice anomaly: closely related to doubling problem
And so is the anomalous transport

Strong IR sensitivity of anomalous transport

Anomalous transport coefficients: proper definition in linear response theory, constant fields not so nice

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Overlap fermions good for CME and CSE
For CVE we need Weyl lattice fermions,
A very difficult problem + IR sensitivity

Open problems for the lattice:

- Anomalous transport and SChSB [1408.4573]
- Radiative QED-like corrections (cond-mat!!!) [1304.4606,1307.3234,1407.3282,1408.4573]