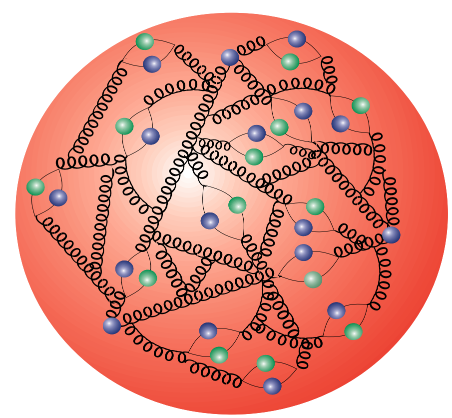
Introduction to Lattice QCD II

Karl Jansen



- **Task: compute the proton mass**

  – need an action
  – need an algorithm
  – need an observable
  – need a supercomputer

- **Anomalous magnetic moment of Muon**

Monte Carlo Method

$$\langle f(x) \rangle = \int dx f(x) e^{-x^2} / \int dx e^{-x^2}$$

$\rightarrow$ solve numerically:

- generate succesively Gaussian random numbers x_i
- do this N-times

$$\Rightarrow \langle f(x) \rangle \approx \frac{1}{N} \sum_i f(x_i) \pm O(1/\sqrt{N})$$

- but, what if I have a distribution $e^{-S(\phi)}$?
  (Φ representing generic degrees of freedom)

find a transition probality $W(\phi, \phi')$ that brings us from a set of generic fields $\{\phi\} \rightarrow \{\phi'\}$ and which satisfies

- $W(\phi, \phi') > 0$ strong ergodicity ($W \geq 0$ is weak ergodicity)
- $\int d\phi' W(\phi, \phi') = 1$
- $W(\phi, \phi') = \int d\phi'' W(\phi, \phi'') W(\phi'', \phi')$ (Markov chain)
- $W(\phi, \phi')$ is measure preserving, $d\phi' = d\phi$

under these conditions, we are guaranteed

– to converge to a unique equilibrium distribution P^{eq} namely the Boltzmann distribution e^{-S}

– that this is independent from the initial conditions

$\rightarrow$ proof: (Creutz and Freedman; Lüscher, Cargese lectures)

Markov chain condition

$$W(\phi,\phi') = \int d\phi'' W(\phi,\phi'') W(\phi'',\phi')$$

can be rephrased when taking the equilibrium distribution itself

$$P(\phi') = \int d\phi W(\phi',\phi) P(\phi)$$

to fulfill (most of) our conditions it is *sufficient not neccessary* that W fulfills the detailled balance condition:

$$\frac{W(\phi,\phi')}{W(\phi',\phi)} = \frac{P(\phi')}{P(\phi)}$$

for example

$$\begin{aligned} \int d\phi P(\phi) W(\phi,\phi') &= \int d\phi P(\phi) \frac{P(\phi')}{P(\phi)} W(\phi',\phi) \\ &= \int d\phi P(\phi') W(\phi',\phi) = P(\phi') \end{aligned}$$

in the following discuss particular choices for W for problems of interest

Metropolis Algorithms

$$W_{\mathrm{Metro}}(\Phi,\Phi') = \Theta(S(\Phi(x)) - S(\Phi'(x)))$$
$$+\exp\left(-\Delta S(\Phi',\Phi)\right)\Theta(S(\Phi'(x)) - S(\Phi(x)))$$

$\Delta S(\Phi'(x),\Phi(x)) = S(\Phi'(x) - S(\Phi(x))$, $\Theta()$ Heavyside function

How this works:

$i)$ generate uniformly distributed $\Phi'(x)$ in a neighbourhood of $\Phi(x)$
– discretized quantum mechanics: $x'_i \in [x_i - \Omega, x_i + \Omega]$
– $SU(N)$: $U'_{n,\mu} = RU_{n,\mu}$, elements of R "close to" $U_{n,\mu}$

$ii)$ if $\Delta S(\Phi'(x),\Phi(x)) \leq 0)$ accept $\Phi'(x)$

$iii)$ if $\Delta S(\Phi'(x),\Phi(x)) > 0)$ accept with probability $\exp(-\Delta S(\Phi'(x),\Phi(x))$

- steps $i) - iii)$ are repeated N-times
- step $iii)$ can be realized by uniformly choosing randon number $r \in [0,1]$ and accept, if $\exp(-\Delta S) > r$

Metropolis Algorithms

- very general algorithm, can be used for many physical systems
- shows, however, often very long autocorrelation times
- much too costly for fermionic systems (why?)

Hybrid Monte Carlo Algorithm

expectation values in lattice field theory

$$\langle O \rangle = \frac{\int \mathcal{D}\Phi O e^{-S}}{\int \mathcal{D}\Phi e^{-S}}$$

do not change if field independent contributions are added to the action

$$\langle O \rangle = \frac{\int \mathcal{D}\Phi \int \mathcal{D}\pi O e^{-\frac{1}{2}\pi^2 - S}}{\int \mathcal{D}\Phi \int \mathcal{D}\pi e^{-\frac{1}{2}\pi^2 - S}}$$

field configurations are generated chronologically in a fictitious (computer) time τ

take π's Gaussian distributed, satisfying

$$< \pi(\tau) >= 0, \;\; < \pi(\tau)\pi(\tau') >= \delta(\tau - \tau')$$

consider a 4-dimensional Hamiltonian

$$H = \tfrac{1}{2}\pi^2 + S$$

consider quantum mechanical action: $S = \sum_n (x(n+a) - x(n))^2 + m^2 x^2(n)$

in ficititious time τ the sytem develops according to
Hamilton's equations of motion

$$\frac{\partial}{\partial\tau}\pi(n) = -\frac{\partial}{\partial\mathbf{x(n)}}S \equiv \mathrm{force}, \quad \frac{\partial}{\partial\tau}\mathbf{x(n)} = \pi(n)$$

$\Rightarrow$ conservation of energy

in practise, equations are integrated numerically up to time $T = 1$

divide T into N intervals of length $\delta\tau$ such that $T = N\delta\tau$: leap-frog scheme

$$\begin{aligned}
\pi(\delta\tau/2) &= \pi(0) - \frac{\delta\tau}{2}\frac{\partial}{\partial\mathbf{x}}S\Big|_{\mathbf{x}(0)} \\
\mathbf{x}(\delta\tau) &= \mathbf{x}(0) + \pi(\delta\tau/2)\delta\tau \\
\pi(3\delta\tau/2) &= \pi(\delta\tau/2) - \delta\tau\frac{\partial}{\partial\mathbf{x}}S\Big|_{\mathbf{x}(\delta\tau)} \\
&\vdots \\
\pi(T) &= \pi(N\delta\tau/2) - \frac{\delta\tau}{2}\frac{\partial}{\partial\mathbf{x}}S\Big|_{\mathbf{x}((N-1)\delta\tau)}
\end{aligned}$$

leap-frog scheme has a *finite* step-size $\delta\tau \Rightarrow$ energy is no longer conserved

$$H(\mathbf{x}_{\mathrm{ini}}, \pi_{\mathrm{ini}}) \neq H(\mathbf{x}_{\mathrm{end}}, \pi_{\mathrm{end}})$$

introduce a Metropolis like accept/reject step

accept new field configuration $\{\mathbf{x}_{\mathrm{end}}, \pi_{\mathrm{end}}\}$ with a probability

$$P_{\mathrm{acc}} = \min\left(1, e^{H(\mathbf{x}_{\mathrm{ini}},\pi_{\mathrm{ini}}) - H(\mathbf{x}_{\mathrm{end}},\pi_{\mathrm{end}})}\right)$$

Hybrid Monte Carlo Algorithm

general transition matrix:

$$W(\Phi', \Phi) \quad = \quad \frac{1}{\mathcal{Z}_\pi} \int \mathrm{D}[\pi]\, \mathrm{e}^{-\frac{1}{2}(\pi,\pi)} \Big\{ P_{\mathrm{acc}}(\pi, \Phi) \prod_x \delta(\Phi'(x), \Phi_\tau(x))$$

$$+ (1 - P_{\mathrm{acc}}(\pi, \Phi)) \prod_x \delta(\Phi'(x), \Phi(x)) \Big\}$$

- fulfills detailled balance proof: (Lüscher, Cargese lectures)
  $\Leftarrow$ needs *reversibility* of the leap-frog integrator
- preserves measure
- Ergodicity?

The case of Lattice QCD

action for two flavors of fermions (up and down quark)

$$S = a^4 \sum_x \bar{\psi} M^\dagger M \psi$$

path integral

$$\mathcal{Z} = \prod_x d\bar{\psi}(x) d\Psi(x) e^{-S} = \prod_x d\Phi^\dagger(x) d\Phi(x) e^{-\Phi^\dagger \left[M^\dagger M\right]^{-1} \Phi}$$

interaction of the scalar fields is very complicated: inverse fermion matrix $\left[M^\dagger M\right]^{-1}$ couples all points on the lattice with each other
How would a Metropolis algorithm work?

simulate with Hybrid Monte Carlo algorithm

$$\begin{aligned} \frac{d}{d\tau}\pi &= -\frac{dS}{d\Phi^\dagger} = \left[M^\dagger M\right]^{-1} \Phi \equiv \text{force} \\ \frac{d}{d\tau}\Phi &= \pi \end{aligned}$$

update of the momenta $\pi(x)$ is completely independent of update of Φ -field, non-locality of the action is not a problem to update the momenta, have to compute the vector

$$X = \left[M^\dagger M\right]^{-1} \Phi$$

$\Rightarrow$ solve an equation

$$\left[M^\dagger M\right] X = \Phi$$

Exercise:

estimate the number of flops to apply the Wilson operator on a vector

assume you want to have 2000 thermalization and 5000 measurement steps on a $48^3 \cdot 96$ lattice

assume number of iterations to solve $\left[M^\dagger M\right] X = \Phi$ is 500

assume number of time steps in the HMC is 100

How long would the program run on your laptop?
(assume –unrealistic– efficiency of 50%)

If you save the 5000 configurations, would this fit on your laptop disk?

autocorrelation times

generating field configurations as a Markov process,
$\Rightarrow$ configurations are not independent from each other

free field theory $S = \int d^4x \Phi(x)[\nabla_m u^* \nabla_\mu + m_0^2]\Phi(x)$ in Fourier space

$$S = \int d^4k \Phi(k)[k^2 + m_0^2]\Phi(k)$$

Langevin equation $\leftrightarrow$ HMC algorithm

$$\frac{d}{d\tau}\Phi(k,\tau) = -[k^2 + m_0^2]\Phi(k,\tau) + \eta(k,\tau)$$

$$< \eta(\tau) >= 0, \;\; < \eta(\tau)\eta(\tau') >= \delta(\tau - \tau')$$

then a solution may be written down

$$\Phi(k,\tau) = \int^\tau ds \exp\left\{-(\tau - s)[k^2 + m_0^2]\eta(k,s)\right\}$$

compute correlation of fields at $\tau = 0$ with fields at τ

consider the autocorrelation function

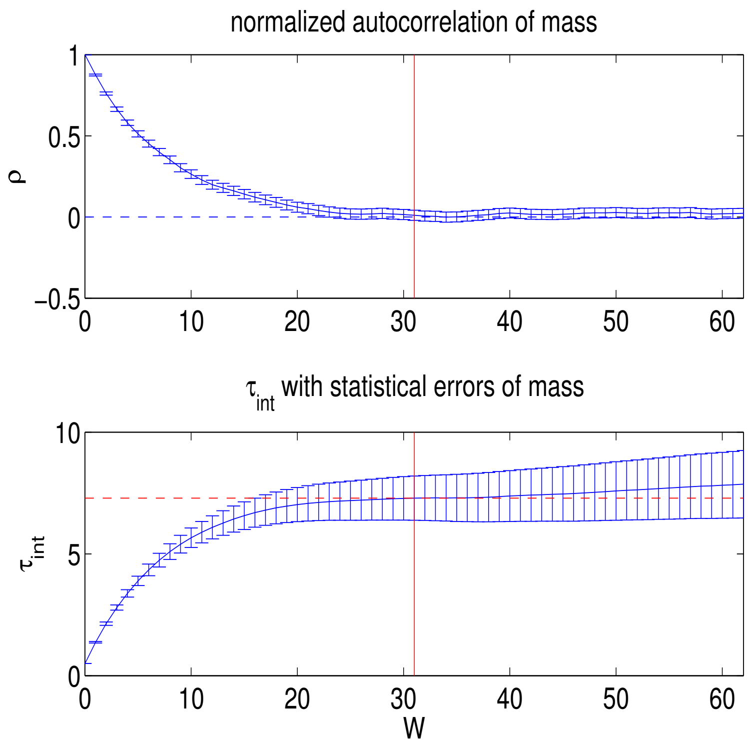
$$
\begin{aligned}
C(k,\tau) &= \Phi(k,0)\Phi(k,\tau) \\
&= \int ds_1 ds_2 \exp\left\{[k^2+m_0^2]s_1\left(-(\tau-s)[k^2+m_0^2]\right)\right. \\
&\qquad\qquad\qquad \left.\eta(k,s_1)\eta(k,s_2)\right\} \\
&\propto \frac{e^{-[k^2+m_0^2]\tau}}{k^2+m_0^2} \equiv \frac{e^{-\tau/\tau_0}}{k^2+m_0^2}
\end{aligned}
$$

- the autocorrelation function $C(k,\tau)$ decays exponentially autocorrelation time τ_0
- decay is lowest for the zero mode $k=0$
- $\tau \propto 1/m_0^2 \quad \Rightarrow$ the correlations become stronger closer to the critical point $m_0 = 0 \rightarrow$ *critical slowing down*
- scaling law $\tau_0 \propto 1/m^z$, $\quad z$ *the dynamical critical exponent*

How to deal with the autocorrelation?

measure it:

$\Gamma(\tau) = \langle x(\tau) \cdot x(0) \rangle / \langle x(0) \rangle^2 \propto e^{-\tau/\tau_{\text{int}}}$



Comment: *integrated auto correlation time* τ_{int} observable dependent

A consequence from autocorrelations: Errors

measure average position of quantum mechanical particle $\bar{x}$ from N measurements

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

This has a <u>variance</u>

$$\sigma = \frac{1}{N-1}\left(\bar{x^2} - \bar{x}^2\right)$$

and a standard deviation

$$\Delta_0 \equiv \sqrt{\sigma} \propto 1/\sqrt{N} \ \text{ for } \ N \gg 1$$

If we have an autocorrelation time $\tau \Rightarrow$ statistics reduces to $n = N/\tau$

$\Rightarrow \Delta_{\mathrm{true}} \propto 1/\sqrt{n} = \sqrt{\tau}/\sqrt{N} = \sqrt{\tau}\Delta_0$

Can we improve the error scaling for evaluating pathintegrals?

reminder:

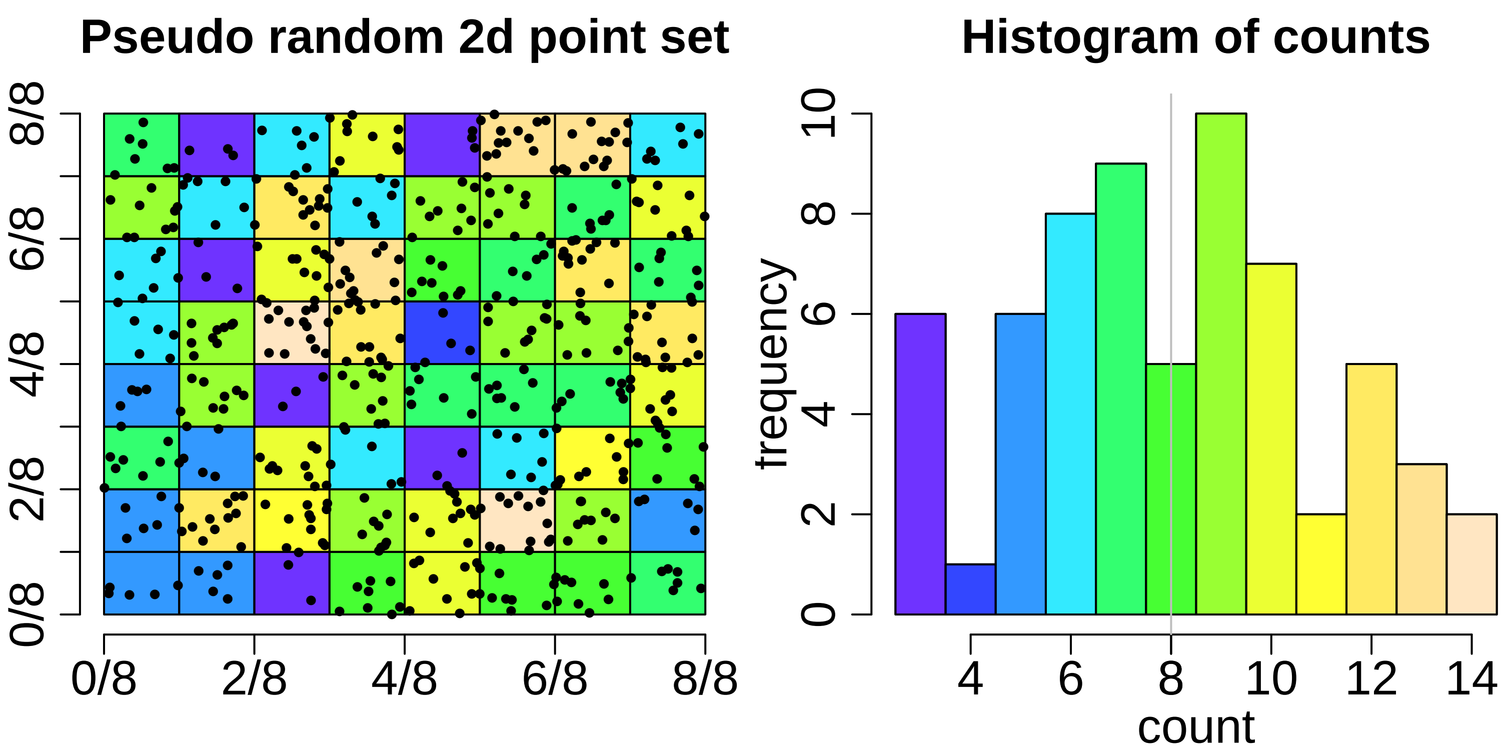
$$Z = \int \mathcal{D}x \, e^{-S[x]} \, ; \quad x = (x_1, \ldots, x_d) \, ; \mathcal{D}x = \prod_{i=1}^{d} x_i$$

$$\langle O \rangle = Z^{-1} \int \mathcal{D}x \, e^{-S[x]} \, O[x]$$

- stochastically through Markov chain Monte Carlo methods
- finite Markov chain: $x_1, \ldots, x_N \to N$ samples of O: $O_1, \ldots, O_N$
- estimate $\overline{\langle O \rangle} = \frac{1}{N} \sum_{i=1}^{N} O_i$ has standard error $\Delta \overline{\langle O \rangle} = \frac{\sigma_O}{\sqrt{N}}$

Distribution of integration variables

Example: 512 two-dimensional pseudo-random points



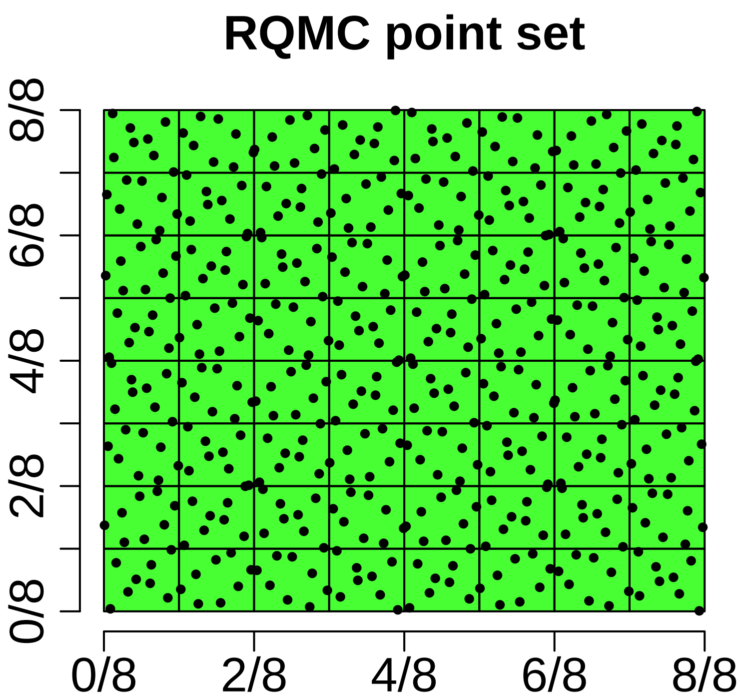
- sample 512 points grid of 8×8 equal squares
- count number of points in each square
- count occurrence of $1, 2, \ldots$ points in a square
- $\approx$ Poisson distribution $\rightarrow$ uneven sampling $\rightarrow$ large stochastic error

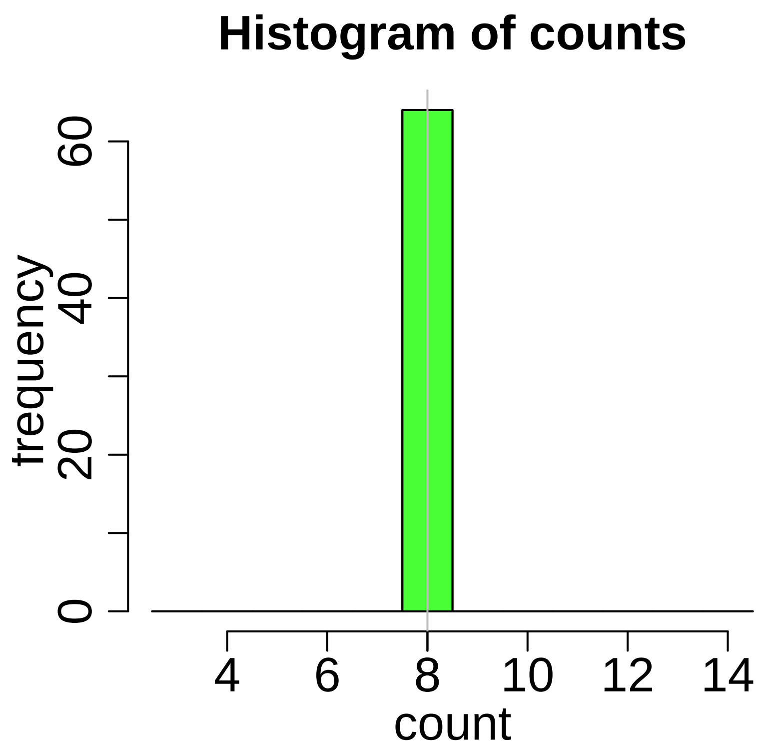
Improved integration: Quasi Monte Carlo

- construction of deterministic low-discrepancy point-sets
- low-discrepancy $\rightarrow$ "more uniform"
- promises N^{-1} asymptotic error behaviour for suitable integrands
- applied successfully to financial problems

Distribution of integration points

QMC point set (2d Sobol samples):





- each square contains same number of points $\rightarrow$ delta distribution
- even coverage $\rightarrow$ less stochastic fluctuations
- makes QMC a most promising tool
- randomisation possible (RQMC): $\rightarrow$ practical error estimation

First investigation for quantum mechanical model in Euclidean time

(K. Jansen, H. Leovey, A. Ammon, A. Griewank, T. Hartung, M. Müller-Preußker)

$$S = a\sum_{i=1}^{d}\left(\frac{M_0}{2}\frac{(x_{i+1}-x_i)^2}{a^2} + \frac{\omega^2}{2}x_i^2 + \lambda x_i^4\right) \ ; \quad x_{d+1} = x_1$$

- M_0: particle mass
- ω: frequency
- a: lattice spacing
- d number of lattice sites, $\rightarrow T = da$ time extent
- $\lambda = 0 \rightarrow$ harmonic oscillator
- $\lambda > 0 \rightarrow$ anharmonic oscillator
  ($\mu^2 < 0 \rightarrow$ double well potential)

Observables considered

Cumulants

$\langle x^2 \rangle = \langle \frac{1}{d} \sum_i x_i^2 \rangle$

$\langle x^4 \rangle = \langle \frac{1}{d} \sum_i x_i^4 \rangle$

Correlator

$\langle x_k x_{k+t} \rangle = \langle \frac{1}{d} \sum_i x_i x_{i+t} \rangle \propto e^{-(E_1 - E_0)t}$

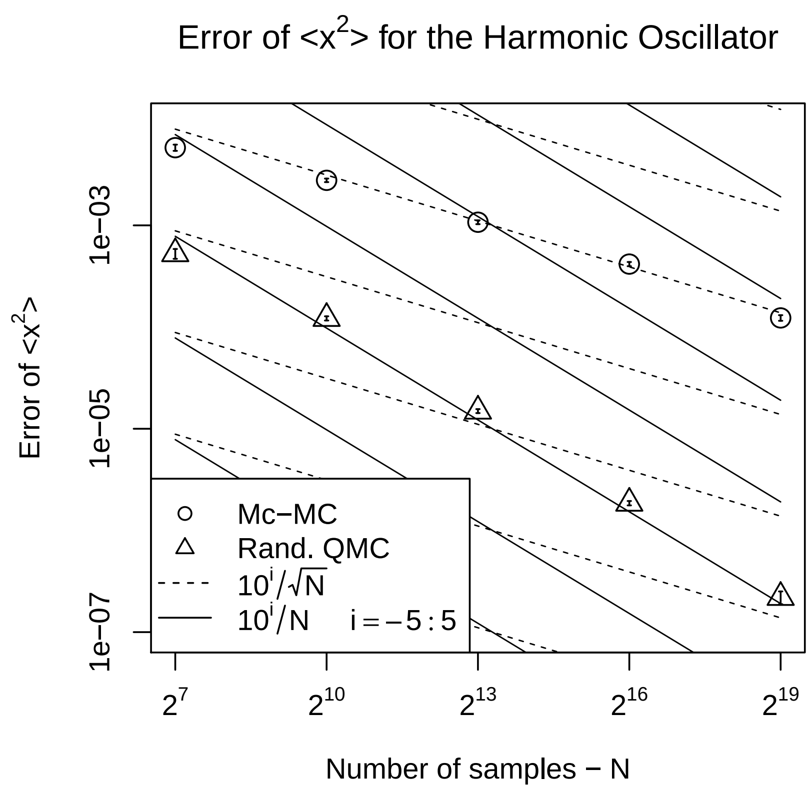
Energies

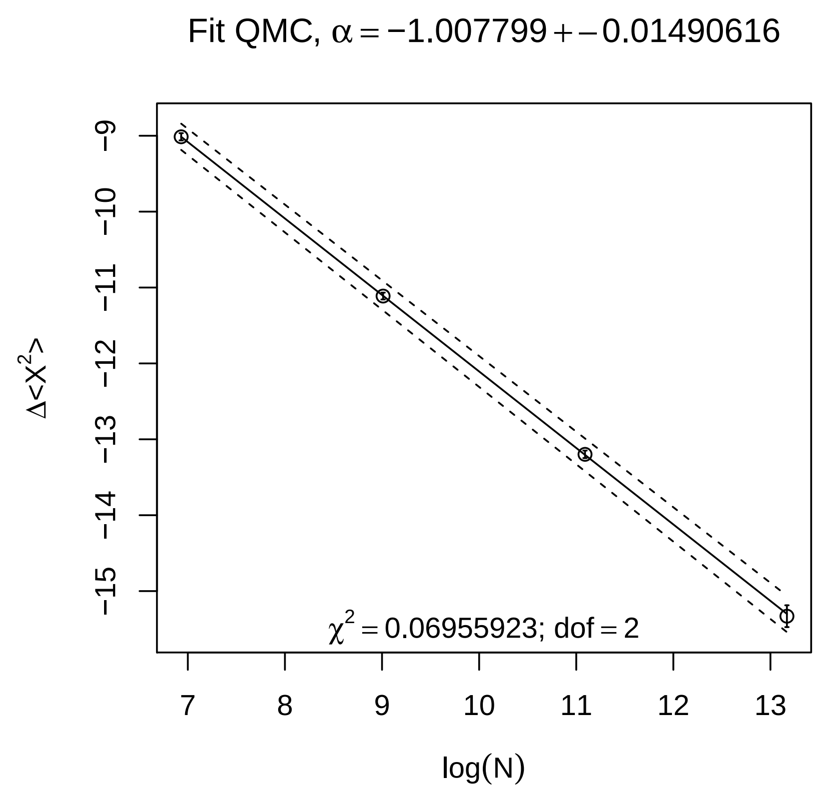
ground state energy: $E_0 = 3\lambda \langle x^4 \rangle + \mu^2 \langle x^2 \rangle + \frac{\omega^4}{16}$

energy gap $E_1 - E_0$

Case of harmonic oscillator

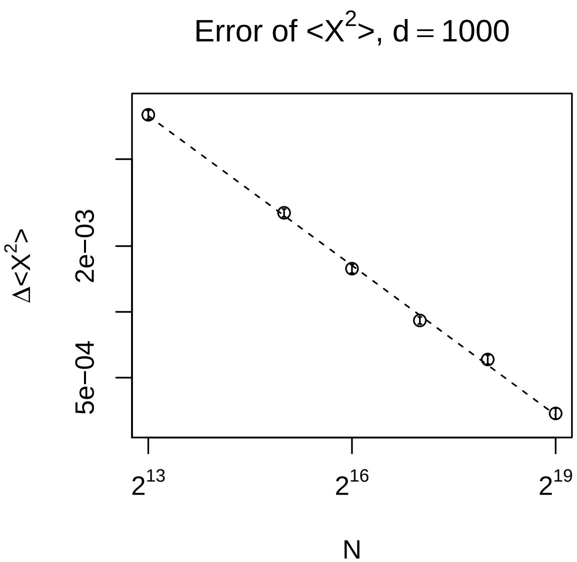
parameters: $\omega^2 = 2.0$, $M_0 = 0.5$, $a = 0.5$, $d = 100$

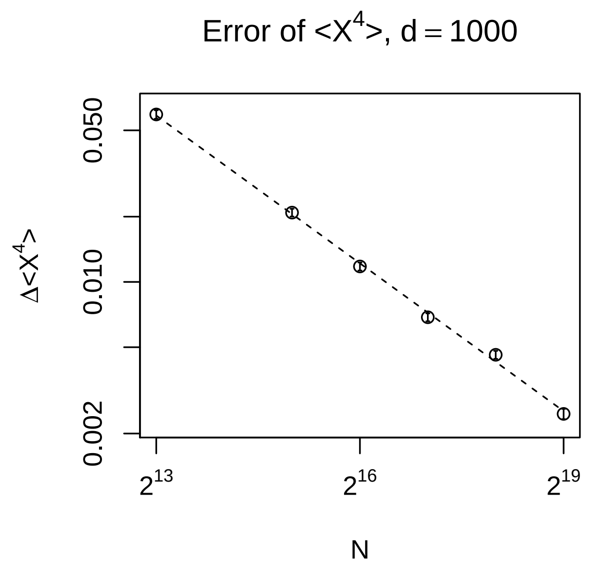


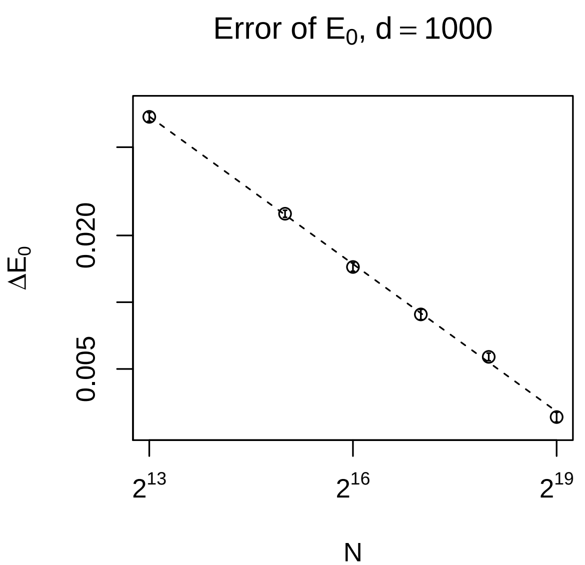


- fit $\Delta\langle x^2\rangle = c \cdot N^\alpha \rightarrow \alpha = -1.007799$
- perfect error scaling $\rightarrow$ QMC at work
- trivial, but successful application to physical problem

Case of anharmonic oscillator







Case of anharmonic oscillator

- parameters: $M_0 = 0.5$, $a = 0.015$, $\omega^2 = -16$
- fit: $\Delta O \sim c \cdot N^{\alpha}$

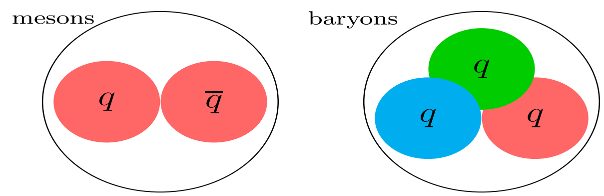
	O	α	$\log C$	χ^2/dof
$d = 100$	X^2	-0.763(8)	2.0(1)	7.9 / 6
	X^4	-0.758(8)	4.0(1)	13.2 / 6
	E_0	-0.737(9)	4.0(1)	8.3 / 6
$d = 1000$	X^2	-0.758(14)	2.0(2)	5.0 / 4
	X^4	-0.755(14)	4.0(2)	5.7 / 4
	E_0	-0.737(13)	4.0(2)	4.0 / 4

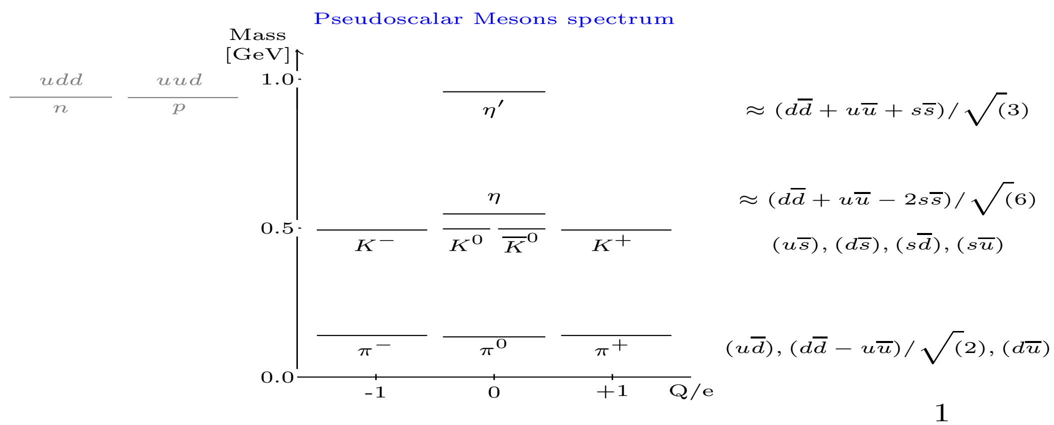
Energy gap

- parameters: $d = 100,\ N = 2^5, 2^8, 2^{11}, 2^{14}$
- 400 Sobol' sequences each $\rightarrow$ error estimate $\rightarrow$ $\alpha = -0.735(13)$

The observable

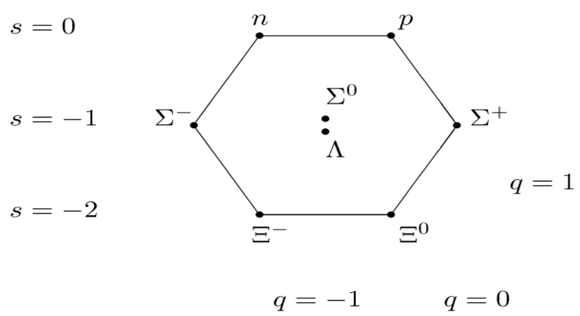
What we learn from mesons and baryons



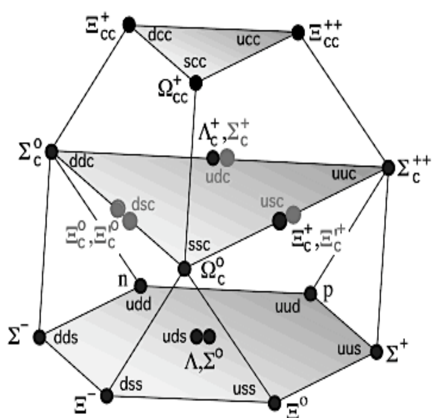


- confinement of quarks
- spontaneous chiral symmetry breaking
- topological effects of gluon field configurations

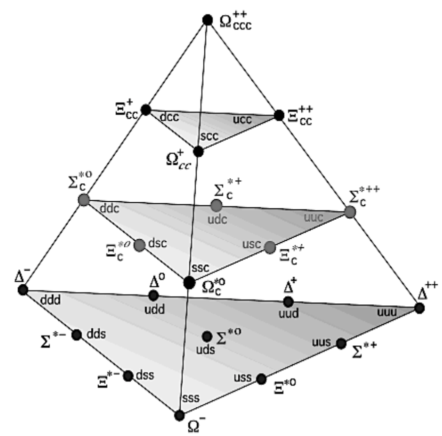
The complicated baryon spectrum



Octet



Decuplett



32-plet

QCD: the Mass Spectrum

goal: **non-perturbative computation of this rich bound state spectrum**

using QCD alone

$\rightarrow$ euclidean correlation functions

Reconstruction theorem relates this to Minkowski space

operator $O(\mathbf{x},t)$ with quantum numbers of a given particle

correlation function decays exponentially: e^{-Et} , $\quad E^2 = m^2 + \mathbf{p}^2$

$\Rightarrow$ mass obtained at zero momemtum

$$O(t) = \sum_{\mathbf{x}} O(\mathbf{x},t)$$

correlation function

$$\begin{aligned} \langle O(0)O(t)\rangle &= \frac{1}{\mathcal{Z}} \sum_n \langle 0|O(0)e^{-\mathbf{H}t}|n\rangle\langle n|O(0)|0\rangle \\ &= \frac{1}{\mathcal{Z}} \sum_n |\langle 0|O(0)|n\rangle|^2 e^{-(E_n-E_0)t} \end{aligned}$$

connected correlation function

$$\lim_{t\to\infty}\left[\langle O(0)O(t)\rangle - |\langle O(0)\rangle|^2\right] \propto e^{-E_1 t}$$

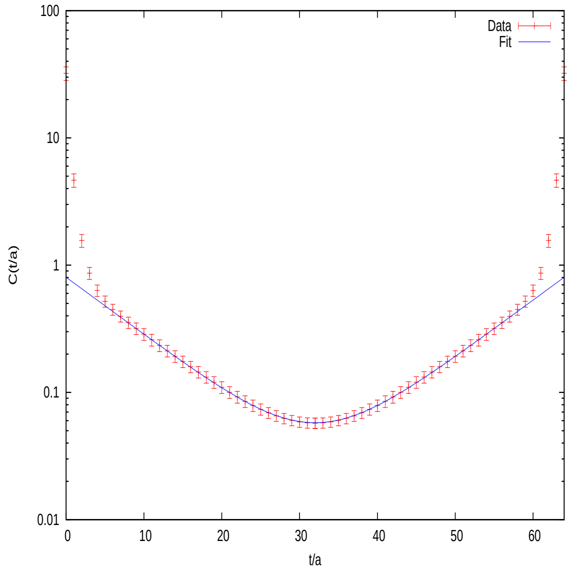
vanishing of connected correlation function at large times

$\rightarrow$ cluster property $\Rightarrow$ *locality of the theory*

periodic boundary conditions

$$\langle O(0)O(t)\rangle_c = \sum_n c_n \left[e^{-E_n t} + e^{-E_n(T-t)}\right]$$

$$1 \ll t \ll T: \quad \langle O(0)O(t)\rangle_c \propto e^{-mt} + e^{-m(T-t)}$$



Hadron Spectrum in QCD

hadrons are bound states in QCD

- mesons pion, kaon, eta, ...
- baryons neutron, proton, Delta, ..

for the computation of the hadon spectrum

– construct operators with the suitable quantum numbers

– compute the connected correlation function

– take the large time limit of the correlation function

Lorentz symmetry, parity and charge conjugation

rotational symmetry $\rightarrow$ hypercubic group: discrete rotations and reflections
classification of operators: irreducible representations R
(note hypercubic group is a subgroup of $SO(3)$)

parity	charge conjugation
$\Psi(\mathbf{x},t) \rightarrow \gamma_0 \Psi(-\mathbf{x},t)$	$\Psi(\mathbf{x},t) \rightarrow C\bar{\Psi}^T(\mathbf{x},t)$
$\bar{\Psi}(\mathbf{x},t) \rightarrow \bar{\Psi}(-\mathbf{x},t)\gamma_0$	$\bar{\Psi}(\mathbf{x},t) \rightarrow -\Psi^T(\mathbf{x},t)C^{-1}$

C charge conjugation matrix $C = \gamma_0\gamma_2$
C satisfies

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T = -\gamma_\mu^*$$

The Proton

Nucleon: baryonic isospin-doublet, $I = \frac{1}{2}$:

proton (uud) $I_3 = +\frac{1}{2}$ and neutron (udd) $I_3 = -\frac{1}{2}$

local interpotating field of proton

$$P(x) = -\epsilon_{abc} \left[u_a^T(x) C \gamma_5 d_b(x) \right] u_c(x) \ , \quad [\ ]\,\text{spin} \ \ \text{trace}$$

u^C charged conjugate quark field

$$\psi^C(x) = C \bar{\psi}^T(x) \ , \ \ \bar{\psi}^C = -\psi^T(x) C^{-1}$$

leading to

$$\Gamma_P(t) = \sum_{\vec{x}} \langle 0 | P(x) \bar{P}(0) | 0 \rangle$$

Exercise:

using the operator

$$P(x) = -\epsilon_{abc}\left[u_a^T(x) C\gamma_5 d_b(x)\right] u_c(x)\ ,\quad [\ \ ]\,\mathrm{spin\ \ trace}$$

will we really get the proton?

$\rightarrow$ check quantum numbers

Contraction

- 2-point-function calculation

$\mathcal{O}_\Gamma(x) = \bar{\psi}\Gamma\psi(0)$

$$\begin{aligned} &\langle \mathcal{O}_\Gamma(x)\mathcal{O}_\Gamma(0)\rangle = \\ &\bar{\psi}(x)\Gamma\bar{\psi}(0)\psi(x)\Gamma\psi(0) \\ &= tr[\Gamma S(x,0)\Gamma S(0,x)] \end{aligned} \tag{1}$$

in terms of eigenvalues and eigenvectors:

$$tr[\Gamma S(x,0)\Gamma S(0,x)] = \sum_{\lambda_i,\lambda_j} \frac{1}{\lambda_i\lambda_j} \sum_{\alpha\beta\gamma\delta} \left[(\phi_j^{\dagger\alpha}(x)\Gamma_{\alpha\beta}\phi_i^{\beta}(x))(\phi_i^{\dagger\gamma}(0)\Gamma_{\gamma\delta}\phi_j^{\delta}(0)) \right]$$

Example: pion operator $\rightarrow$ need speudoscalar operator

$$O_{\mathrm{PS}}(\mathbf{x},t) = \bar{\Psi}(\mathbf{x},t)\gamma_5\Psi(\mathbf{x},t)$$

correlation function

$$\begin{aligned} f_{\mathrm{PS}}(t) \equiv \langle O_{\mathrm{PS}}(0)O_{\mathrm{PS}}(t)\rangle &= \sum_{\mathbf{x}} \left[\bar{\psi}(\mathbf{x},t)\gamma_5\Psi(\mathbf{x},t)\right]\left[\bar{\psi}(0,0)\gamma_5\Psi(0,0)\right] \\ &= \sum_{\mathbf{x}} \mathrm{Tr}\left[S_F(0,0;\mathbf{x},t)\gamma_5 S_F(\mathbf{x},t;0,0)\gamma_5\right] \end{aligned}$$

used Wick's theorem and $S_F = D^{-1}$ the fermion propagator

remark: formula simplifies using γ_5 hermiticity of D_{Wilson}

$\Rightarrow$ need to compute inverse of the fermion matrix

$$a \ll t \ll T: \quad f_{\mathrm{PS}}(t) = \underbrace{\frac{|\langle 0|P|\mathrm{PS}\rangle|^2}{2m_{\mathrm{PS}}}}_{\equiv F_{\mathrm{PS}}^2/2m_{\mathrm{PS}}} \cdot \left(e^{-m_{\mathrm{PS}}t} + e^{-m_{\mathrm{PS}}(T-t)}\right)$$

F_{PS} pion decay constant

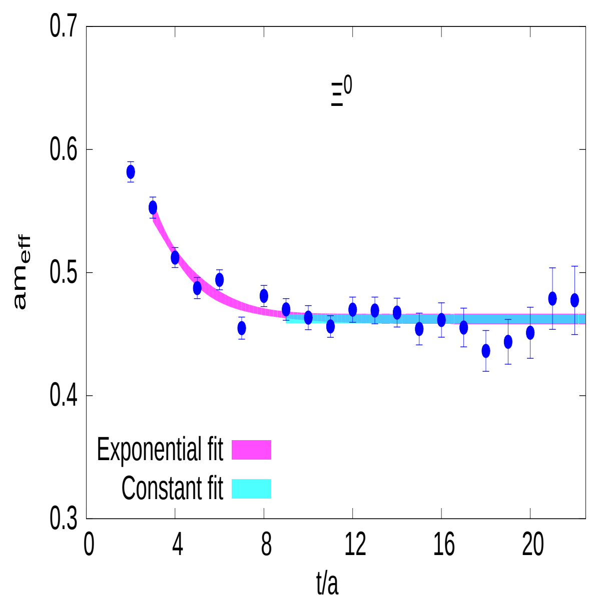
Effective Masses and exponential fits

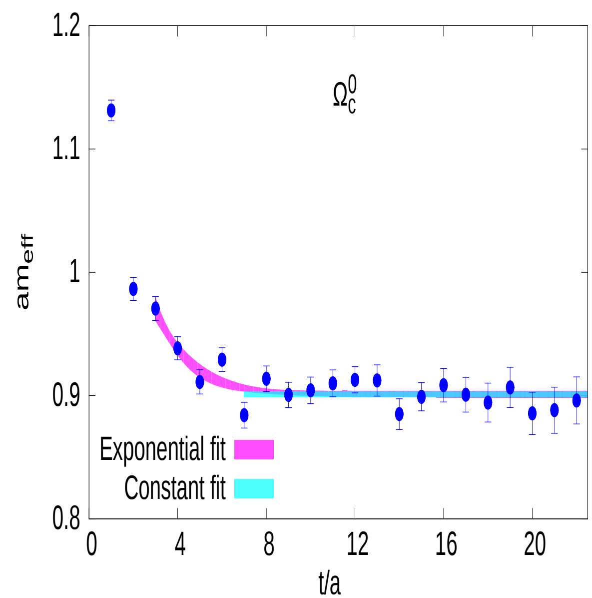
effective mass (neglecting boundary)

$$am_{\text{eff}}^X(t) = \log\left(\frac{C_X(t)}{C_X(t+1)}\right) = am_X + \log\left(\frac{1+\sum_{i=1}^{\infty} c_i e^{-\Delta_i t}}{1+\sum_{i=1}^{\infty} c_i e^{-\Delta_i (t+1)}}\right) \longrightarrow_{t\to\infty} am_X$$

$\Delta_i = m_i - m_X$ mass difference of the excited state i to the ground mass m_X

in practise: $am_{\text{eff}}^X(t) \approx am_X + \log\left(\frac{1+c_1 e^{-\Delta_1 t}}{1+c_1 e^{-\Delta_1 (t+1)}}\right)$

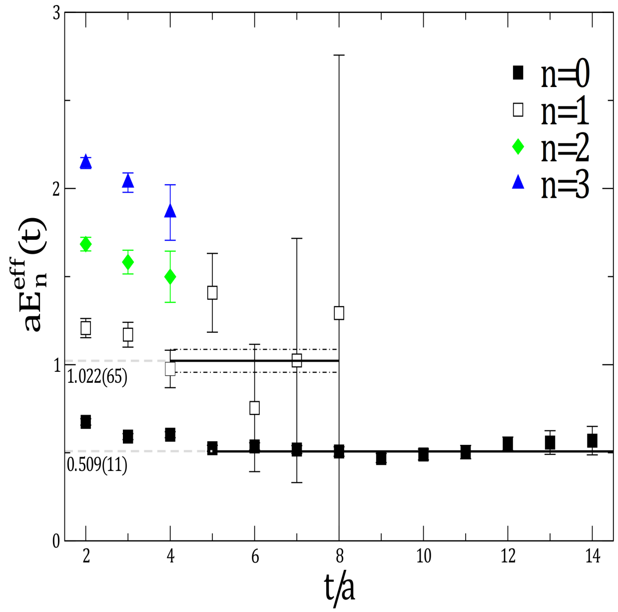




Identifying exicited states

$$am_{\text{eff}}^{X}(t) = \log\left(\frac{C_X(t)}{C_X(t+1)}\right) = am_X + \log\left(\frac{1+\sum_{i=1}^{\infty} c_i e^{-\Delta_i t}}{1+\sum_{i=1}^{\infty} c_i e^{-\Delta_i (t+1)}}\right) \longrightarrow_{t\to\infty} am_X$$

$\Delta_i = m_i - m_X$ mass difference of the excited state i to the ground mass m_X



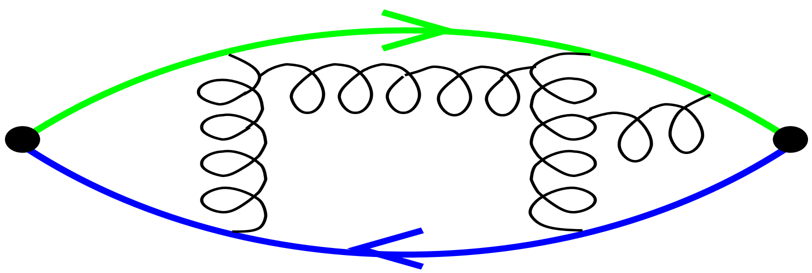
- large errors for excited states
- generalized eigenvalue approach
- need of large operator basis

Quenched approximation

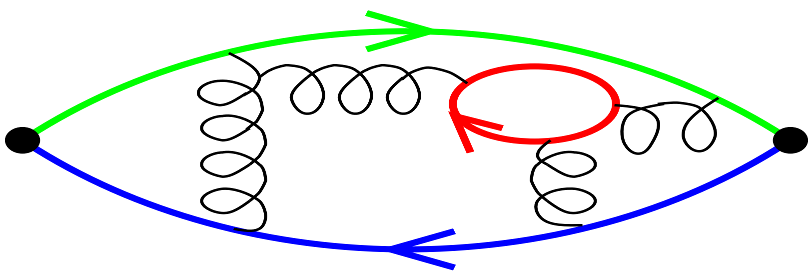
The Quenched Approximation

$\rightarrow$ neglect steady generation of quarks and antiquarks in physical quantum processes

$\Rightarrow$ *truncation*, works surprisingly well however



(A) Quenched QCD: no internal quark loops



(B) full QCD

A short history of proton mass computation

(take example of Japanese group)

1986 (Itoh, Iwasaki, Oyanagi and Yoshie)

quenched approximation, $12^3 \cdot 24$ lattice

$a \approx 0.15\mathrm{fm}$, 30 configurations

Machine: HITAC S810/20 $\rightarrow$ 630 Mflops

$\Rightarrow$ only meson masses, conclusion:
time extent of $T = 24$ too small to extract baryon ground state

1988 (plenary talk by Iwasaki at Lattice symposium at FermiLab)

quenched approximation, $16^3 \cdot 48$ lattice

$a \approx 0.11\mathrm{fm}$, 15 configurations

particle	lattice	experiment
Kaon	470(45)	494
Nucleon	866(108)	938
Ω	1697(89)	1672

The story goes on ...

1992 (Talk Yoshie at Lattice '92 in Amsterdam):

quenched approximation, $24^3 \cdot 54$ lattice

two lattice spacings: $a \approx 0.11\mathrm{fm}$, $a \approx 0.10\mathrm{fm}$, $O(200)$ configurations

Machine: QCDPAX 14 Gflops

$\Rightarrow$ worries about excited state effects

$\Rightarrow$ worries about finite size effects

1995 (paper by QCDPAX collaboration)

		stat.	sys.(fit-range)		sys.(fit-func.)		
$\beta = 6.00$	$m_N = 1.076$	± 0.060	$+0.047$	-0.020	$+0.0$	-0.017	GeV
$\beta = 6.00$	$m_\Delta = 1.407$	± 0.086	$+0.096$	-0.026	$+0.038$	-0.015	GeV

"Even when the systematic errors are included, the baryon masses at $\beta = 6.0$ do not agree with experiment. Our data are consistent with the GF11[1] data at finite lattice spacing, within statistical errors. In order to take the continuum limit of our results, we need data for a wider range of β with statistical and systematic errors much reduced."

[1] GF11 has been a 5.6Gflops machine developed by IBM research.

where the quenched story ends

2003 (Paper by CP-PACS collaboration):

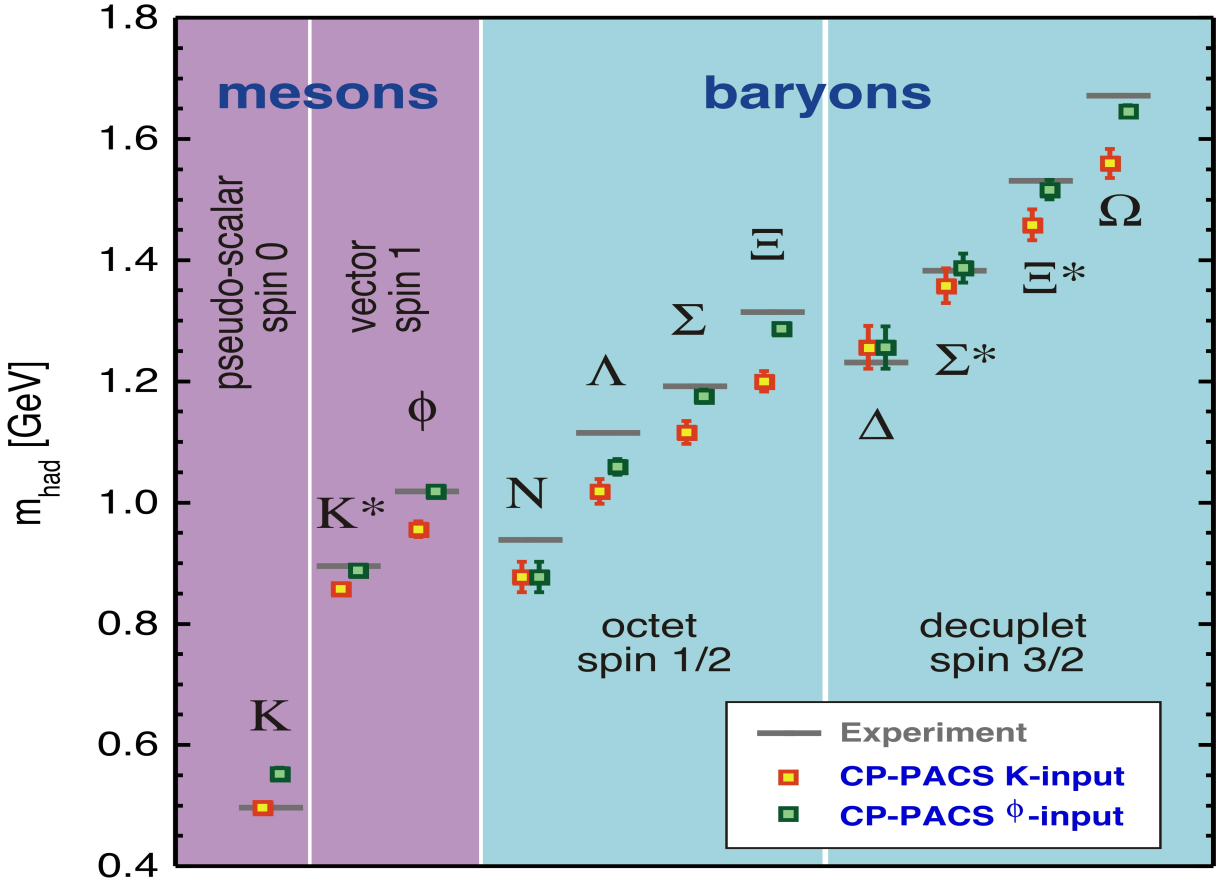
quenched approximation from $32^3 \cdot 56$ to $64^3 \cdot 112$ lattice

two lattice spacings: $a \approx 0.05\text{fm} \; - a \approx 0.10\text{fm}$, $O(150) - O(800)$configurations

Machine: CP-PACS, massively parallel, 2048 processing nodes,
completed september 1996

$\rightarrow$ reached 614Gflops

- control of systematic errors
  - finite size effects
  - lattice spacing
  - chiral extrapolation
  - excited states



quenched

quenched

CP-PACS collaboration

Solution of QCD?

$\rightarrow$ a number of systematic errors

Another example: glueballs

prediction of QCD: the existence of states made out of gluons alone, the glueballs

- hard to detect experimentally
- difficult to compute, of purely non-perturbative nature

$\Rightarrow$ challenge for lattice QCD

transformation laws for gauge links

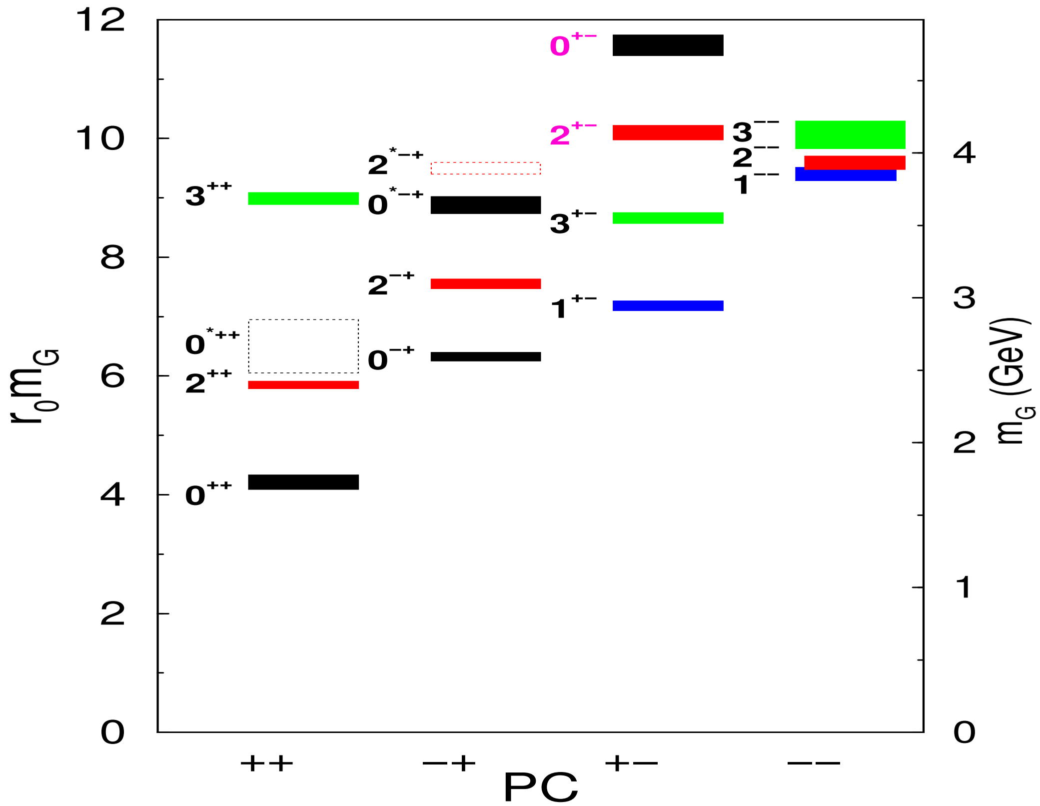
parity	charge conjugation
$U(\mathbf{x},t,4) \to U(-\mathbf{x},t,4)$	$U(\mathbf{x},t,4) \to U^*(\mathbf{x},t,4)$
$U(\mathbf{x},t,i) \to U(-\mathbf{x},t,-i)$	$U(\mathbf{x},t,i) \to U^*(\mathbf{x},t,i)$

example: combination of 1×1 Wilson loops $W(C)_{xy}$

$$O(\mathbf{x},t) = W_{(\mathbf{x},t),12} + W_{(\mathbf{x},t),13} + W_{(\mathbf{x},t),23}$$

invariant under hypercubic group, parity and charge conjugation $\to 0^{++}$

glueball spectrum $\rightarrow$ unique prediction from lattice QCD



quenched

quenched

'

Morningstar and Peardon

start of dynamical (mass-degenerate up and down) quark simulations

1998 (Paper by UKQCD collaboration):

lattices: from $8^3 \cdot 24$ to $16^3 \cdot 24$

$a \approx 0.10\mathrm{fm}$, $m_\pi/m_\rho > 0.7$

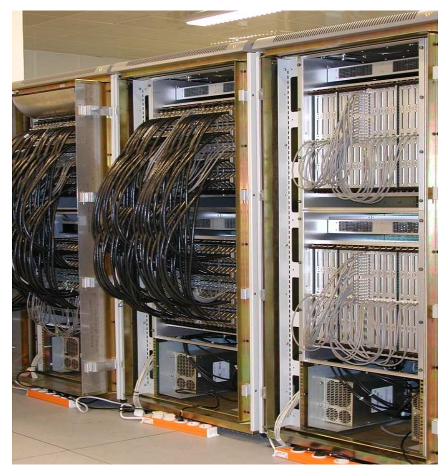
Machine: CRAY T3E $\approx$ 1Tflop

1999 (Paper by SESAM collaboration):

lattice: $16^3 \cdot 32$ lattice

$a \approx 0.10\mathrm{fm}$, $m_\pi/m_\rho > 0.7$

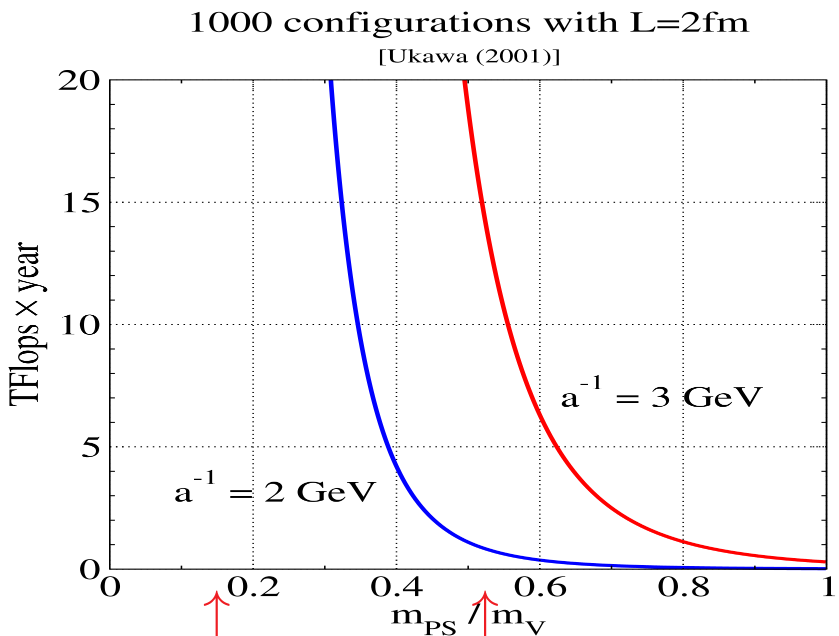
Machine: APE100 $\approx$ 100Gflop



- period of algorithm development
  - improved higher order integrators
  - multiboson algorithm
  - PHMC algorithm

Costs of dynamical fermions simulations, the "Berlin Wall"

see panel discussion in Lattice2001, Berlin, 2001



physical point

contact to χPT (?)

formula $C \propto \left(\frac{m_\pi}{m_\rho}\right)^{-z_\pi} (L)^{z_L} (a)^{-z_a}$

$$z_\pi = 6, \;\; z_L = 5, \;\; z_a = 7$$

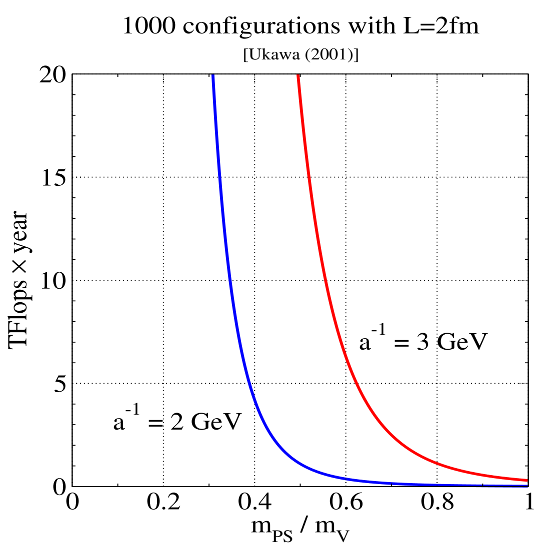
"both a 10^8 increase in computing power AND spectacular algorithmic advances before a useful interaction with experiments starts taking place." (Wilson, 1989)

$\Rightarrow$ need of **Exaflops Computers**

A generic improvement for Wilson type fermions

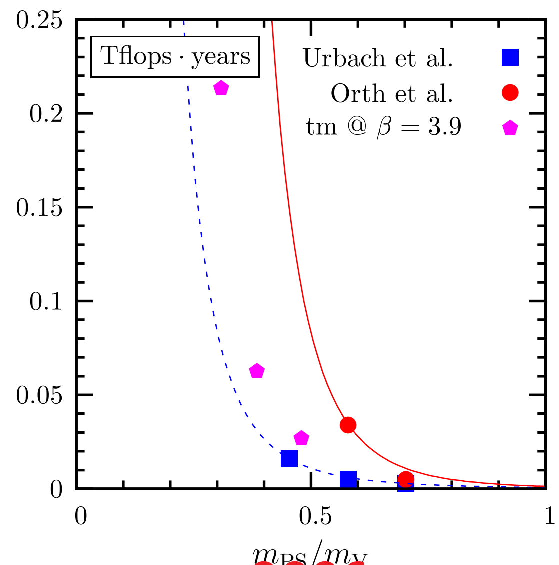
New variant of HMC algorithm (Urbach, Shindler, Wenger, K.J.)
(see also SAP (Lüscher) and RHMC (Clark and Kennedy) algorithms)

- even/odd preconditioning
- (twisted) mass-shift (Hasenbusch trick)
- multiple time steps



2001

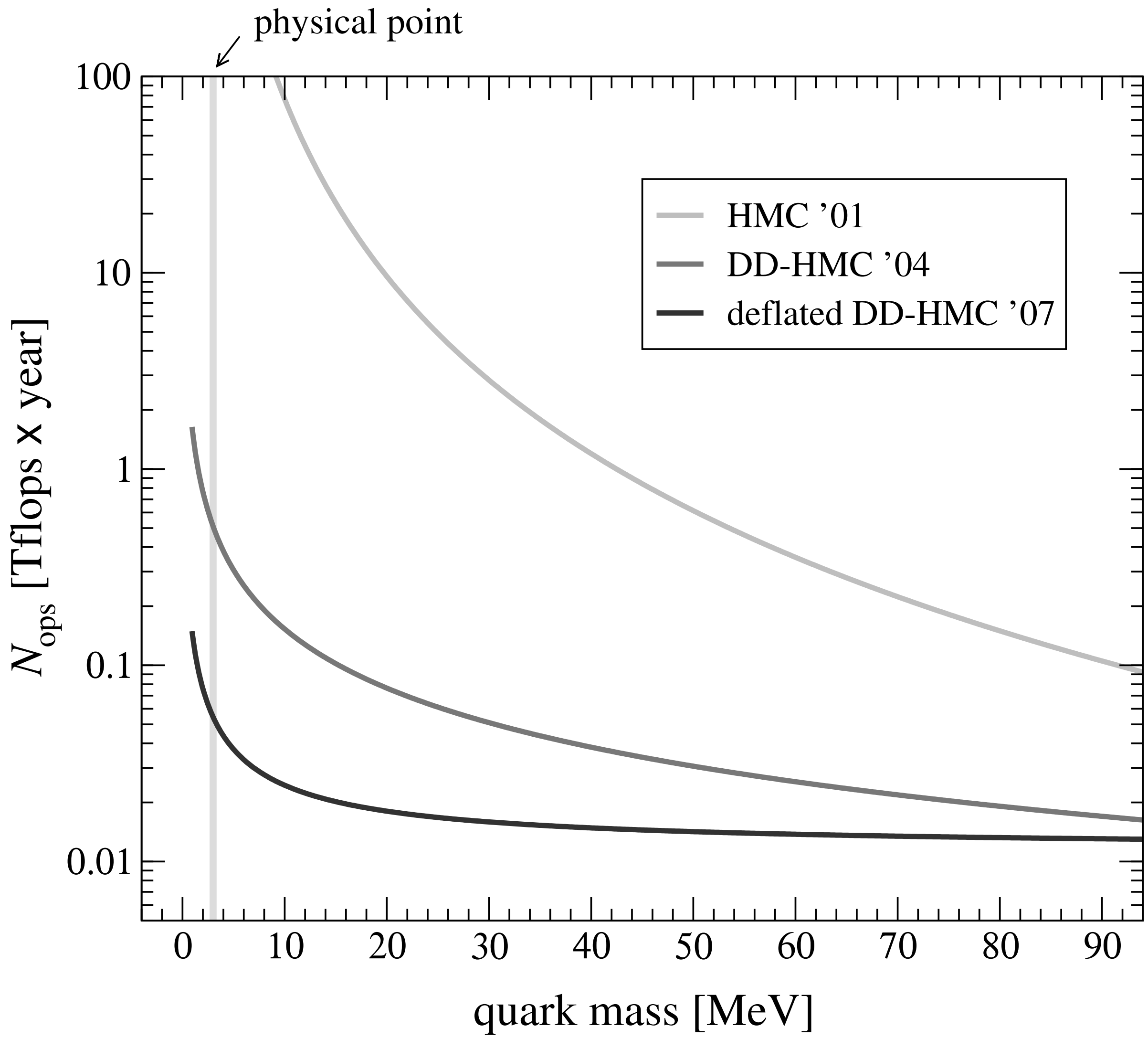
$\longrightarrow$



2006

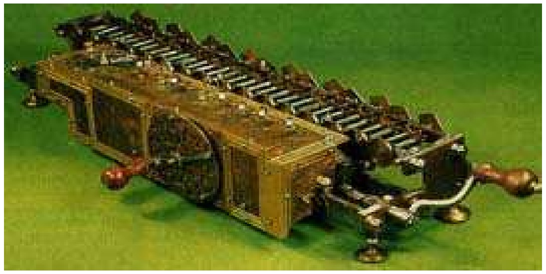
– comparable to staggered
– reach small pseudo scalar masses ≈ 300MeV

Recent pricture



Supercomputer

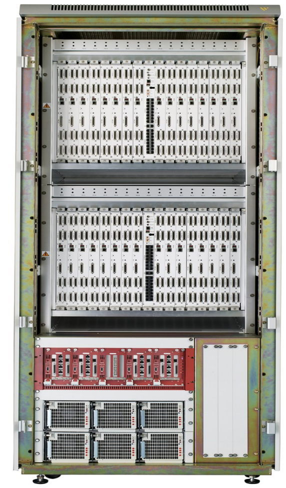
ca. 1700, Leibniz Rechenmaschine



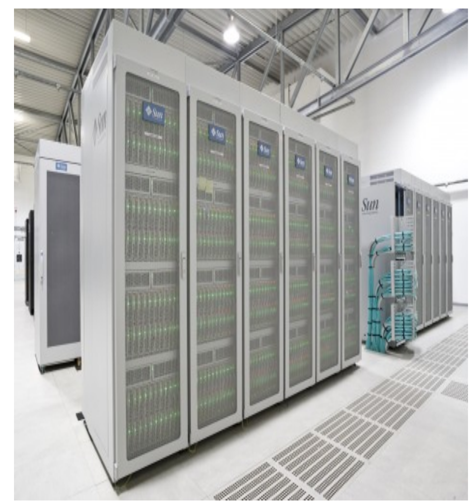
Denn es ist eines ausgezeichneten Mannes nicht würdig, wertvolle Stunden wie ein Sklave im Keller der einfachen Rechnungen zu verbringen. Diese Aufgaben könnten ohne Besorgnis abgegeben werden, wenn wir Maschinen hätten.

Because it is unworthy for an excellent man to spent valuable hours as a slave in the cellar of simple calculations. These tasks can be given away without any worry, if we would have machines.

German Supercomputer Infrastructure



- apeNEXT in Zeuthen 3Teraflops
  and Bielefeld 5Teraflops
  → dedicated to LGT



- NIC 72 racks of BG/P System
  at FZ-Jülich **1 Petaflops**
- 2208 Nehalem processor
  Cluster computer:
  208 Teraflops
- Altix System at LRZ Munic
- SGI Altix ICE 8200 at HLRN (Berlin, Hannover)
  31 Teraflops
  → will be upgraded to a **3 Petaflops system**

State of the art

- **BG/Q**

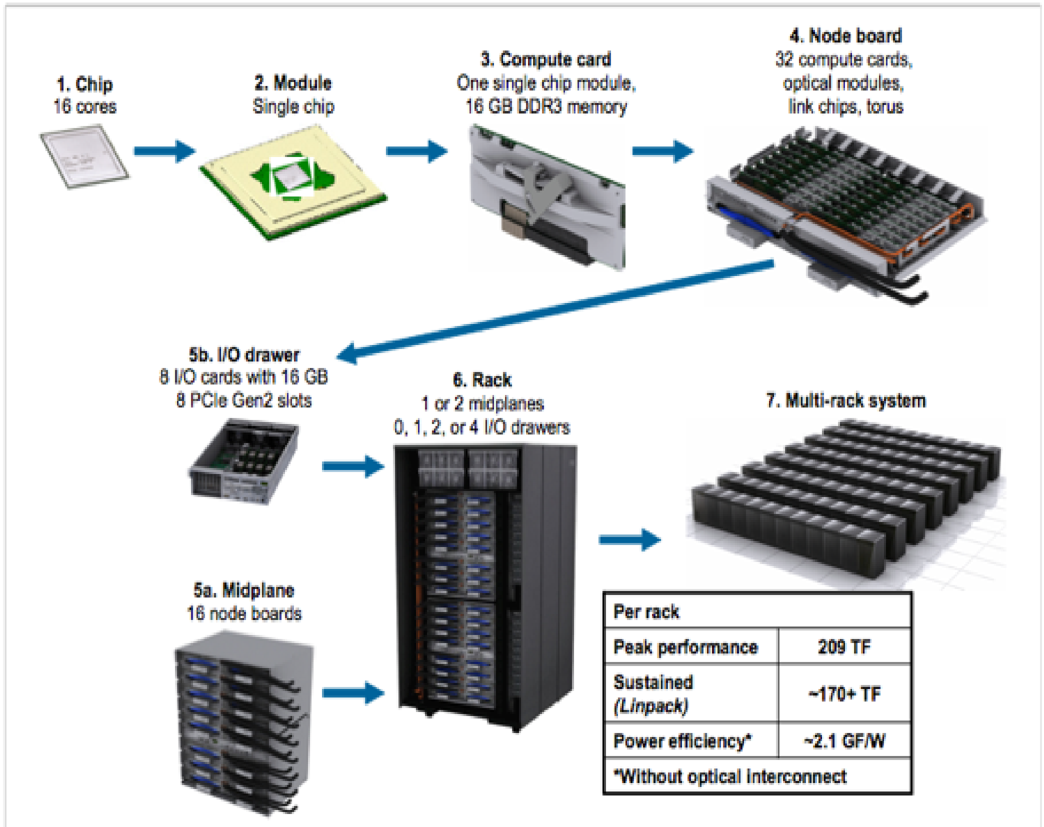
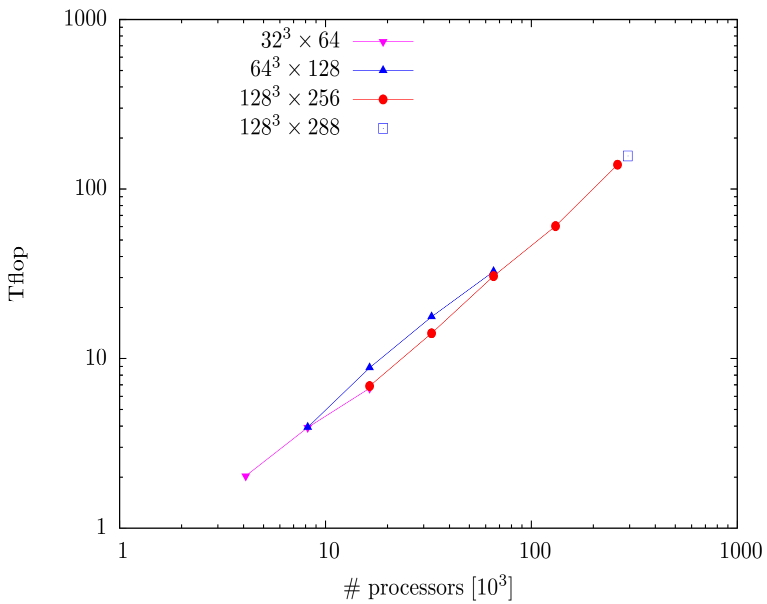


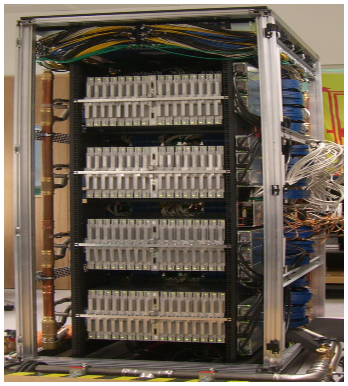
Figure 1-2 Blue Gene/Q hardware overview

Strong Scaling

- Test on 72 racks BG/P installation at supercomputer center Jülich (Gerhold, Herdioza, Urbach, K.J.)
- using tmHMC code (Urbach, K.J.)



Low budget machines



- QPACE 4+4 Racks in Jülich und Wuppertal
  1900 PowerXCell 8i nodes **190 TFlops (peak)**
  based on cell processor
  3-d torus network
  low power consumption 1.5W/Gflop

- **Videocards (NVIDIA Tesla)**
  CUDA programming language (C extension)

- Judge GPU cluster at research center Jülich with 206 nodes

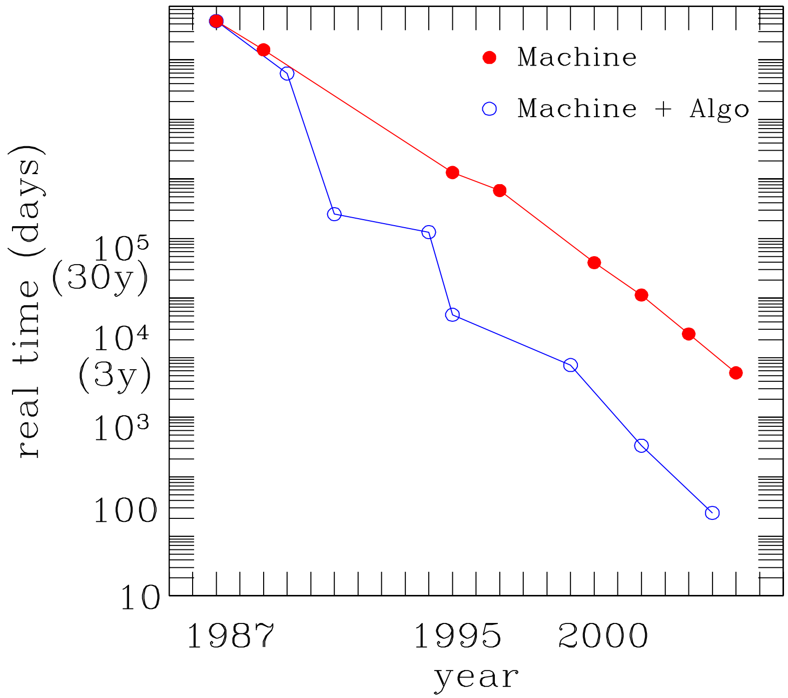
  - GPU: 2 NVIDIA Tesla M2050 (Fermi) 1.15 GHz (448 cores)
  - GPU: 2 NVIDIA Tesla M2070 (Fermi) 1.15 GHz (448 cores)
  - 240 Tflops peak performance

- trend: hybrid archictetures with accelerators

- challenge for 2020: **Exaflop Computing**

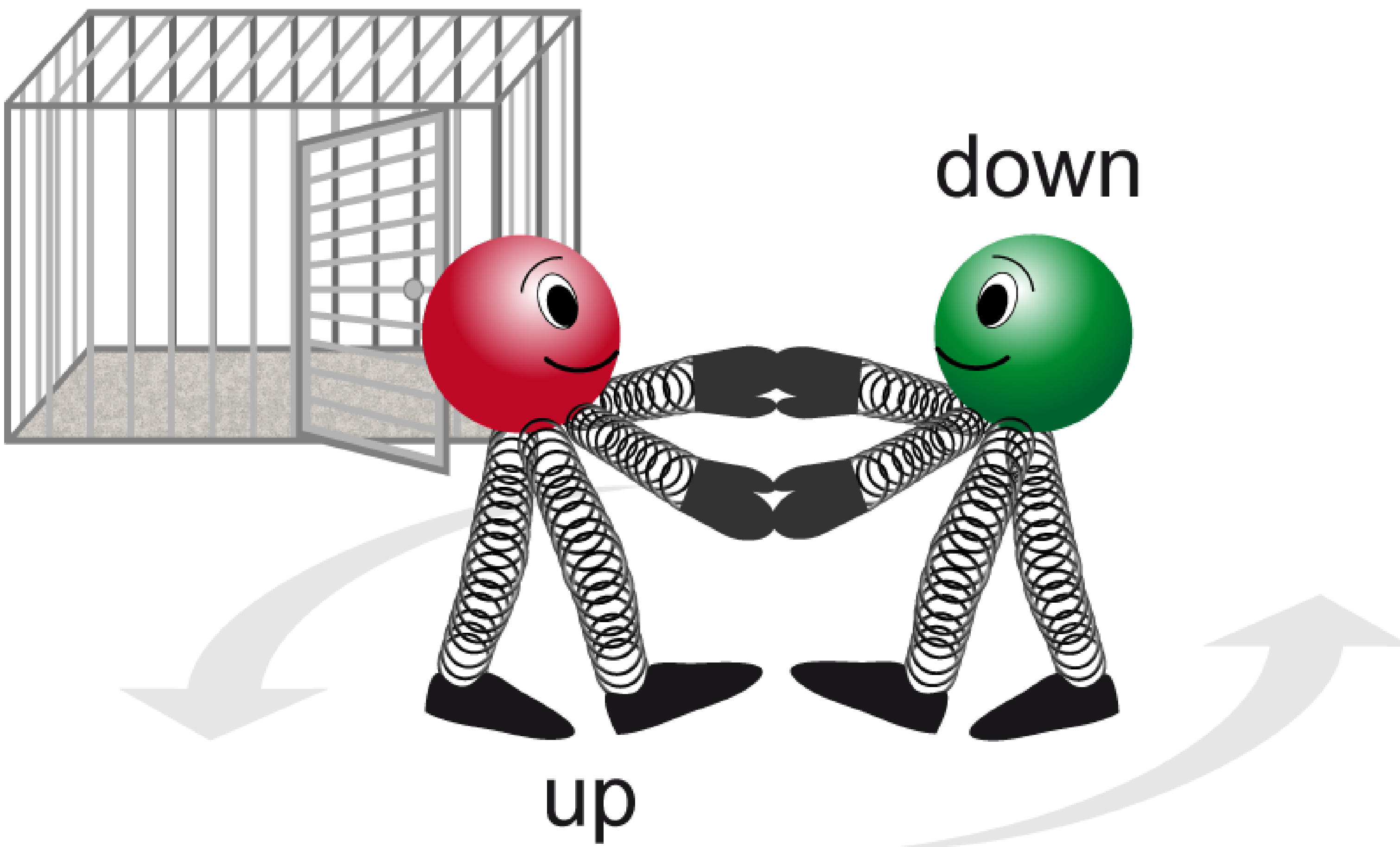
Computer and algorithm development over the years

time estimates for simulating $32^3 \cdot 64$ lattice, 5000 configurations



$\rightarrow$ O(few months) nowadays with a typical collaboration supercomputer contingent

$N_f = 2$ **dynamical flavours**



Examples of present Collaborations (using Wilson fermions)

- CLS collaboration
  Wilson gauge and clover improved Wilson fermions

- ALPHA
  Wilson gauge and clover improved Wilson fermions
  Schrödinger functional

- QCDSF
  tadpole improved Symanzik gauge and clover improved Wilson fermions

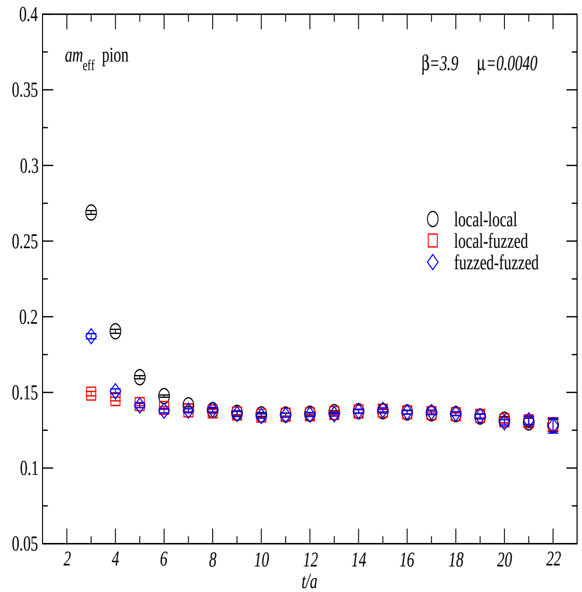
- ETMC
  tree-level Symanzik improved gauge and
  maximally twisted mass Wilson fermions

- RBC
  domain wall fermions

- BMW
  improved gauge and
  non-perturbatively imporved, 6 stout smeared Wilson fermions

Extraction of Masses

- Quark propagator: stochastic, fuzzed sources
- Change the location of the time-slice source: reduce autocorrelations



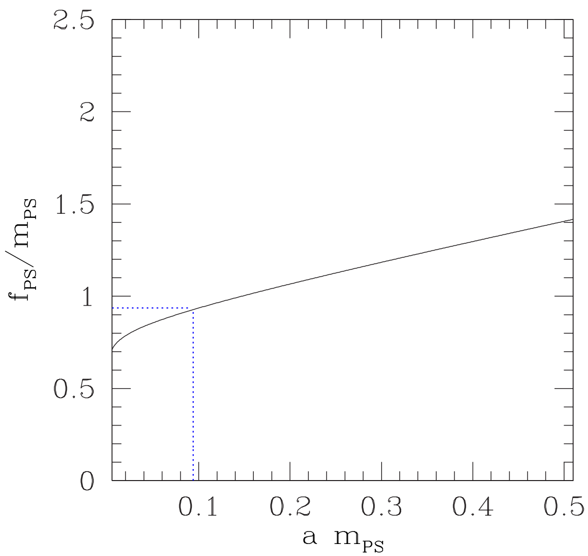
- effective mass of $\pi^{\pm}$
- isolate ground state : small statistical errors

$\Rightarrow$ get a number, but what does it mean? How to get physical units?

Setting the scale

$$m_{\rm PS}^{\rm latt} = a m_{\rm PS}^{\rm phys} \ , \quad f_{\rm PS}^{\rm latt} = a f_{\rm PS}^{\rm phys}$$

$$\frac{f_{\rm PS}^{\rm phys}}{m_{\rm PS}^{\rm phys}} = \frac{f_{\rm PS}^{\rm latt}}{m_{\rm PS}^{\rm latt}} + \mathrm{O}(a^2)$$



$\longrightarrow$ setting $\frac{f_{\rm PS}^{\rm latt}}{m_{\rm PS}^{\rm latt}} = 130.7/139.6$

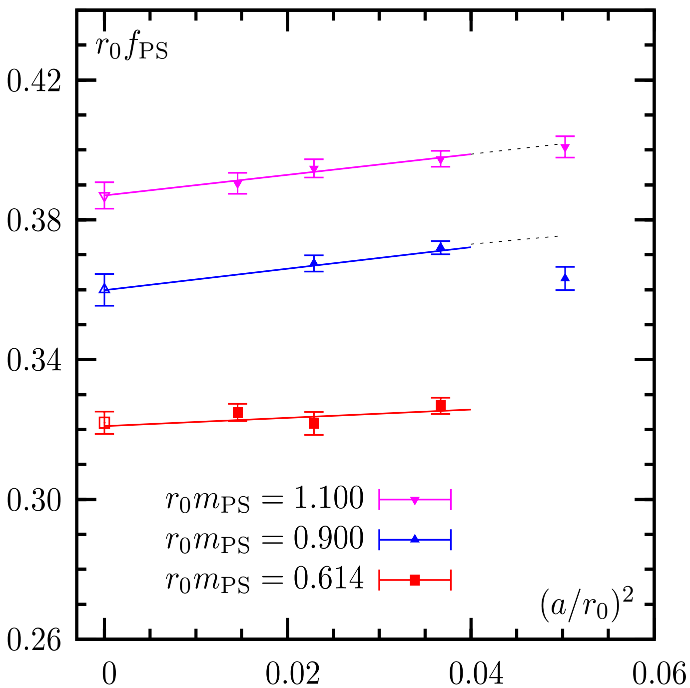
$\longrightarrow$ obtain $m_{\rm PS}^{\rm latt} = a139.6[\mathrm{Mev}]$

$\longrightarrow$ value for lattice spacing a

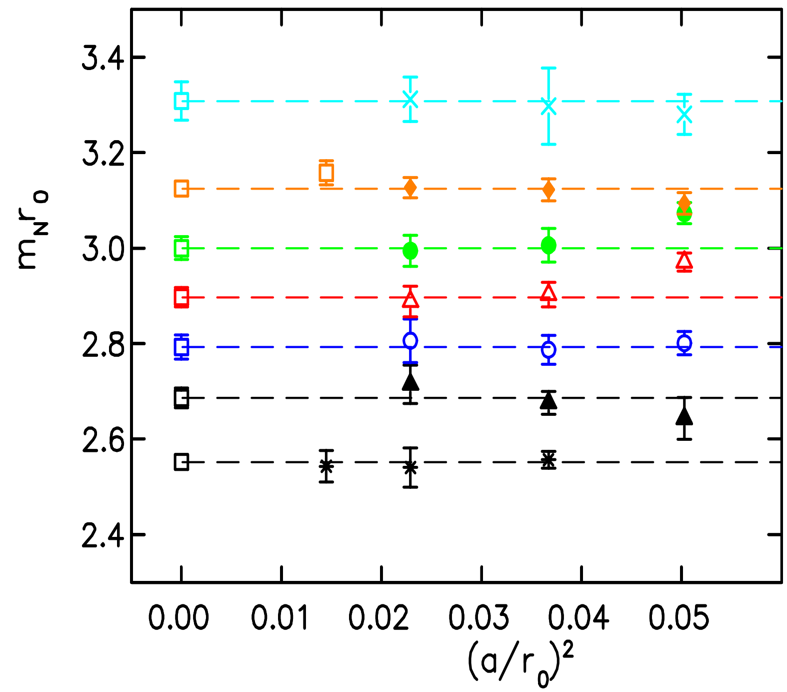
available configurations **(free on ILDG)**

β	a [fm]	$L^3 \cdot T$	L [fm]	$a\mu$	N_{traj} $(\tau = 0.5)$	m_{PS} [MeV]
4.20	~ 0.050	$48^3 \cdot 96$	2.4	0.0020	5200	~ 300
		$32^3 \cdot 64$	2.1	0.0060	5600	~ 420
4.05	~ 0.066	$32^3 \cdot 64$	2.2	0.0030	5200	~ 300
				0.0060	5600	~ 420
				0.0080	5300	~ 480
				0.0120	5000	~ 600
3.9	~ 0.086	$32^3 \cdot 64$	2.8	0.0030	4500	~ 270
				0.0040	5000	~ 300
		$24^3 \cdot 48$	2.1	0.0064	5600	~ 380
				0.0085	5000	~ 440
				0.0100	5000	~ 480
				0.0150	5400	~ 590
3.8	~ 0.100	$24^3 \cdot 48$	2.4	0.0060	4700×2	~ 360
				0.0080	3000×2	~ 410
				0.0110	2800×2	~ 480
				0.0165	2600×2	~ 580

Continuum limit scaling



$f_{\rm PS}$



$M_{\rm N}$

observe small $O(a^2)$ effects

Chiral perturbation theory

$$r_0 f_{\mathrm{PS}} = r_0 f_0 \left[1 - 2\xi \log\left(\frac{\chi_\mu}{\Lambda_4^2}\right) + \ldots \right.$$

$$(r_0 m_{\mathrm{PS}})^2 = \chi_\mu r_0^2 \left[1 + \xi \log\left(\frac{\chi_\mu}{\Lambda_3^2}\right) + \ldots \right.$$

where

$$\xi \equiv 2B_0\mu_q/(4\pi f_0)^2, \quad \chi_\mu \equiv 2B_0\mu_R, \quad \mu_R \equiv \mu_q/Z_{\mathrm{P}}$$

Chiral perturbation theory

$\rightarrow$ add finite volume and lattice spacing dependence:

$$r_0 f_{\rm PS} = r_0 f_0 \left[1 - 2\xi \log\left(\frac{\chi_\mu}{\Lambda_4^2}\right) + D_{f_{\rm PS}} a^2/r_0^2 + T_f^{\rm NNLO} \right] K_f^{\rm CDH}(L)$$

$$(r_0 m_{\rm PS})^2 = \chi_\mu r_0^2 \left[1 + \xi \log\left(\frac{\chi_\mu}{\Lambda_3^2}\right) + D_{m_{\rm PS}} a^2/r_0^2 + T_m^{\rm NNLO} \right] K_m^{\rm CDH}(L)^2$$

$$r_0/a(a\mu_q) = r_0/a + D_{r_0}(a\mu_q)^2$$

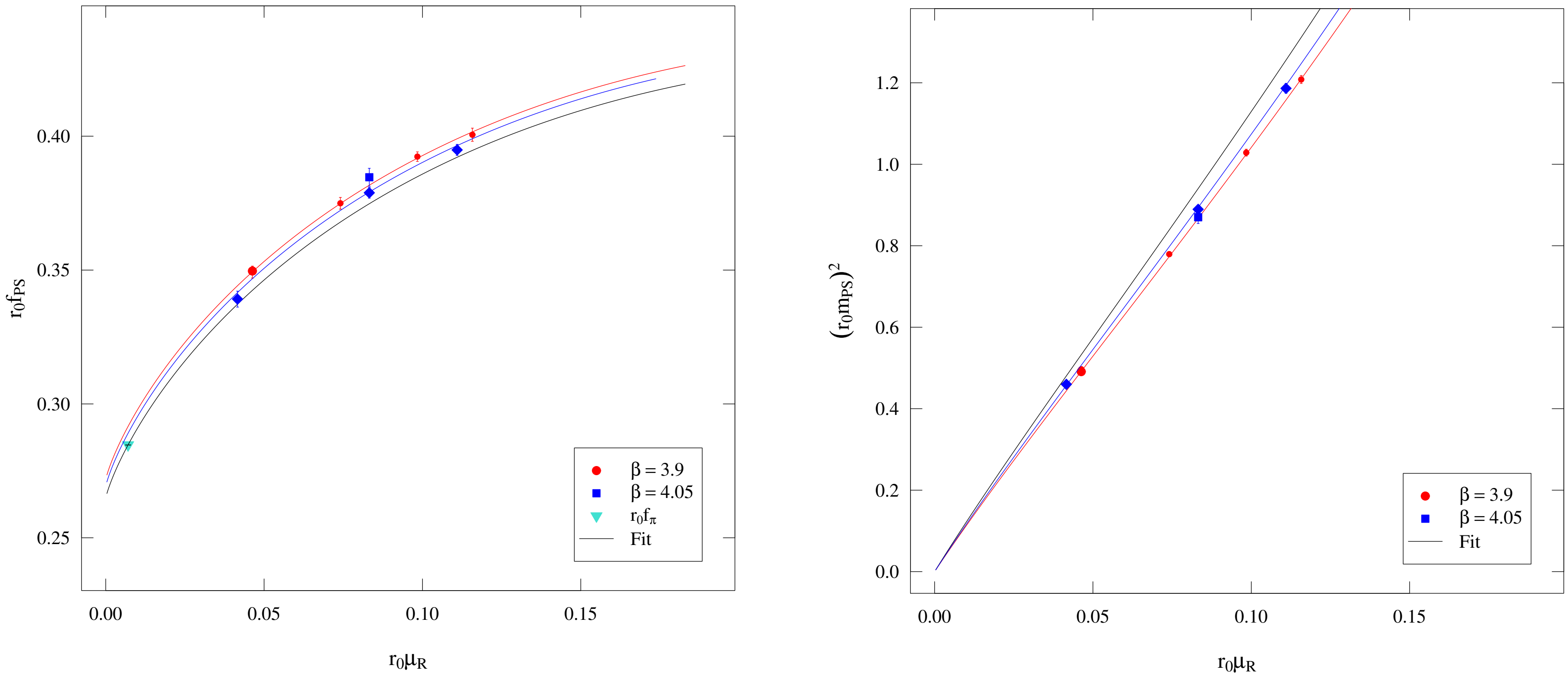
where

$$\xi \equiv 2B_0\mu_q/(4\pi f_0)^2, \quad \chi_\mu \equiv 2B_0\mu_R, \quad \mu_R \equiv \mu_q/Z_{\rm P}$$

- $D_{f,m}$ parametrize lattice artefacts
- $K_{f,m}^{\rm CDH}(L)$ Finite size corrections Colangelo *et al.*, 2005
- $T_{f,m}^{\rm NNLO}$ NNLO correction

Chiral perturbation theory

- Fit B: NLO continuum χPT, $T^{\mathrm{NNLO}}_{m,f} \equiv 0$, $D_{m_{\mathrm{PS}},f_{\mathrm{PS}}}$ fitted



two values of the lattice spacing $\rightarrow$ $D_m = -1.07(97)$, $D_f = 0.71(57)$

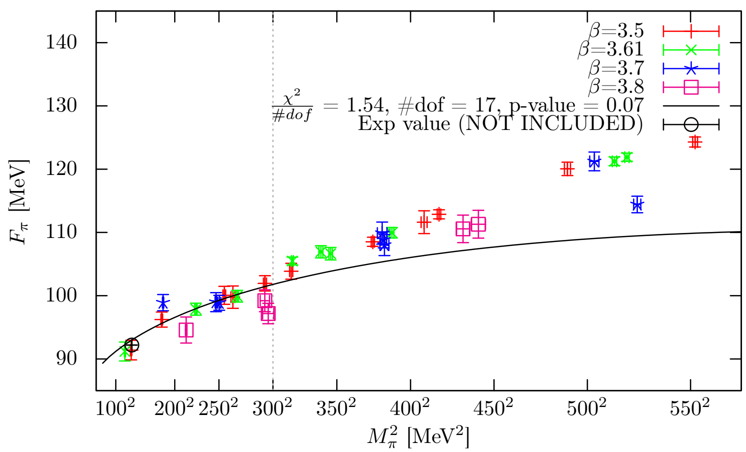
Chiral perturbation theory: results

quantity	value
$m_{u,d}$ [MeV]	3.37(23)
$\bar{\ell}_3$	3.49(19)
$\bar{\ell}_4$	4.57(15)
f_0 [MeV]	121.75(46)
B_0 [GeV]	2774(190)
r_0 [fm]	0.433(14)
$\langle r^2 \rangle_s$	0.729(35)
$\Sigma^{1/3}$ [MeV]	273.9(6.0)
f_π / f_0	1.0734(40)

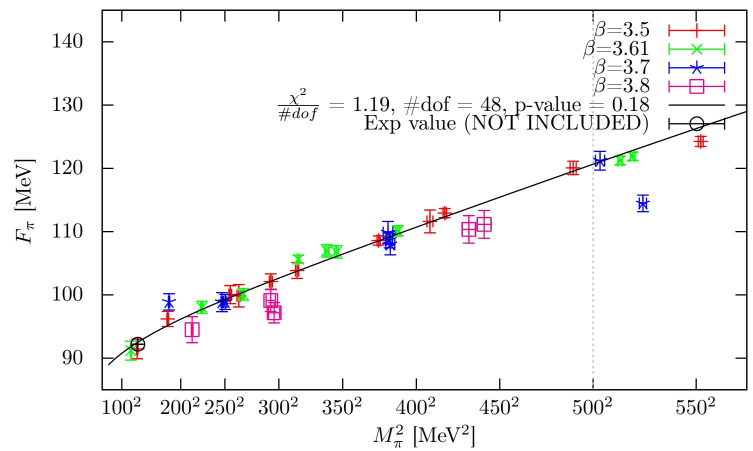
- averaged over many fit results, weighted with confidence levels
- B_0, Σ, $m_{u,d}$ renormalised in $\overline{\mathrm{MS}}$ scheme at scale $\mu = 2$ GeV
- LECs can be used to compute further quantities: scattering lengths

A more rigorous approach to chiral perturbation theory

BMW collaboration $\rightarrow$ pion masses down to 120MeV



NLO, fit for $M_\pi < 300\text{MeV}$



NNLO, fit for $M_\pi < 500\text{MeV}$

- NLO: only applicable for $M_\pi \leq 300\text{MeV}$
- NNLO: wider applicability $M_\pi \lesssim 300\text{MeV}$
  but need very precise data for stable fit

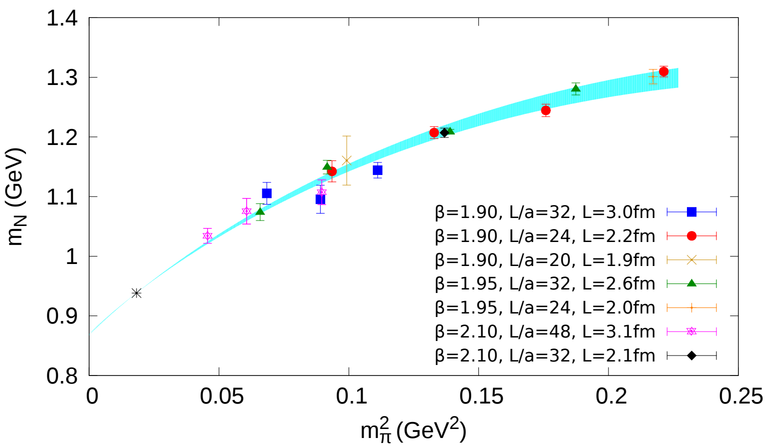
$\Rightarrow$ better work directly at physical point

What about chiral extrapolations for the nucleon mass?

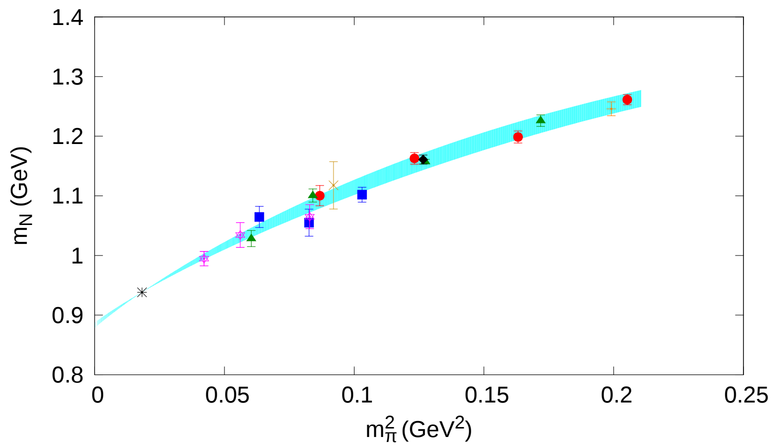
simplest, $O(p^3)$ formula $m_N = m_N^{(0)} - 4c_1 m_\pi^2 - \frac{3g_A^2}{32\pi f_\pi^2} m_\pi^3$

next order, $O(p^4)$

$$
\begin{aligned}
m_N \quad = \quad & m_N^0 - 4c_1 m_\pi^2 - \frac{3g_A^2}{32\pi f_\pi^2} m_\pi^3 - 4E_1(\lambda) m_\pi^4 - \frac{3\left(g_A^2+3c_A^2\right)}{64\pi^2 f_\pi^2 m_N^0} m_\pi^4 - \frac{\left(3g_A^2+10c_A^2\right)}{32\pi^2 f_\pi^2 m_N^0} m_\pi^4 \log \\
- \quad & \frac{c_A^2}{3\pi^2 f_\pi^2}\left(1+\frac{\Delta}{2m_N^0}\right)\left[\frac{\Delta}{4} m_\pi^2 + \left(\Delta^3 - \frac{3}{2} m_\pi^2 \Delta\right)\log\left(\frac{m_\pi}{2\Delta}\right) + \left(\Delta^2 - m_\pi^2\right) R\left(m_\pi\right)\right]
\end{aligned}
$$



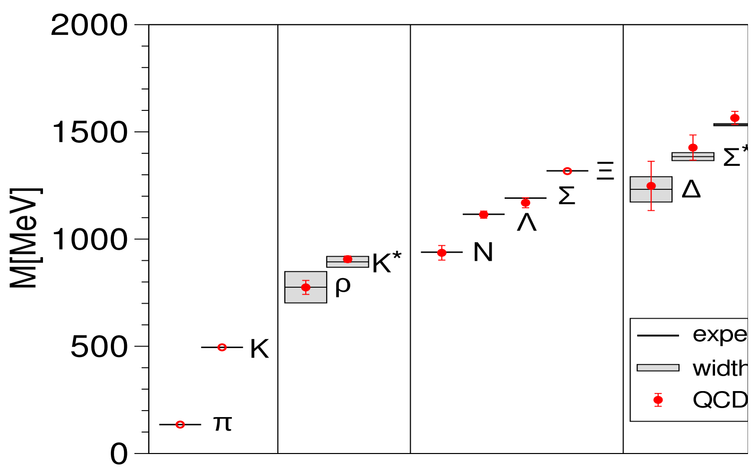
$O(p^3)$ fit

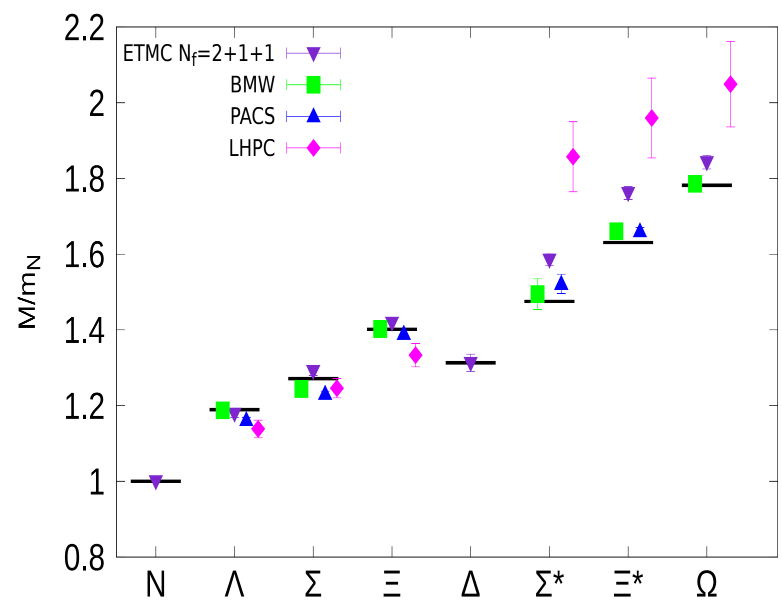


$O(p^4)$ fit

The lattice QCD benchmark calculation: the spectrum

spectrum for $N_f = 2+1$ and $2+1+1$ flavours





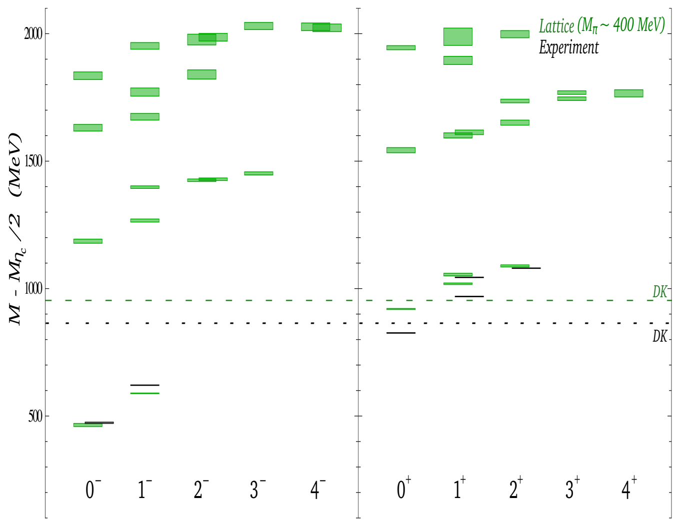
first spectrum calculation BMW

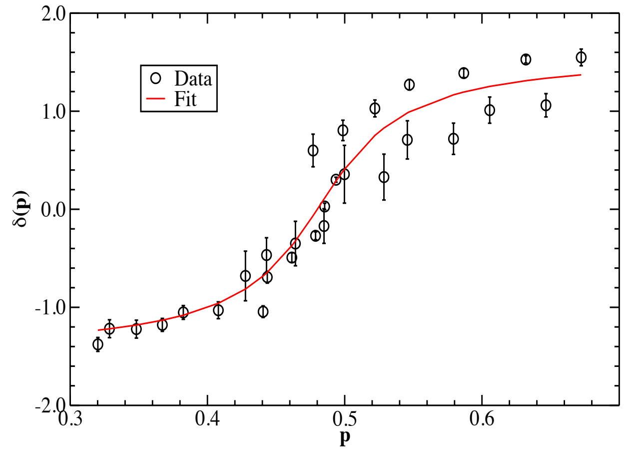
repeated by other collaborations

(ETMC: C. Alexandrou, M. Constantinou, V. Drach, G. Koutsou, K.J.)

- spectrum for $N_f = 2$, $N_f = 2+1$ and $N_f = 2+1+1$ flavours
  $\rightarrow$ no flavour effects for light baryon spectrum

We can go further





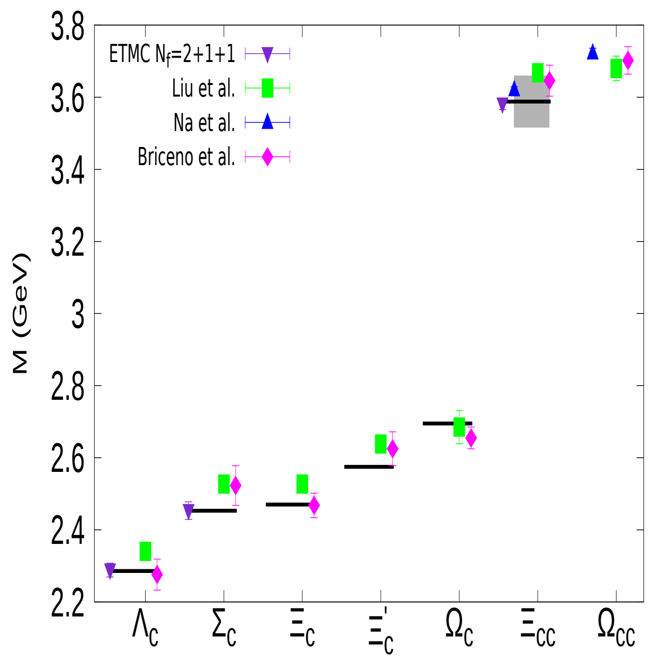
charmed spectrum

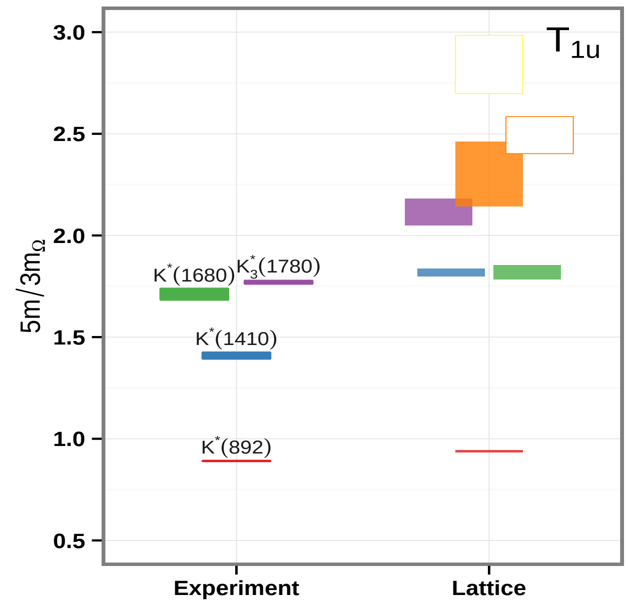
(G. Moir, M. Peardon, S. Ryan
C.Thomas, L. Liu)

Phase shifts (model calculation)

(P. Giudice, D. McManus, M. Peardon)

We can go further





charmed spectrum

(C. Alexandrou, V. Drach,
K. Hadiyiannakou, C. Kallidonis, K.J.)

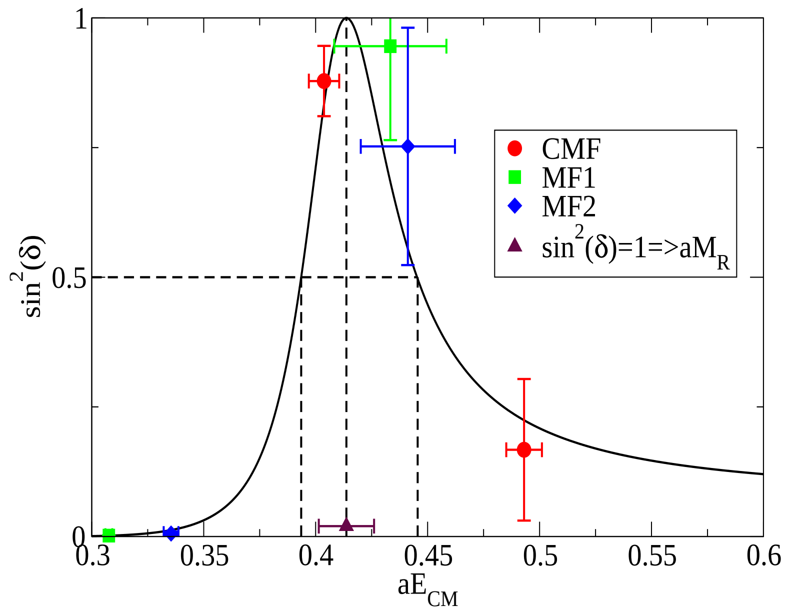
excited K meson spectrum

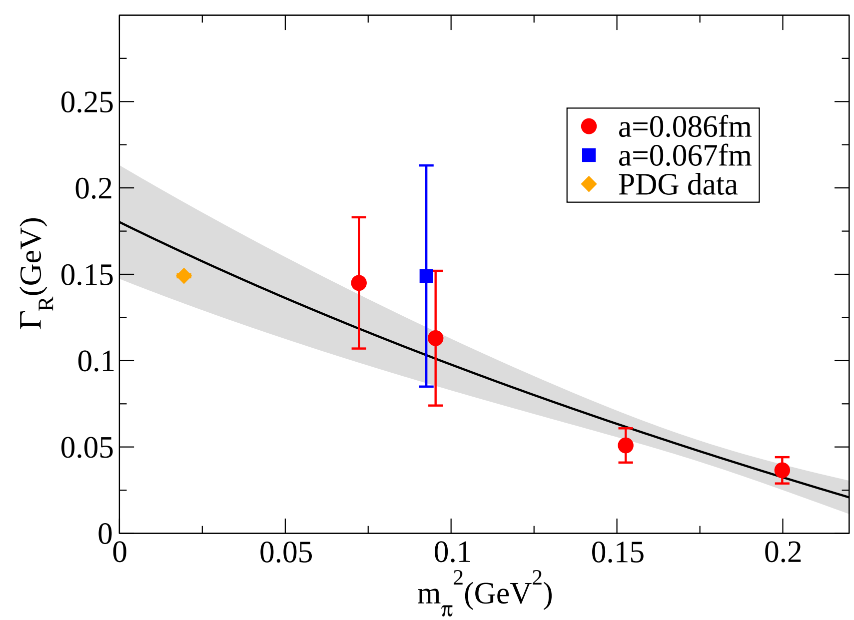
(J. Bulava, B. Fahy, J. Foley, Y. Jhang
K. Juge, D. Lenkner, C. Morningstar, H. Wong)

The ρ-meson resonance: dynamical quarks at work

(X. Feng, D. Renner, K.J., 2011)

- usage of three Lorentz frames



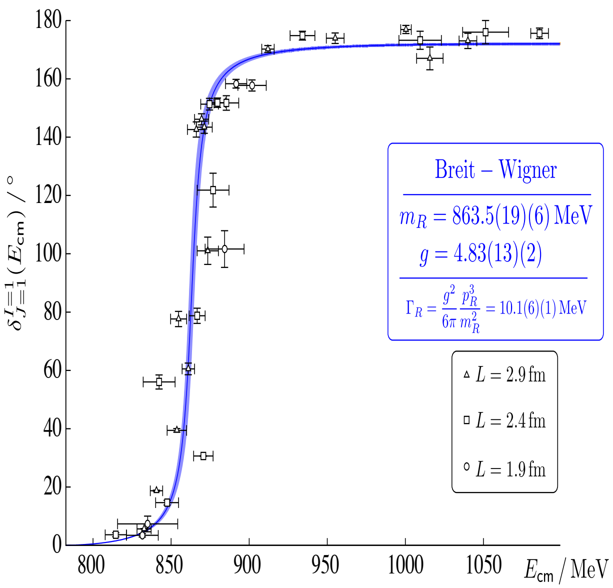


$m_{\pi^+} = 330$ MeV, $a = 0.079$ fm, $L/a = 32$

fitting $z = (M_\rho + i\frac{1}{2}\Gamma_\rho)^2$

$m_\rho = 1033(31)$ MeV, $\Gamma_\rho = 123(43)$ MeV

We can go further

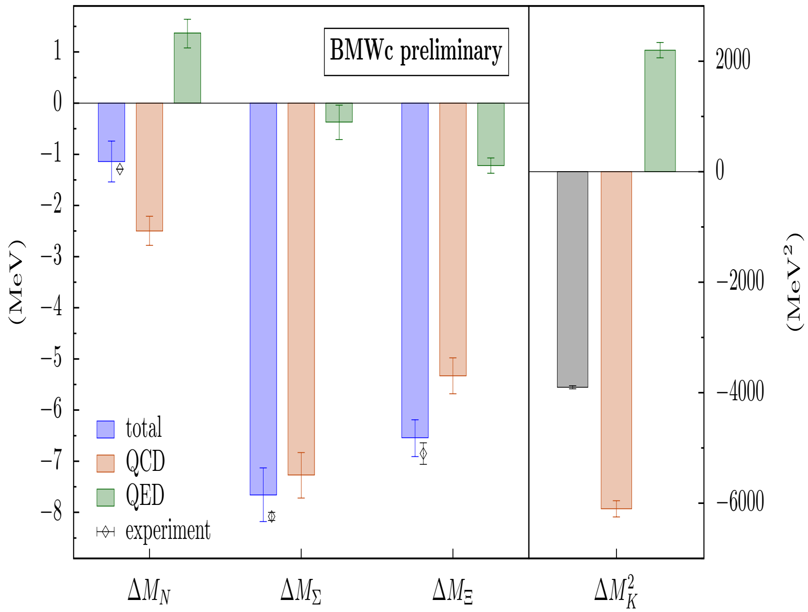


QCD phase shift

(J. Dudek, R. Edwards, C. Thomas, 2013)

Even isospin and electromagnetic mass splitting

(BMW collaboration)



baryon spectrum with mass splitting

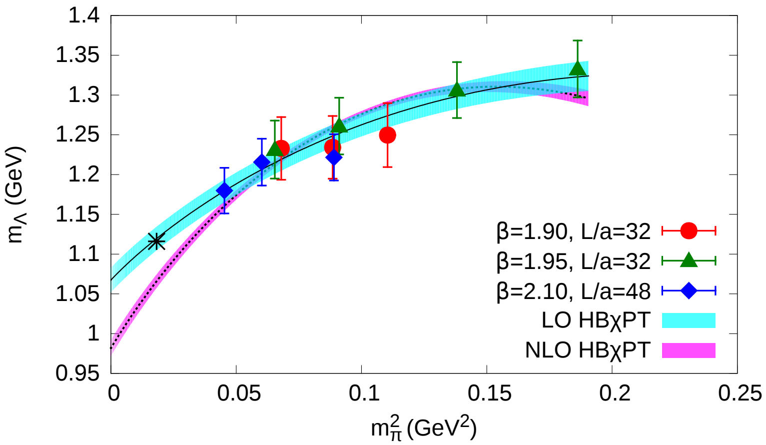
- nucleon: isospin and electromagnetic effects with opposite signs
- nevertheless physical splitting reproduced

NLO for other masses

$$
\begin{aligned}
m_{\Lambda}^{\mathrm{NLO}}(m_\pi) &= m_{\Lambda}^{(0)} - 4c_{\Lambda}^{(1)} m_\pi^2 - \frac{g_{\Lambda\Sigma}^2}{(4\pi f_\pi)^2}\mathcal{F}(m_\pi, \Delta_{\Lambda\Sigma}, \lambda) - \frac{4g_{\Lambda\Sigma^*}^2}{(4\pi f_\pi)^2}\mathcal{F}(m_\pi, \Delta_{\Lambda\Sigma^*}, \lambda) \\
m_{\Sigma}^{\mathrm{NLO}}(m_\pi) &= m_{\Sigma}^{(0)} - 4c_{\Sigma}^{(1)} m_\pi^2 - \frac{2g_{\Sigma\Sigma}^2}{16\pi f_\pi^2} m_\pi^3 - \frac{g_{\Lambda\Sigma}^2}{3(4\pi f_\pi)^2}\mathcal{F}(m_\pi, -\Delta_{\Lambda\Sigma}, \lambda) - \frac{4g_{\Sigma^*\Sigma}^2}{3(4\pi f_\pi)^2}\mathcal{F}(\\
m_{\Xi}^{\mathrm{NLO}}(m_\pi) &= m_{\Xi}^{(0)} - 4c_{\Xi}^{(1)} m_\pi^2 - \frac{3g_{\Xi\Xi}^2}{16\pi f_\pi^2} m_\pi^3 - \frac{2g_{\Xi^*\Xi}^2}{(4\pi f_\pi)^2}\mathcal{F}(m_\pi, \Delta_{\Xi\Xi^*}, \lambda) \\
m_{\Delta}^{\mathrm{NLO}}(m_\pi) &= m_{\Delta}^{(0)} - 4c_{\Delta}^{(1)} m_\pi^2 - \frac{25}{27}\frac{g_{\Delta\Delta}^2}{16\pi f_\pi^2} m_\pi^3 - \frac{2g_{\Delta N}^2}{3(4\pi f_\pi)^2}\mathcal{F}(m_\pi, -\Delta_{N\Delta}, \lambda) \\
m_{\Sigma^*}^{\mathrm{NLO}}(m_\pi) &= m_{\Sigma^*}^{(0)} - 4c_{\Sigma^*}^{(1)} m_\pi^2 - \frac{10}{9}\frac{g_{\Sigma^*\Sigma^*}^2}{16\pi f_\pi^2} m_\pi^3 - \frac{2}{3(4\pi f_\pi)^2}\left[g_{\Sigma^*\Sigma}^2\mathcal{F}(m_\pi, -\Delta_{\Sigma\Sigma^*}, \lambda) + g\right. \\
m_{\Xi^*}^{\mathrm{NLO}}(m_\pi) &= m_{\Xi^*}^{(0)} - 4c_{\Xi^*}^{(1)} m_\pi^2 - \frac{5}{3}\frac{g_{\Xi^*\Xi^*}^2}{16\pi f_\pi^2} m_\pi^3 - \frac{g_{\Xi^*\Xi}^2}{(4\pi f_\pi)^2}\mathcal{F}(m_\pi, -\Delta_{\Xi\Xi^*}, \lambda) \\
m_{\Omega}^{\mathrm{NLO}}(m_\pi) &= m_{\Omega}^{(0)} - 4c_{\Omega}^{(1)} m_\pi^2
\end{aligned}
$$

$$
\mathcal{F}(m,\Delta,\lambda) = (m^2-\Delta^2)\sqrt{\Delta^2-m^2+i\epsilon}\ \log\left(\frac{\Delta-\sqrt{\Delta^2-m^2+i\epsilon}}{\Delta+\sqrt{\Delta^2-m^2+i\epsilon}}\right) - \frac{3}{2}\Delta\ m^2\log\left(\frac{m^2}{\lambda^2}\right) - \Delta^3\log\left(\frac{4\Delta^2}{m^2}\right)
$$

Effect of applying LO and NLO chiral perturbation theory



Baryon	$\mathcal{O}(p^3)$	NLO
N	64.9(1.5)	45.3(4.3)
Λ	46.0(1.8)	74.5(1.8)
Σ^+	55.6(2.1)	65.3(2.2)
Σ^0	54.7(1.7)	64.5(1.8)
Σ^-	58.3(1.9)	68.3(1.9)
Δ	79.9(3.0)	100.3(3.1)
Σ^*	45.1(2.8)	68.6(2.7)
Ξ^*	20.8(2.2)	38.2(2.2)

nucleon σ-terms

- chiral extrapolation dominating systematic error

$\Rightarrow$ need physical point simulations

Simulation at physical pion mass

- Lattice spacing: $a = 0.1\text{fm}$
- Pion mass: $m_\pi = 140\text{MeV}$
- Suppression of finite size effects: $L \cdot m_\pi > 5$
- Requirement
  - $L \approx 5\text{fm} \rightarrow 48^3 \cdot 96$ lattice ($\leftarrow$ present standard)
  - for $a = 0.05\text{fm} \rightarrow 96^3 \cdot 192$ lattice
  - grows of simulation cost $\propto a^{-6}$

Examples of present Collaborations (using Wilson fermions)

- BMW
  E.g., low lying baryon spectrum, B_K,
  quark masses, QCD thermodynamics, several lattice spacings

- PACS-CS
  E.g., charmed baryons, charm physics

- MILC collaboration
  E.g.: kaon semileptonic formfactor

- ETMC
  Meson masses and decay constants, nucleon structure,
  hadronic contributions to electroweak observables

- RBC
  domain wall fermions

Summary

- wanted to show basic step for proton mass computation
- 25 years effort
  - conceptual developments: $O(a)$-improved actions
  - algorithm developments
  - machine developments
- mission of hadron spectrum benchmark calculation completed
- ready for more complicated observables
  $\longrightarrow$ (see other lectures at this school

General articles

Lectures, review articles

- R. Gupta
  Introduction to Lattice QCD, hep-lat/9807028

- C. Davies
  Lattice QCD, hep-ph/0205181

- M. Lüscher
  Advanced Lattice QCD, hep-lat/9802029
  Simulation strategies, hep-lat/xxxxxxx
  Chiral gauge theories revisited, hep-th/0102028

- A.D. Kennedy
  Algorithms for Dynamical Fermions, hep-lat/0607038

Books about Lattice Field Theory

- **C. Gattringer and C. Lang**
  Quantum Chromodynamics on the Lattice
  Lecture Notes in Physics 788, Springer, 2010
- **T. DeGrand and C. DeTar**
  Lattice methods for Quantum Chromodynamics
  World Scientific, 2006
- **H.J. Rothe**
  Lattice gauge theories: An Introduction
  World Sci.Lect.Notes Phys.74, 2005
- **J. Smit** *Introduction to quantum fields on a lattice: A robust mate*
  Cambridge Lect.Notes Phys.15, 2002
- **I. Montvay and G. Münster**
  Quantum fields on a lattice
  Cambridge, UK: Univ. Pr., 1994
- **Yussuf Saad**
  Iterative Methods for sparse linear systems
  Siam Press, 2003