Introduction to Lattice QCD II

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- **Task: compute proton mass**
  - need an action
  - need an algorithm
  - need an observable
  - need a supercomputer
- **the muon anomalous magnetic moment**

Quantum Chromodynamics

Pathintegral

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{-S_{\text{gauge}}-S_{\text{ferm}}}$$

gauge action

Fermion action

The actions

- fermion action

$$S_{\text{ferm}} = \int d^4x \bar{\Psi}(x) \left[\gamma_\mu D_\mu + m\right] \Psi(x)$$

gauge covariant derivative

$$D_\mu \Psi(x) \equiv (\partial_\mu - i g_0 A_\mu(x)) \Psi(x)$$

A_μ gauge potential

g_0 coupling m quark mass

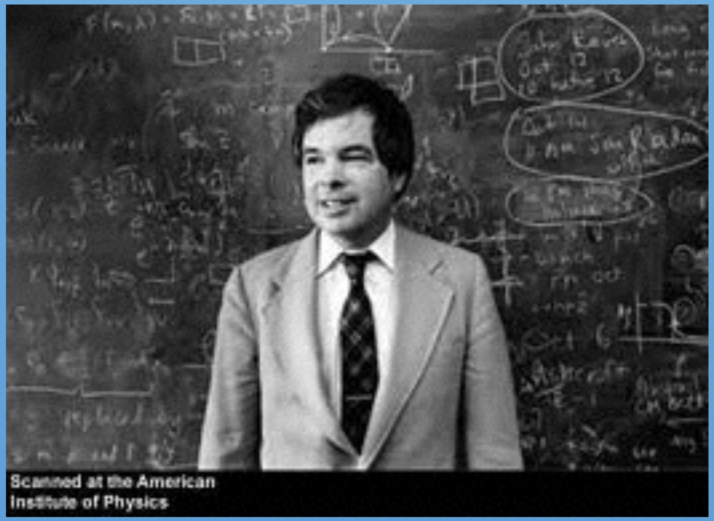
- gauge action

$$S_{\text{gauge}} = \int d^4x F_{\mu\nu} F_{\mu\nu} \ , \ F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i \left[A_\mu(x), A_\nu(x)\right]$$

QCD had to be invented

asymptotic freedom	confinement
distances << 1fm	distances O(1fm)
world of quarks and gluons	world of hadrons
perturbative description	non-perturbative methods

As Wilson said it:



Unfortunately, it is not known yet whether the quarks in quantum chromodynamics actually form the required bound states. To establish whether these bound states exist one must solve a strong coupling problem and present methods for solving field theories don't work for strong coupling.

(Wilson, Cargese Lecture notes 1976)

Going to the lattice

quark fields

$$\Psi(x) \quad \bar{\Psi}(x) \qquad\qquad x = (t, \mathbf{x}) \ \text{ integers}$$

discretized action

$$S \to a^4 \sum_x \bar{\Psi}(x) \left[\gamma_\mu \partial_\mu - r \underbrace{\partial_\mu^2}_{\nabla_\mu^* \nabla_\mu} + m\right] \Psi(x)$$

$$\partial_\mu \to \frac{1}{2} \left[\nabla_\mu^* + \nabla_\mu\right]$$

$$\nabla_\mu \Psi(x) = \frac{1}{a} \left[\Psi(x + a\hat{\mu}) - \Psi(x)\right]$$

$$\nabla_\mu^* \Psi(x) = \frac{1}{a} \left[\Psi(x) - \Psi(x - a\hat{\mu})\right]$$

Nielsen-Ninomiya theorem

clash between chiral symmetry
and fermion proliferation

for any lattice Dirac operator D

- D is local; bounded by $Ce^{-\gamma/a|x|}$
- $\tilde{D}(p) = i\gamma_\mu p_\mu + \mathrm{O}(ap^2)$
- D is invertible for all $p \neq 0$
- chiral symmetric: $\gamma_5 D + D\gamma_5 = 0$

cannot be fulfilled simultaneously

The theorem simply states the fact that the Chern number is a cobordism invariant
(Friedan)

Gauge fields

Introduce group valued fields $U(x,\mu) \in \mathrm{SU}(3)$

relation to gauge potential

$$U(x,\mu) = \exp(iaA_\mu^b(x)T^b) = 1 + iaA_\mu^b(x)T^b + \dots$$

discretization of field strength tensor

$$U(x,\mu)U(x+a\mu,\nu) - U(x,\nu)U(x+a\hat{\nu},\mu) = ia^2 F_{\mu\nu}(x) + O(a^3)$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i[A_\mu(x), A_\nu(x)$$

lattice action

$$S_{\mathrm{w}}(U) = \sum_{\Box} \beta \left\{ 1 - \frac{1}{3} Re\mathrm{Tr}(U_\Box) \right\} \ a \to 0 \ \frac{1}{2g^2} a^4 \sum_x \mathrm{Tr}(F_{\mu\nu}(x)^2) + O(a^6)$$

$$\beta = 6/g^2$$

Physical observables

$$\underbrace{\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{\text{fields}} \mathcal{O} e^{-S}}$$

$\downarrow$ lattice discretization

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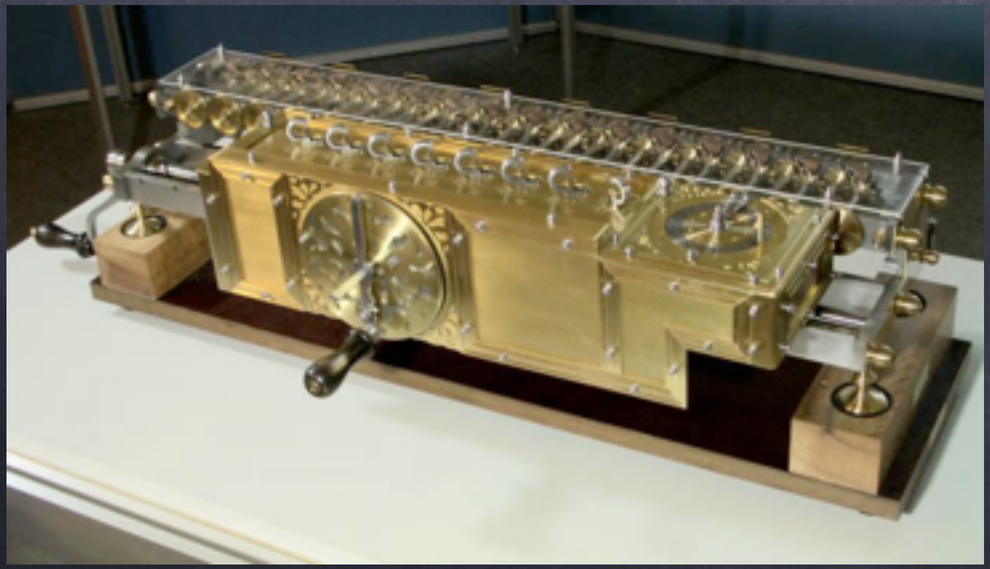
$\downarrow$



Monte Carlo Method

$$\langle f(x)\rangle = \int dx\, f(x) e^{-x^2}$$

integration points $x_i, i = 1, \cdots, N$

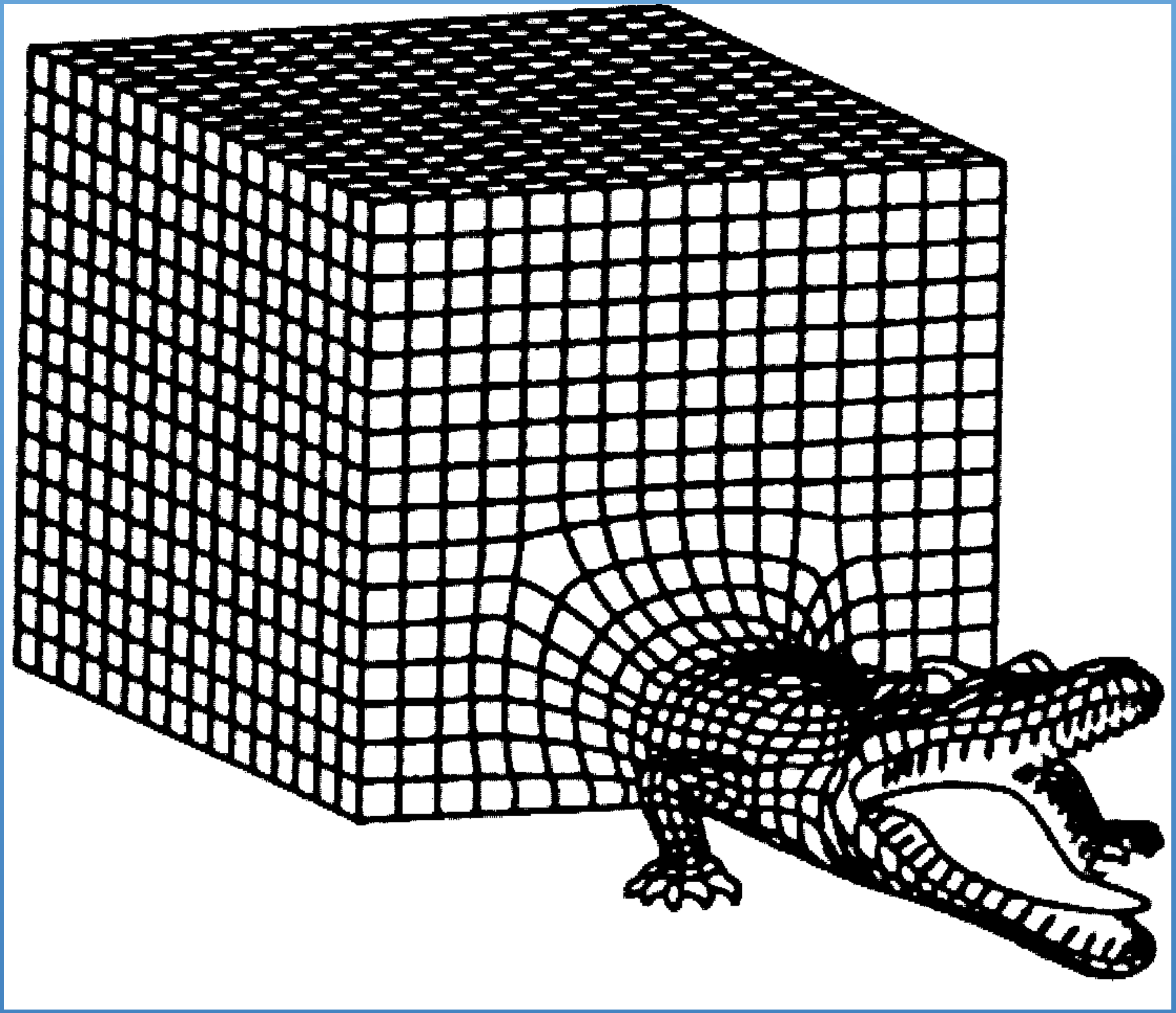
taken from distribution e^{-x^2}

$$\Rightarrow \langle f(x)\rangle \approx \frac{1}{N}\sum_i f(x_i)$$

error $\propto 1/\sqrt{N}$

QMC method (maybe later): $\propto 1/N$

There are dangerous animals



Another look at the Wilson action

$$S = \underbrace{S_{\mathrm{G}}}_{\mathrm{O}(a^2)} + \underbrace{S_{\mathrm{naive}}}_{\mathrm{O}(a^2)} + \underbrace{S_{\mathrm{wilson}}}_{\mathrm{O}(a)}$$

linear lattice artefacts:

⟹ need small lattice spacings

⟹ need large volumes

⟸ want L=aN = 1fm

killing O(a) effects

clover-improved Wilson fermions

maximally twisted mass Wilson fermions

overlap/domainwall fermions

⟹ exact lattice chiral symmetry

Realizing O(a)-improvement

continuum action: $S = \bar{\Psi}\left[m + \gamma_\mu D_\mu\right]\Psi$

axial transformation

$$\Psi \rightarrow e^{i\omega\gamma_5\tau_3/2}\Psi \qquad \bar{\Psi} \rightarrow \bar{\Psi}e^{i\omega\gamma_5\tau_3/2}$$

$$m \rightarrow m e^{i\omega\gamma_5\tau_3} \equiv m' + i\mu\gamma_5\tau_3$$

polar mass m and twist angle ω

$$m = \sqrt{m'^2 + \mu^2}\,, \quad \tan\omega = \mu/m$$

$\omega = 0$ standard QCD action

$\omega = \pi/2 :$ $\quad S = \bar{\Psi}\left[i\mu\gamma_5\tau_3 + \gamma_\mu D_\mu\right]$

(maximal twist)

Repeat this on the lattice

$$D_{\mathrm{tm}} = m_q + i\mu\tau_3\gamma_5 + \frac{1}{2}\gamma_\mu\left[\nabla_\mu + \nabla^*_\mu\right] - ar\frac{1}{2}\nabla^*_\mu\nabla_\mu$$

difference to continuum situation:

Wilson term not invariant under axial transformations

free fermion propagator

$$\left[m_q + i\gamma_\mu \sin p_\mu a + \frac{r}{a}\sum_\mu(1-\cos p_\mu a) + i\mu\tau_3\gamma_5\right]^{-1}$$

$$\propto (\sin p_\mu a)^2 + \left[m_q + \frac{r}{a}\sum_\mu(1-\cos p_\mu a)\right]^2 + \mu^2$$

$$\lim_{a\to 0}: \quad p_\mu^2 + m_q^2 + \mu^2 + \underbrace{am_q\sum_\mu p_\mu}_{\mathrm{O}(a)}$$

$m_q = 0 \ \ (\omega = \pi/2)$ kills O(a) effects

A general argument

Symanzik expansion

$$\langle \mathcal{O} \rangle|_{(m_q,r)} = [\xi(r) + a m_q \eta(r)]\ \langle \mathcal{O} \rangle|_{m_q}^{\text{cont}} + a\chi(r)\ \langle \mathcal{O}_1 \rangle|_{m_q}^{\text{cont}}$$

$$\langle \mathcal{O} \rangle|_{(-m_q,-r)} = [\xi(-r) - a m_q \eta(-r)]\ \langle \mathcal{O} \rangle|_{-m_q}^{\text{cont}} + a\chi(-r)\ \langle \mathcal{O}_1 \rangle|_{-m_q}^{\text{cont}}$$

Symmetry

$$R_5 \times (r \to -r) \times (m_q \to -m_q)\ ,\ R_5 = e^{i\omega\gamma_5\tau^3}$$

automatic O(a) improvement

through mass averaging

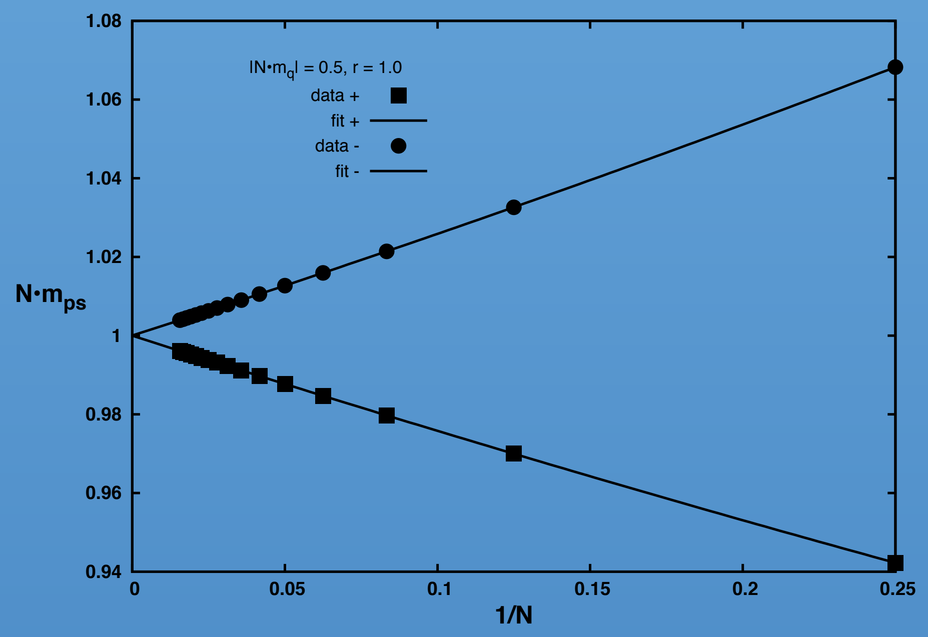
$$\frac{1}{2}\Big[\langle \mathcal{O} \rangle|_{m_q,r} + \langle \mathcal{O} \rangle|_{-m_q,r}\Big] = \langle \mathcal{O} \rangle|_{m_q}^{\text{cont}} + O(a^2)$$

special case: $m_q = 0$

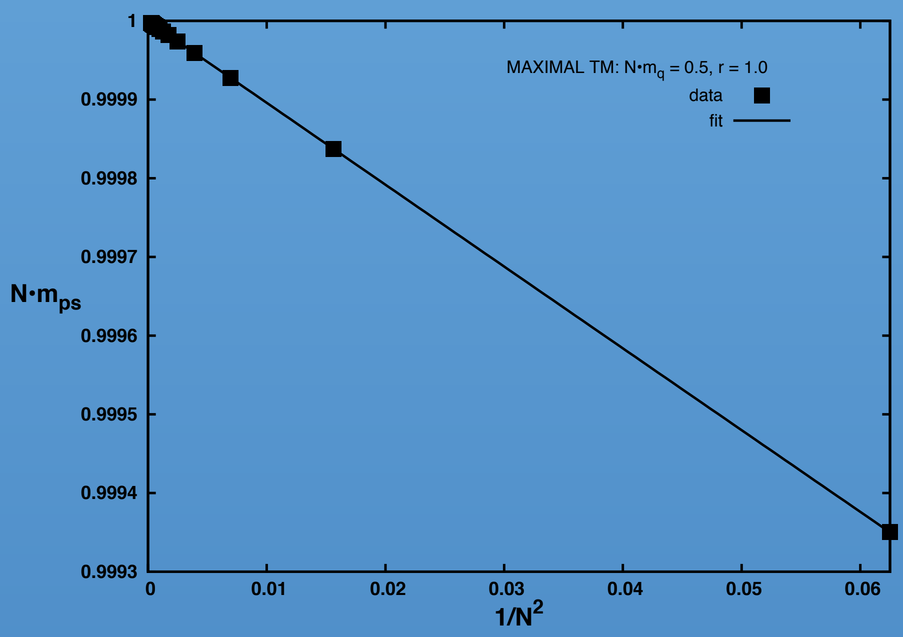
$$\langle \mathcal{O} \rangle|_{m_q=0,r} = \langle \mathcal{O} \rangle|_{m_q}^{\text{cont}} + O(a^2)$$

A test at tree-level

positive and negative mass



mass average



$$1/N \propto a$$

$$1/N^2 \propto a^2$$

a lattice spacing

N lattice size

$m_{\rm PS}$ "Pion Mass"

Exact lattice chiral symmetry

Starting point: Ginsparg-Wilson relation

$$\gamma_5 D + D\gamma_5 = 2aD\gamma_5 D$$

$$\Rightarrow D^{-1}\gamma_5 + \gamma_5 D^{-1} = 2a\gamma_5$$

for any lattice Dirac operator D, satisfying the Ginsparg Wilson relation, the action

$$S = \bar{\psi} D \psi$$

is invariant under

$$\delta\psi = \gamma_5(1 - \frac{1}{2}aD)\psi \ , \quad \delta\bar{\psi} = \bar{\psi}(1 - \frac{1}{2}aD)\gamma_5$$

one solution of GW-relation

Neuberger's overlap operator

$$\boxed{D_{\rm ov} = \left[1 - A(A^\dagger A)^{-1/2}\right]}$$

$$A = 1 + s - D_{\rm w}(m_q = 0)$$

advantages of overlap operator

- often trivial renormalisation constants
- index theorem fulfilled
- continuum like behaviour

dis-advantage

- computationally very expensive

No free lunch theorem

V, m_π	Overlap	Wilson TM	rel. factor
12^4, 720Mev	48.8(6)	2.6(1)	18.8
12^4, 390Mev	142(2)	4.0(1)	35.4
16^4, 720Mev	225(2)	9.0(2)	25.0
16^4, 390Mev	653(6)	17.5(6)	37.3
16^4, 230Mev	1949(22)	22.1(8)	88.6

- timings on PC cluster

Actions

ACTION	ADVANTAGES	DISADVANTAGES
clover improved Wilson	computationally fast	breaks chiral symmetry needs operator improvement
twisted mass fermions	computationally fast automatic improvement	breaks chiral symmetry violation of isospin
staggered	computationally fast	fourth root problem complicated contraction
domain wall	improved chiral symmetry	computationally demanding needs tuning
overlap fermions	exact chiral symmetry	computationally expensive

All actions O(a)-improvement:

$$\langle O^{\text{latt}}_{\text{phys}} \rangle = \langle O^{\text{latt}}_{\text{cont}} \rangle + O(a^2)$$

In the following: twisted mass fermions