Introduction to Lattice QCD II
A dedicated project

Karl Jansen

- **Leading order hadronic contribution to the muon anomalous magnetic moment**
  - motivation
  - describe the computation
  - new method for the lattice
  - results

A success of Quantum Field Theories: Electromagnetic Interaction

Quantum Electrodynamics (QED)

coupling of the electromagnetic interaction is small
$\Rightarrow$ perturbation theory 4-loop calculation

magnetic moment of the electron

$$\vec{\mu} = g_e \frac{e\hbar}{2m_e c} \vec{S}$$

$\vec{S}$ spin vector

deviation from $g_e = 2$: $a_e = (g_e - 2)/2$

$$\begin{aligned} a_e(\text{theory}) &= 1159652201.1(2.1)(27.1) \cdot 10^{-12} \\ a_e(\text{experiment}) &= 1159652188.4(4.3) \cdot 10^{-12} \end{aligned}$$

The question of the muon

magnetic moment of the muon

$$\vec{\mu} = g_\mu \frac{e\hbar}{2m_\mu c}\vec{S}$$

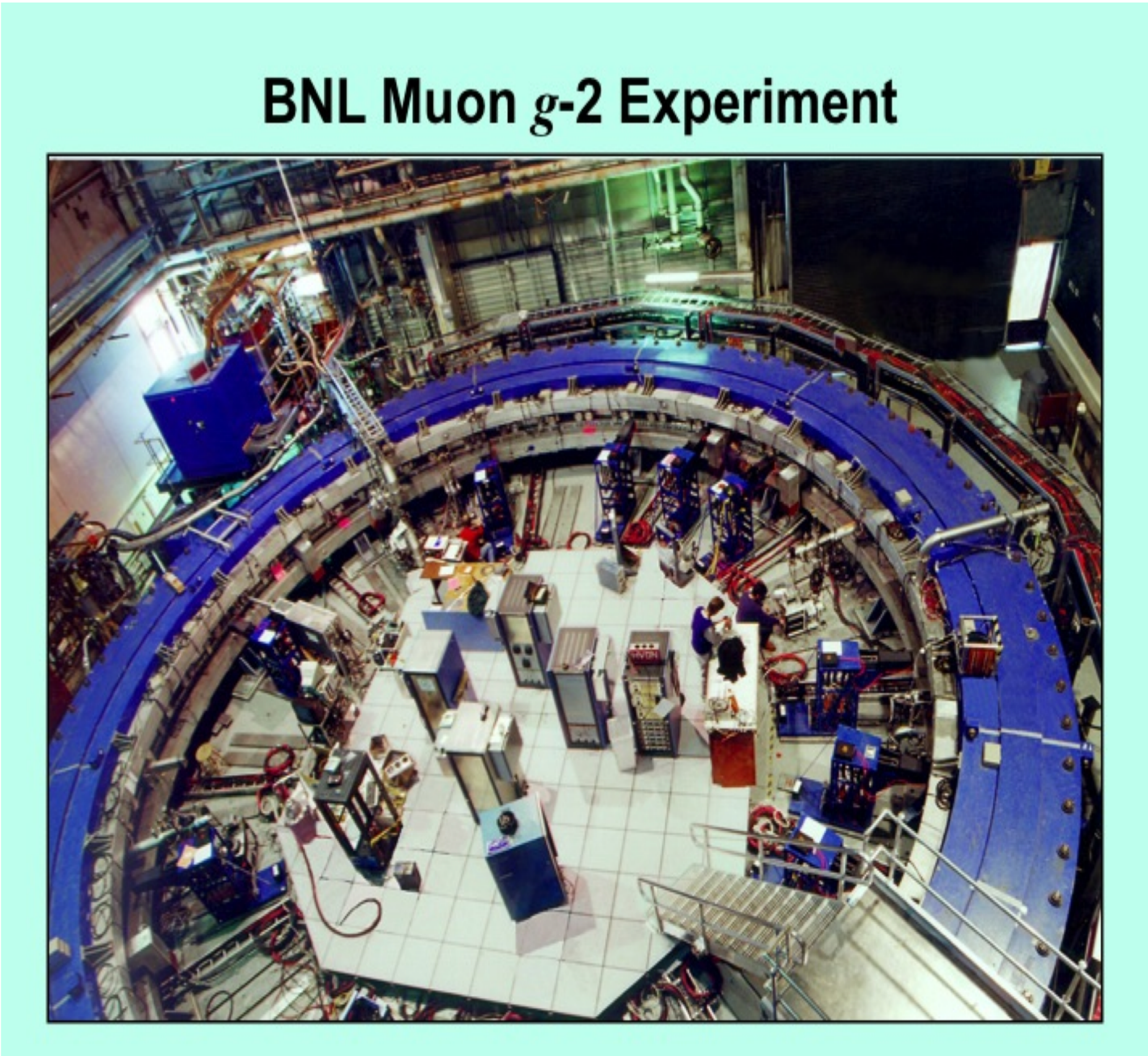
deviation from $g_\mu = 2$: $a_\mu = (g_\mu - 2)/2$

$$\begin{aligned} a_\mu(\text{theory}) &= 1.16591790(65) \cdot 10^{-3} \\ a_\mu(\text{experiment}) &= 1.16592080(63) \cdot 10^{-3} \end{aligned}$$

$\rightarrow$ there is a 3.2σ discrepancy

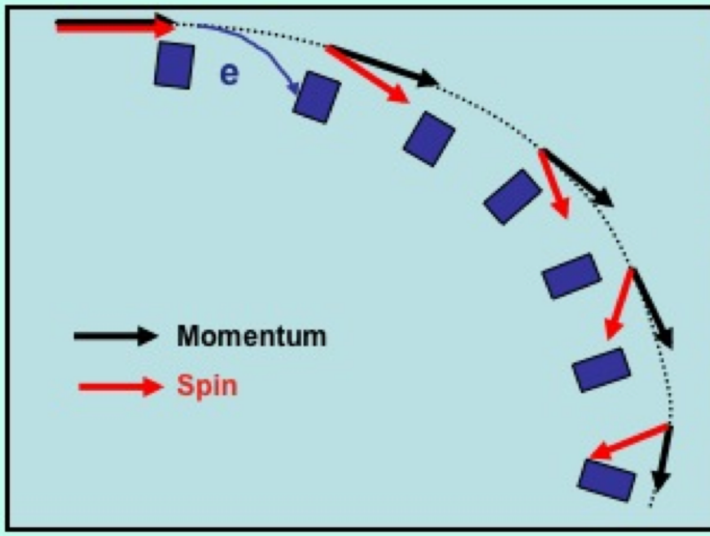
$$a_\mu(\text{experiment}) - a_\mu(\text{theory}) = 2.90(91) \cdot 10^{-9}$$

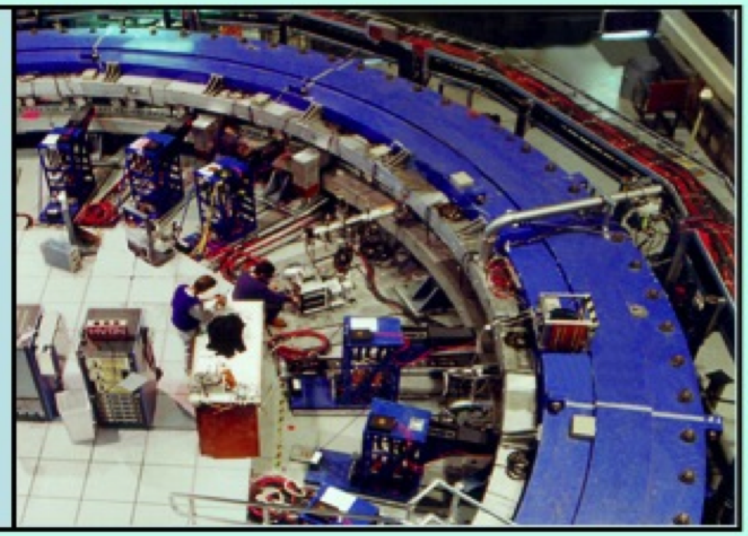
The storagering at BNL



The storagering at BNL

- a_μ is proportional to the difference between the spin precession and the rotation rate

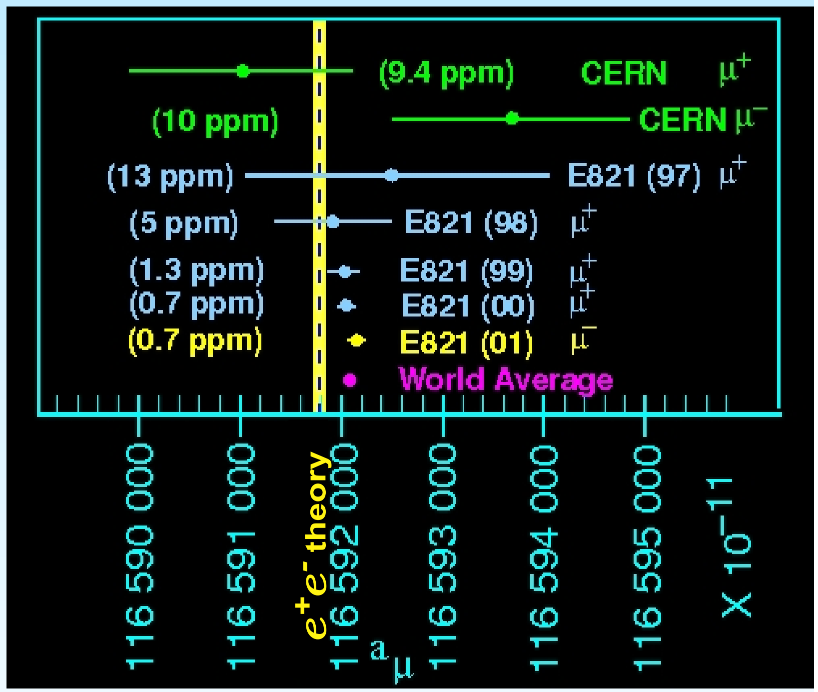




$$\Delta\omega = \omega_a = \left(\frac{g-2}{2}\right)\frac{eB}{mc}$$

Motivation

E821 achieved ± 0.54 ppm. The e^+e based theory is at the ~0.4 ppm level. Difference is ~3.6 σ



$$a_\mu^{exp} = 116\,592\,089(63) \times 10^{-11} \;\; (0.54\,\mathrm{ppm})$$

$$\Delta a_\mu \equiv a_\mu^{exp} - a_\mu^{SM} = (287 \pm 80) \times 10^{-11}$$

Theory: arXiv:1010.4180v1 [hep-ph] Davier, Hoecker, Malaescu, and Zhang, Tau2010

Why is this interesting?

$\rightarrow$ new proposed experiments at Fermilab (USA) and JPARC (Japan) ≈ 2015

brings down error

$\sigma_{\rm ex} = 6.3 \cdot 10^{-10} \rightarrow 1.6 \cdot 10^{-10}$

challenge: bring down theoretical error to same level

$\Rightarrow$ possibility

$$a_\mu(\text{experiment}) - a_\mu(\text{theory}) > 5\sigma$$

- possible breakdown of standard model?
- $a_\mu = a_\mu^{\rm QED} + a_\mu^{\rm weak} + a_\mu^{\rm QCD} + a_\mu^{\rm NP}$

$a_\mu^{NP} \propto m^2_{\rm lepton}/m^2_{\rm new}$

$\rightarrow$ muon ideal to discover new physics

for τ experimental results too imprecise

Is the lattice important?

(Snowmass report, T. Blum et.al.)

$\rightarrow$ size of various contributions to a_μ

	Value ($\times 10^{-11}$) units
QED ($\gamma + \ell$)	$116\,584\,718.951 \pm 0.009 \pm 0.019 \pm 0.007 \pm 0.077_\alpha$
HVP(lo) (Davier11)	$6\,923 \pm 42$
HVP(lo) (Hagiwara11)	$6\,949 \pm 43$
HVP(ho) (Hagiwara:2011)	-98.4 ± 0.7
HLbL	105 ± 26
EW	154 ± 1
Total SM (Davier11)	$116\,591\,802 \pm 42_{\text{H-LO}} \pm 26_{\text{H-HO}} \pm 2_{\text{other}}\ (\pm 49_{\text{tot}})$
Total SM (Hagiwara:2011)	$116\,591\,828 \pm 43_{\text{H-LO}} \pm 26_{\text{H-HO}} \pm 2_{\text{other}}\ (\pm 50_{\text{tot}})$

$\Rightarrow$ non-perturbative hadronic contributions are most important

Most recent experimemtal determination: **E821 collaboration**

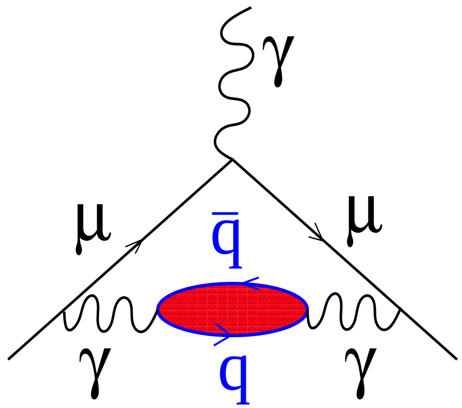
$$a_\mu^{\text{E821}} = (116\,592\,089 \pm 63) \times 10^{-11} \quad (0.54\,\text{ppm})$$

comparison

$$\Delta a_\mu(\text{E821} - \text{SM}) = (287 \pm 80) \times 10^{-11}\ (\text{Davier11})$$
$$\Delta a_\mu(\text{E821} - \text{SM}) = (261 \pm 78) \times 10^{-11}\ (\text{Hagiwara11})$$

$\Rightarrow$ larger than $3 - \sigma$ discrepancy

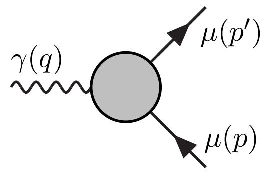
First principle calculation of leading order hadronic contribution, $a_\mu^{\rm had}$



- $a_\mu^{\rm had}$ contributes almost 60% of theoretical error
- needs to be computed non-perturbatively
- computation of vacuum polarization tensor

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2)$$

The contributions to $g_\mu - 2$



The vertex

– p, p' incoming and outgoing momenta

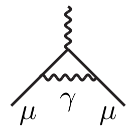
– $q = p - p'$ photon momentum

- put muon in magnetic field $\vec{B}$ interaction Hamiltonian $H = \vec{\mu}\vec{B}$

- measure interaction of muon with photon (magnetic field)

The simplest, QED contribution to $g_l - 2$

Schwinger, 1948



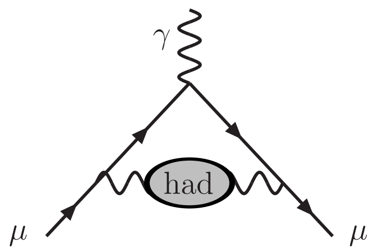
$$\Rightarrow \; a_\mu^{\mathrm{QED}(1)} = \frac{\alpha}{2\pi} \, ,$$

α QED fine structure constant

- Schwinger's calculation: $g_e = 2.00232$
- experimental result: $g_e = 2.00238(10)$ (Foley, 1948)

First success of quantum field theory

The leading order hadronic contribution, a_μ^{had}



- a_μ^{had} contributes almost 60% of theoretical error
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$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2)$$

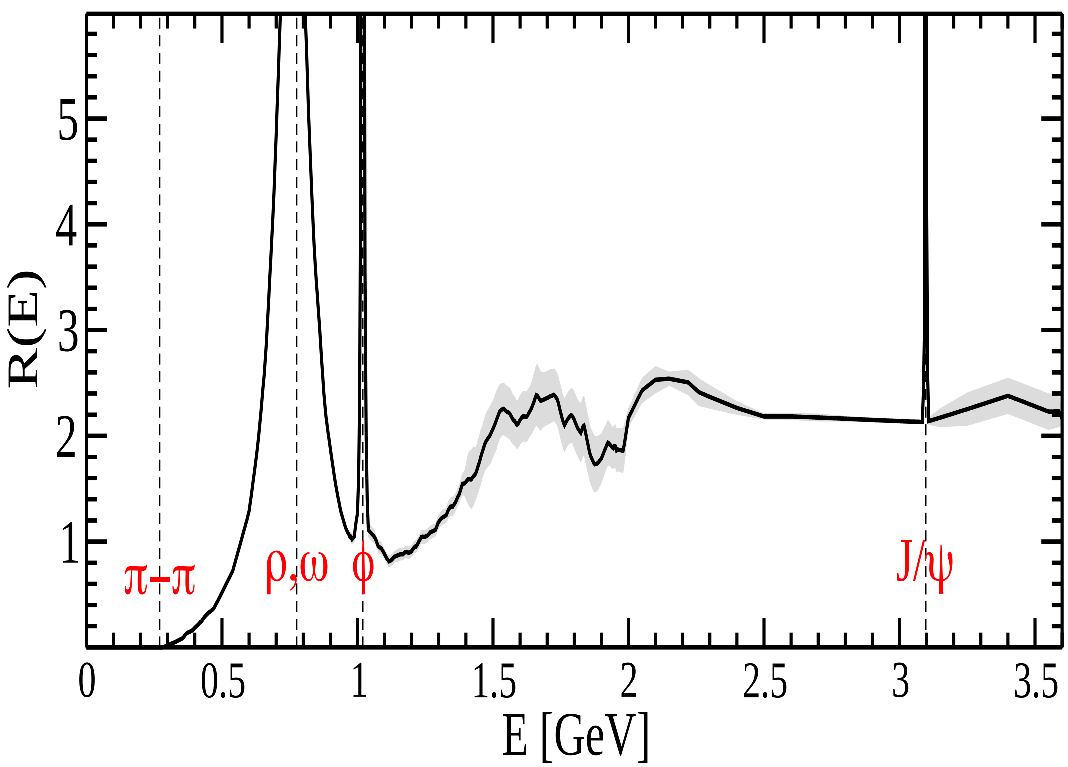
Relation to experimental extraction

connection between real and imaginary part of $\Pi(q^2)$

$$\Pi(q^2) - \Pi(0) = \frac{q^2}{\pi} \int_0^\infty ds \frac{\mathrm{Im}\Pi(s)}{s(s-q^2)}$$

$\mathrm{Im}\Pi(s)$ related to experimental data of total cross section in e^+e^- annihilation

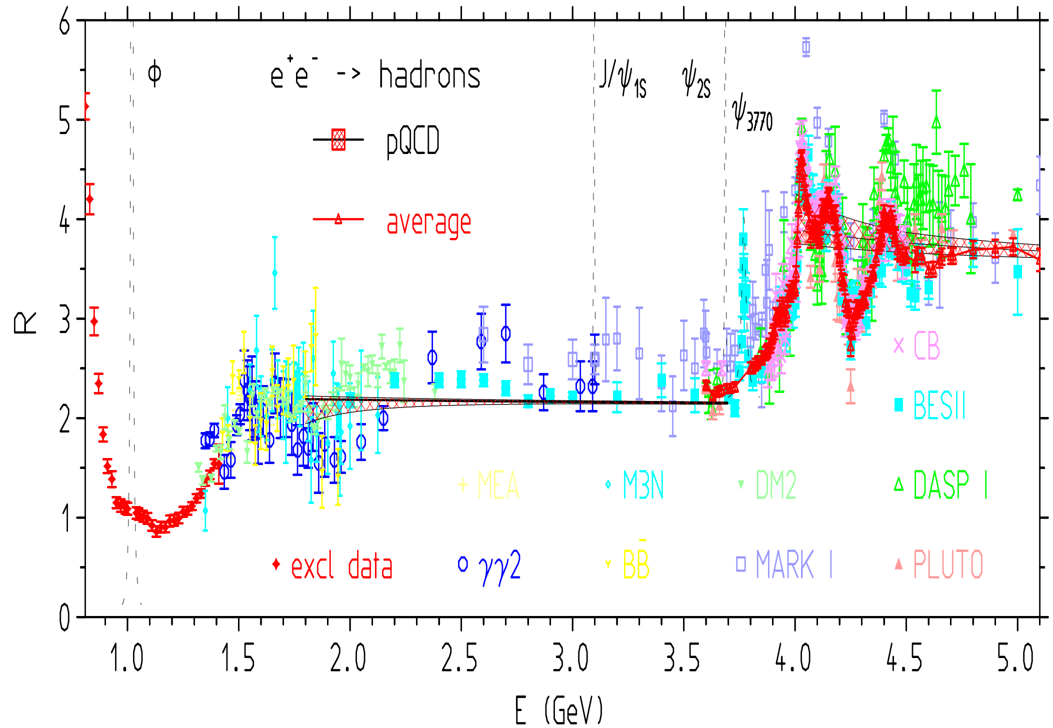
$$\mathrm{Im}\Pi(s) = \frac{\alpha}{3} R(s)$$



important contributions

- $\rho, \omega \ (N_f = 2)$
- $\Phi \ (N_f = 2+1)$
- $J/\Psi \ (N_f = 2+1+1)$

How the data really look like



- demanding analysis of $O(1000)$ channels
  Jegerlehner, Nyffeler, Phys.Rep.

Euclidean expression for hadronic contribution

$$a_\mu^{\rm had} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2} F\left(\frac{Q^2}{m_\mu^2}\right) \left(\Pi(Q^2) - \Pi(0)\right)$$

with $F\left(\frac{Q^2}{m_\mu^2}\right)$ a known function T. Blum

$\rightarrow$ need to compute vacuum polaization function

Continuum:

$$\Pi_{\mu\nu}(Q) = i \int d^4x e^{iQ\cdot(x-y)} \langle 0|T J_\mu(x) J_\nu(y)|0\rangle$$

J_μ hadronic electromagnetic current

$$J_\mu(x) = \sum_f e_f \bar{\psi}_f(x)\gamma_\mu\psi_f(x) = \frac{2}{3}\bar{u}(x)\gamma_\mu u(x) - \frac{1}{3}\bar{d}(x)\gamma_\mu d(x) + \cdots$$

local vector current given is conserved

$$\partial_\mu J_\mu(x) = 0 \qquad \Rightarrow \qquad Q_\mu \Pi(Q)_{\mu\nu} = 0$$

eliminating the factor $Q_\mu Q_\nu - Q^2\delta_{\mu\nu}$

$\Rightarrow$ obtain $\Pi(Q^2)$

Going to the lattice

- work with $N_f = 2$ **twisted mass fermions**

$$\mathcal{S}_{tm} = \sum_x \bar{\chi}(x) \left[D_W + m_0 + i\mu\gamma_5\tau_3 \right] \chi(x)$$

- use the conserved lattice current

Exercize: derive conserved current

use vector transformation

$$\delta_V \chi(x) = i\epsilon_V(x)\tau\chi(x) \; , \quad \delta_V \bar{\chi}(x) = -i\bar{\chi}(x)\tau\epsilon_V(x)$$

$$\tau = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} = \tfrac{1}{6}\mathbf{1} + \tfrac{1}{2}\tau^3$$

Can we also use the local current?

Conserved lattice current

$$J_\mu^{tm}(x) = \frac{1}{2}\left(\bar{\chi}(x)\tau(\gamma_\mu - r)U_\mu(x)\chi(x+\hat{\mu}) + \bar{\chi}(x+\hat{\mu})\tau(\gamma_\mu + r)U_\mu^\dagger(x)\chi(x)\right)$$

satisfying

$$\partial_\mu^* J_\mu^{tm}(x) = 0$$

with ∂_μ^* backward lattice derivative

Lattice vacuum polarization tensor

Fourier transformation of conserved vector current:

$$J_\mu^{tm}(\hat{Q}) = \sum_x e^{iQ\cdot(x+\hat{\mu}/2)} J_\mu^{tm}(x) \ , \quad \hat{Q}_\mu = 2\sin\left(\frac{Q_\mu}{2}\right)$$

leading to

$$\Pi_{\mu\nu}(\hat{Q}) = \frac{1}{V}\sum_{x,y} e^{iQ\cdot(x+\hat{\mu}/2-y-\hat{\nu}/2)} \langle J_\mu^{tm}(x) J_\nu^{tm}(y)\rangle$$

from which we extract the vacuum polarization function

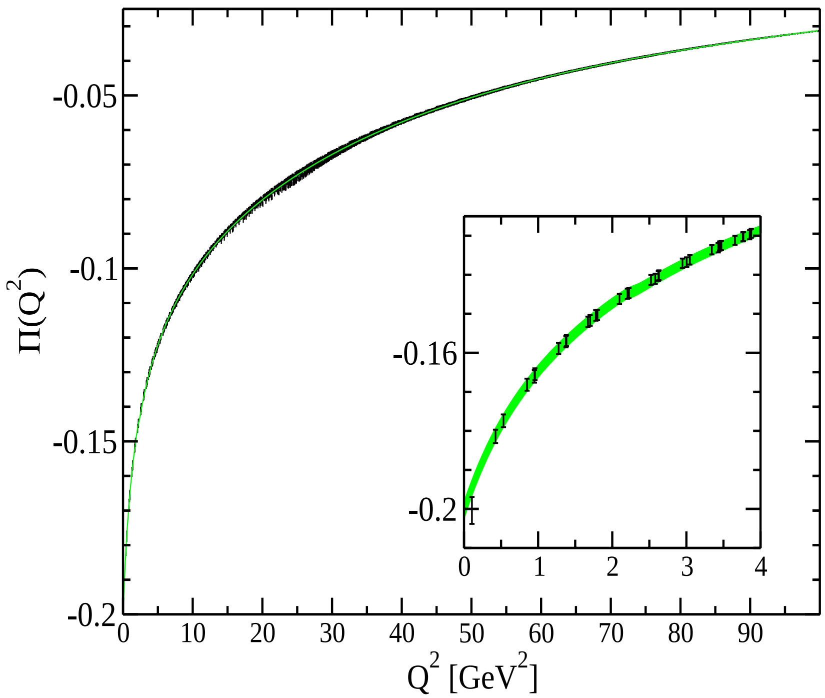
$$\Pi_{\mu\nu}(\hat{Q}) = (\hat{Q}_\mu\hat{Q}_\nu - \hat{Q}^2\delta_{\mu\nu})\Pi(\hat{Q}^2)$$

Fit to vacuum polarization function

Fit function

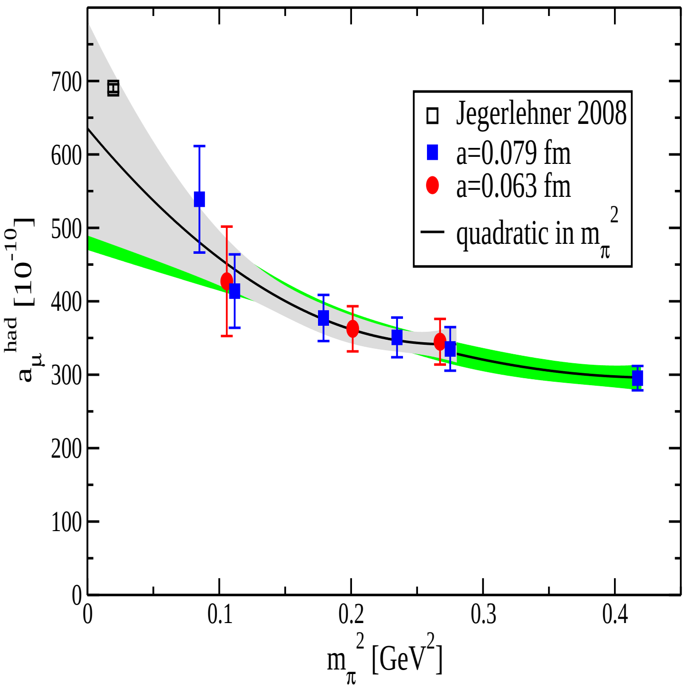
$$\Pi_{M,N}(Q^2) = -\frac{5}{9}\sum_{i=1}^{M}\frac{m_i^2}{Q^2+m_i^2} + \sum_{n=0}^{N} a_n (Q^2)^n$$

typically $i = 1, 2, 3$, $n = 0, 1, 2, 3$ (systematic error)



Do we control hadronic vacuum polarisation?

(Xu Feng, Dru Renner, Marcus Petschlies, K.J.; Lattice 2010)



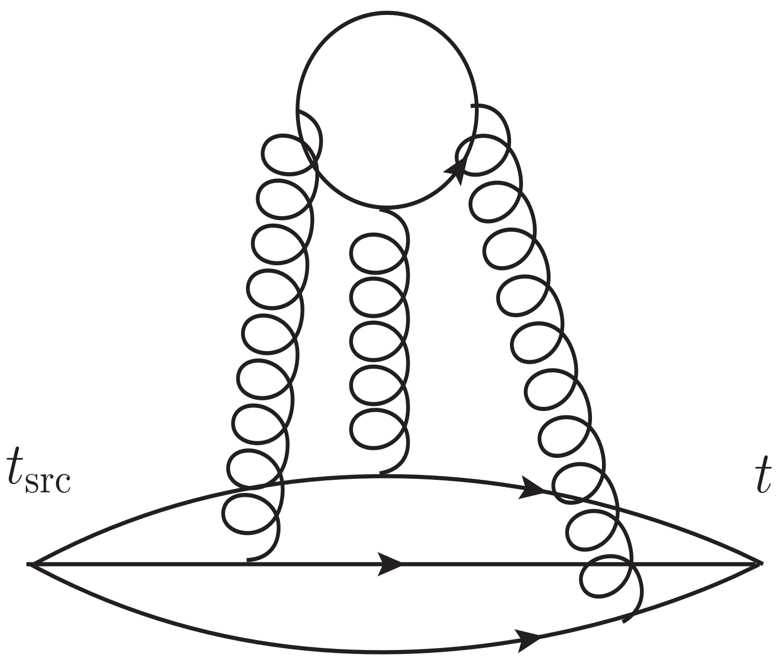
- experiment: $a_{\mu,N_f=2}^{\mathrm{hvp,exp}} = 5.66(05)10^{-8}$
- lattice: $a_{\mu,N_f=2}^{\mathrm{hvp,old}} = 2.95(45)10^{-8}$

$\rightarrow$ misses the experimental value
$\rightarrow$ order of magnitude larger error

- have used different volumes
- have used different values of lattice spacing

Dis-connected contribution

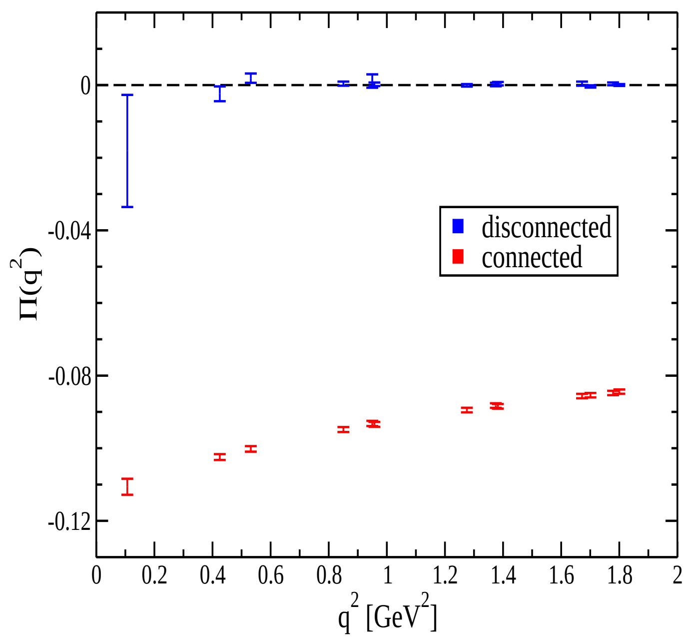
a graph representing dis-conected contributions



$\rightarrow$ has been basically always be neglected

Can it be the dis-connected contribution?

(Xu Feng, Dru Renner, Marcus Petschlies, K.J.)



- dedicated effort
- have included dis-connected contributions for first time
- smallness consistent with chiral perturbation theory (Della Morte, Jüttner)

Different extrapolation to the physical point

lattice: simulations at unphysical quark masses, demand only

$$\lim_{m_{\mathrm{PS}}\to m_\pi} a_l^{\mathrm{hvp,latt}} = a_l^{\mathrm{hvp,phys}}$$

$\Rightarrow$ flexibility to define $a_l^{\mathrm{hvp,latt}}$

standard definitions in the continuum

$$a_l^{\mathrm{hvp}} = \alpha^2 \int_0^\infty dQ^2 \tfrac{1}{Q^2}\omega(r)\Pi_R(Q^2)$$

$$\Pi_R(Q^2) = \Pi(Q^2) - \Pi(0)$$

$$\omega(r) = \frac{64}{r^2\left(1+\sqrt{1+4/r}\right)^4\sqrt{1+4/r}}$$

with $r = Q^2/m_l^2$

Redefinition of $a_l^{\mathrm{hvp,latt}}$

redefinition of r for lattice computations

$$r_{\mathrm{latt}} = Q^2 \cdot \frac{H^{\mathrm{phys}}}{H}$$

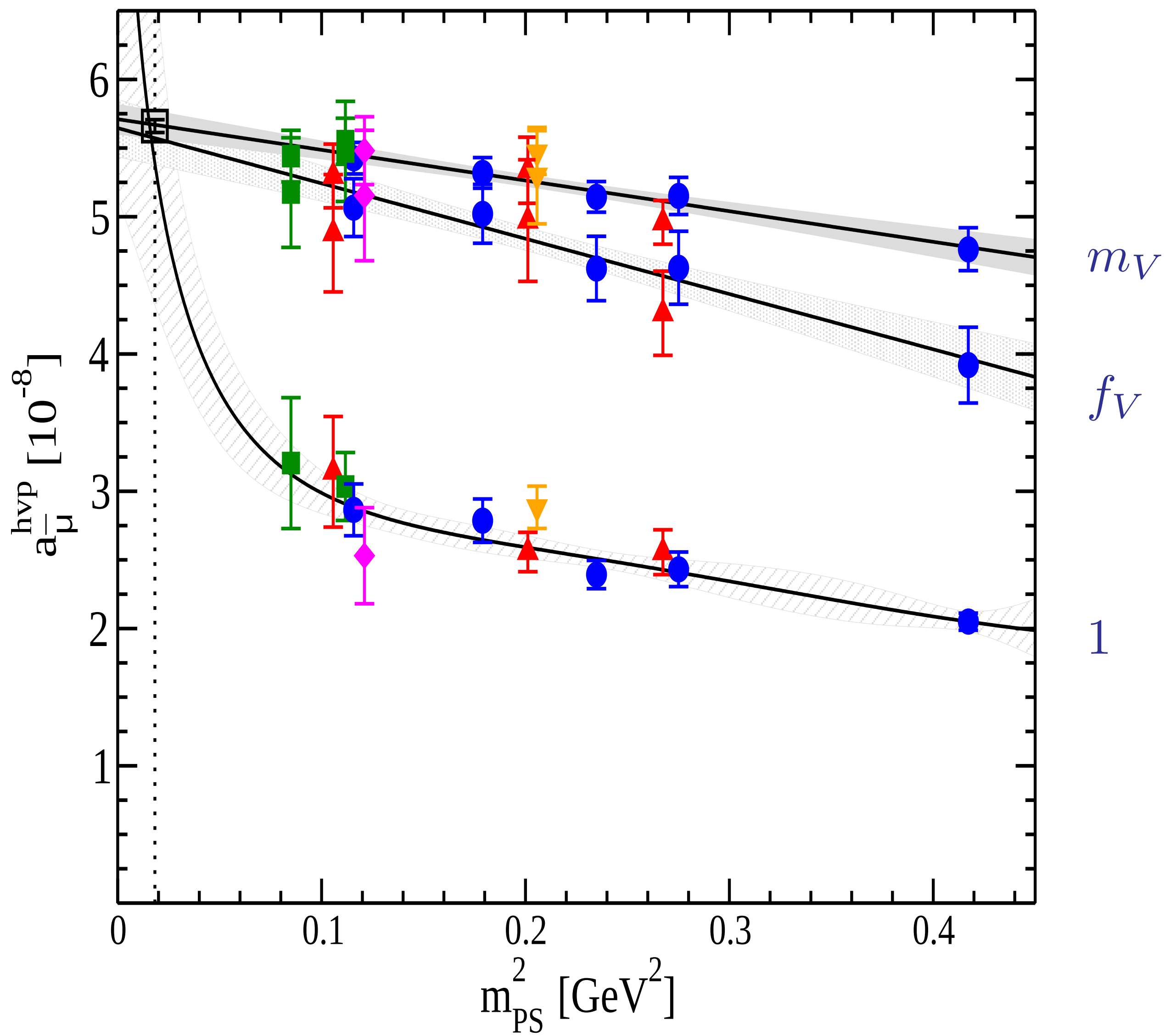
choices

- r_1: $H = 1$; $H^{\mathrm{phys}} = 1/m_l^2$
- r_2: $H = m_V^2(m_{\mathrm{PS}})$; $H^{\mathrm{phys}} = m_\rho^2/m_l^2$
- r_3: $H = f_V^2(m_{\mathrm{PS}})$; $H^{\mathrm{phys}} = f_\rho^2/m_l^2$

each definition of r will show a different dependence on m_{PS} but agree *by construction* at the physical point

remark: strategy often used in continuum limit extrapolations, e.g. charm quark mass determination

comparison using r_1, r_2, r_3



A new result from the lattice

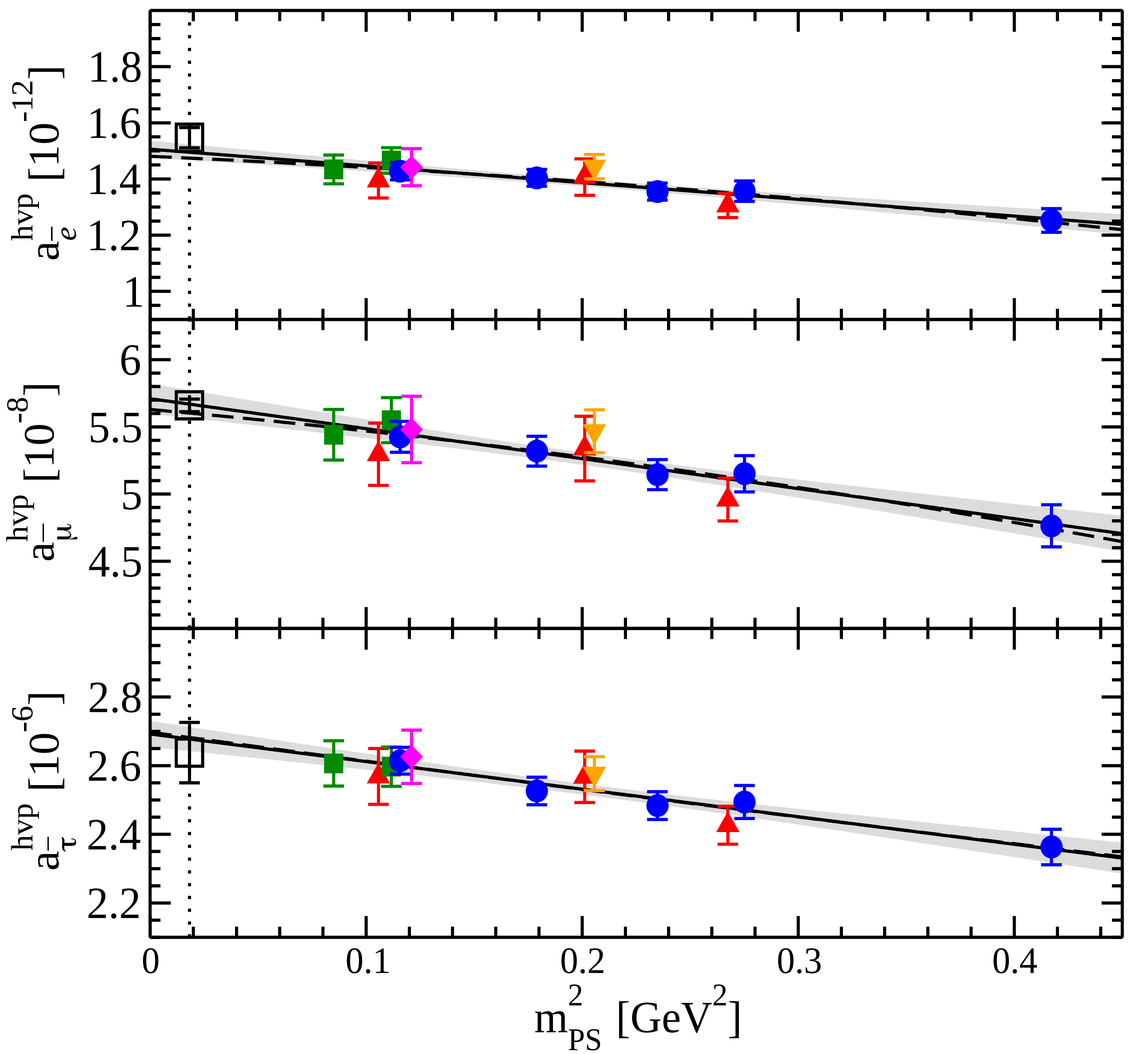
- experimental value: $a_{\mu,N_f=2}^{\mathrm{hvp,exp}} = 5.66(05)10^{-8}$
- from our old analysis: $a_{\mu,N_f=2}^{\mathrm{hvp,old}} = 2.95(45)10^{-8}$

$\longrightarrow$ misses the experimental value
$\longrightarrow$ order of magnitude larger error

- from our new analysis: $a_{\mu,N_f=2}^{\mathrm{hvp,new}} = 5.66(11)10^{-8}$

$\longrightarrow$ error (including systematics) almost matching experiment

Anomalous magnetic moments, a check

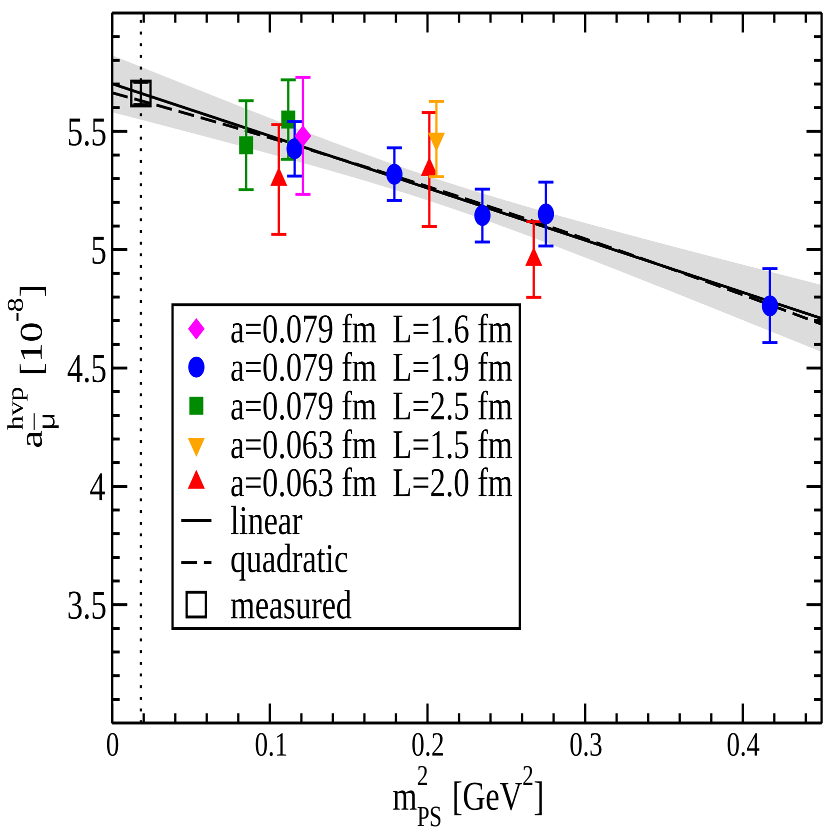


$$a_e^{\rm hlo} = 1.513\,(43)\cdot 10^{-12} \quad \text{LQCD}$$
$$a_e^{\rm hlo} = 1.527\,(12)\cdot 10^{-12} \quad \text{PHENO}$$

$$a_\mu^{\rm hlo} = 5.72\,(16)\cdot 10^{-8} \quad \text{LQCD}$$
$$a_\mu^{\rm hlo} = 5.67\,(05)\cdot 10^{-8} \quad \text{PHENO}$$

$$a_\tau^{\rm hlo} = 2.650\,(54)\cdot 10^{-6} \quad \text{LQCD}$$
$$a_\tau^{\rm hlo} = 2.638\,(88)\cdot 10^{-12} \quad \text{PHENO}$$

anomalous magnetic moment of muon



- have used different volumes
- have used different values of lattice spacing
- have included dis-connected contributions

⇒ can control systematic effects

Why it works: fitting the Q^2 dependence

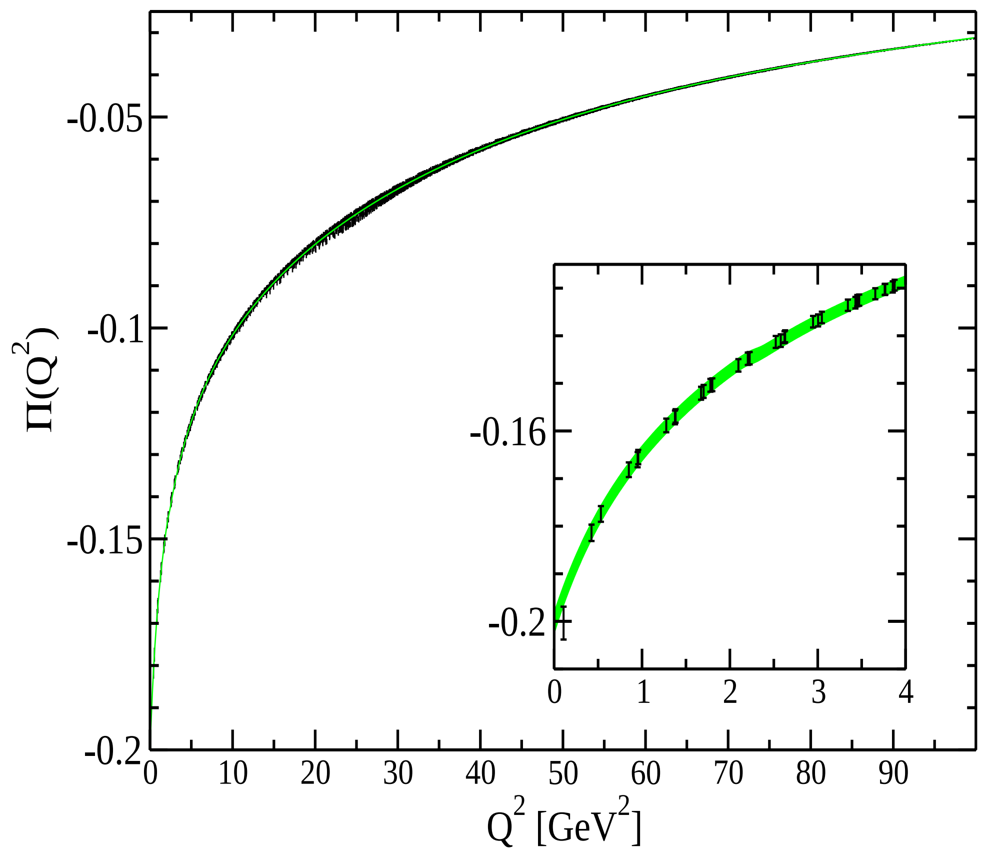
Fit function

$$\Pi_{M,N}(Q^2) = -\tfrac{5}{9}\sum_{i=1}^{M} \frac{m_i^2}{Q^2+m_i^2} + \sum_{n=0}^{N} a_n (Q^2)^n$$

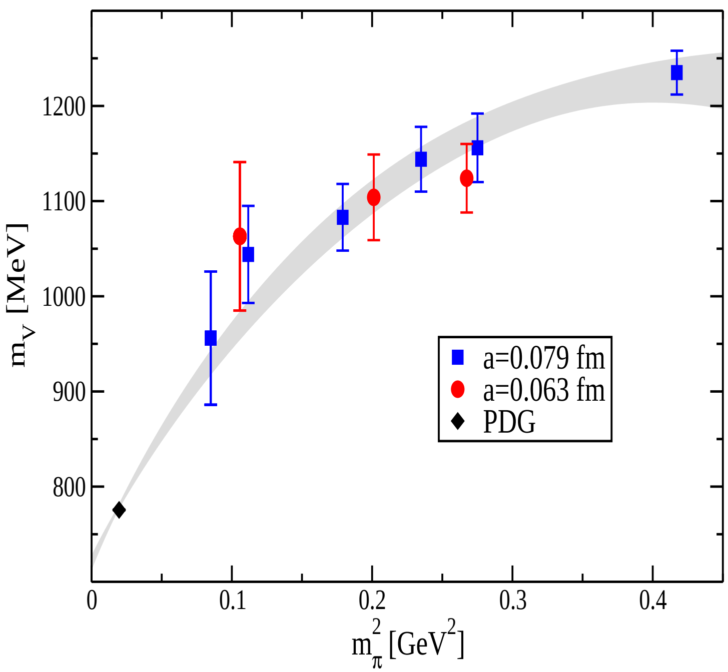
$i=1$: ρ-meson $\rightarrow$ dominant contribution $\propto 5.010^{-8}$

$i=2$: ω-meson $\propto 3.710^{-9}$

$i=3$: ϕ-meson $\propto 3.410^{-9}$



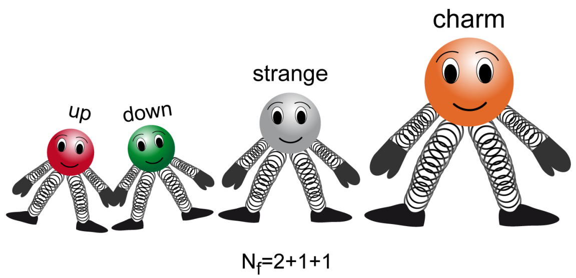
Why it works



- $a_\mu^{\rm hvp} \approx \frac{4}{3}\alpha^2 g_V^2 \frac{m_\mu^2}{m_V^2}, \quad \frac{m_\mu^2}{m_V^2} \ll 1$
- m_V consistent with resonance analysis (Feng, Renner, K.J.)
- strong dependence on $m_{\rm PS}$

Extension to the first two quark families

(F. Burger, X. Feng, G. Hotzel, M. Petschlies, D. Renner, K.J.)

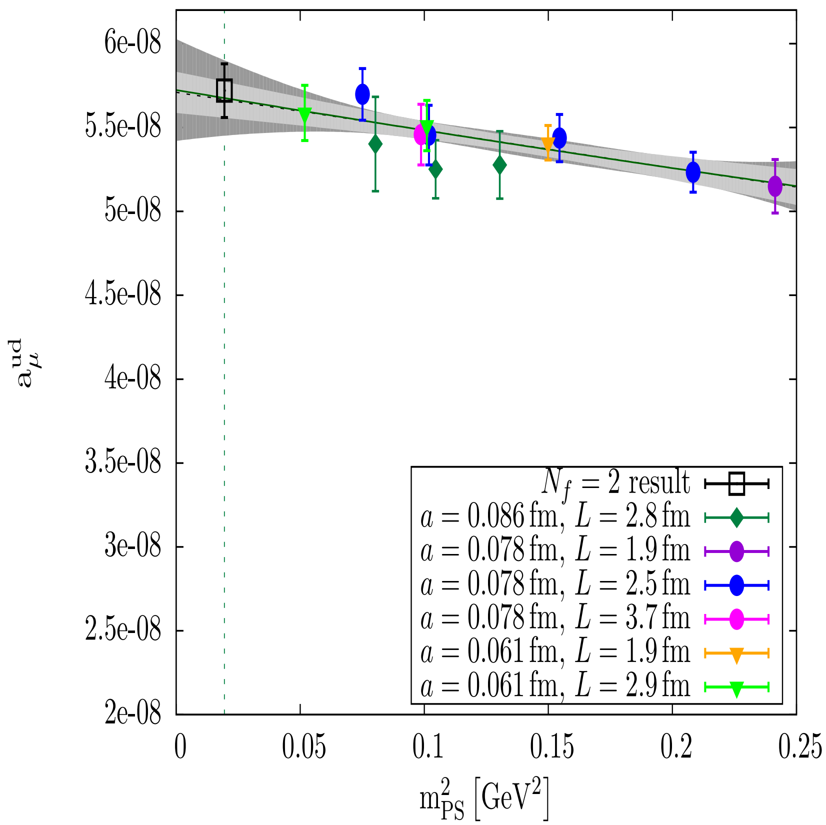


- previous work: only mass-degenerate up and down quarks

- new extension: including strange and charm:
  - no ambiguity to disentangle flavour contributions
  - charm contribution $a_{\mu,c}^{had} \propto O(140 \cdot 10^{-11})$
    $\Rightarrow$ same order of magnitude as light-by-light contribution $(105 \cdot 10^{-11})$

- first calculation with 4 flavours of active quarks

The light quark contribution with active sea strange and charm quarks

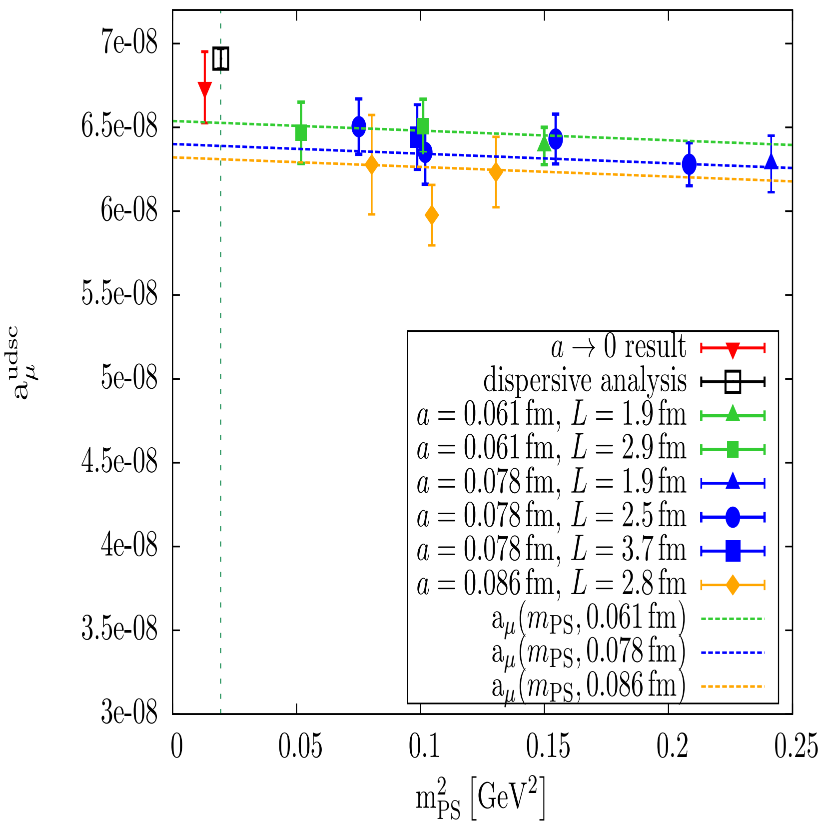


- improved method works als here
- light quark contribution:
  no visible effect of strange and charm

$$a_{\mu,\mathrm{ud}}^{\mathrm{hvp}} = 5.67(11) \cdot 10^{-8} \quad (N_f = 2+1+1)$$

$$a_{\mu,\mathrm{ud}}^{\mathrm{hvp}} = 5.72(16) \cdot 10^{-8} \quad (N_f = 2)\,,$$

The full four-flavour contribution



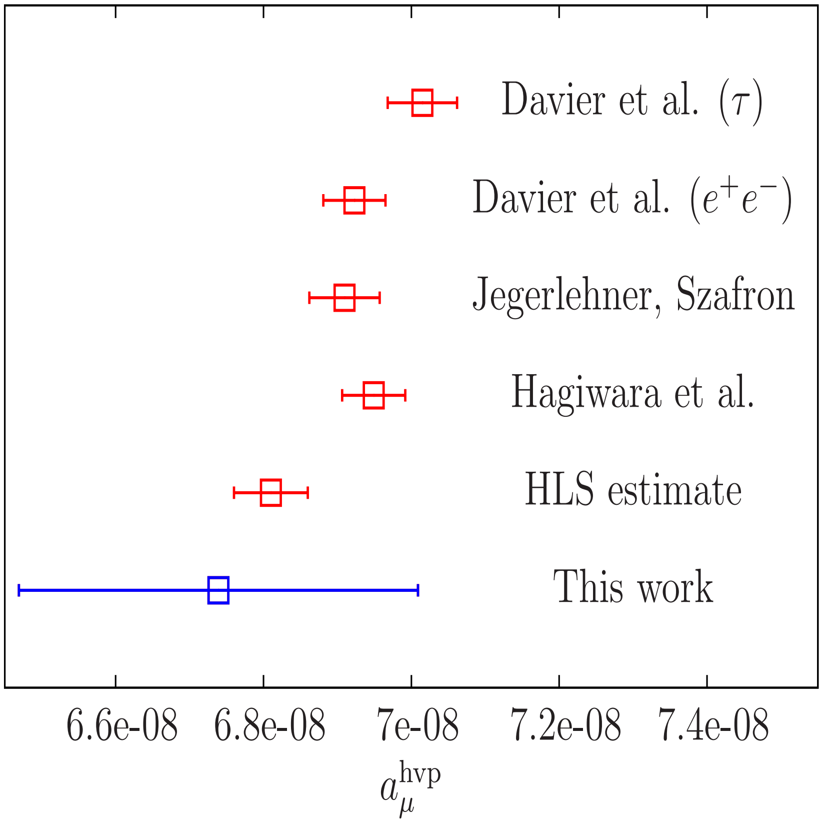
- fit function:

$$a_\mu(m_{\rm PS}, a) = A + B\ m_{PS}^2 + C\ a^2$$

$$a_\mu^{\rm hvp} = 6.74(21)(18) \cdot 10^{-8} \quad (N_f = 2+1+1)$$

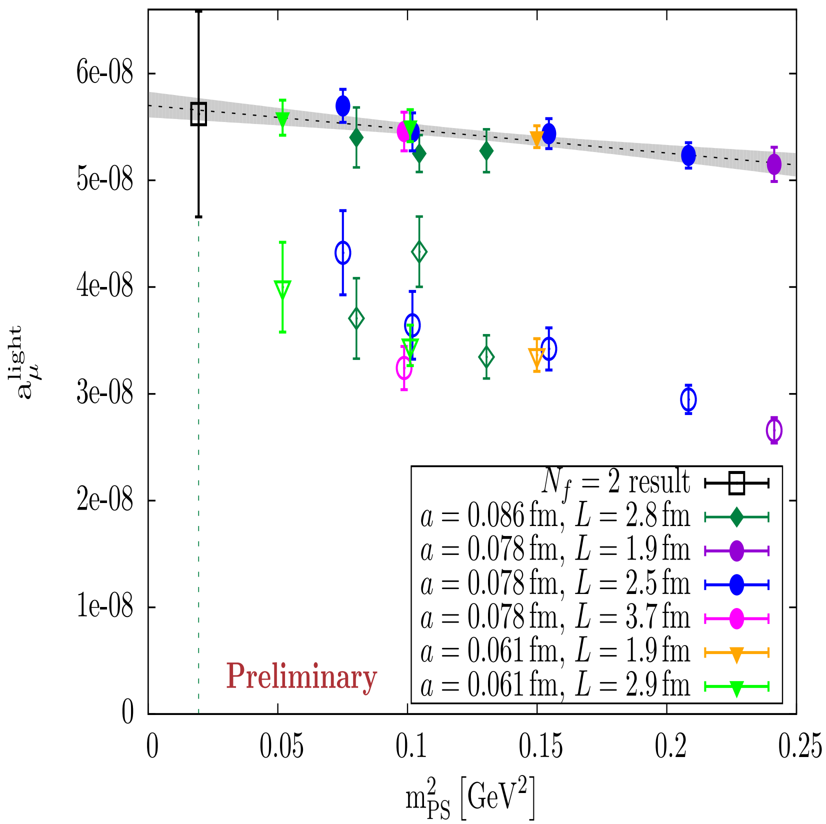
$$a_\mu^{\rm hvp} = 6.91(01)(05) \cdot 10^{-8} \quad \text{(dispersive analysis)}\ .$$

Comparison to phenomenological analyses



- lattice can provide 4-flavour result
- accuracy does not (yet) match phenomenological results
- still first principle result from purely QCD
- no model assumptions, no experimental input

Muon anomalous magnetic moment at the physical point



improved method

standard method

- first results by **ETMC** at the physical pion mass
- light quark contribution to a_μ^{had}
  $\rightarrow$ confirms extrapolation of improved method
- computations ongoing
  $\rightarrow$ increasing statistics

Open questions

- Fit function: do we really have small momentum region under control?
- how can we handle the instable ρ-meson
  (analytical continuation? moment method?)
- generalized boundary conditions:
- when do the dis-connected contributions become important?

Next steps

Fit function

$$\Pi_{M,N}(Q^2) = -\frac{5}{9}\sum_{i=1}^{M} \frac{m_i^2}{Q^2+m_i^2} + \sum_{n=0}^{N} a_n (Q^2)^n$$

add $i = 4$: J/Ψ, $i = 5 \ldots$

- simulations with dynamical
  up, down, strange and charm quarks
  $\rightarrow$ unique opportunity

- avoids ambiguity with experiment comparison
  (what counts for $N_f = 2$?)

- generalized boundary conditions: $\Psi(L + a\hat{\mu}) = e^{i\theta}\Psi(x)$
  $\rightarrow$ θ *continuous* momentum
  $\rightarrow$ allows to realize arbitrary momenta on the lattice

The transport to FermiLab

What is expected

Fermilab E989: Approved January 2011

- Re-locate the $(g - 2)$ storage ring to Fermilab
- Use the many proton storage rings to form the ideal proton beam
- Use one of the antiproton rings as a 900 m decay line to produce a pure muon beam
- Accumulate 21 times the statistics
- Improve the systematic errors
- Final goal: At least a factor of 4 more precise over E821

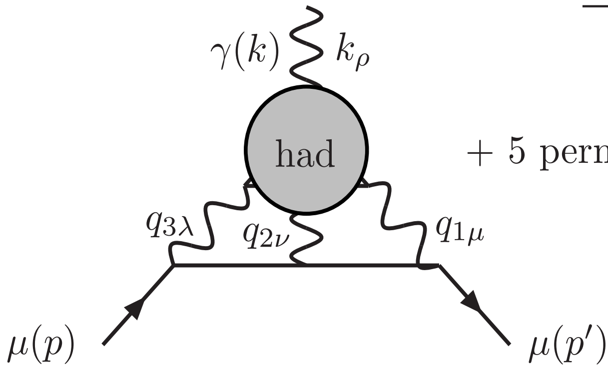
The accuracy question

We need a precision $< 1\%$

- include explicit isospin breaking
- include electromagnetism
- need computation of light-by-light contribution
- reach small quark mass $\rightarrow$ physical point

Much ado for young people

Next, α_s^3, contribution



$+ 5$ permutations of the q_i

termed: *light-by-light scattering*

involves 4-point function

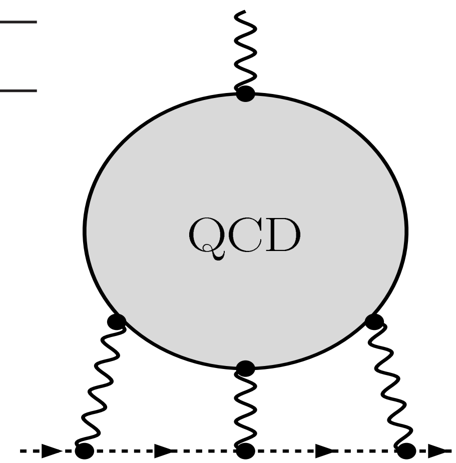
$$\Pi_{\mu\nu\alpha\beta}(q_1, q_2, q_3) = \int_{xyz} e^{iq_1\cdot x + iq_2\cdot y + iq_3\cdot z} \langle j_\mu(0) j_\nu(x) j_\alpha(y) j_\beta(z)\rangle$$

j_μ electromagnetic quark current

$$j_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c$$

Importance of light-by-light contribution

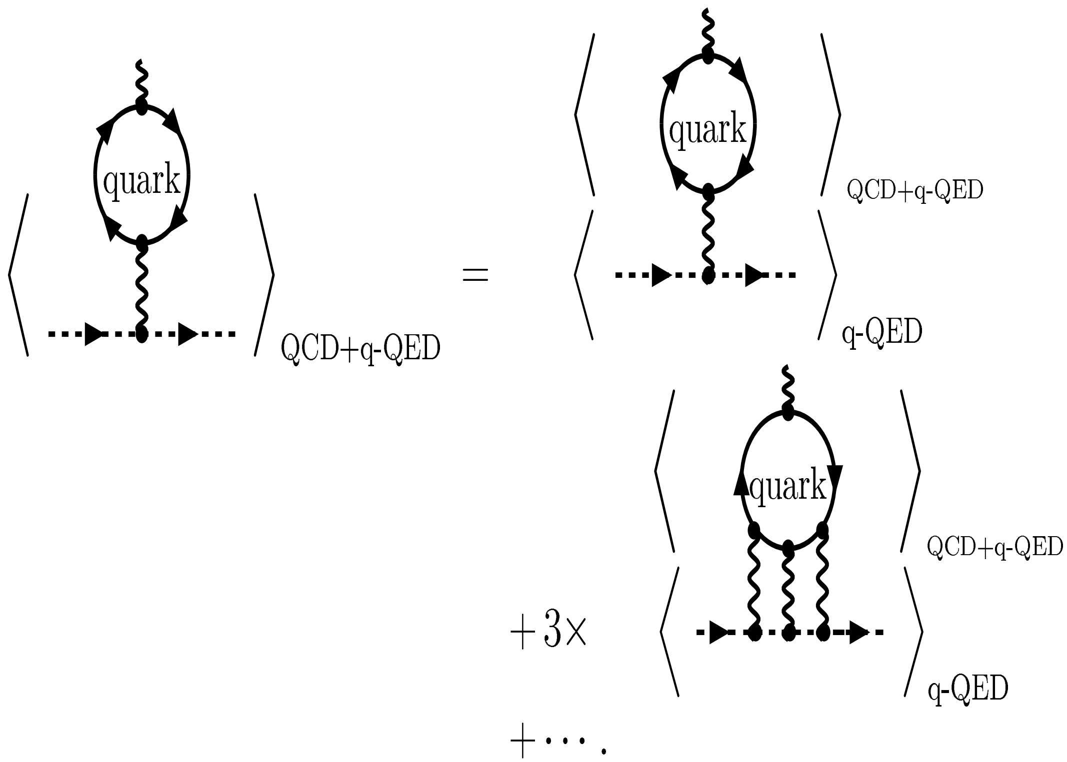
QED	116 584 71.8 951 (9)(19)(7)(77)
EW	15.4 (2)
QCD	LO HVP
	692.3 (4.2)
	694.91 (3.72) (2.10)
	701.5 (4.7)
NLO HVP	-9.79 (9)
HLbL	10.5 (2.6)



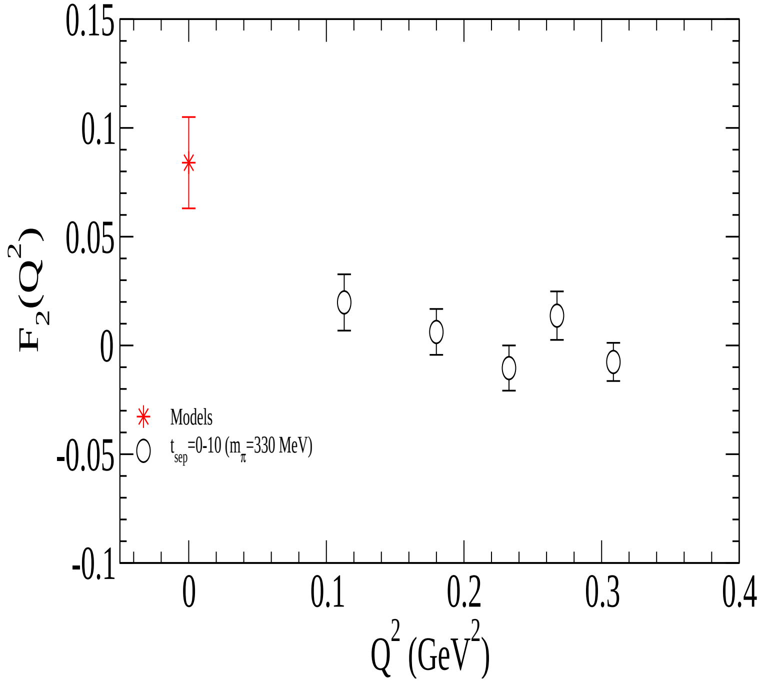
units 10^{-10}

One idea: reduce to difference of 3-point functions

(T. Blum, S. Chowdhury, M. Hayakawa, T. Izubuchi)



First promising results



- seem to get a signal
- not so far from model calculations
- need further tests

Summary

- lattice calculation of muon anomalous magnetic moment
- looked hopeless first $\leftarrow$ order of magnitude larger error than experiment
- introduced new method $\rightarrow$ start to match experimental accuracy
- outlook
  - include first two quark generations
  - include isospin breaking and electromagnetism
  - attack light-by-light scattering