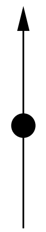
Hadron structure from lattice QCD

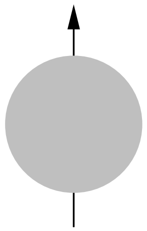
M. Göckeler

Universität Regensburg

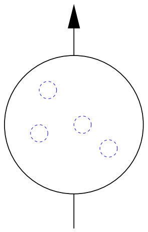
Investigating the internal structure of the nucleon



magnetic moments of proton and neutron are not those of a (structureless) Dirac particle
Nobel prize 1943 (Stern)

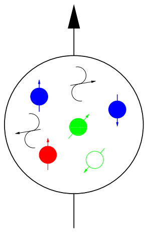


finite radius (≈ 0.86 fm), electromagnetic form factors, charge distribution
Nobel prize 1961 (Hofstadter)



deep inelastic scattering (DIS): scaling → pointlike free constituents (partons)
Nobel prize 1990 (Friedman, Kendall, Taylor)

quark model: proton as a uud bound state



scaling violations in DIS: QCD – the (asymptotically free) quantum field theory of quarks and gluons
Nobel prize 2004 (Gross, Politzer, Wilczek)

long term goal: structure of hadrons from QCD
nonperturbative problem

R.P. Feynman:

Now we were in a position that's different in history than any other time in physics, that's always different. We have a theory, a complete and definite theory of all these hadrons, and we have an enormous number of experiments and lots and lots of details, so why can't we test the theory right away to find out whether it's right or wrong? Because what we have to do is calculate the consequences of the theory. If the theory is right, what should happen, and has that happened? Well, this time the difficulty is in the first step. If the theory is right, what should happen is very hard to figure out. The mathematics needed to figure out what the consequences of this theory are have turned out to be, at the present time, insuperably difficult. At the present time–all right? And therefore it's obvious what my problem is–my problem is to try to develop a way of getting numbers out of this theory, to test it really carefully, not just qualitatively, to see if it might give the right result.

The pleasure of finding things out

approach from first principles: lattice QCD

Topics

- Prelude: pion decay constant
  detailed description of (elementary) techniques and relations
- Hadron structure on the lattice (general remarks)
- How to describe the internal structure of a hadron?
- Distribution amplitude of the pion
- Distribution amplitude of the nucleon
- Nucleon structure functions (parton distribution functions in the nucleon)
- Evaluation of matrix elements of local operators between nucleon states
  needed for (generalised) parton distribution functions
  - How are the required three-point functions computed?
  - How are the desired matrix elements extracted from the three-point functions?
- Electromagnetic form factors of the nucleon (mostly results)
- Generalised parton distributions (GPDs) (formalism)
- Lattice results for GPDs: distributions in impact parameter space
- Lattice results for GPDs: transverse spin structure
- Lattice results for GPDs: quark angular momentum in the nucleon
- Renormalisation of composite operators
- Disconnected contributions

- subjective selection of topics biased by own work
- only few references given (far from complete!)
- more emphasis on (fundamental) techniques than on results

general reference: Ph. Hägler, Phys. Rep. 490 (2010) 49 [arXiv:0912.5483]

Note

impossible to describe all the technical tricks needed in a state-of-the-art calculation ...

Systematic problems

bare lattice results $\rightarrow\rightarrow\rightarrow$ value to be compared with experiment

- renormalisation (and mixing)
  $\rightarrow$ dependence on renormalisation scale μ
  perturbative $\leftrightarrow$ nonperturbative
- projection onto the desired state
  excited states sufficiently suppressed?
- finite size effects
  volume large enough?
- chiral extrapolation (in m_π)
  quark masses in the simulations larger than in reality
- continuum extrapolation
  lattice spacing small enough?
  physical value of the lattice spacing?

chiral perturbation theory
if applicable

Prelude: pion decay constant

our lattice: spacing a, time extent $L_t = aN_t$, spatial extent $L_s = aN_s$

what can be computed: correlation functions of (gauge invariant) observables $\mathcal{O}_1$, …, $\mathcal{O}_n$ ("operators") in (Euclidean) time

$$\langle \mathcal{O}_1(t_1) \ldots \mathcal{O}_n(t_n) \rangle \qquad (t_1 > t_2 > \ldots > t_n)$$

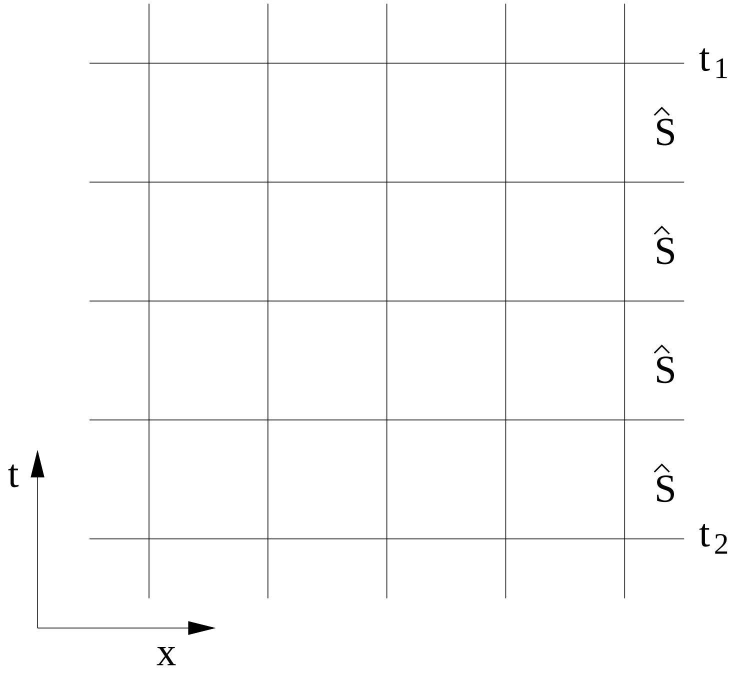
representation as a trace in the Hilbert space $\mathcal{H}$ of the theory (for $n = 2$)

$$\langle \mathcal{O}_1(t_1 = ak_1) \mathcal{O}_2(t_2 = ak_2) \rangle = \frac{1}{Z} \operatorname{Tr} \hat{S}^{N_t - k_1} \hat{\mathcal{O}}_1 \hat{S}^{k_1 - k_2} \hat{\mathcal{O}}_2 \hat{S}^{k_2}$$

- $Z = \operatorname{Tr} \hat{S}^{N_t}$ (partition function)
- $\hat{\mathcal{O}}_i$: operators in $\mathcal{H}$ corresponding to the fields $\mathcal{O}_i(t)$
- $\hat{S}$: transfer matrix, positive definite, selfadjoint, bounded operator in $\mathcal{H}$
  (for Wilson fermions and the standard one-plaquette gauge action)

boundary conditions in Euclidean time: periodic for boson fields
antiperiodic for fermion fields

$$\langle \mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\rangle = \frac{1}{Z}\,\mathrm{Tr}\,\hat{S}^{(L_t-t_1)/a}\hat{\mathcal{O}}_1\hat{S}^{(t_1-t_2)/a}\hat{\mathcal{O}}_2\hat{S}^{t_2/a}$$



transfer matrix $\hat{S}$:
evolution over one time step in Euclidean time

define a Hamilton operator $\hat{H}$ by $\hat{S} = \mathrm{e}^{-\hat{H}a}$

compare with the Minkowski space expression (Heisenberg picture)

$$\langle 0|\hat{\mathcal{O}}_1(t_1)\hat{\mathcal{O}}_2(t_2)|0\rangle = \langle 0|\mathrm{e}^{\mathrm{i}\hat{H}t_1}\hat{\mathcal{O}}_1(0)\mathrm{e}^{-\mathrm{i}\hat{H}t_1}\mathrm{e}^{\mathrm{i}\hat{H}t_2}\hat{\mathcal{O}}_2(0)\mathrm{e}^{-\mathrm{i}\hat{H}t_2}|0\rangle = \langle 0|\hat{\mathcal{O}}_1(0)\mathrm{e}^{-\mathrm{i}\hat{H}(t_1-t_2)}\hat{\mathcal{O}}_2(0)|0\rangle \;(\hat{H}|0\rangle = 0)$$

$\rightarrow$ $\hat{S}$ replaces the (unitary) Minkowski time evolution operator

finite volume: spectrum of $\hat{H}$ and $\hat{S}$ discrete

$L_t \rightarrow \infty$ (temp. $\rightarrow$ 0): different boundary conditions possible, e.g., open boundary conditions

$|\nu\rangle$: complete set of eigenstates of $\hat{S}$ (energy eigenstates)

$$\langle\nu|\mu\rangle = \delta_{\nu\mu} \quad , \quad \hat{S}|\nu\rangle = \mathrm{e}^{-a\hat{H}}|\nu\rangle = \mathrm{e}^{-aE_\nu}|\nu\rangle \quad , \quad E_0 < E_1 \leq E_2 \leq \ldots$$

states in “lattice (finite volume) normalisation”!

$$\rightarrow \qquad Z = \mathrm{Tr}\,\hat{S}^{N_t} = \mathrm{Tr}\,\mathrm{e}^{-L_t\hat{H}} = \sum_\nu \langle\nu|\mathrm{e}^{-L_t\hat{H}}|\nu\rangle = \sum_\nu \mathrm{e}^{-L_t E_\nu}$$

In the limit $L_t \to \infty$ only the state $|0\rangle$ with lowest energy (vacuum) survives, choose $E_0 = 0$.
$\Rightarrow Z = 1$ in this limit (always assumed in the following)

Similarly for correlation functions:

$$\begin{aligned}\langle\mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\rangle &= \frac{1}{Z}\mathrm{Tr}\,\mathrm{e}^{-(L_t-t_1)\hat{H}}\hat{\mathcal{O}}_1\mathrm{e}^{-(t_1-t_2)\hat{H}}\hat{\mathcal{O}}_2\mathrm{e}^{-t_2\hat{H}} \\ &= \sum_{\nu,\mu}\mathrm{e}^{-E_\mu(L_t-t_1)}\langle\mu|\hat{\mathcal{O}}_1|\nu\rangle\mathrm{e}^{-E_\nu(t_1-t_2)}\langle\nu|\hat{\mathcal{O}}_2|\mu\rangle\mathrm{e}^{-E_\mu t_2} \\ &= \sum_{\nu,\mu}\mathrm{e}^{-E_\mu(L_t-t_1+t_2)}\langle\mu|\hat{\mathcal{O}}_1|\nu\rangle\mathrm{e}^{-E_\nu(t_1-t_2)}\langle\nu|\hat{\mathcal{O}}_2|\mu\rangle\end{aligned}$$

depend on $t_1 - t_2$ only (invariance under translations in time) without loss of generality $t_1 = t$, $t_2 = 0$:

$$\langle \mathcal{O}_1(t)\mathcal{O}_2(0)\rangle = \frac{1}{Z}\mathrm{Tr}\,\mathrm{e}^{-(L_t-t)\hat{H}}\hat{\mathcal{O}}_1\mathrm{e}^{-t\hat{H}}\hat{\mathcal{O}}_2 = \sum_{\nu,\mu}\mathrm{e}^{-E_\mu(L_t-t)}\langle\mu|\hat{\mathcal{O}}_1|\nu\rangle\mathrm{e}^{-E_\nu t}\langle\nu|\hat{\mathcal{O}}_2|\mu\rangle$$

t fixed, $L_t \to \infty$: $\quad \langle \mathcal{O}_1(t)\mathcal{O}_2(0)\rangle = \sum_\nu \langle 0|\hat{\mathcal{O}}_1|\nu\rangle\langle\nu|\hat{\mathcal{O}}_2|0\rangle\mathrm{e}^{-E_\nu t}$

$t \to \infty$ (t large): vacuum and the lowest (one-particle) state(s) coupling to $\hat{\mathcal{O}}_i$ dominate

$$\langle \mathcal{O}_1(t)\mathcal{O}_2(0)\rangle = \underbrace{\langle 0|\hat{\mathcal{O}}_1|0\rangle\langle 0|\hat{\mathcal{O}}_2|0\rangle}_{\text{often 0}} + \langle 0|\hat{\mathcal{O}}_1|1\rangle\langle 1|\hat{\mathcal{O}}_2|0\rangle\mathrm{e}^{-E_1 t} + \cdots$$

$\hat{\mathcal{O}}_2$ creates the particle from the vacuum, $\hat{\mathcal{O}}_1$ annihilates it

related to physical quantities: energies and matrix elements between energy eigenstates

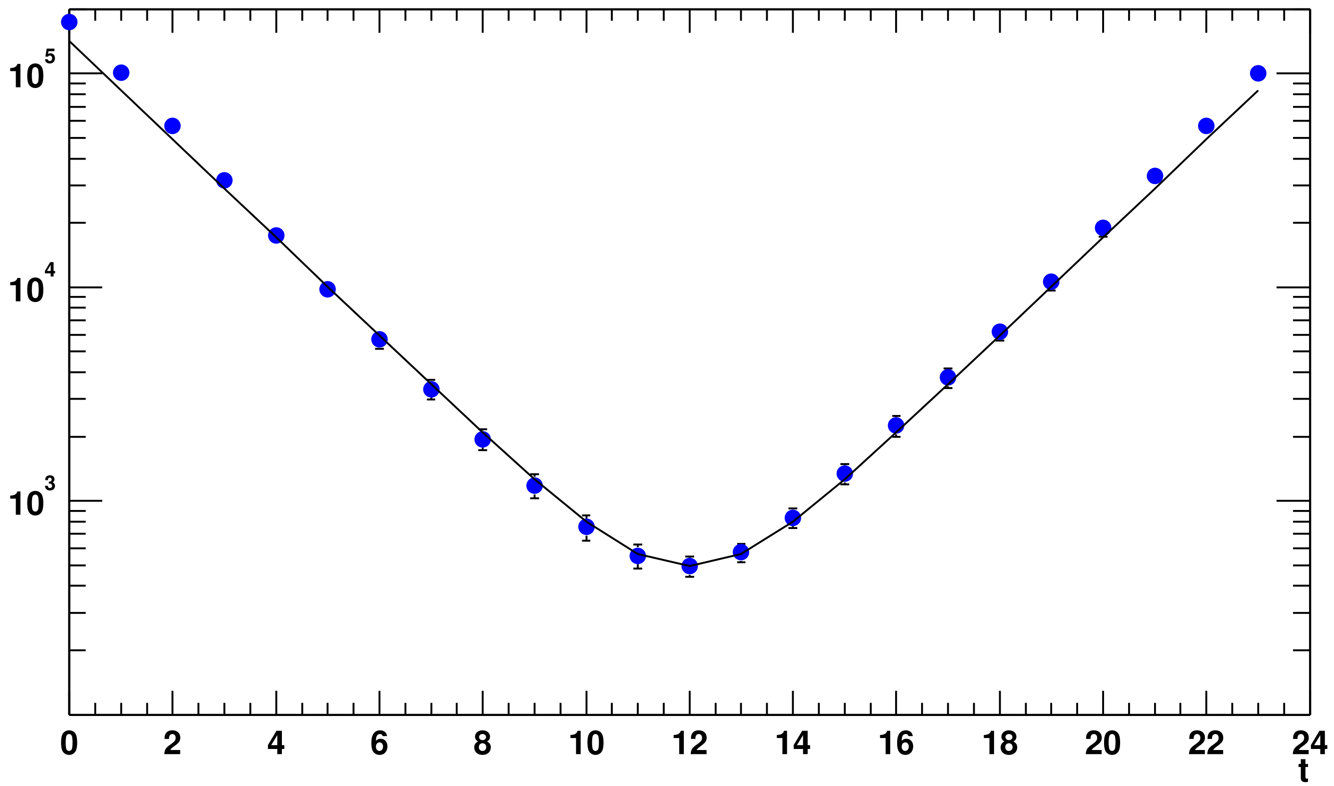
contributing states are those “coupling to the chosen operators $\mathcal{O}_i$” (restricted by symmetries)

energies (masses) from eigenvalues of the transfer matrix
→ independent of the operators (within a symmetry sector)

note: $\quad L_t - t$ fixed, $L_t \to \infty$: $\langle \mathcal{O}_1(t)\mathcal{O}_2(0)\rangle = \sum_\mu \langle\mu|\hat{\mathcal{O}}_1|0\rangle\langle 0|\hat{\mathcal{O}}_2|\mu\rangle\mathrm{e}^{-E_\mu(L_t-t)}$

$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle = A\mathrm{e}^{-Et} + A\mathrm{e}^{-E(L_t-t)} = 2A\mathrm{e}^{-EL_t/2}\cosh\left(E(t-L_t/2)\right)$$

$\mathrm{e}^{-m_\pi t}$ $\mathrm{e}^{-m_\pi (L_t-t)}$



pion propagator with single-cosh fit

example: pion decay constant f_π

"wave function" at the origin in position space
determines the rate of the decay $\pi^+ \to \mu^+ \nu_\mu$

definition in Minkowski space (and for renormalised operator):

$$\langle 0|\bar{d}(0)\gamma_\mu^{\rm M}\gamma_5^{\rm M} u(0)|\pi^+(p)\rangle_{\rm c} = {\rm i} f_\pi p_\mu$$

in particular: $\langle 0|\bar{d}(0)\gamma_0^{\rm M}\gamma_5^{\rm M} u(0)|\pi^+(p)\rangle_{\rm c} = {\rm i} f_\pi p_0 = {\rm i} f_\pi E_\pi(\mathbf{p})$

experiment: $f_\pi \approx 131 {\rm MeV}$

beware of factor $\sqrt{2}$!

$|\pi^+(p)\rangle_{\rm c}$: π^+ with momentum p in continuum (infinite volume) normalisation
phase such that f_π real and positive

remember: $\langle \mathcal{O}_1(t)\mathcal{O}_2(0)\rangle = \langle 0|\hat{\mathcal{O}}_1|1\rangle\langle 1|\hat{\mathcal{O}}_2|0\rangle {\rm e}^{-E_1 t} + \cdots$

needed:

- field $\mathcal{O}_1$ representing the nonsinglet axial-vector current in Euclidean space
- interpolating field $\mathcal{O}_2$ "creating" a π^+ from the vacuum

take

- nonsinglet axial-vector current (in Euclidean space), e.g.

$$\mathcal{O}_1(\mathbf{p},t) = a^3 \sum_{x,\, x_4=t} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\, \bar{d}^i_\alpha(x)(\gamma_4\gamma_5)_{\alpha\beta} u^i_\beta(x)$$

(no smearing!)

(extension fixed in lattice units)

- operator creating $\pi^+(p)$ from the vacuum, e.g., the nonsinglet pseudoscalar density

$$\overline{\mathcal{O}}_2(\mathbf{p},t) = -a^3 \sum_{x,\, x_4=t} \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\, \bar{u}^i_\alpha(x)(\gamma_5)_{\alpha\beta} d^i_\beta(x)$$

(smearing allowed)

(extension fixed in physical units)

such that $\hat{\overline{\mathcal{O}}}_2(\mathbf{p}) = \hat{\mathcal{O}}_2(\mathbf{p})^\dagger$ with $\mathcal{O}_2(\mathbf{p},t) = a^3 \sum_{x,\, x_4=t} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\, \bar{d}^i_\alpha(x)(\gamma_5)_{\alpha\beta} u^i_\beta(x)$

$i, j, \ldots$: colour indices

$\alpha, \beta, \ldots$: Dirac (spin) indices

for these operators: $\langle 0|\hat{\mathcal{O}}_1(\mathbf{p})|0\rangle = \langle 0|\hat{\overline{\mathcal{O}}}_2(\mathbf{p})|0\rangle = 0$

volume finite → momenta discrete: $p_j = \frac{2\pi}{L_s} n_j$

for periodic boundary conditions in space

$n_j = 0, 1, \ldots, N_s - 1$

or

$n_j = -\frac{1}{2}N_s + 1, -\frac{1}{2}N_s + 2, \ldots, \frac{1}{2}N_s$

for these operators:

$$\begin{aligned}\langle \mathcal{O}_1(\mathbf{p},t)\overline{\mathcal{O}}_2(\mathbf{p},0)\rangle &= \langle 0|\hat{\mathcal{O}}_1(\mathbf{p})|1\rangle\langle 1|\hat{\overline{\mathcal{O}}}_2(\mathbf{p})|0\rangle \mathrm{e}^{-E_1 t} + \cdots \\ &= \langle 0|\hat{\mathcal{O}}_1(\mathbf{p})|\pi^+(\mathbf{p})\rangle\langle \pi^+(\mathbf{p})|\hat{\mathcal{O}}_2(\mathbf{p})^\dagger|0\rangle \mathrm{e}^{-E_\pi(\mathbf{p})t} + \cdots\end{aligned}$$

normalisation on a finite lattice: $\langle \pi^+(\mathbf{p})|\pi^+(\mathbf{q})\rangle = \delta_{\mathbf{p},\mathbf{q}}$

normalisation in the infinite volume (continuum): ${}_\mathrm{c}\langle \pi^+(\mathbf{p})|\pi^+(\mathbf{q})\rangle_\mathrm{c} = 2E_\pi(\mathbf{p})(2\pi)^3\delta(\mathbf{p}-\mathbf{q})$

hence $|\pi^+(\mathbf{p})\rangle_\mathrm{c} = \sqrt{L_s^3 \cdot 2E_\pi(\mathbf{p})}|\pi^+(\mathbf{p})\rangle$ and

$$\langle \mathcal{O}_1(\mathbf{p},t)\overline{\mathcal{O}}_2(\mathbf{p},0)\rangle = \frac{1}{L_s^3 \cdot 2E_\pi(\mathbf{p})}\langle 0|\hat{\mathcal{O}}_1(\mathbf{p})|\pi^+(\mathbf{p})\rangle_\mathrm{c}\langle 0|\hat{\mathcal{O}}_2(\mathbf{p})|\pi^+(\mathbf{p})\rangle_\mathrm{c}^*\,\mathrm{e}^{-E_\pi(\mathbf{p})t} + \cdots$$

operators are integrals (sums) of local densities: $\hat{\mathcal{O}}(\mathbf{p}) = a^3 \sum_{\mathbf{x}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\hat{\mathcal{O}}(\mathbf{x})$

using $\quad \hat{\mathcal{O}}(\mathbf{p}) = a^3 \sum_{\mathbf{x}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \hat{\mathcal{O}}(\mathbf{x})$

invariance under spatial translations $\Rightarrow \langle 0|\hat{\mathcal{O}}(\mathbf{x})|\pi^+(\mathbf{p})\rangle = \langle 0|\hat{\mathcal{O}}(\mathbf{x}=\mathbf{0})|\pi^+(\mathbf{p})\rangle \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}$

$$\begin{aligned}\Rightarrow \langle 0|\hat{\mathcal{O}}(\mathbf{p})|\pi^+(\mathbf{p})\rangle &= a^3 \sum_{\mathbf{x}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \langle 0|\hat{\mathcal{O}}(\mathbf{x})|\pi^+(\mathbf{p})\rangle \\ &= a^3 \sum_{\mathbf{x}} \langle 0|\hat{\mathcal{O}}(\mathbf{x}=\mathbf{0})|\pi^+(\mathbf{p})\rangle = L_s^3 \langle 0|\hat{\mathcal{O}}(\mathbf{x}=\mathbf{0})|\pi^+(\mathbf{p})\rangle\end{aligned}$$

hence we get from

$$\langle \mathcal{O}_1(\mathbf{p},t)\overline{\mathcal{O}}_2(\mathbf{p},0)\rangle = \frac{1}{L_s^3 \cdot 2E_\pi(\mathbf{p})} \langle 0|\hat{\mathcal{O}}_1(\mathbf{p})|\pi^+(\mathbf{p})\rangle_{\mathrm{c}} \langle 0|\hat{\mathcal{O}}_2(\mathbf{p})|\pi^+(\mathbf{p})\rangle_{\mathrm{c}}^* \, \mathrm{e}^{-E_\pi(\mathbf{p})t} + \cdots$$

the result

$$\langle \mathcal{O}_1(\mathbf{p},t)\overline{\mathcal{O}}_2(\mathbf{p},0)\rangle = \frac{L_s^3}{2E_\pi(\mathbf{p})} \langle 0|\hat{\mathcal{O}}_1(\mathbf{x}=\mathbf{0})|\pi^+(\mathbf{p})\rangle_{\mathrm{c}} \langle 0|\hat{\mathcal{O}}_2(\mathbf{x}=\mathbf{0})|\pi^+(\mathbf{p})\rangle_{\mathrm{c}}^* \, \mathrm{e}^{-E_\pi(\mathbf{p})t} + \cdots$$

choosing $\mathbf{p} = \mathbf{0}$:

$$\frac{1}{L_s^3} \langle \mathcal{O}_1(\mathbf{p}=\mathbf{0},t)\overline{\mathcal{O}}_2(\mathbf{p}=\mathbf{0},0)\rangle = \frac{1}{2m_\pi} \langle 0|\hat{\mathcal{O}}_1(\mathbf{x}=\mathbf{0})|\pi^+\rangle_{\mathrm{c}} \langle 0|\hat{\mathcal{O}}_2(\mathbf{x}=\mathbf{0})|\pi^+\rangle_{\mathrm{c}}^* \, \mathrm{e}^{-m_\pi t} + \cdots$$

remembering that $\mathcal{O}_1 \leftrightarrow A_4$, $\mathcal{O}_2 \leftrightarrow P$:

$$\frac{1}{L_s^3}\langle \mathcal{O}_1(\mathbf{p}=\mathbf{0},t)\overline{\mathcal{O}}_2(\mathbf{p}=\mathbf{0},0)\rangle = \underbrace{\frac{1}{2m_\pi}\langle 0|\hat{\mathcal{O}}_1(\mathbf{x}=\mathbf{0})|\pi^+\rangle_{\rm c}\langle 0|\hat{\mathcal{O}}_2(\mathbf{x}=\mathbf{0})|\pi^+\rangle_{\rm c}^*}_{A(A_4,P)}\,{\rm e}^{-m_\pi t}+\cdots$$

similarly:

$$\frac{1}{L_s^3}\langle \mathcal{O}_2(\mathbf{p}=\mathbf{0},t)\overline{\mathcal{O}}_2(\mathbf{p}=\mathbf{0},0)\rangle = \frac{1}{2m_\pi}\left|\langle 0|\hat{\mathcal{O}}_2(\mathbf{x}=\mathbf{0})|\pi^+\rangle_{\rm c}\right|^2{\rm e}^{-m_\pi t}+\cdots = A(P,P){\rm e}^{-m_\pi t}+\cdots$$

These 2-point functions are real, choose the phase of $|\pi^+\rangle_{\rm c}$ such that $\langle 0|\hat{\mathcal{O}}_1(\mathbf{x}=\mathbf{0})|\pi^+\rangle_{\rm c}$ is real and positive:

$$\frac{|A(A_4,P)|}{\sqrt{A(P,P)}} = \frac{1}{\sqrt{2m_\pi}}\frac{|\langle 0|\hat{\mathcal{O}}_1(\mathbf{x}=\mathbf{0})|\pi^+\rangle_{\rm c}|\,|\langle 0|\hat{\mathcal{O}}_2(\mathbf{x}=\mathbf{0})|\pi^+\rangle_{\rm c}|}{|\langle 0|\hat{\mathcal{O}}_2(\mathbf{x}=\mathbf{0})|\pi^+\rangle_{\rm c}|} = \frac{1}{\sqrt{2m_\pi}}\langle 0|\hat{\mathcal{O}}_1(\mathbf{x}=\mathbf{0})|\pi^+\rangle_{\rm c}$$

Z_A = renormalisation factor of the axial-vector current, i.e., of $\hat{\mathcal{O}}_1$:

$$Z_A\frac{|A(A_4,P)|}{\sqrt{A(P,P)}} = \frac{1}{\sqrt{2m_\pi}}Z_A\langle 0|\hat{\mathcal{O}}_1(\mathbf{x}=\mathbf{0})|\pi^+\rangle_{\rm c} = \frac{m_\pi f_\pi}{\sqrt{2m_\pi}}$$

note: explicit breaking of the chiral symmetry for Wilson-like fermions ⇒ $Z_A \neq 1$
for Ginsparg-Wilson fermions: $Z_A = 1$

to be evaluated in a Monte Carlo simulation:

$$\langle \mathcal{O}_1(\mathbf{p},t)\overline{\mathcal{O}}_2(\mathbf{p},0)\rangle = -a^6 \sum_{\substack{x,\,x_4=t\\ y,\,y_4=0}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\langle \bar{d}(x)\gamma_4\gamma_5 u(x)\,\bar{u}(y)\gamma_5 d(y)\rangle$$

$$= -a^6 \sum_{\substack{x,\,x_4=t\\ y,\,y_4=0}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}(\gamma_4\gamma_5)_{\alpha\beta}(\gamma_5)_{\alpha'\beta'}\,\langle \bar{d}^j_\alpha(x) u^j_\beta(x)\,\bar{u}^{j'}_{\alpha'}(y) d^{j'}_{\beta'}(y)\rangle$$

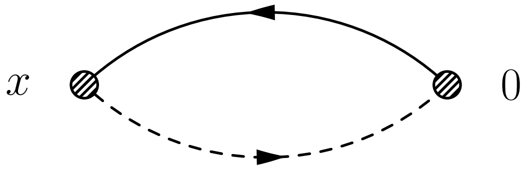
$$= -L_s^3 a^3 \sum_{x,\,x_4=t} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}(\gamma_4\gamma_5)_{\alpha\beta}(\gamma_5)_{\alpha'\beta'}\,\langle \bar{d}^j_\alpha(x) u^j_\beta(x)\,\bar{u}^{j'}_{\alpha'}(0) d^{j'}_{\beta'}(0)\rangle$$

$$= L_s^3 a^3 \sum_{x,\,x_4=t} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}(\gamma_4\gamma_5)_{\alpha\beta}(\gamma_5)_{\alpha'\beta'}\,\langle G_u(x,0)^{jj'}_{\beta\alpha'}\,G_d(0,x)^{j'j}_{\beta'\alpha}\rangle_g$$

integration over the gauge fields only

$$= L_s^3 a^3 \sum_{x,\,x_4=t} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,\langle \mathrm{tr}_{\mathrm{DC}}\,\gamma_4\gamma_5 G_u(x,0)\gamma_5 G_d(0,x)\rangle_g$$

pictorially:



γ_5 hermiticity of the lattice Dirac operator: $G_q(0,x) = \gamma_5 G_q(x,0)^\dagger \gamma_5$

$$\begin{aligned}\langle \mathcal{O}_1(\mathbf{p},t)\overline{\mathcal{O}}_2(\mathbf{p},0)\rangle &= L_s^3 a^3 \sum_{x,\, x_4=t} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \langle \mathrm{tr}_{\mathrm{DC}}\, \gamma_4\gamma_5 G_u(x,0)\gamma_5 G_d(0,x)\rangle_g \\ &= L_s^3 a^3 \sum_{x,\, x_4=t} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \langle \mathrm{tr}_{\mathrm{DC}}\, \gamma_4\gamma_5 G_u(x,0)\gamma_5\gamma_5 G_d(x,0)^\dagger\gamma_5\rangle_g \\ &= -L_s^3 a^3 \sum_{x,\, x_4=t} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \langle \mathrm{tr}_{\mathrm{DC}}\, \gamma_4 G_u(x,0) G_d(x,0)^\dagger\rangle_g\end{aligned}$$

a single source point is sufficient!

Hadron structure on the lattice

main steps:

- find out which quantities contain information about hadron structure, e.g. f_π

- express these quantities (if possible!) in terms of matrix elements of local operators
  e.g. $\langle 0|\bar{d}\gamma_\mu^{\mathrm{M}}\gamma_5^{\mathrm{M}} u|\pi^+(p)\rangle_{\mathrm{c}} = \mathrm{i} f_\pi p_\mu$

- relate hadron matrix elements and Euclidean correlation functions (transfer matrix!)
  e.g. $\langle \mathcal{O}_1(\mathbf{p},t)\overline{\mathcal{O}}_2(\mathbf{p},0)\rangle = \langle 0|\hat{\mathcal{O}}_1(\mathbf{p})|\pi^+(\mathbf{p})\rangle\langle\pi^+(\mathbf{p})|\hat{\mathcal{O}}_2(\mathbf{p})^\dagger|0\rangle \mathrm{e}^{-E_\pi(\mathbf{p})t} + \cdots$
  interpolating fields for creating particles and operators to be studied

- compute Euclidean n-point functions within a Monte Carlo simulation, i.e., on given gauge field configurations
  e.g. $\langle \mathcal{O}_1(\mathbf{p},t)\overline{\mathcal{O}}_2(\mathbf{p},0)\rangle = -a^6 \sum_{\substack{x,\, x_4=t \\ y,\, y_4=0}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\langle \bar{d}(x)\gamma_4\gamma_5 u(x)\,\bar{u}(y)\gamma_5 d(y)\rangle = \cdots$
  expressed in terms of quark propagators
  computational effort: quark propagators and contractions

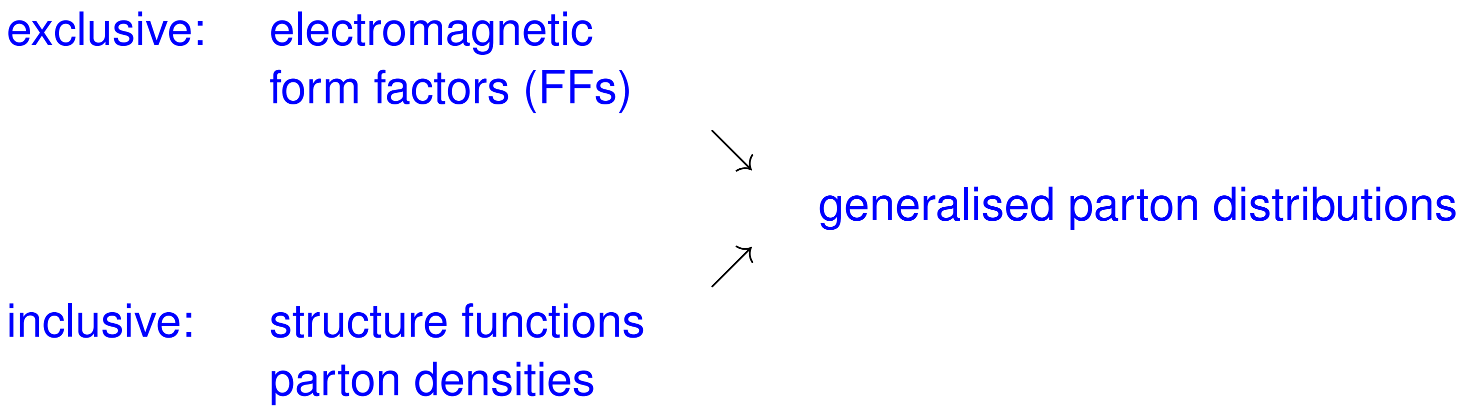
- renormalise the local operators on the lattice e.g. compute Z_A

How to describe the internal structure of a hadron?

hydrogen atom: proton + electron	proton: 3 valence quarks + 1, 2, . . . quark-antiquark pairs + 1, 2, . . . gluons
nonrelativistic quantum mechanics	quantum field theory
wave functions	distribution amplitudes, (generalised) parton distributions, . . .

distribution amplitudes (DAs) (of leading twist)
describe hadrons in terms of valence quark Fock states at small transverse separation
→ (semi)exclusive processes

generalised parton distributions (GPDs)
encode information from, e.g., lepton-nucleon scattering experiments



exclusive: electromagnetic form factors (FFs)

↘

generalised parton distributions

↗

inclusive: structure functions parton densities

Distribution amplitude of the pion

description in terms of pion-to-vacuum matrix elements (in Minkowski space)

$$\langle 0|\bar{d}(-z)\gamma_\mu^{\mathrm{M}}\gamma_5^{\mathrm{M}}\underset{\substack{\uparrow\\ \text{Wilson line}}}{[-z,z]}u(z)|\pi^+(p)\rangle_{\mathrm{c}} = \mathrm{i}f_\pi p_\mu \int_{-1}^{1}\mathrm{d}\xi\,\mathrm{e}^{-\mathrm{i}\xi p\cdot z}\underset{\substack{\uparrow\\ \text{distribution amplitude}}}{\phi(\xi,\mu^2)}$$

z_μ: light-like vector ($z^2 = 0$) f_π: pion decay constant μ: renormalisation scale

$$\begin{aligned} x &= \tfrac{1}{2}(1+\xi) \\ 1-x &= \tfrac{1}{2}(1-\xi) \end{aligned}$$ fraction of the momentum carried by the quark / antiquark

normalisation: $\int_{-1}^{1}\mathrm{d}\xi\,\phi(\xi,\mu^2) = 1$

expansion in terms of Gegenbauer polynomials: $\phi(\xi,\mu^2) = \dfrac{3}{4}(1-\xi^2)\left(1+\sum_{n=1}^{\infty} a_n(\mu^2)C_n^{3/2}(\xi)\right)$

$a_n = 0$ for odd n in case of exact isospin symmetry

asymptotically: $\phi(\xi,\mu^2\to\infty) = \dfrac{3}{4}(1-\xi^2)$

moments $\langle \xi^n \rangle(\mu^2) = \int_{-1}^{1} \mathrm{d}\xi \, \xi^n \phi(\xi, \mu^2)$

are related to pion-to-vacuum matrix elements of local (renormalised) operators:

$$\langle 0 | \mathrm{i}^n \bar{d}(0) \gamma_5^{\mathrm{M}} \gamma_{(\mu_0}^{\mathrm{M}} \overset{\leftrightarrow}{D}_{\mu_1} \dots \overset{\leftrightarrow}{D}_{\mu_n)} u(0) | \pi^+(p) \rangle_{\mathrm{c}} = -\mathrm{i} f_\pi p_{(\mu_0} \dots p_{\mu_n)} \langle \xi^n \rangle$$

$\overset{\leftrightarrow}{D}_\mu = \vec{D}_\mu - \overset{\leftarrow}{D}_\mu$ $(\cdots)$: symmetrisation of all indices and subtraction of traces

example $(n = 1)$: $\mathcal{O}_{(\mu\nu)} = \frac{1}{2}\mathcal{O}_{\mu\nu} + \frac{1}{2}\mathcal{O}_{\nu\mu} - \frac{1}{4} g_{\mu\nu} \sum_\lambda \mathcal{O}_\lambda{}^\lambda$

matrix elements $\langle 0 |$local operator$| \pi^+(p) \rangle$ in principle accessible on the lattice

compare computation of f_π

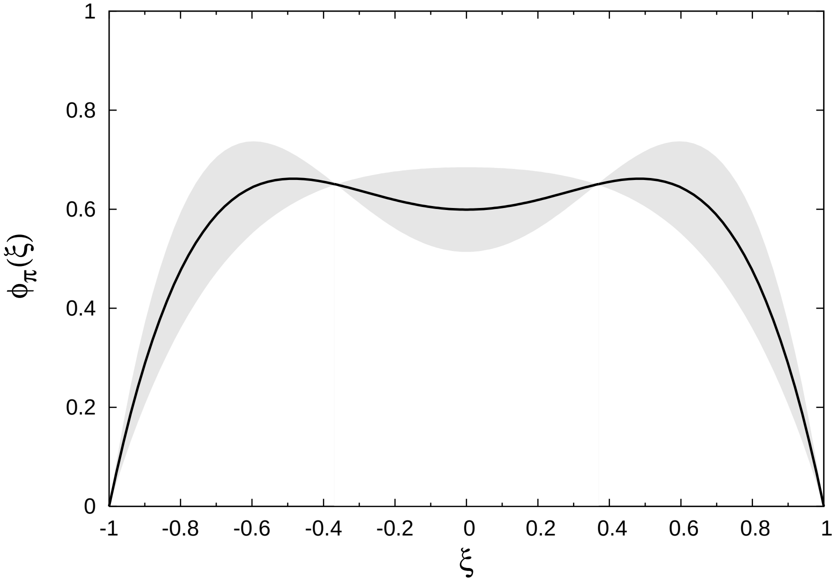
relation to the Gegenbauer moments a_n: $a_2 = \frac{7}{12} \left(5 \langle \xi^2 \rangle - 1 \right)$ etc.

QCDSF, Phys. Rev. D74 (2006) 074501:
mild dependence of the MC data on quark mass and lattice spacing

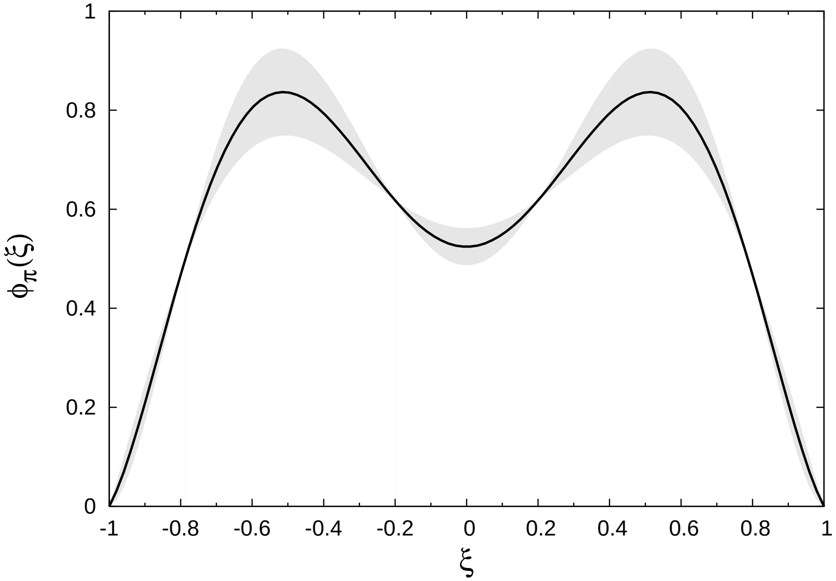
$$\langle \xi^2 \rangle^{\overline{\mathrm{MS}}}(\mu^2 = 4\,\mathrm{GeV}^2) = 0.269(39) > \langle \xi^2 \rangle_{\mathrm{asymptotic}} = 0.2 \quad \Rightarrow \quad a_2(\mu^2 = 4\,\mathrm{GeV}^2) = 0.201(114)$$

consistent with QCD sum rule estimates

plot $\phi(\xi,\mu^2) = \frac{3}{4}(1-\xi^2)\left(1 + a_2(\mu^2)C_2^{3/2}(\xi) + a_4(\mu^2)C_4^{3/2}(\xi)\right)$ $(\mu^2 = 4\,\mathrm{GeV}^2)$



$0.201 - 0.114 < a_2 < 0.201 + 0.114,\ a_4 = 0$



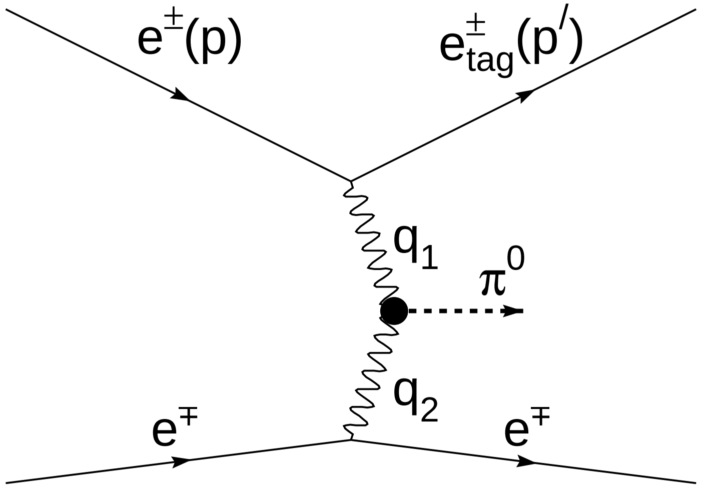
$a_2 = 0.201,\ -0.15 < a_4 < -0.05$

"double hump structure"

preliminary results from RQCD point towards a smaller value of a_2

Experimental information:

measurement of the $\gamma\gamma^* \to \pi^0$ transition form factor



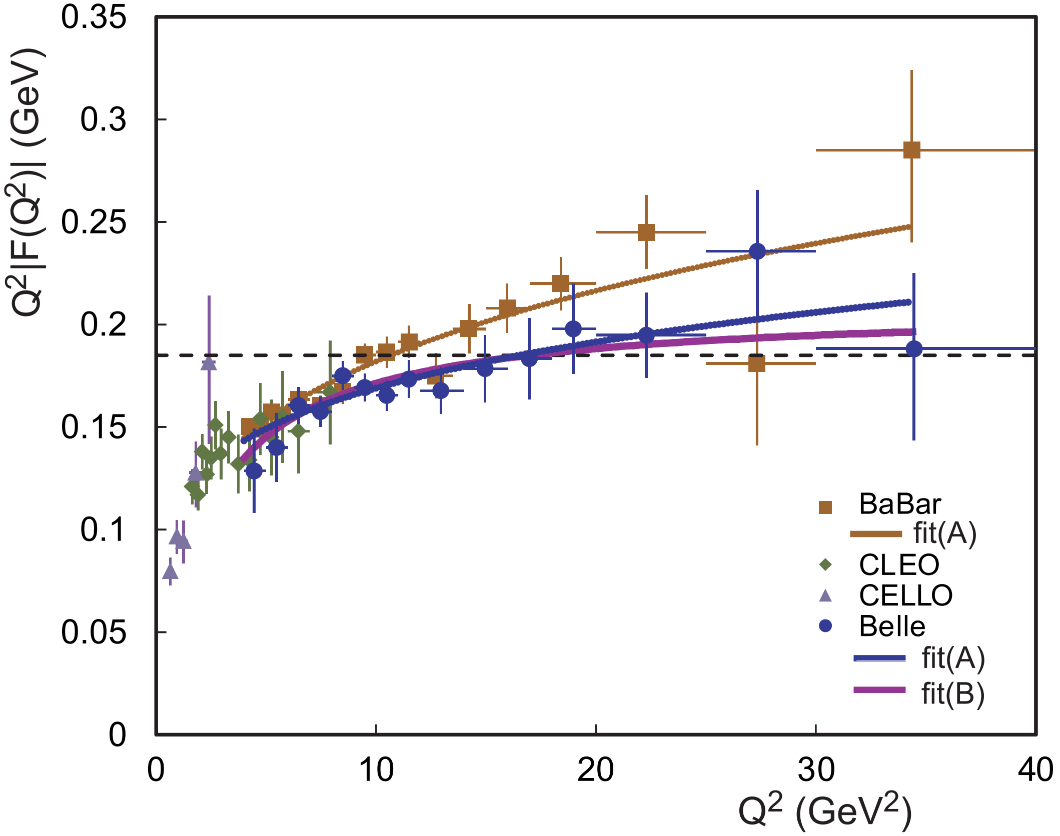
$q_1^2 = -Q^2$ large, $q_2^2 \approx 0$

$\gamma\gamma^* \to \pi^0$ described by one form factor $F(Q^2)$
$\to$ differential cross section for π^0 production

perturbative QCD:
transition form factor $F(Q^2)$ = convolution of a hard scattering amplitude ($\gamma\gamma^* \to q\bar{q}$)
with the pion DA $\varphi(x,Q^2) = 2\phi(2x-1,Q^2)$

$$Q^2F(Q^2) = \frac{\sqrt{2}f_\pi}{3}\int_0^1 \frac{\mathrm{d}x}{x}\varphi(x,Q^2) + O(\alpha_s) + O\left(\frac{\Lambda^2_{\mathrm{QCD}}}{Q^2}\right) \xrightarrow{Q^2\to\infty} \sqrt{2}f_\pi$$

recent measurements: BABAR Collaboration (Phys. Rev. D80 (2009) 052002)
Belle Collaboration (Phys. Rev. D86 (2012) 092007)



$\leftarrow \sqrt{2} f_\pi$ (asymptotic value)

BABAR data inconsistent with expected asymptotic behaviour?
can (only?) be described by an essentially flat (no zeros at the endpoints) DA

more accurate lattice results desirable! difficult to get. . .

Distribution amplitude of the nucleon

describes the nucleon in terms of valence quark Fock states at small transverse separation

leading twist: $\varphi(x_1, x_2, x_3, \mu^2)$

x_i: longitudinal momentum fraction carried by the i-th quark ($0 \leq x_i \leq 1$, $\sum_i x_i = 1$)

proton state:

$$\begin{aligned} |p, \uparrow\rangle &= f_N \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, \frac{\delta(1 - x_1 - x_2 - x_3)\varphi(x_i)}{2\sqrt{24 x_1 x_2 x_3}} \left\{ |u^\uparrow(x_1) u^\downarrow(x_2) d^\uparrow(x_3)\rangle - |u^\uparrow(x_1) d^\downarrow(x_2) u^\uparrow(x_3)\rangle \right\} \\ &= f_N \int_0^1 [\mathrm{d}x] \, \frac{\varphi(x_i)}{2\sqrt{24 x_1 x_2 x_3}} \left\{ |u^\uparrow(x_1) u^\downarrow(x_2) d^\uparrow(x_3)\rangle - |u^\uparrow(x_1) d^\downarrow(x_2) u^\uparrow(x_3)\rangle \right\} \end{aligned}$$

$$[\mathrm{d}x] = \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \, \delta(1 - x_1 - x_2 - x_3)$$

arrows indicate the helicities

f_N: normalisation constant (“wave function” at the origin in position space)

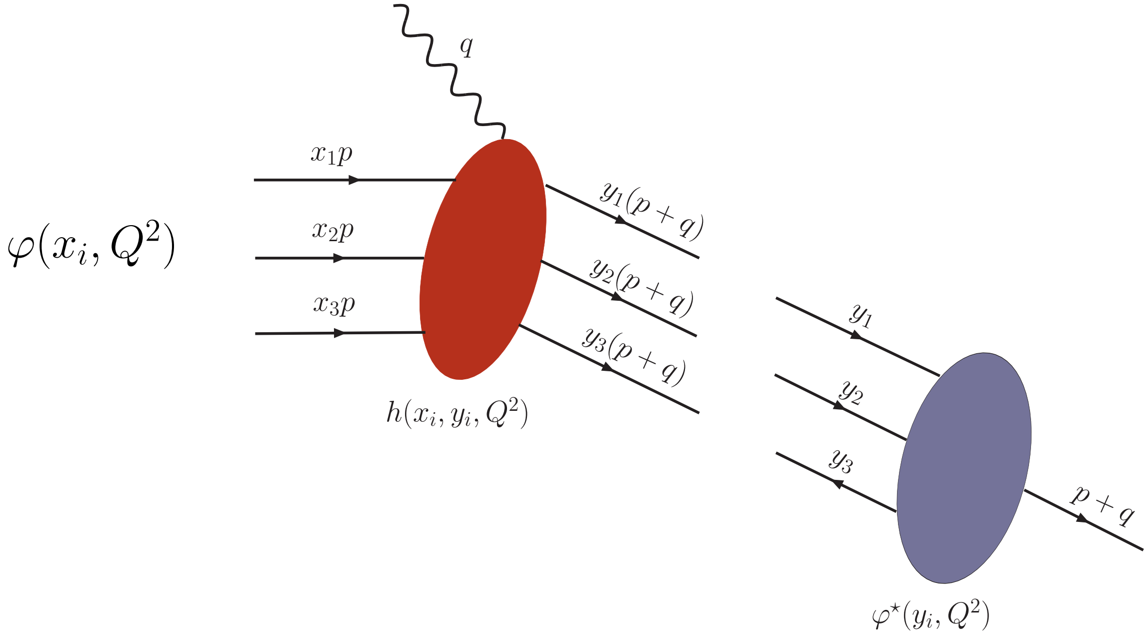
$$\int [dx] \, \varphi(x_1, x_2, x_3) = 1$$

DA related to hard (semi)exclusive processes for example: magnetic form factor $G_M(Q^2)$ (measured in elastic electron-nucleon scattering)

= convolution of a hard scattering kernel $h(x_i, y_i, Q^2)$ with the nucleon DA

for very large $Q^2 = -q^2$

$$G_M(Q^2) = f_N^2 \int_0^1 [\mathrm{d}x] \int_0^1 [\mathrm{d}y] \, \varphi^*(y_i, Q^2) \, h(x_i, y_i, Q^2) \, \varphi(x_i, Q^2)$$



similar to the case of the pion DA:

moments of φ
$\updownarrow$
matrix elements of three-quark operators between the vacuum and a nucleon state
$\updownarrow$
two-point functions on the lattice

however, several complications as compared to the pion:

- three-quark operators instead of quark-antiquark operators
- three types of distribution amplitudes
  can be expressed in terms of a single amplitude φ due to symmetries
- distribution amplitudes depend on three momentum fractions x_1, x_2, x_3 with $x_1 + x_2 + x_3 = 1$
- nucleon spin
  operators have explicit spin index
- interpolating fields for baryons couple to both parities

→ more detailed look at nucleon two-point functions interpolating field for a proton (neutron by interchanging u and d)

$$B_\alpha(t,\mathbf{p}) = a^3 \sum_{x,\, x_4=t} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \epsilon_{ijk} u^i_\alpha(x) u^j_\beta(x) (C^{-1}\gamma_5)_{\beta\gamma} d^k_\gamma(x)$$

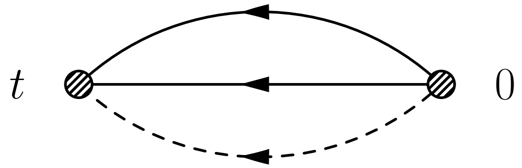
$$\bar{B}_\alpha(t,\mathbf{p}) = -a^3 \sum_{x,\, x_4=t} \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \epsilon_{ijk} \bar{d}^i_\beta(x) (\gamma_5 C)_{\beta\gamma} \bar{u}^j_\gamma(x) \bar{u}^k_\alpha(x)$$

with $C\gamma_\mu^{\mathrm{T}} C^{-1} = -\gamma_\mu \Leftarrow$

two-point function with quark fields integrated out:

$$\begin{aligned}
\langle B_\alpha(t,\mathbf{p}) \bar{B}_\beta(0,\mathbf{p}) \rangle &= -a^6 \sum_{\substack{x \\ x_4=t}} \sum_{\substack{y \\ y_4=0}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \epsilon_{ijk}\epsilon_{i'j'k'} (C^{-1}\gamma_5)_{\alpha'\alpha''} (\gamma_5 C)_{\beta'\beta''} \\
&\times \left\langle G_d(x,y)^{ki'}_{\alpha''\beta'} \left(G_u(x,y)^{jj'}_{\alpha'\beta''} G_u(x,y)^{ik'}_{\alpha\beta} - G_u(x,y)^{ij'}_{\alpha\beta''} G_u(x,y)^{jk'}_{\alpha'\beta} \right) \right\rangle_g \\
&= -L_s^3 a^3 \sum_{\substack{x \\ x_4=t}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \epsilon_{ijk}\epsilon_{i'j'k'} (C^{-1}\gamma_5)_{\alpha'\alpha''} (\gamma_5 C)_{\beta'\beta''} \\
&\times \left\langle G_d(x,0)^{ki'}_{\alpha''\beta'} \left(G_u(x,0)^{jj'}_{\alpha'\beta''} G_u(x,0)^{ik'}_{\alpha\beta} - G_u(x,0)^{ij'}_{\alpha\beta''} G_u(x,0)^{jk'}_{\alpha'\beta} \right) \right\rangle_g
\end{aligned}$$

pictorially:



projection on (positive) parity for $\mathbf{p}=\mathbf{0}$:

$$\frac{1}{2}\sum_{\alpha,\beta}(1+\gamma_4)_{\beta\alpha}\left\langle B_\alpha(t,\mathbf{0})\bar{B}_\beta(0,\mathbf{0})\right\rangle = \sum_{\nu,\alpha}\langle 0|\left(\tfrac{1}{2}(1+\gamma_4)\hat{B}(\mathbf{0})\right)_\alpha|\nu\rangle\langle\nu|\left(\hat{\bar{B}}(\mathbf{0})\tfrac{1}{2}(1+\gamma_4)\right)_\alpha|0\rangle \mathrm{e}^{-E_\nu t}$$

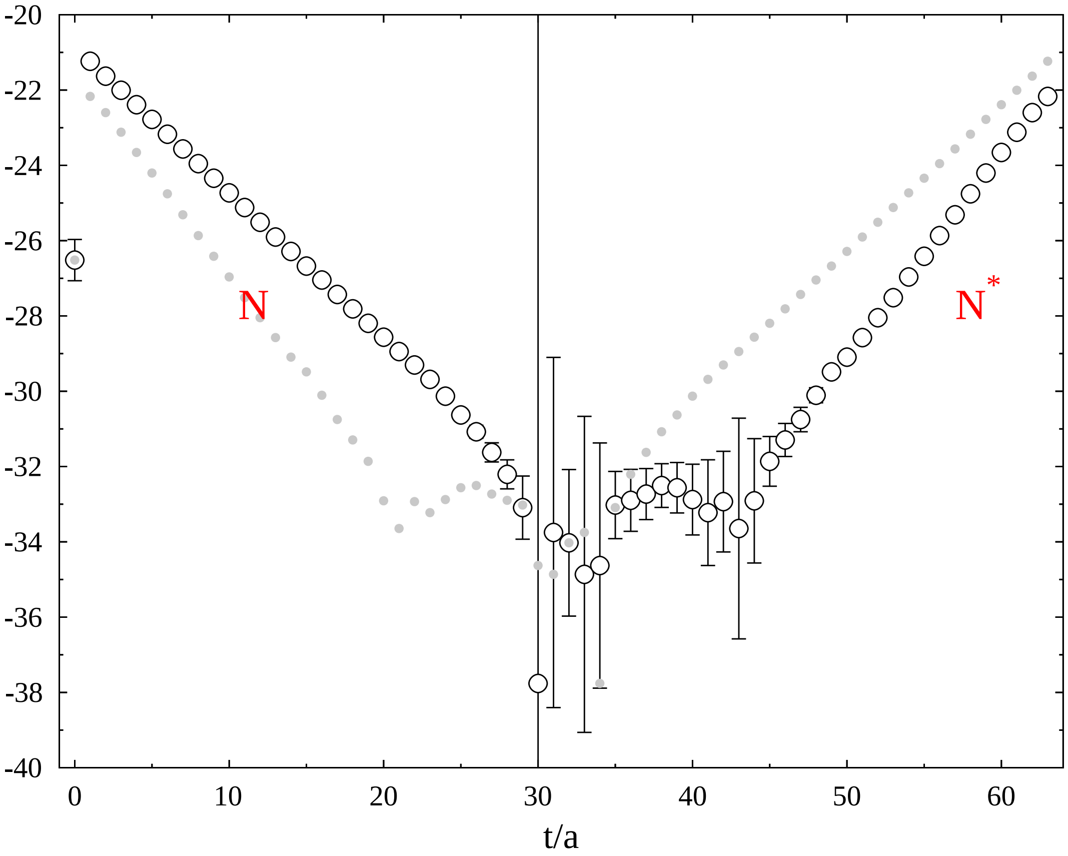
$$+\sum_{\nu,\alpha}\langle\nu|\left(\tfrac{1}{2}(1+\gamma_4)\hat{B}(\mathbf{0})\right)_\alpha|0\rangle\langle 0|\left(\hat{\bar{B}}(\mathbf{0})\tfrac{1}{2}(1+\gamma_4)\right)_\alpha|\nu\rangle \mathrm{e}^{-E_\nu(L_t-t)}$$

	first sum	second sum
	unless	
	$\cdots\hat{\bar{B}}\cdots\|0\rangle=0$	$\cdots\hat{B}\cdots\|0\rangle=0$
parity	$+1$	$+1$
baryon number	$+1$	-1
lowest energy eigenstate in the sum	proton	antiparticle of (excited proton with parity -1)

excited proton with negative parity identified with the N(1535) S_{11} resonance N(1650)?

→ results for states of negative parity come automatically along with the nucleon results (logarithm of the absolute value of the) nucleon 2-point function:

nucleon (N) and its (backward propagating) parity partner N^*



2 flavours, $\beta = 5.29$, $\kappa = 0.13632$, $40^3 \times 64$ lattice $m_\pi \approx 290\,\mathrm{MeV}$, $a \approx 0.07\,\mathrm{fm}$

one-particle (one-proton) states: $|N,\mathbf{p},\sigma\rangle_{\rm c} = \sqrt{2L_s^3\,E_N(\mathbf{p})}|N,\mathbf{p},\sigma\rangle$

with $\sigma = \pm\frac{1}{2}$ (3-component of the nucleon spin if $\mathbf{p}=\mathbf{0}$)

nucleon operators:

$$\hat{B}_\alpha(\mathbf{p}) = a^3 \sum_{\mathbf{x}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,\hat{B}_\alpha(\mathbf{x}) \quad , \quad \hat{\bar{B}}_\beta(\mathbf{p}) = a^3 \sum_{\mathbf{x}} \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,\hat{\bar{B}}_\beta(\mathbf{x})$$

assume (up to lattice artefacts)

$$\langle 0|\hat{B}_\alpha(\mathbf{x})|N,\mathbf{p},\sigma\rangle_{\rm c} = \sqrt{Z(\mathbf{p})}\,U_\alpha(N,\mathbf{p},\sigma)\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}$$
$$_{\rm c}\langle N,\mathbf{p},\sigma|\hat{\bar{B}}_\beta(\mathbf{x})|0\rangle = \sqrt{Z(\mathbf{p})}\,\bar{U}_\beta(N,\mathbf{p},\sigma)\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}$$

- $Z(\mathbf{p})$ dimensionful
- momentum dependence of $Z(\mathbf{p})$ due to smeared sources

normalisation of the Dirac spinors:

$$\bar{U}(N,\mathbf{p},\sigma)U(N,\mathbf{p},\sigma') = 2m_N\,\delta_{\sigma\sigma'}$$
$$\sum_\sigma U_\alpha(N,\mathbf{p},\sigma)\bar{U}_\beta(N,\mathbf{p},\sigma) = (E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N)_{\alpha\beta}$$

$$C_{\alpha\beta}(t;\mathbf{p}) := \langle B_\alpha(t,\mathbf{p})\bar{B}_\beta(0,\mathbf{p})\rangle = L_s^3 a^3 \sum_{\mathbf{x}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \frac{\mathrm{Tr}\,\mathrm{e}^{-(L_t-t)\hat{H}}\hat{B}_\alpha(\mathbf{x})\mathrm{e}^{-t\hat{H}}\hat{\bar{B}}_\beta(\mathbf{x}=\mathbf{0})}{\mathrm{Tr}\,\mathrm{e}^{-L_t\hat{H}}}$$

for $L_t \to \infty$ and large t (keeping only the groundstate)

$$C_{\alpha\beta}(t;\mathbf{p}) = L_s^3 \sum_\sigma \langle 0|\hat{B}_\alpha(\mathbf{x}=\mathbf{0})|N,\mathbf{p},\sigma\rangle_{\mathrm{c}}\ {}_{\mathrm{c}}\langle N,\mathbf{p},\sigma|\hat{\bar{B}}_\beta(\mathbf{x}=\mathbf{0})|0\rangle \frac{\mathrm{e}^{-E_N(\mathbf{p})t}}{2E_N(\mathbf{p})} + \cdots$$

$$= L_s^3\, Z(\mathbf{p}) \sum_\sigma U_\alpha(N,\mathbf{p},\sigma)\bar{U}_\beta(N,\mathbf{p},\sigma)\frac{\mathrm{e}^{-E_N(\mathbf{p})t}}{2E_N(\mathbf{p})} + \cdots$$

$$= L_s^3\, Z(\mathbf{p})\,(E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N)_{\alpha\beta}\ \frac{\mathrm{e}^{-E_N(\mathbf{p})t}}{2E_N(\mathbf{p})} + \cdots$$

$$\Gamma_{\beta\alpha}C_{\alpha\beta}(t;\mathbf{p}) = L_s^3\, Z(\mathbf{p})\,\mathrm{tr}\Big(\Gamma\,(E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N)\Big)\ \frac{\mathrm{e}^{-E_N(\mathbf{p})t}}{2E_N(\mathbf{p})} + \cdots$$

in particular: $\mathrm{tr}\Big(\tfrac{1}{2}(1+\gamma_4)\,(E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N)\Big) = 2\,(m_N + E_N(\mathbf{p}))$

$$\tfrac{1}{2}(1+\gamma_4)_{\beta\alpha}C_{\alpha\beta}(t;\mathbf{p}) = L_s^3\, Z(\mathbf{p})\,\frac{m_N + E_N(\mathbf{p})}{E_N(\mathbf{p})}\,\mathrm{e}^{-E_N(\mathbf{p})t}$$

back to the nucleon DA!

three types of DAs: $V(x_1, x_2, x_3)$, $A(x_1, x_2, x_3)$, $T(x_1, x_2, x_3)$

due to symmetries expressible in terms of $\varphi(x_1, x_2, x_3) = V(x_1, x_2, x_3) - A(x_1, x_2, x_3)$

moments $\varphi^{lmn} = \int_0^1 [\mathrm{d}x]\, x_1^l x_2^m x_3^n\, \varphi(x_1, x_2, x_3)$

related to matrix elements of local three-quark operators between the vacuum and the nucleon

can be extracted from two-point functions

basic correlation function:

$$\langle \epsilon_{ijk} [D_{\lambda_1} \dots D_{\lambda_l} u(x)]^i_\alpha [D_{\mu_1} \dots D_{\mu_m} u(x)]^j_\beta [D_{\nu_1} \dots D_{\nu_n} d(x)]^k_\gamma \bar{B}_\tau(y) \rangle$$

with $\bar{B}_\tau(x) = -\epsilon_{ijk} \bar{d}^i_\beta(x) (\gamma_5 C)_{\beta\gamma} \bar{u}^j_\gamma(x) \bar{u}^k_\tau(x)$

projection on momentum $\mathbf{p}$:

$$\mathcal{O}_\tau(t, \mathbf{p}) = a^3 \sum_{x,\, x_4 = t} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} \mathcal{O}_\tau(x) \quad \leftrightarrow \quad \hat{\mathcal{O}}_\tau(\mathbf{p}) = a^3 \sum_{\mathbf{x}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\, \hat{\mathcal{O}}_\tau(\mathbf{x})$$

τ: spin index

two-point function in terms of matrix elements:

$$\langle \mathcal{O}_\tau(t,\mathbf{p})\bar{B}_{\tau'}(0,\mathbf{p})\rangle = L_s^3 \sum_\sigma \langle 0|\hat{\mathcal{O}}_\tau(\mathbf{x}=\mathbf{0})|N,\mathbf{p},\sigma\rangle_{\mathrm{c}}\ {}_{\mathrm{c}}\langle N,\mathbf{p},\sigma|\hat{\bar{B}}_{\tau'}(\mathbf{x}=\mathbf{0})|0\rangle \frac{\mathrm{e}^{-E_N(\mathbf{p})t}}{2E_N(\mathbf{p})}$$

$$= L_s^3 \frac{\sqrt{Z(\mathbf{p})}}{2E_N(\mathbf{p})} \sum_\sigma \langle 0|\hat{\mathcal{O}}_\tau(\mathbf{x}=\mathbf{0})|N,\mathbf{p},\sigma\rangle_{\mathrm{c}}\, \bar{U}_{\tau'}(N,\mathbf{p},\sigma)\, \mathrm{e}^{-E_N(\mathbf{p})t}$$

matrix elements ↔ moments φ^{lmn}:

$$\langle 0|\hat{\mathcal{O}}_\alpha^{\mu\nu\cdots}|N,\mathbf{p},\sigma\rangle_{\mathrm{c}} \propto f_N \varphi^{lmn} p^\mu p^\nu \cdots U_\alpha(N,\mathbf{p},\sigma)$$

$Z(\mathbf{p})$:

from the nucleon propagator $\frac{1}{2}(1+\gamma_4)_{\beta\alpha} C_{\alpha\beta}(t;\mathbf{p}) = L_s^3\, Z(\mathbf{p})\, \frac{m_N + E_N(\mathbf{p})}{E_N(\mathbf{p})}\, \mathrm{e}^{-E_N(\mathbf{p})t}$

alternatively ratios like $\dfrac{\left(\frac{1}{2}(1+\gamma_4)_{\beta\alpha}\langle \mathcal{O}_\alpha(t,\mathbf{p})\bar{B}_\beta(0,\mathbf{p})\rangle\right)^2}{\frac{1}{2}(1+\gamma_4)_{\beta\alpha} C_{\alpha\beta}(t;\mathbf{p})}$ ($Z(\mathbf{p})$ drops out!)

or combined fits or …

"backward propagating" states in the nucleon 2-point function:

parity partners (N^*) of the nucleon (N(1535), N(1650), . . .)

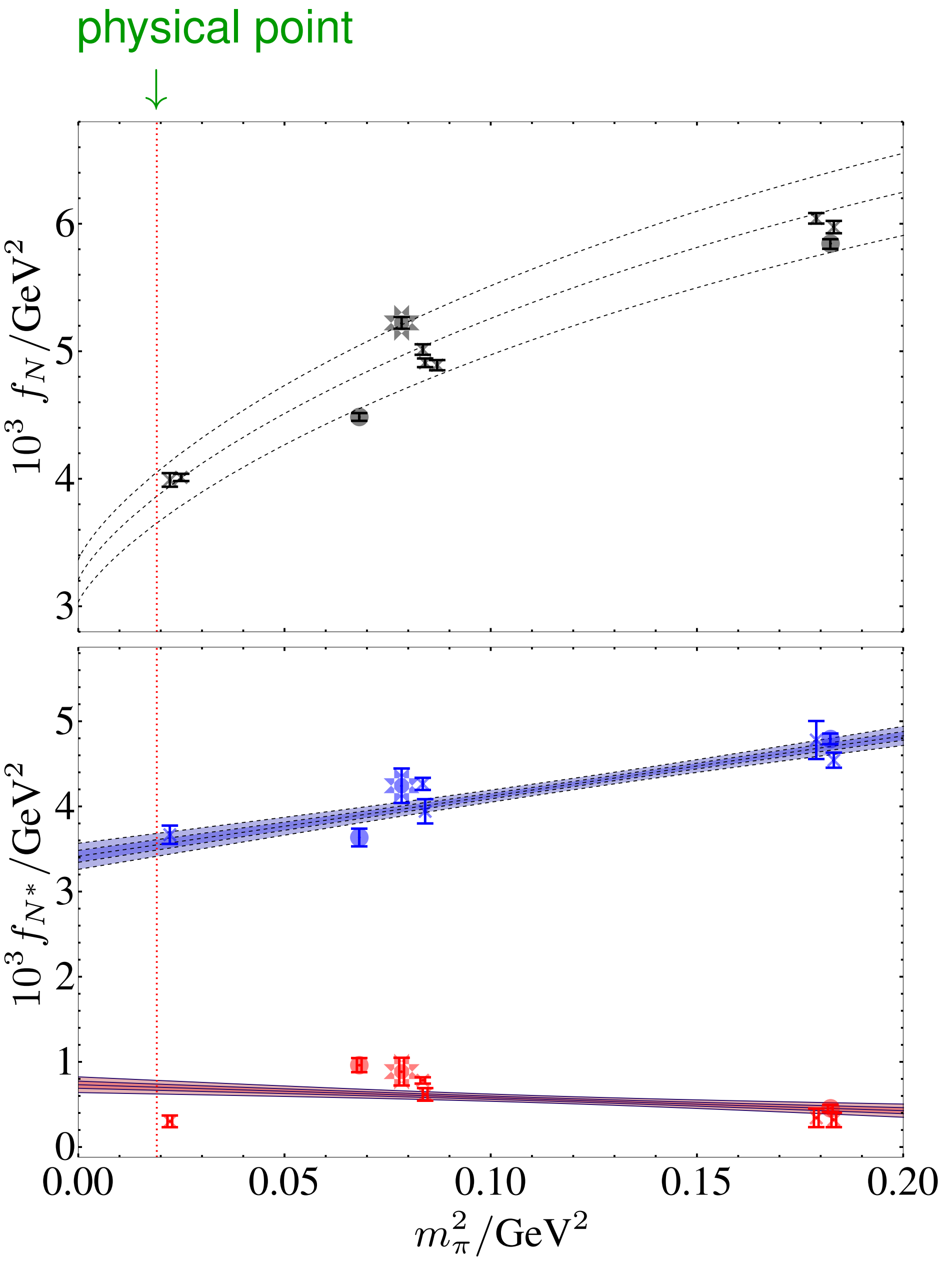
→ results for N and N^* from the same correlation functions

different interpolators → distinguish N(1535) and N(1650) difficult!

recent results: V.M. Braun et al., Phys. Rev. D89 (2014) 094511 [arXiv:1403.4189]

tentative identification of two negative parity states:

$N^*(1535?)$	N(1535)
$N^*(1650?)$	N(1650) contaminated by nucleon-pion scattering states



stars: $\beta = 5.20$ $(a = 0.081\mathrm{fm})$
crosses: $\beta = 5.29$ $(a = 0.071\mathrm{fm})$
circles: $\beta = 5.40$ $(a = 0.060\mathrm{fm})$

curves: chiral extrapolation
(chiral perturbation theory)

$N^*(1650?)$ $f_{N^*} \approx f_N$

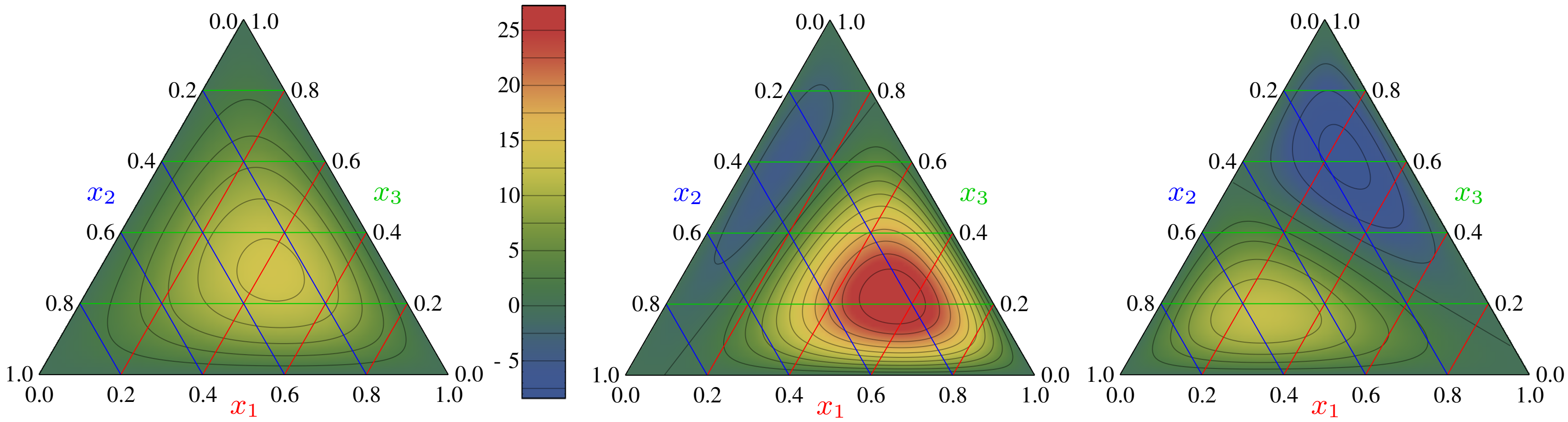
$N^*(1535?)$ $f_{N^*} \ll f_N$

expand $\varphi(x_1, x_2, x_3, \mu^2)$ in contributions renormalising multiplicatively (in 1-loop approximation)

(analogous to the expansion in Gegenbauer polynomials for the pion DA)

$$\varphi(x_1, x_2, x_3, \mu^2) = 120 x_1 x_2 x_3 \Big\{ 1 + c_{10}(x_1 - 2x_2 + x_3) L^{8/(3\beta_0)} + c_{11}(x_1 - x_3) L^{20/(9\beta_0)} + c_{20} \left[1 + 7(x_2 - 2x_1 x_3 - 2x_2^2) \right] L^{14/(3\beta_0)} + \cdots \Big\} \qquad L = \alpha_s(\mu)/\alpha_s(\mu_0)$$

“shape parameters” c_{ki} ↔ kth moments of φ ↔ operators with k derivatives



nucleon

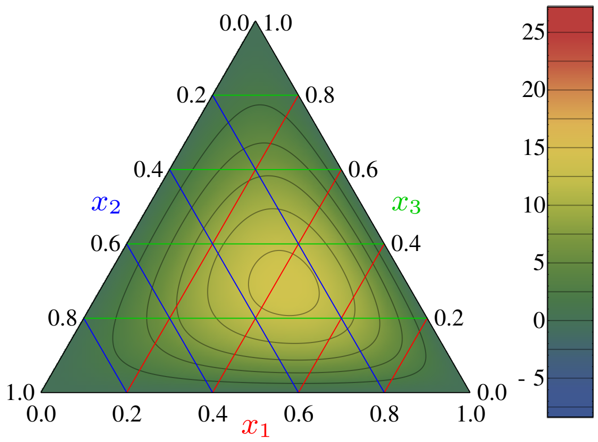
$N^*(1650?)$

$N^*(1535?)$

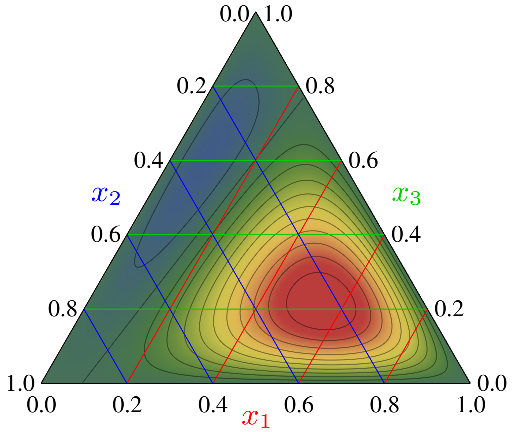
lines of constant x_1, x_2, x_3 parallel to the sides of the triangle labelled by x_2, x_3, x_1

$\mu^2 = 2\,\mathrm{GeV}^2$, only first moments used, factor f_N included

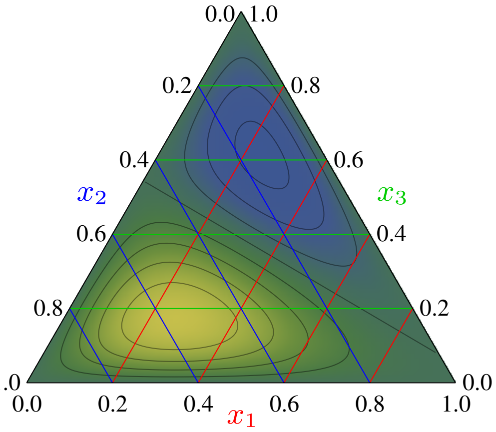
$$\varphi(x_1, x_2, x_3, \mu^2) = 120 x_1 x_2 x_3 \left\{ 1 + c_{10}(x_1 - 2x_2 + x_3) L^{8/(3\beta_0)} + c_{11}(x_1 - x_3) L^{20/(9\beta_0)} + \cdots \right\}$$



nucleon

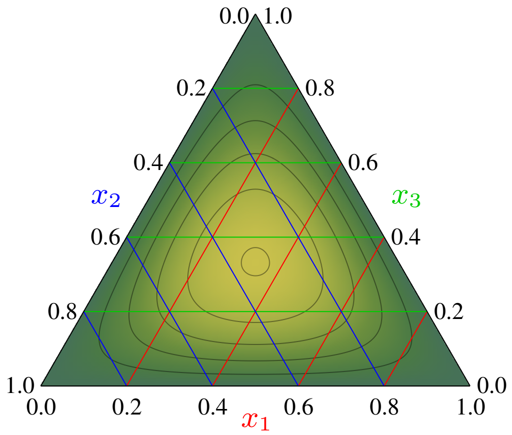


$N^*(1650?)$



$N^*(1535?)$

lines of constant x_1, x_2, x_3 parallel to the sides of the triangle labelled by x_2, x_3, x_1



asymptotic

DA of $N^*(1650?)$ $\sim$ DA of the nucleon (at the scale $\mu^2 = 2\,\mathrm{GeV}^2$)
but with larger deviations from $\varphi_{\mathrm{asy}}(x_1, x_2, x_3) = 120 x_1 x_2 x_3$
approximately symmetric under the exchange of x_2 and x_3
(quarks in the diquark)

DA of $N^*(1535?)$ qualitatively different (at the scale $\mu^2 = 2\,\mathrm{GeV}^2$)
very small value of the wave function at the origin $f_{N^*} \ll f_N$
approximately antisymmetric under the exchange of x_2 and x_3

lowest moments of the next-to-leading twist DAs ($\leftrightarrow$ operators without derivatives):
two additional constants λ_1, λ_2

$$\mathcal{L}_\tau(x) = \epsilon_{ijk} \left[u^i(x)^{\mathrm{T}} C\gamma^\rho u^j(x)\right] \times (\gamma_5\gamma_\rho d^k(x))_\tau$$
$$\mathcal{M}_\tau(x) = \epsilon_{ijk} \left[u^i(x)^{\mathrm{T}} C\sigma^{\mu\nu} u^j(x)\right] \times (\gamma_5\sigma_{\mu\nu} d^k(x))_\tau$$

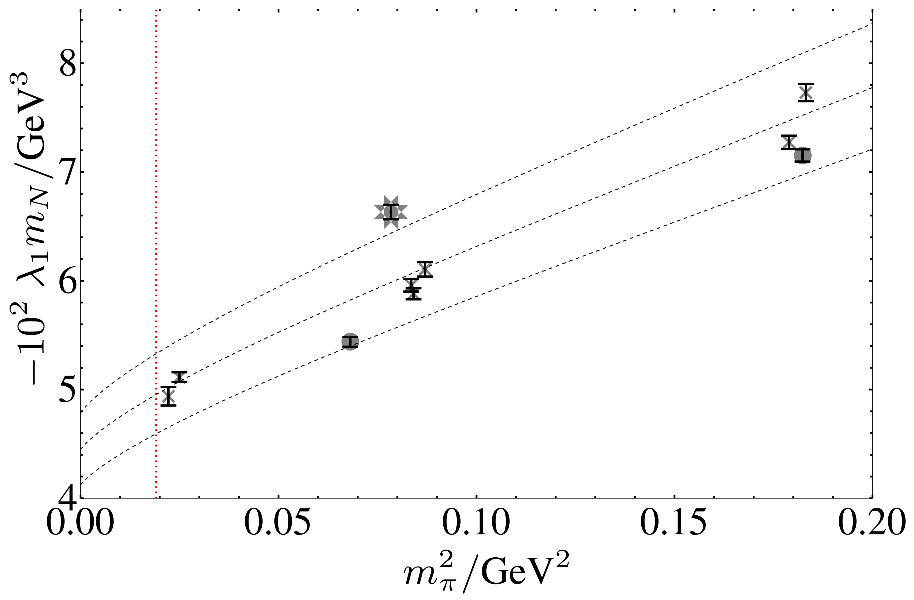
(Minkowski space)

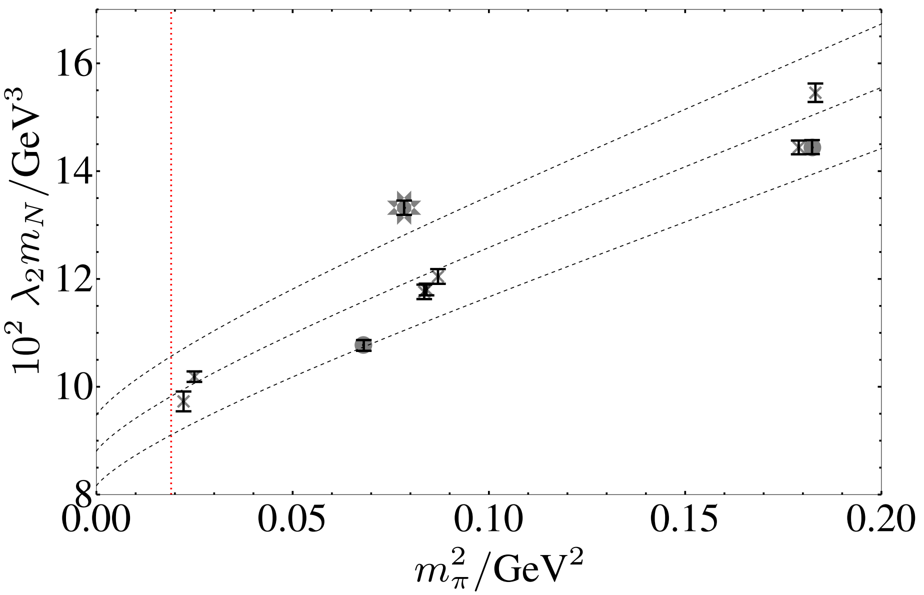
$$\langle 0|\hat{\mathcal{L}}_\tau(0)|N, \mathbf{p}, \sigma\rangle_{\mathrm{c}} = \lambda_1 m_N U_\tau(N, \mathbf{p}, \sigma)$$
$$\langle 0|\hat{\mathcal{M}}_\tau(0)|N, \mathbf{p}, \sigma\rangle_{\mathrm{c}} = \lambda_2 m_N U_\tau(N, \mathbf{p}, \sigma)$$

operators $\hat{\mathcal{L}}$, $\hat{\mathcal{M}}$ appear also in the low-energy effective action of generic GUT models:

matrix elements $\langle\pi|\hat{\mathcal{L}}|p\rangle$ and $\langle\pi|\hat{\mathcal{M}}|p\rangle$ give rise to proton decay

relevant factors in the decay amplitude for the $p \to \pi^0$ decay $\propto \alpha$, β
with $\alpha = m_N\lambda_1/4$, $\beta = m_N\lambda_2/8$
(from soft pion theorems or to leading order in chiral perturbation theory)



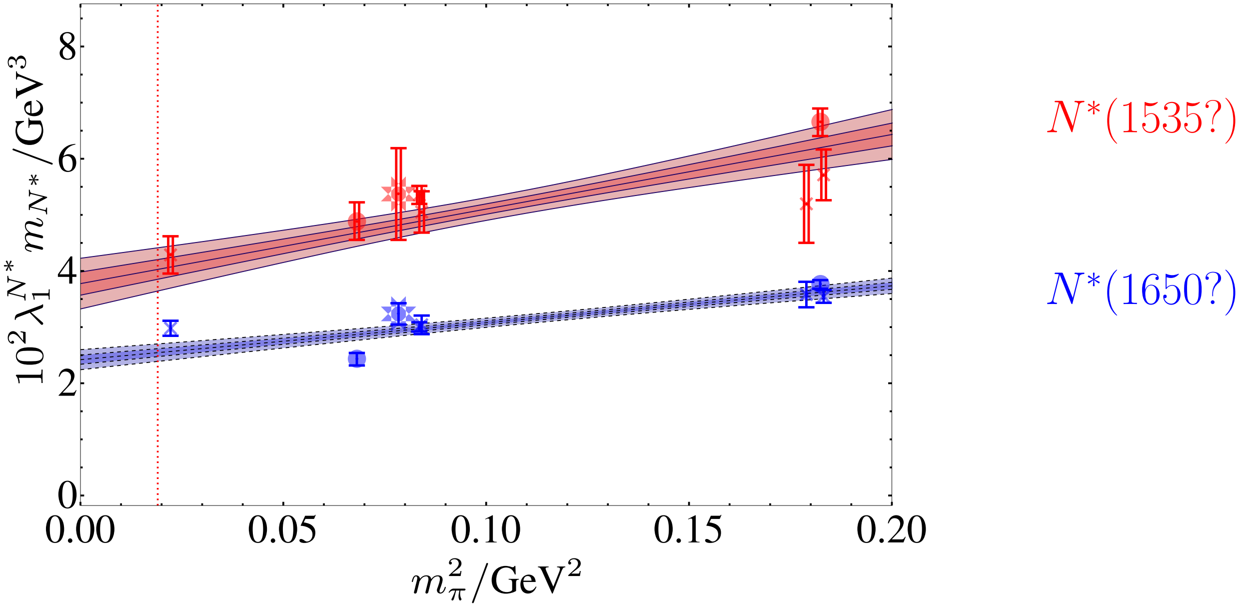


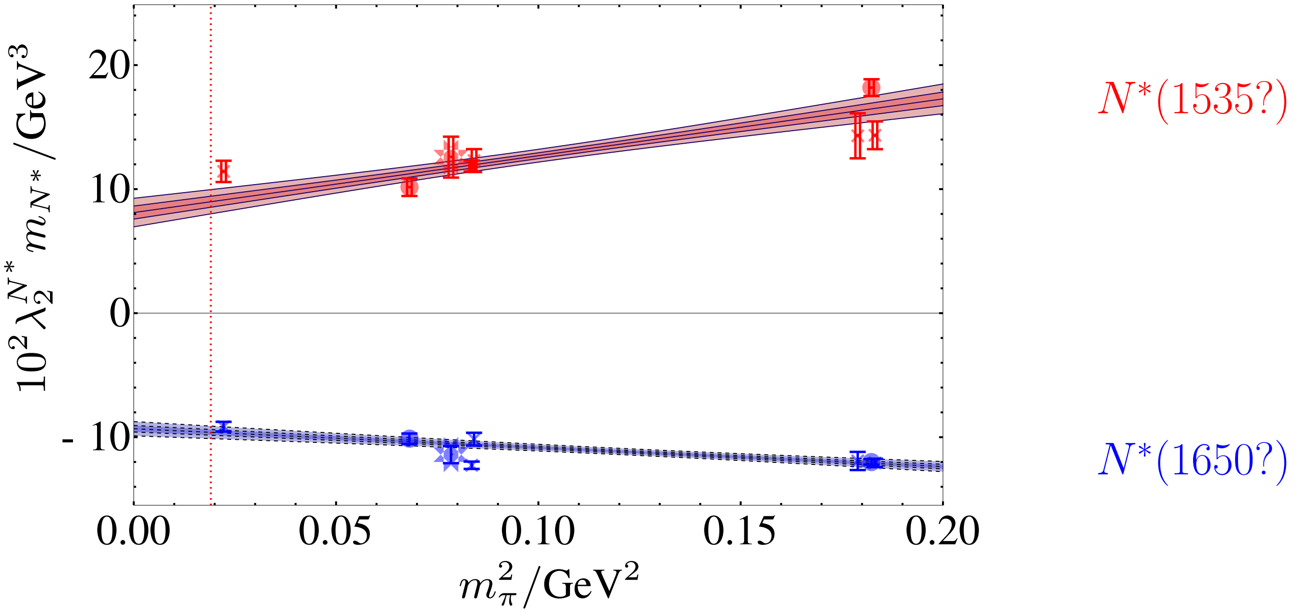
stars:	$\beta = 5.20$	$(a = 0.081\mathrm{fm})$
crosses:	$\beta = 5.29$	$(a = 0.071\mathrm{fm})$
circles:	$\beta = 5.40$	$(a = 0.060\mathrm{fm})$

curves: chiral extrapolation
(chiral perturbation theory)

nonrelativistic limit: $\lambda_2 = -2\lambda_1$
in the nucleon

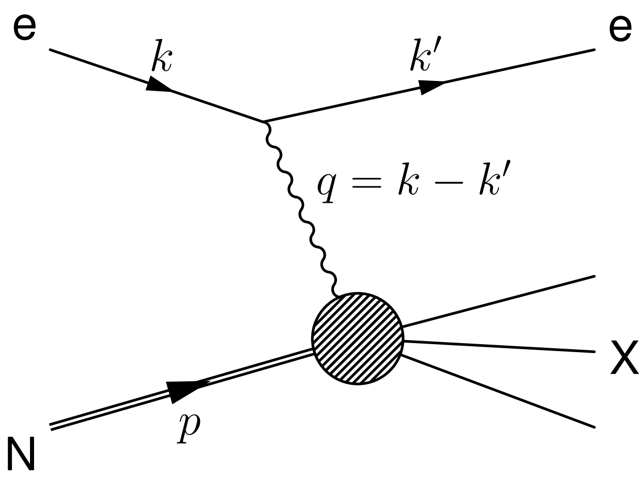
anomalous dimensions agree


$10^2 \lambda_1^{N^*} m_{N^*} / \mathrm{GeV}^3$
8
6
4
2
0
0.00
0.05
0.10
0.15
0.20
m_π^2 / GeV^2
$N^*(1535?)$
$N^*(1650?)$


$10^2 \lambda_2^{N^*} m_{N^*} / \mathrm{GeV}^3$
20
10
0
- 10
0.00
0.05
0.10
0.15
0.20
m_π^2 / GeV^2
$N^*(1535?)$
$N^*(1650?)$

Nucleon structure functions

deep-inelastic scattering (DIS):



kinematic variables: $Q^2 = -q^2$, $x = Q^2/(2p \cdot q)$ ($0 \leq x \leq 1$)

$q = k - k'$

$$\left.\begin{array}{ll} \text{unpolarised structure functions} & F_1(x,Q^2)\,,\ F_2(x,Q^2) \\ \text{polarised structure functions} & g_1(x,Q^2)\,,\ g_2(x,Q^2) \end{array}\right\}\ \text{from inclusive cross section}$$

deep-inelastic limit: $Q^2 \to \infty$, x fixed $\Rightarrow$ scaling consider moments in the deep-inelastic limit

$$2\int_0^1 \mathrm{d}x\, x^{n-1} F_1(x,Q^2) = \sum_{q=u,d} \underset{\substack{\uparrow \\ \text{Wilson coefficients}}}{c_{1,n}^{(q)}}(Q^2/\mu^2, g(\mu))\, v_n^{(q)}\underset{\substack{\uparrow \\ \text{renormalisation scale}}}{(\mu)} + \cdots \qquad (n = 2, 4, \ldots)$$

Wilson coefficients $c_{1,n}^{(q)}$
calculated in perturbation theory

$v_n^{(q)}(\mu) \leftrightarrow$ proton (forward) matrix elements
nonperturbative quantities, accessible on the lattice

in covariant notation: $s^2 = -m_N^2$

$$\frac{1}{2}\sum_s {}_{\mathrm{c}}\langle p, s|\mathcal{O}^q_{(\mu_1\cdots\mu_n)}|p, s\rangle_{\mathrm{c}} = 2 v_n^{(q)} p_{(\mu_1}\cdots p_{\mu_n)}$$

$$\mathcal{O}^q_{\mu_1\cdots\mu_n} = (\mathrm{i}/2)^{n-1}\, \bar{q}\gamma^{\mathrm{M}}_{\mu_1}\overset{\leftrightarrow}{D}_{\mu_2}\cdots\overset{\leftrightarrow}{D}_{\mu_n} q \quad \text{with} \quad \overset{\leftrightarrow}{D}_\mu = \vec{D}_\mu - \overset{\leftarrow}{D}_\mu \qquad \text{(covariant derivative)}$$

$(\cdots)$: symmetrisation and subtraction of trace terms → twist-2 operators

- dominating contributions in the deep-inelastic limit
- multiplet of twist-2 operators with given n transforming irreducibly under the Lorentz group (representation $D^{(n/2,n/2)}$)

---

parton model: interpretation in terms of parton distribution functions (PDFs) $q(x)$, $\bar{q}(x)$

$$v_n^{(q)} = \int_0^1 \mathrm{d}x\, x^{n-1} \left(q(x) + (-1)^n \bar{q}(x)\right) = \langle x^{n-1} \rangle_q$$

“probability” to find a quark (antiquark) with momentum fraction x

***

additionally in the polarised case: operators $\mathcal{O}^{q,5}_{\mu_1\cdots\mu_n} = (\mathrm{i}/2)^{n-1}\ \bar{q}\gamma^{\mathrm{M}}_{\mu_1}\gamma^{\mathrm{M}}_5 \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n} q$

with matrix elements ${}_{\mathrm{c}}\langle p, s | \mathcal{O}^{q,5}_{(\mu_1\cdots\mu_n)} | p, s \rangle_{\mathrm{c}} = a^{(q)}_{n-1} s_{(\mu_1} p_{\mu_2} \cdots p_{\mu_n)}$

$$a_n^{(q)} = 2 \int_0^1 \mathrm{d}x\, x^n \Big[\underbrace{q_+(x) - q_-(x)}_{\Delta q(x)} + (-1)^n (\bar{q}_+(x) - \bar{q}_-(x)) \Big] = 2\langle x^n \rangle_{\Delta q}$$

$q_+(x)$ ($q_-(x)$): “probability” to find a quark with momentum fraction x and helicity equal (opposite) to that of the nucleon

in particular: $\frac{1}{2} a_0^{(q)} = \langle 1 \rangle_{\Delta q} = \Delta q$

fraction of the nucleon spin carried by the spin of the quarks of flavour q

note: $\Delta u - \Delta d = g_A$ (axial coupling constant of the nucleon) (moments of) polarised structure functions in the deep-inelastic limit:

$$\int_0^1 dx\, x^n g_1(x,Q^2) = \frac{1}{4}\sum_{q=u,d} e_{1,n}^{(q)}(Q^2/\mu^2, g(\mu))a_n^{(q)}(\mu)$$
$$\int_0^1 dx\, x^n g_2(x,Q^2) = \frac{1}{4}\frac{n}{n+1}\sum_{q=u,d} \Big[e_{2,n}^{(q)}(Q^2/\mu^2, g(\mu))d_n^{(q)}(\mu) - e_{1,n}^{(q)}(Q^2/\mu^2, g(\mu))a_n^{(q)}(\mu)\Big]$$

for even n and $n \geq 0$ ($n \geq 2$) for g_1 (g_2)

$d_n^{(q)}(\mu)$: twist 3, but not power suppressed

***

in addition transversity distribution $h_1^q(x)$ (not measurable in DIS):

moments related to operators $(\mathrm{i}/2)^n\ \bar{q}\mathrm{i}\sigma_{\mu\nu}\overset{\leftrightarrow}{D}_{\mu_1}\cdots\overset{\leftrightarrow}{D}_{\mu_n}q$

“probability” weighted by quark transverse-spin projection relative to nucleon transverse-spin direction

Evaluation of matrix elements of local operators between nucleon states

aim: compute $\langle N, \mathbf{p}, \sigma | \text{local operator} | N, \mathbf{p}', \sigma' \rangle$ ($\mathbf{p}$ may be different from $\mathbf{p}'$!)

tool: three-point correlation functions

$$C_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q}) = \langle B_\alpha(t,\mathbf{p}) J^{(q)}(\tau,\mathbf{q}) \bar{B}_\beta(0,\mathbf{p}') \rangle \qquad \text{with} \qquad \mathbf{p}' = \mathbf{p} - \mathbf{q}$$

operator represented by $J^{(q)}(\tau,\mathbf{q}) = a^3 \sum_{x,\, x_4=\tau} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}} J^{(q)}(x)$ with $q = u, d, s, \ldots$

examples: $J^{(q)}(x) = \bar{q}(x)\Gamma q(x)$, $J^{(q)}(x) = \bar{q}(x)\Gamma D_\nu q(x)$, ... ($\Gamma = \gamma_\mu, \gamma_\mu\gamma_5, \ldots$)

for $0 < \tau < t$: ($L_t \to \infty$, keeping only the lowest contributing state)

$$C_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q}) = L_s^3 a^6 \sum_{\mathbf{x}} \sum_{\mathbf{y}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}+\mathrm{i}\mathbf{q}\cdot\mathbf{y}} \frac{\mathrm{Tr}\, \mathrm{e}^{-(L_t-t)\hat{H}} \hat{B}_\alpha(\mathbf{x}) \mathrm{e}^{-(t-\tau)\hat{H}} \hat{J}(\mathbf{y}) \mathrm{e}^{-\tau\hat{H}} \hat{\bar{B}}_\beta(\mathbf{x}=\mathbf{0})}{\mathrm{Tr}\, \mathrm{e}^{-L_t\hat{H}}}$$

$$= L_s^3 a^6 \sum_{\mathbf{x}} \sum_{\mathbf{y}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}+\mathrm{i}\mathbf{q}\cdot\mathbf{y}}$$

$$\times \sum_{\sigma,\sigma'} \langle 0 | \hat{B}_\alpha(\mathbf{x}) | N, \mathbf{p}, \sigma \rangle \langle N, \mathbf{p}, \sigma | \hat{J}(\mathbf{y}) | N, \mathbf{p}', \sigma' \rangle \langle N, \mathbf{p}', \sigma' | \hat{\bar{B}}_\beta(\mathbf{x}=\mathbf{0}) | 0 \rangle \mathrm{e}^{-E_N(\mathbf{p})(t-\tau) - E_N(\mathbf{p}')\tau} + \ldots$$

How are the required three-point functions computed?

represent the “current” J as $$J^{(q)}(x) = \sum_{z,z'} \bar{q}^i_\alpha(z) J^{(q)}(z,z';x)^{ij}_{\alpha\beta}\, q^j_\beta(z')$$

examples:

$$J^{(u)}(x) = \bar{u}(x)\Gamma u(x) \Rightarrow J^{(u)}(z,z';x)^{ij}_{\alpha\beta} = \delta_{ij}\Gamma_{\alpha\beta}\delta_{z,x}\delta_{z',x}$$

$$J^{(d)}(x) = \bar{d}(x)\Gamma D_\mu d(x) = \frac{1}{2a}\big(\bar{d}(x)\Gamma U(x,\mu) d(x+\hat{\mu}) - \bar{d}(x)\Gamma U^\dagger(x-\hat{\mu},\mu) d(x-\hat{\mu})\big)$$

$$\Rightarrow J^{(d)}(z,z';x)^{ij}_{\alpha\beta} = \frac{1}{2a}\Gamma_{\alpha\beta}\delta_{z,x}\big(U_{ij}(z,\mu)\delta_{z',x+\hat{\mu}} - U_{ji}(z',\mu)^*\delta_{z',x-\hat{\mu}}\big)$$

gauge field dependence of J suppressed!

$\hat{\mu}$: vector of length a in direction μ

three-point function: sum of a (quark-line) connected and a disconnected contribution

$$C_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q}) = C^{(q)}_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q})^{\text{con}} + C^{(q)}_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q})^{\text{dis}}$$

schematically: $\langle u(t)u(t)d(t)\bar{q}(\tau)q(\tau)\bar{u}(0)\bar{u}(0)\bar{d}(0)\rangle$

$$C^{(q)}_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q})^{\mathrm{dis}} = -L_s^3a^6 \sum_{\substack{x,z,z'\\x_4=\tau}} \sum_{\substack{y\\y_4=t}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{y}+\mathrm{i}\mathbf{q}\cdot\mathbf{x}} \epsilon_{ijk}\epsilon_{i'j'k'}(C^{-1}\gamma_5)_{\gamma\delta}(\gamma_5C)_{\gamma'\delta'}$$

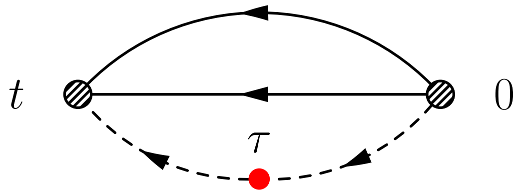
$$\times \left\langle \mathrm{tr}_{\mathrm{DC}}\left(J^{(q)}(z,z';x)G_q(z',z)\right) G_d(y,0)^{ki'}_{\delta\gamma'} \left(G_u(y,0)^{ij'}_{\alpha\delta'}\,G_u(y,0)^{jk'}_{\gamma\beta} - G_u(y,0)^{jj'}_{\gamma\delta'}\,G_u(y,0)^{ik'}_{\alpha\beta}\right)\right\rangle_g$$

$$C^{(d)}_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q})^{\mathrm{con}} = -L_s^3a^6 \sum_{\substack{x,z,z'\\x_4=\tau}} \sum_{\substack{y\\y_4=t}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{y}+\mathrm{i}\mathbf{q}\cdot\mathbf{x}} \epsilon_{ijk}\epsilon_{i'j'k'}(C^{-1}\gamma_5)_{\gamma\delta}(\gamma_5C)_{\gamma'\delta'}$$

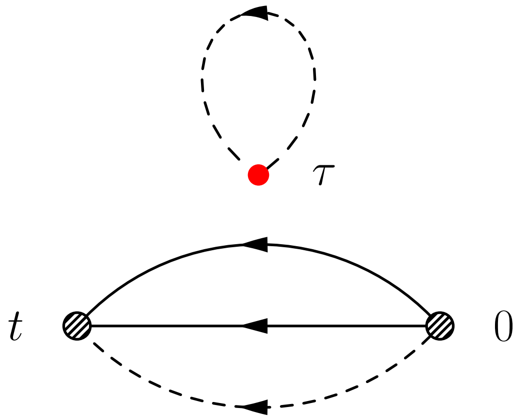
$$\times \left\langle \left(G_d(y,z)J^{(d)}(z,z';x)G_d(z',0)\right)^{ki'}_{\delta\gamma'} \left(G_u(y,0)^{jj'}_{\gamma\delta'}\,G_u(y,0)^{ik'}_{\alpha\beta} - G_u(y,0)^{ij'}_{\alpha\delta'}\,G_u(y,0)^{jk'}_{\gamma\beta}\right)\right\rangle_g$$

$C^{(u)}_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q})^{\mathrm{con}}$: analogous, but slightly more complicated

connected:



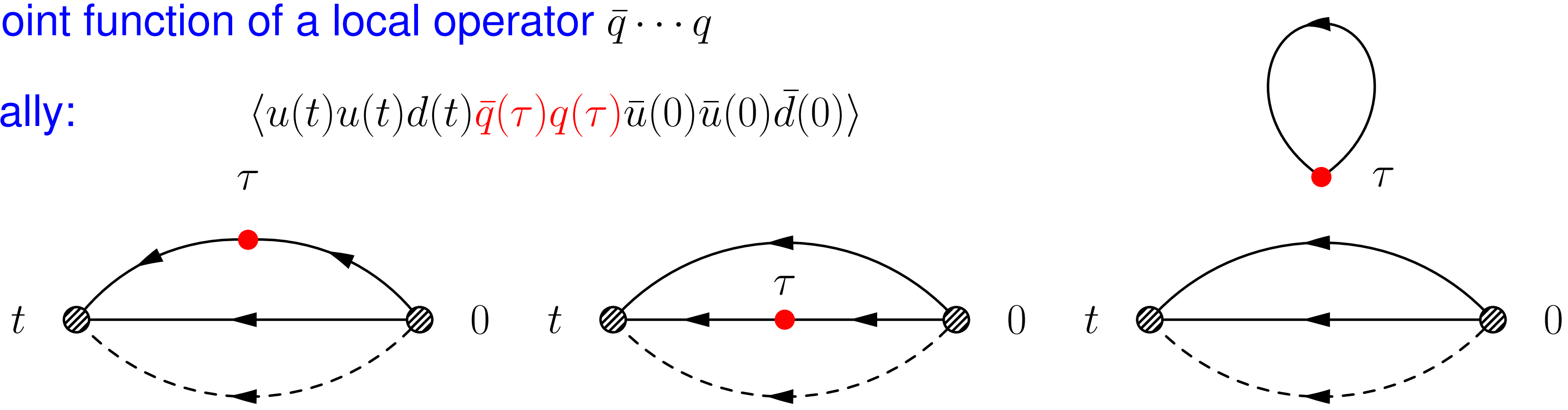
disconnected:



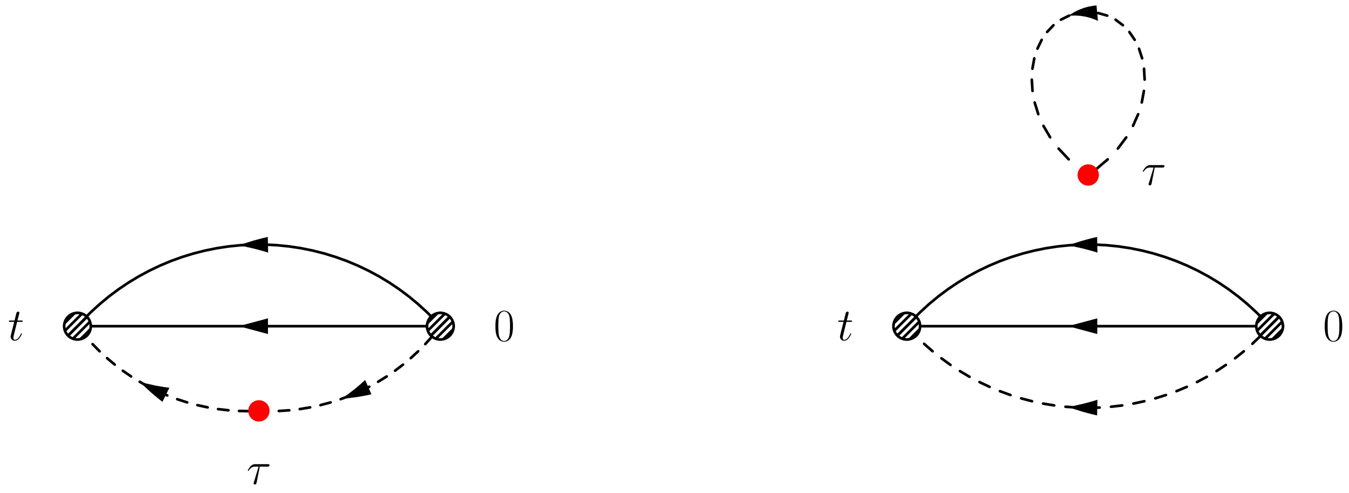
proton 3-point function of a local operator $\bar{q} \cdots q$

schematically: $\langle u(t)u(t)d(t)\bar{q}(\tau)q(\tau)\bar{u}(0)\bar{u}(0)\bar{d}(0)\rangle$

$\bar{u} \cdots u$



$\bar{d} \cdots d$



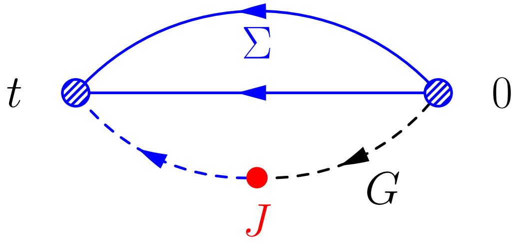
$\bar{s} \cdots s$ only disconnected contribution

(quark-line) disconnected contributions drop out in isovector quantities ($\bar{u} \cdots u - \bar{d} \cdots d$)
if isospin invariance is exact ($m_u = m_d$)

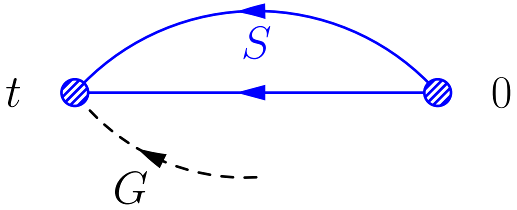
standard method for the evaluation of connected contributions: sequential sources or sequential propagators

$$\Gamma_{\beta\alpha} C^{(q)}_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q})^{\mathrm{con}} = L_s^3 a^3 \sum_{\substack{x,z,z' \\ x_4=\tau}} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}} \left\langle \mathrm{tr}_{\mathrm{DC}} \left(\Sigma^{(q)}_{\Gamma}(z;\mathbf{p},t) J^{(q)}(z,z';x) G_q(z',0) \right) \right\rangle_g$$

for $q = d$:



$$\Sigma^{(q)}_{\Gamma}(z;\mathbf{p},t) = a^3 \sum_{y,y_4=t} S^{(q)}_{\Gamma}(y;\mathbf{p}) G_q(y,z)$$



$(q = d)$

$$S^{(d)}_{\Gamma}(y;\mathbf{p})^{i'i}_{\alpha\beta} = \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{y}} \epsilon_{ijk}\epsilon_{i'j'k'} \Big[\gamma_5 C \left(G_u(y,0)^{kk'} \right)^{\mathrm{T}} \Gamma^{\mathrm{T}} \left(G_u(y,0)^{jj'} \right)^{\mathrm{T}} C^{-1}\gamma_5 + \mathrm{tr}_{\mathrm{D}} \left(\Gamma G_u(y,0)^{jj'} \right) \left(C^{-1}\gamma_5 G_u(y,0)^{kk'} \gamma_5 C \right)^{\mathrm{T}} \Big]_{\alpha\beta}$$

$S_\Gamma^{(u)}(y;\mathbf{p})$: analogous, but slightly more complicated

crucial point:

$\Sigma_\Gamma^{(q)}(z;\mathbf{p},t) = a^3 \sum_{y,y_4=t} S_\Gamma^{(q)}(y;\mathbf{p}) G_q(y,z)$ can be computed from the linear system of equations

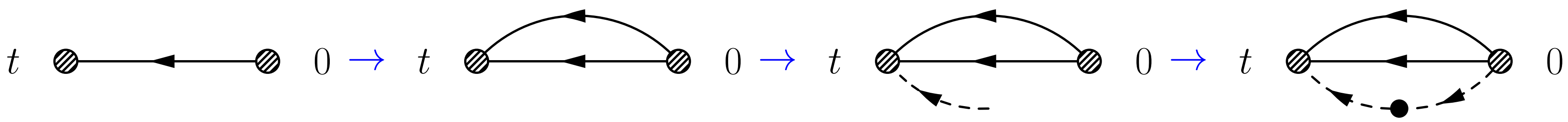
$$a^8 \sum_z \underset{\substack{\uparrow \\ \text{fermion matrix}}}{M_q(x,z)} \gamma_5 \, \Sigma_\Gamma^{(q)}(z;\mathbf{p},t)^\dagger = \gamma_5 S_\Gamma^{(q)}(x;\mathbf{p})^\dagger \delta_{x_4,t}$$

$\Sigma_\Gamma^{(q)}(z;\mathbf{p},t)$: sequential propagator

based on a source $S_\Gamma^{(q)}(x;\mathbf{p})$ constructed from two ordinary propagators placed at time t

computation of $C(t,\tau;\mathbf{p},\mathbf{q})^{\mathrm{con}}$ proceeds in several steps:

1. compute propagator $G(x,0)$
2. construct sources $S^{(u)}$ and $S^{(d)}$
3. compute sequential propagators $\Sigma^{(u)}$ and $\Sigma^{(d)}$ (second inversion of the fermion matrix)
4. multiply $\Sigma^{(u)}$, $\Sigma^{(d)}$ with G and J (representing the operator under study)



advantage:
without additional inversions arbitrary operators, arbitrary momenta $\mathbf{q}$, arbitrary times τ

disadvantage:
additional inversions for each Γ (polarised or unpolarised nucleon), momentum $\mathbf{p}$, time t
(sequential propagators depend on Γ, $\mathbf{p}$, t)

$\Rightarrow$ varying t (in addition to τ) is expensive. . .   but has become feasible in recent years

alternative: stochastic sources

How are the desired matrix elements extracted from the three-point functions?

remember ($0 < \tau < t$):

$$C_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q}) = L_s^3 a^6 \sum_{\mathbf{x}} \sum_{\mathbf{y}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}+\mathrm{i}\mathbf{q}\cdot\mathbf{y}}$$
$$\times \sum_{\sigma,\sigma'} \langle 0|\hat{B}_\alpha(\mathbf{x})|N,\mathbf{p},\sigma\rangle\langle N,\mathbf{p},\sigma|\hat{J}(\mathbf{y})|N,\mathbf{p}',\sigma'\rangle\langle N,\mathbf{p}',\sigma'|\hat{\bar{B}}_\beta(\mathbf{x}=\mathbf{0})|0\rangle \mathrm{e}^{-E_N(\mathbf{p})(t-\tau)-E_N(\mathbf{p}')\tau} + \cdots$$

expressing the nucleon matrix element of the operator under study as

$${}_{\mathrm{c}}\langle N,\mathbf{p},\sigma|\hat{J}(\mathbf{0})|N,\mathbf{p}',\sigma'\rangle_{\mathrm{c}} = \bar{U}(N,\mathbf{p},\sigma)M(\mathbf{p},\mathbf{p}')U(N,\mathbf{p}',\sigma')$$

one gets

$$\Gamma_{\beta\alpha}C_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q}) = L_s^3 \frac{\sqrt{Z(\mathbf{p})Z(\mathbf{p}')}}{4E_N(\mathbf{p})E_N(\mathbf{p}')} \mathrm{e}^{-E_N(\mathbf{p})(t-\tau)-E_N(\mathbf{p}')\tau}$$
$$\times \operatorname{tr}\big(\Gamma(E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N)M(\mathbf{p},\mathbf{p}')(E_N(\mathbf{p}')\gamma_4 - \mathrm{i}\mathbf{p}'\cdot\boldsymbol{\gamma} + m_N)\big) + \cdots$$

$Z(\mathbf{p})$ can be extracted from the two-point function (see above)

$$\Gamma_{\beta\alpha}C_{\alpha\beta}(t;\mathbf{p}) = L_s^3\, Z(\mathbf{p}) \operatorname{tr}\big(\Gamma\,(E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N)\big) \frac{\mathrm{e}^{-E_N(\mathbf{p})t}}{2E_N(\mathbf{p})} + \cdots$$

in the special case $\mathbf{p} = \mathbf{p}'$ ($\Rightarrow \mathbf{q} = \mathbf{0}$) $Z(\mathbf{p})$ cancels in the ratio of the three-point function

$$\Gamma'_{\beta\alpha} C_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q}=\mathbf{0}) = L_s^3 \, \frac{Z(\mathbf{p})}{4E_N(\mathbf{p})^2} \, \mathrm{e}^{-E_N(\mathbf{p})t} \\ \times \operatorname{tr}\Big(\Gamma'(E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N) M(\mathbf{p},\mathbf{p}) (E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N)\Big) + \cdots$$

over the two-point function

$$\Gamma_{\beta\alpha} C_{\alpha\beta}(t;\mathbf{p}) = L_s^3 \, Z(\mathbf{p}) \operatorname{tr}\Big(\Gamma\,(E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N)\,\Big) \, \frac{\mathrm{e}^{-E_N(\mathbf{p})t}}{2E_N(\mathbf{p})} + \cdots$$

which reads

$$\frac{\Gamma'_{\beta\alpha} C_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q}=\mathbf{0})}{\Gamma_{\beta\alpha} C_{\alpha\beta}(t;\mathbf{p})} = \frac{\operatorname{tr}\Big(\Gamma'(E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N) M(\mathbf{p},\mathbf{p}) (E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N)\Big)}{2E_N(\mathbf{p}) \operatorname{tr}\Big(\Gamma\,(E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N)\,\Big)} + \cdots$$

remember for $\Gamma = \frac{1}{2}(1+\gamma_4)$: $\operatorname{tr}\Big(\frac{1}{2}(1+\gamma_4)\,(E_N(\mathbf{p})\gamma_4 - \mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma} + m_N)\,\Big) = 2\,(m_N + E_N(\mathbf{p}))$

ratio independent of τ (for $0 < \tau < t$) if only the lowest state contributes

→ look for a plateau in τ (t fixed)

example

details for the special case $v_2^{(q)} = \langle x \rangle_q$

for general n we have in Minkowski space

$$ {}_{\rm c}\langle N,\mathbf{p},\sigma|\mathcal{O}^q_{(\mu_1\cdots\mu_n)}|N,\mathbf{p},\sigma'\rangle_{\rm c} = v_n^{(q)}\,\bar{U}(N,\mathbf{p},\sigma)\gamma^{\rm M}_{(\mu_1}p_{\mu_2}\cdots p_{\mu_n)}U(N,\mathbf{p},\sigma') $$

with $\mathcal{O}^q_{\mu_1\cdots\mu_n} = ({\rm i}/2)^{n-1}\ \bar{q}\gamma^{\rm M}_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}q$

in particular for $n=2$: $\mathcal{O}^q_{(01)} = \frac{1}{2}\left(\mathcal{O}^q_{01}+\mathcal{O}^q_{10}\right) = \frac{\rm i}{4}\bar{q}\left(\gamma^{\rm M}_0\overleftrightarrow{D}_1+\gamma^{\rm M}_1\overleftrightarrow{D}_0\right)q$

$$ \Rightarrow \frac{\rm i}{4}\,{}_{\rm c}\langle N,\mathbf{p},\sigma|\bar{q}\left(\gamma^{\rm M}_0\overleftrightarrow{D}_1+\gamma^{\rm M}_1\overleftrightarrow{D}_0\right)q|N,\mathbf{p},\sigma'\rangle_{\rm c} = v_2^{(q)}\cdot\tfrac{1}{2}\bar{U}(N,\mathbf{p},\sigma)\left(\gamma^{\rm M}_0 p_1+\gamma^{\rm M}_1 p_0\right)U(N,\mathbf{p},\sigma') $$

Euclidianisation ($\gamma^{\rm M}_0 = \gamma^{\rm E}_4$, $\gamma^{\rm M}_1 = -{\rm i}\gamma^{\rm E}_1$, $D^{\rm M}_0 = {\rm i}D^{\rm E}_4$, $D^{\rm M}_1 = D^{\rm E}_1$) leads to

$$ \frac{\rm i}{4}\,{}_{\rm c}\langle N,\mathbf{p},\sigma|\bar{q}\left(\gamma^{\rm E}_4\overleftrightarrow{D}_1+\gamma^{\rm E}_1\overleftrightarrow{D}_4\right)q|N,\mathbf{p},\sigma'\rangle_{\rm c} = v_2^{(q)}\cdot\tfrac{1}{2}\bar{U}(N,\mathbf{p},\sigma)\left(-\gamma^{\rm E}_4 p_1-{\rm i}\gamma^{\rm E}_1 E_N(\mathbf{p})\right)U(N,\mathbf{p},\sigma') $$

(marking Euclidean objects for once by an $\rm E$)

comparing

$$\frac{\mathrm{i}}{4}\;{}_{\mathrm{c}}\langle N,\mathbf{p},\sigma|\bar{q}\left(\gamma_4\overset{\leftrightarrow}{D}_1+\gamma_1\overset{\leftrightarrow}{D}_4\right)q|N,\mathbf{p},\sigma'\rangle_{\mathrm{c}}=v_2^{(q)}\cdot\tfrac{1}{2}\bar{U}(N,\mathbf{p},\sigma)\left(-\gamma_4 p_1-\mathrm{i}\gamma_1 E_N(\mathbf{p})\right)U(N,\mathbf{p},\sigma')$$

with $\quad {}_{\mathrm{c}}\langle N,\mathbf{p},\sigma|\hat{J}(\mathbf{0})|N,\mathbf{p}',\sigma'\rangle_{\mathrm{c}}=\bar{U}(N,\mathbf{p},\sigma)M(\mathbf{p},\mathbf{p}')U(N,\mathbf{p}',\sigma')$

we have $\quad M(\mathbf{p},\mathbf{p})=\dfrac{1}{Z}\dfrac{1}{2}v_2^{(q)}\left(\mathrm{i}\gamma_4 p_1-\gamma_1 E_N(\mathbf{p})\right)$

if the renormalised operator $\quad \frac{1}{4}\bar{q}\left(\gamma_4\overset{\leftrightarrow}{D}_1+\gamma_1\overset{\leftrightarrow}{D}_4\right)q \quad$ corresponds to $Z\hat{J}(\mathbf{0})$

Z: renormalisation factor

i.e. $J(x)=\frac{1}{4}\bar{q}(x)\left(\gamma_4\overset{\leftrightarrow}{D}_1+\gamma_1\overset{\leftrightarrow}{D}_4\right)q(x)$ with a suitable discretisation of the derivatives

then taking $\Gamma'=\Gamma=\frac{1}{2}(1+\gamma_4)$:

$$\frac{\Gamma_{\beta\alpha}C_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q}=\mathbf{0})}{\Gamma_{\beta\alpha}C_{\alpha\beta}(t;\mathbf{p})}=\frac{\mathrm{tr}\Big(\Gamma(E_N(\mathbf{p})\gamma_4-\mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma}+m_N)M(\mathbf{p},\mathbf{p})(E_N(\mathbf{p})\gamma_4-\mathrm{i}\mathbf{p}\cdot\boldsymbol{\gamma}+m_N)\Big)}{4E_N(\mathbf{p})\left(m_N+E_N(\mathbf{p})\right)}$$

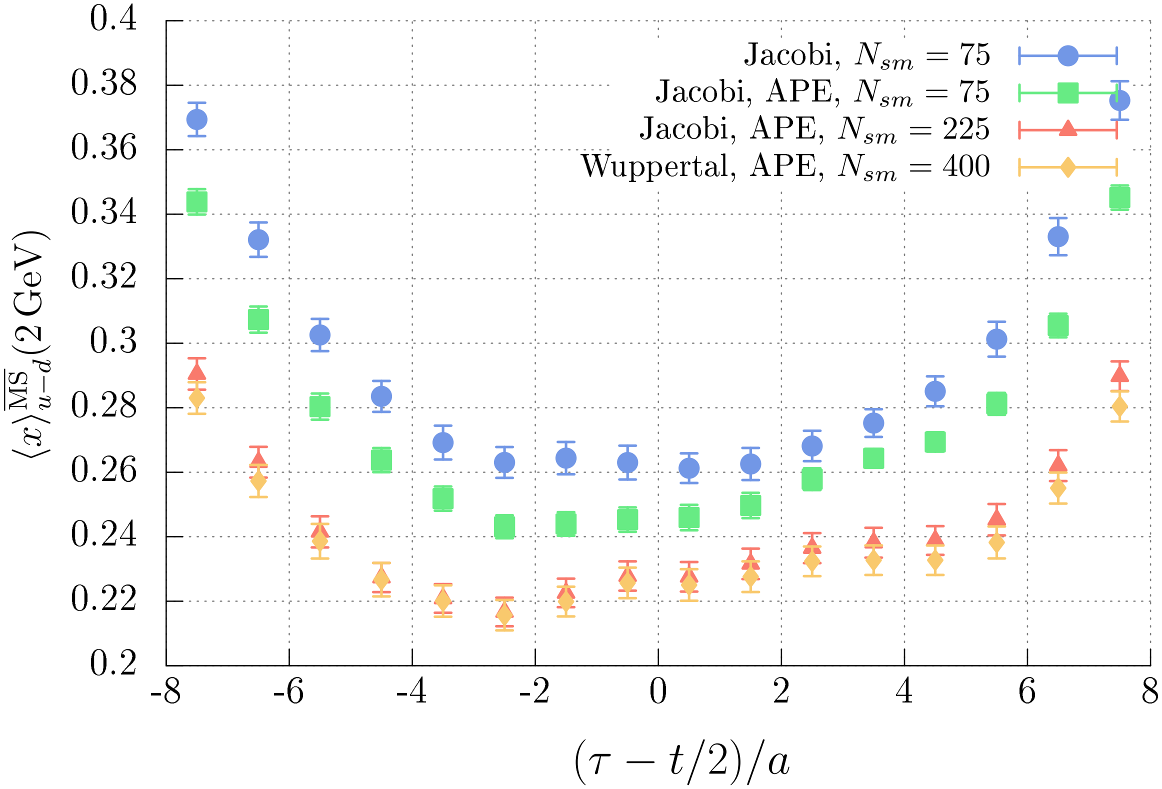
$$=\frac{\mathrm{i}p_1}{Z}v_2^{(q)}$$

- one needs $p_1\neq 0$ (disadvantage of the particular operator)
- quark fields are assumed to be normalised as in the continuum
- polarisation would require $\Gamma'\neq\frac{1}{2}(1+\gamma_4)$

details (and pitfalls) of the analysis: $\langle x \rangle_{u-d} = \langle x \rangle_u - \langle x \rangle_d = v_2^{(u-d)}$ (forward matrix element)

G.S. Bali et al. (RQCD), in preparation

ratio $\Gamma_{\beta\alpha} C_{\alpha\beta}(t, \tau; \mathbf{p}, \mathbf{q} = \mathbf{0}) / \Gamma_{\beta\alpha} C_{\alpha\beta}(t; \mathbf{p})$ (times appropriate factor) for different smearings:



nonperturbatively improved Wilson fermions

$\beta = 5.40$, $\kappa = 0.13640$
$V = 32^3 \times 64$

$n_f = 2$
$a \approx 0.06\,\text{fm}$
$m_\pi \approx 490\,\text{MeV}$

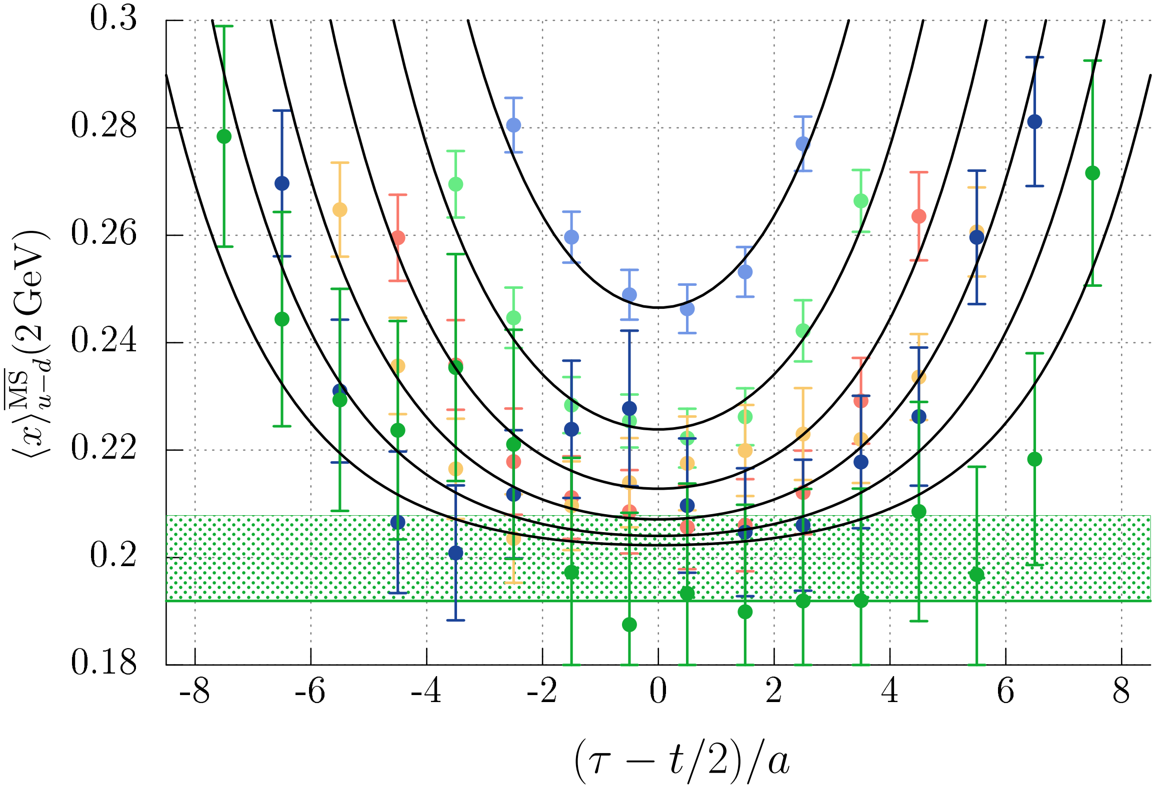
sufficient smearing required for improving the overlap with the ground state investigate remaining excited state contamination by varying t

simultaneous fits to the two- and three-point functions for all ts including the first excited state:

$$\Gamma_{\beta\alpha} C_{\alpha\beta}(t;\mathbf{p}) = A_0 \mathrm{e}^{-m_N \cdot t} + A_1 \mathrm{e}^{-(m_N+\Delta m)\cdot t}$$

τ independent!

$$\Gamma_{\beta\alpha} C_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q}=\mathbf{0}) = A_0 \mathrm{e}^{-m_N \cdot t}\left[B_0 + B_1\left(\mathrm{e}^{-\Delta m\cdot(t-\tau)} + \mathrm{e}^{-\Delta m \cdot \tau}\right) + \overset{\downarrow}{B_2 \mathrm{e}^{-\Delta m\cdot t}}\right]$$

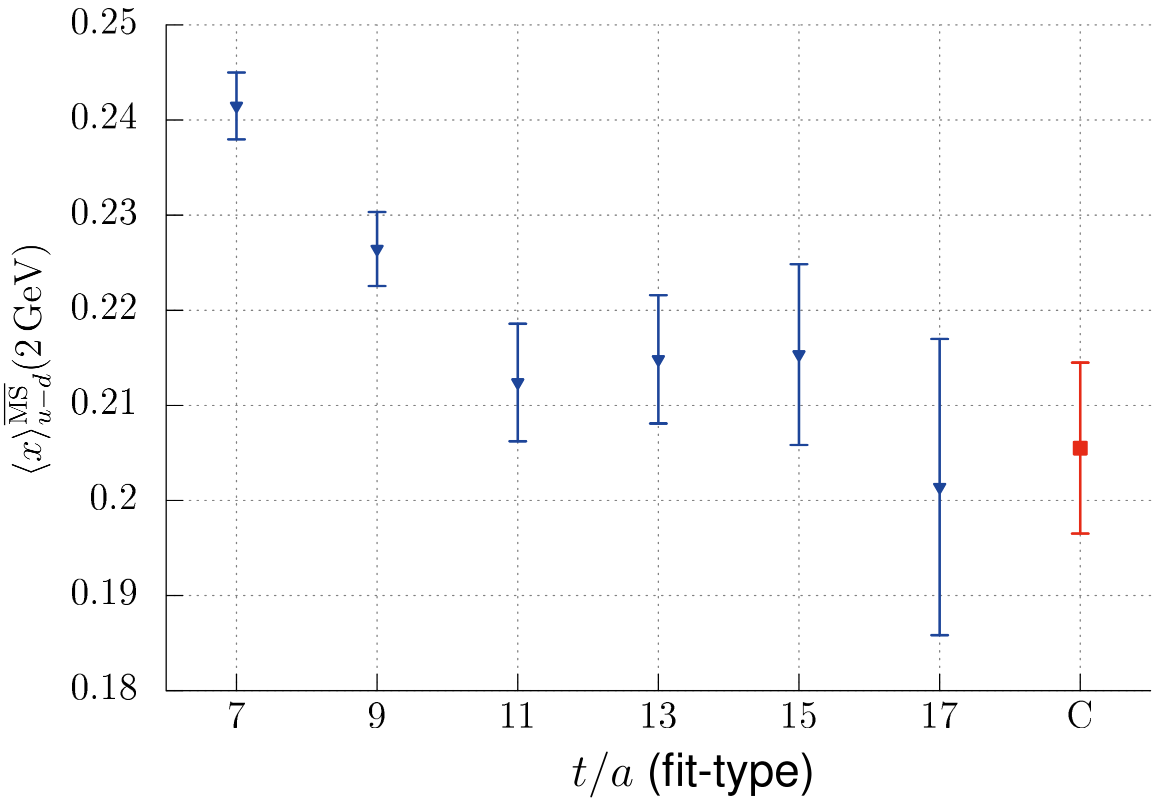


$\beta = 5.29,\ \kappa = 0.13632$
$V = 32^3 \times 64$

$n_f = 2$
$a \approx 0.07\,\mathrm{fm}$
$m_\pi \approx 290\,\mathrm{MeV}$
$t/a = 7, 9, 11, 13, 15, 17$

plotted: data $\Gamma_{\beta\alpha} C_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q}=\mathbf{0})/(A_0 \mathrm{e}^{-m_N\cdot t})$ (times factor) and fitted value $\langle x\rangle_{u-d}$

fitting the ratio $\Gamma_{\beta\alpha} C_{\alpha\beta}(t, \tau; \mathbf{p}, \mathbf{q} = \mathbf{0}) / \Gamma_{\beta\alpha} C_{\alpha\beta}(t; \mathbf{p})$ for fixed t to a constant B_0:

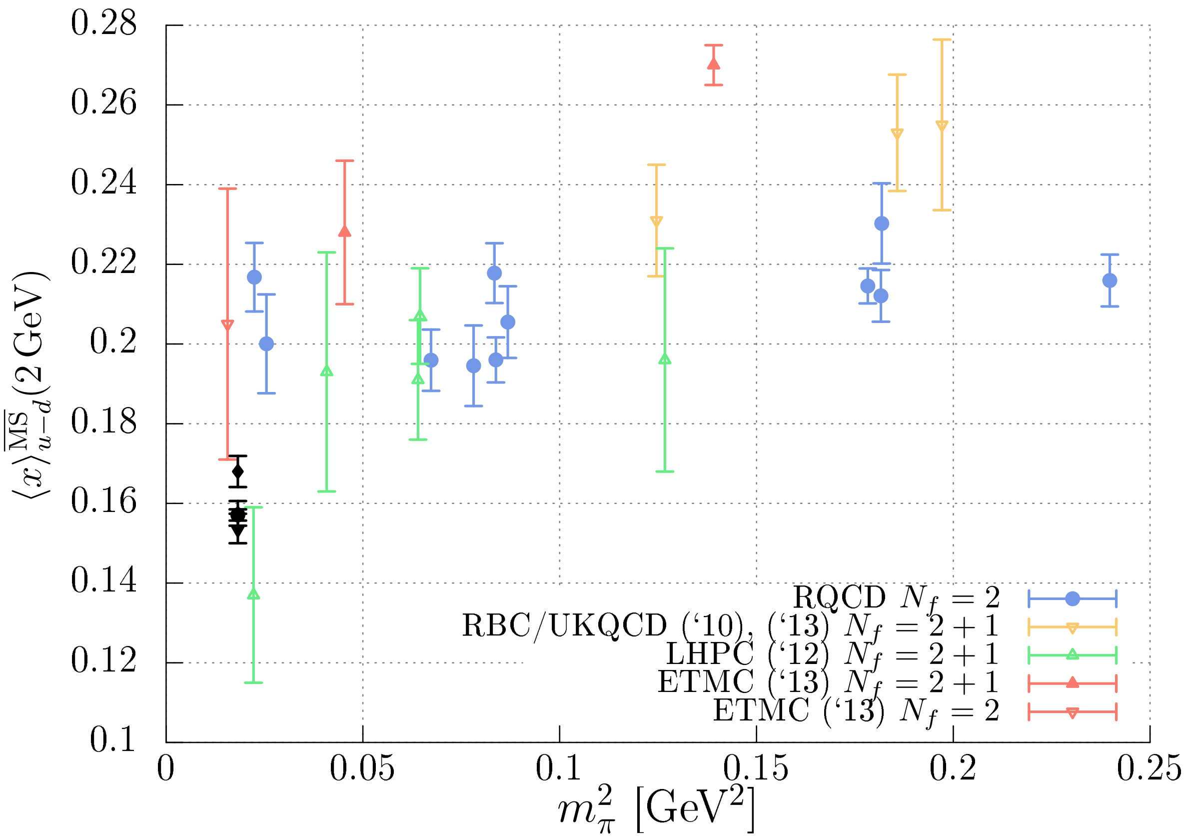


$\beta = 5.29$, $\kappa = 0.13632$
$V = 32^3 \times 64$

$n_f = 2$
$a \approx 0.07\,\mathrm{fm}$
$m_\pi \approx 290\,\mathrm{MeV}$

results consistent with the combined fits (C) for $t \geq 11a$

comparison of results for $\langle x \rangle_{u-d}$



LHPC: $n_f = 2+1$
(tree-level improved clover fermions, link smearing)

RBC/UKQCD: $n_f = 2+1$
(domain-wall fermions)

ETMC: $n_f = 2+1+1$
$n_f = 2$
(twisted mass fermions)

consistency within the (large) errors (except for one high statistics ETMC point)
higher precision needed to see effects of strange quarks, discretisation, finite size, . . .

agreement with phenomenology?
LHPC at almost physical pion mass: predominantly coarse ($a \approx 0.12\,\mathrm{fm}$) lattices

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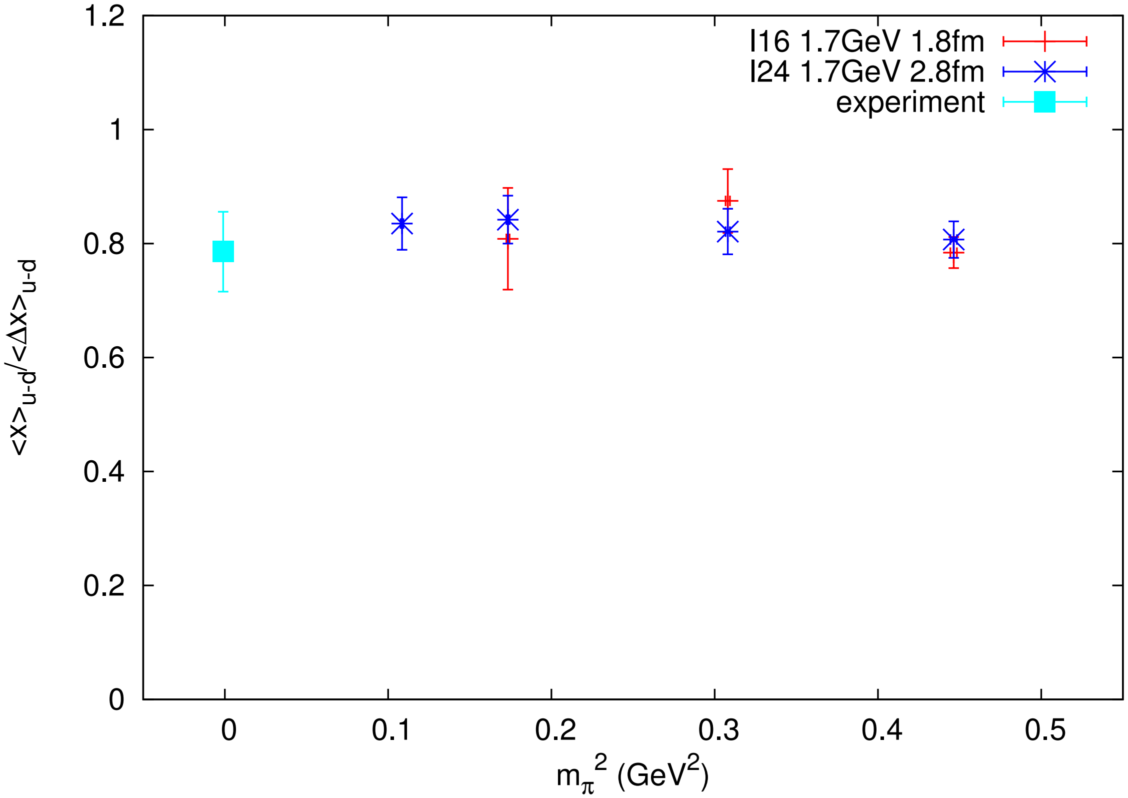
RBC-UKQCD collaborations: DWF fermions, $a \approx 0.12\,\mathrm{fm}$



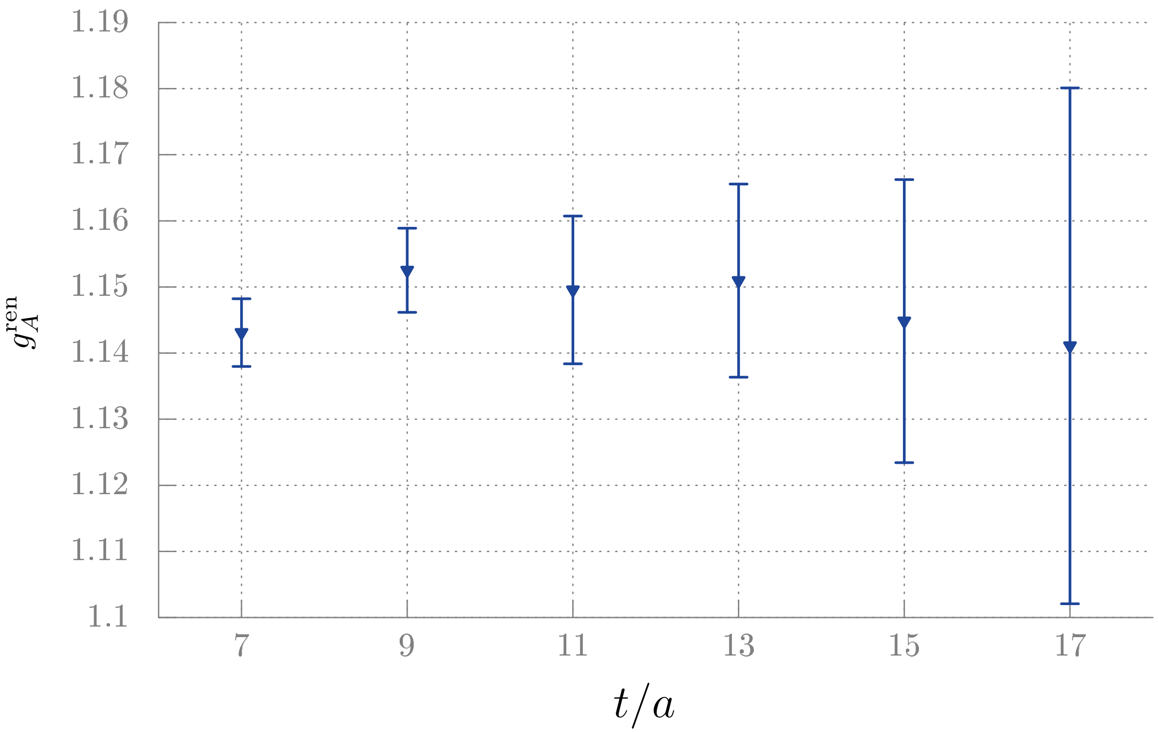
S. Ohta, arXiv:1102.0551 [hep-lat]

ratio $\dfrac{\langle x\rangle_{u-d}}{\langle x\rangle_{\Delta u-\Delta d}}$ “naturally renormalised” for DWF fermions: renormalisation factors cancel

$\langle x\rangle = v_2\,,\ \langle \Delta x\rangle = a_1/2$

no dependence on lattice size or pion mass seen

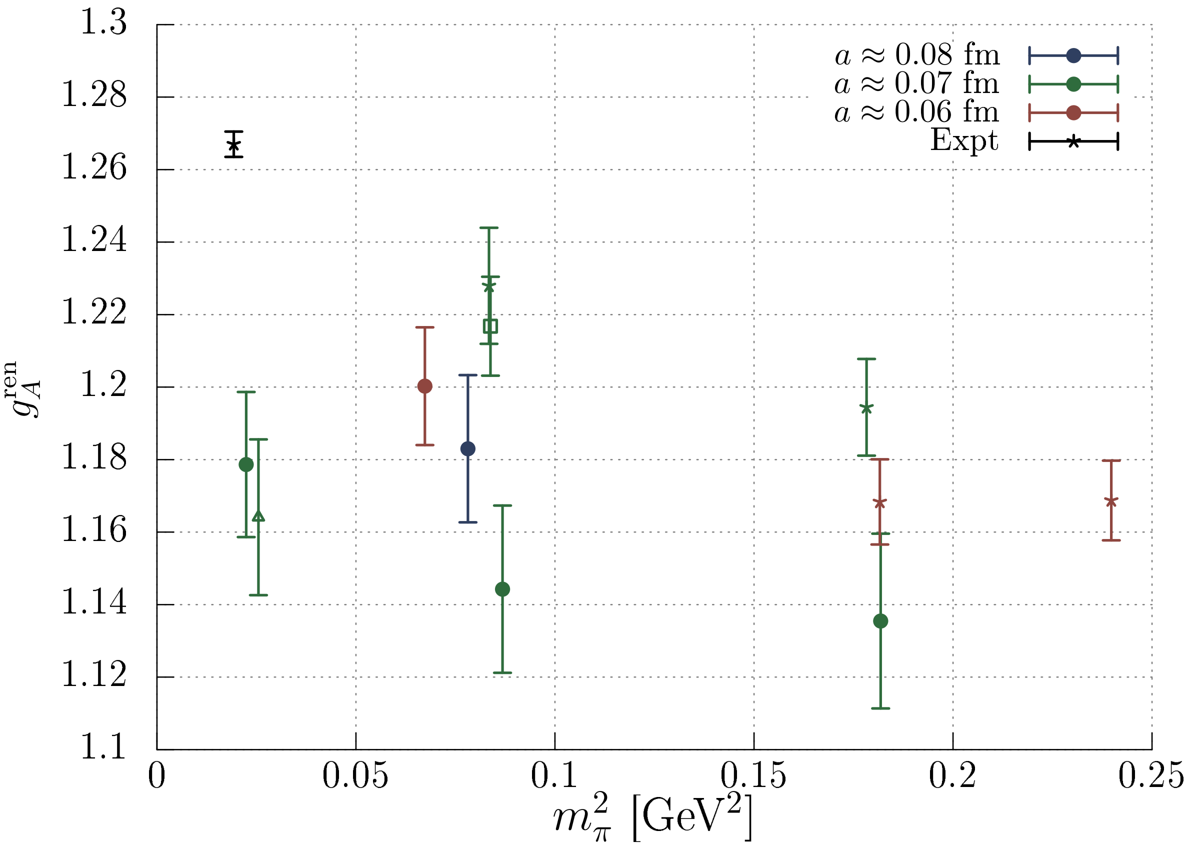
a look at g_A



$\beta = 5.29$, $\kappa = 0.13632$
$V = 32^3 \times 64$

$n_f = 2$
$a \approx 0.07\,\mathrm{fm}$
$m_\pi \approx 290\,\mathrm{MeV}$

excited states effectively suppressed in the ratio by smearing



box: $m_\pi L = 6.7$
crosses: $m_\pi L = 4.2 - 4.9$
filled circles: $m_\pi L = 3.4 - 3.8$
triangle: $m_\pi L = 2.8$

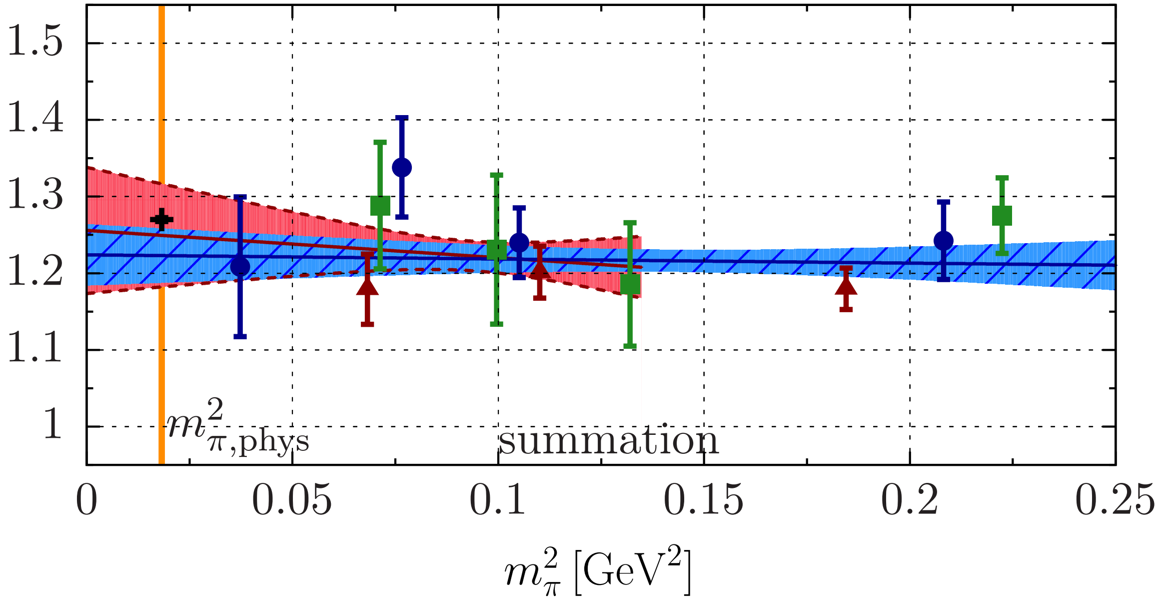
discretisation effects hardly significant for similar volumes

$m_\pi \approx 415\,\mathrm{MeV}$: increase by $\approx 5\%$ for $m_\pi L$ increasing from 3.7 to 4.9
$m_\pi \approx 290\,\mathrm{MeV}$: increase by $\approx 6\%$ for $m_\pi L$ increasing from 3.4 to 4.2 (6.7)
$m_\pi \approx 150\,\mathrm{MeV}$: no difference between $m_\pi L = 2.8$ and $m_\pi L = 3.5$

increase by 6% at $m_\pi \approx 150\,\mathrm{MeV}$ would bring us close to the experimental value. . .

However, the Mainz group obtains agreement with experiment.

B. Jäger et al., arXiv:1311.5804 (Lattice 2013)



green squares: $a = 0.079\,\mathrm{fm}$
blue circles: $a = 0.063\,\mathrm{fm}$
read triangles: $a = 0.050\,\mathrm{fm}$

black cross: experiment

blue band: linear fit (all masses)

red band: linear fit ($m_\pi < 360\,\mathrm{MeV}$)

summation method for suppressing excited states

nonperturbatively improved Wilson fermions ($n_f = 2$, CLS ensembles)

Electromagnetic form factors of the nucleon

experimentally: electromagnetic form factors from electron-nucleon scattering

theoretically: matrix elements of the electromagnetic (vector) current

$$J^\mu = \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d + \cdots$$

decomposition of the nucleon matrix element (in Minkowski space)

$${}_{\rm c}\langle p', s'|J^\mu|p, s\rangle_{\rm c} = \bar{U}(p', s')\left[\gamma_\mu F_1(q^2) + \mathrm{i}\sigma^{\mu\nu}\frac{q_\nu}{2m_N}F_2(q^2)\right]U(p, s)$$

form factors: Dirac Pauli

$q = p' - p$ with $Q^2 = -q^2 \geq 0$

values at $q^2 = 0$ for the proton:

$F_1^p(0) = 1$ vector current conserved

$F_2^p(0) = \mu^p - 1$ anomalous magnetic moment (in units of $e/(2m_N)$)

similarly for the neutron: $F_1^n(0) = 0$, $F_2^n(0) = \mu^n$

Sachs form factors:
$$G_e(q^2) = F_1(q^2) + \frac{q^2}{(2m_N)^2} F_2(q^2)$$
$$G_m(q^2) = F_1(q^2) + F_2(q^2)$$

experimental results (until recently) compatible with (dipole) fits:

$$G_e^p(q^2) \sim \frac{G_m^p(q^2)}{|\mu^p|} \sim \frac{G_m^n(q^2)}{|\mu^n|} \sim \left(1 - q^2/m_V^2\right)^{-2}$$
$$G_e^n(q^2) \sim 0 \qquad m_V \sim 0.82\,\mathrm{GeV} \qquad \mu^p \sim 2.79 \qquad \mu^n \sim -1.91$$

with respect to flavour SU(2): decomposition into isovector and isoscalar components

$$G_e^v(q^2) = G_e^p(q^2) - G_e^n(q^2) \qquad G_m^v(q^2) = G_m^p(q^2) - G_m^n(q^2) \qquad \text{etc.}$$

such that $G_m^v(0) = G_m^p(0) - G_m^n(0) = \mu^p - \mu^n = \kappa_v + 1$

$\kappa_v =$ isovector anomalous magnetic moment ~ 3.71

alternative (used in the actual simulations):

$$\langle \text{proton}|\frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d|\text{proton}\rangle - \langle \text{neutron}|\frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d|\text{neutron}\rangle = \langle \text{proton}|\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d|\text{proton}\rangle$$

$\to \kappa_v = \kappa_{u-d}$ etc.

in the following: QCDSF results for F_1, F_2, Dirac and Pauli radii, anomalous magnetic moment (in the isovector channel)

S. Collins et al., Phys. Rev. D84 (2011) 074507 [arXiv:1106.3580]

definition of the radii:

mean square radii ↔ slopes of the form factors at $Q^2 = 0$

$$\langle r^2 \rangle_i = -\frac{6}{F_i(0)} \frac{\mathrm{d}F_i(Q^2)}{\mathrm{d}Q^2}\bigg|_{Q^2=0}$$

normalisation of the anomalous magnetic moment:

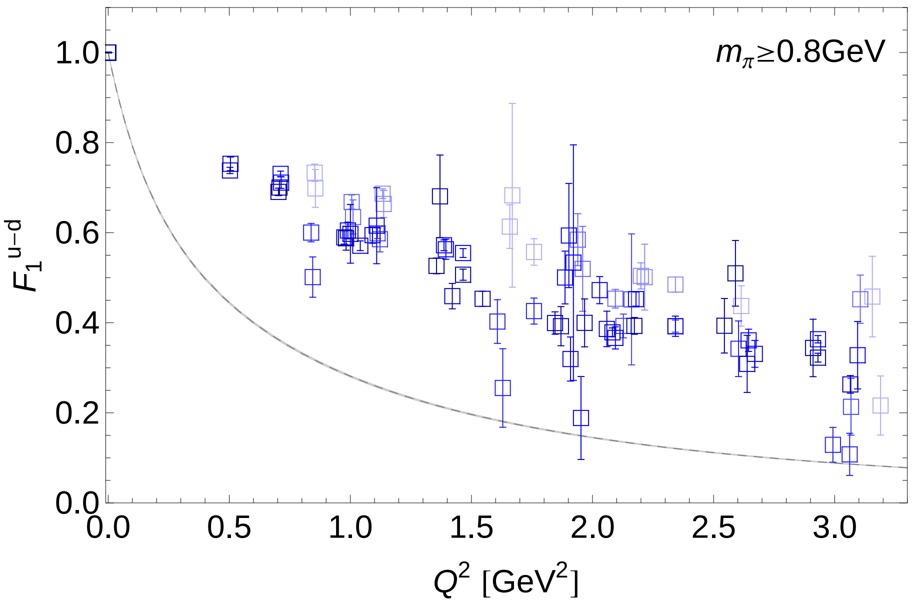
κ_v is the magnetic moment in units of the nuclear magneton

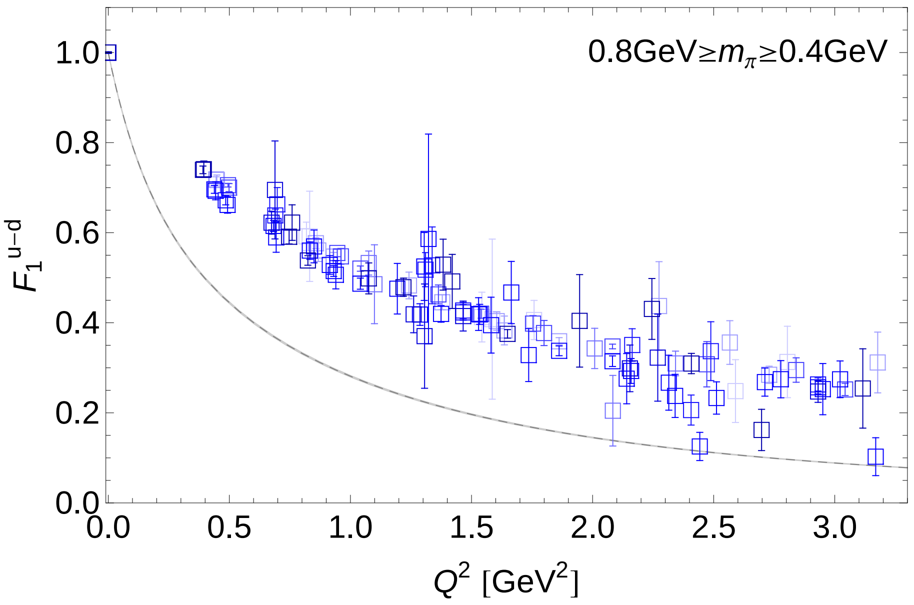
dimensionful: $\kappa_v \frac{e}{2m_N}$

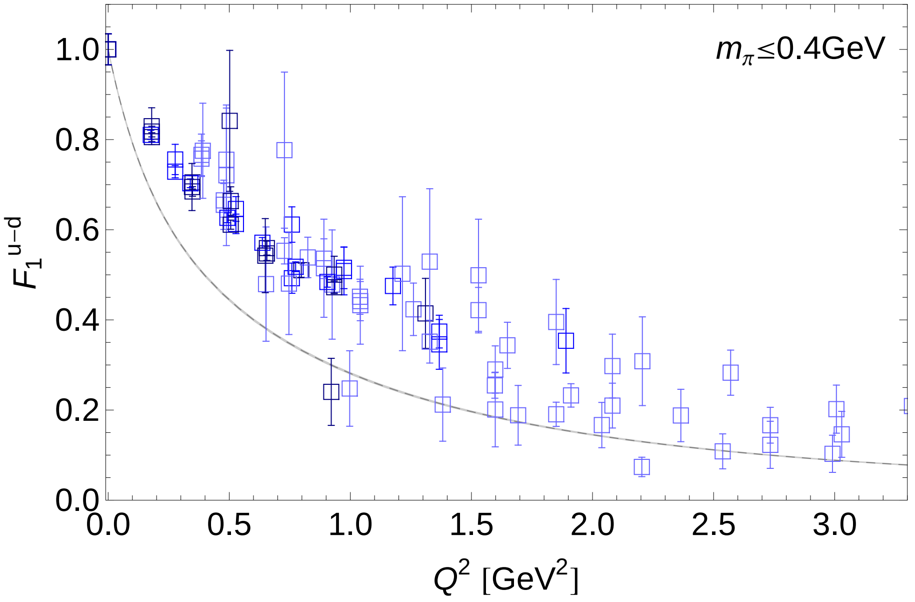
↖ nucleon mass for the quark mass considered

“normalised” κ_v^{norm} referring to the physical nuclear magneton:

$$\kappa_v^{\mathrm{norm}} \frac{e}{2m_N^{\mathrm{phys}}} = \kappa_v \frac{e}{2m_N} \quad \Rightarrow \quad \kappa_v^{\mathrm{norm}} = \kappa_v \frac{m_N^{\mathrm{phys}}}{m_N}$$



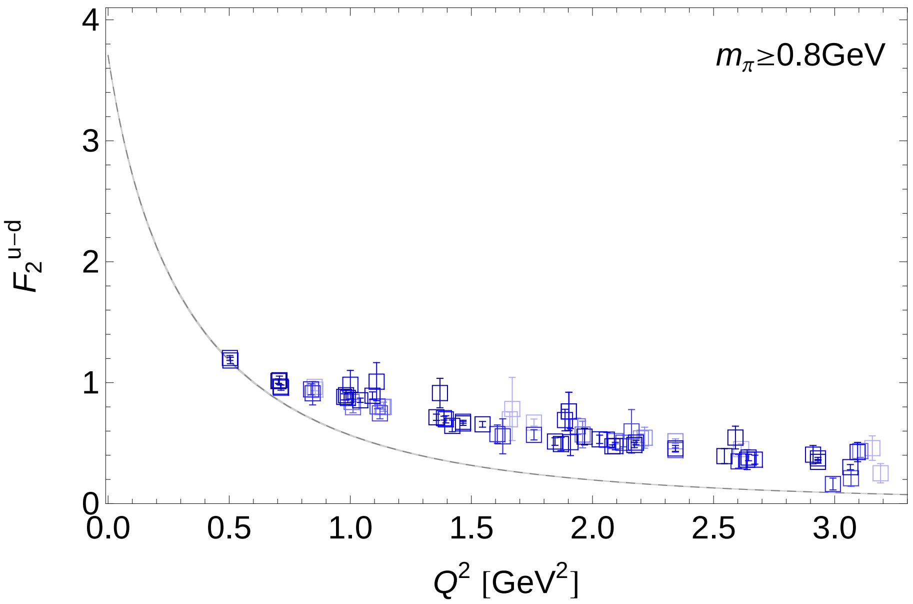


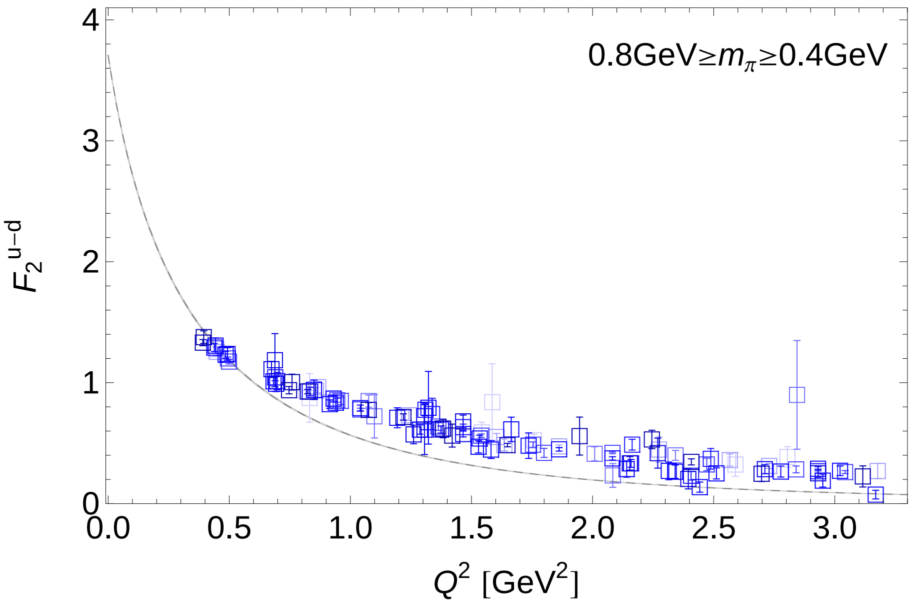


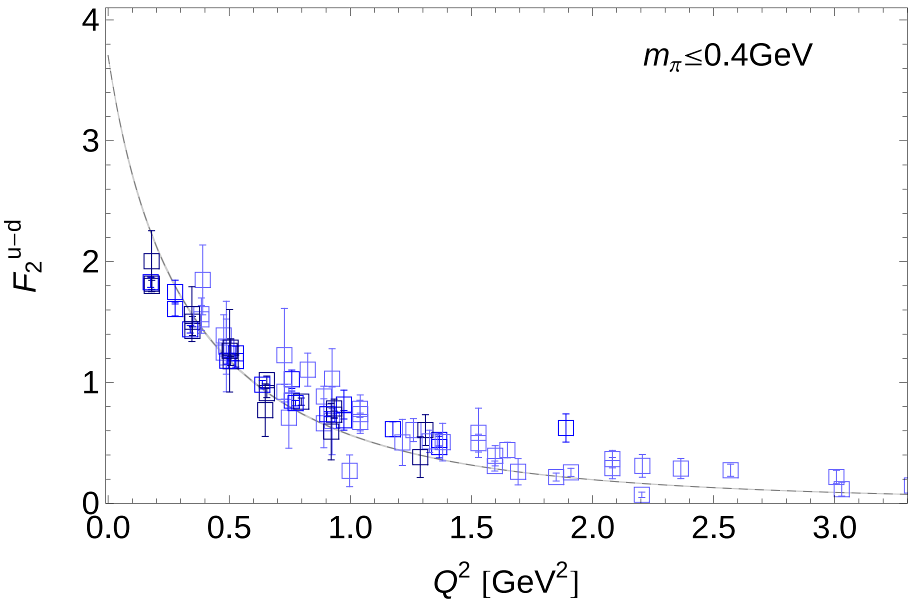
isovector Dirac form factor (QCDSF, $n_f = 2$)

darker colours $\leftrightarrow$ smaller pion masses

gray shaded band:
parametrisation of the experimental data





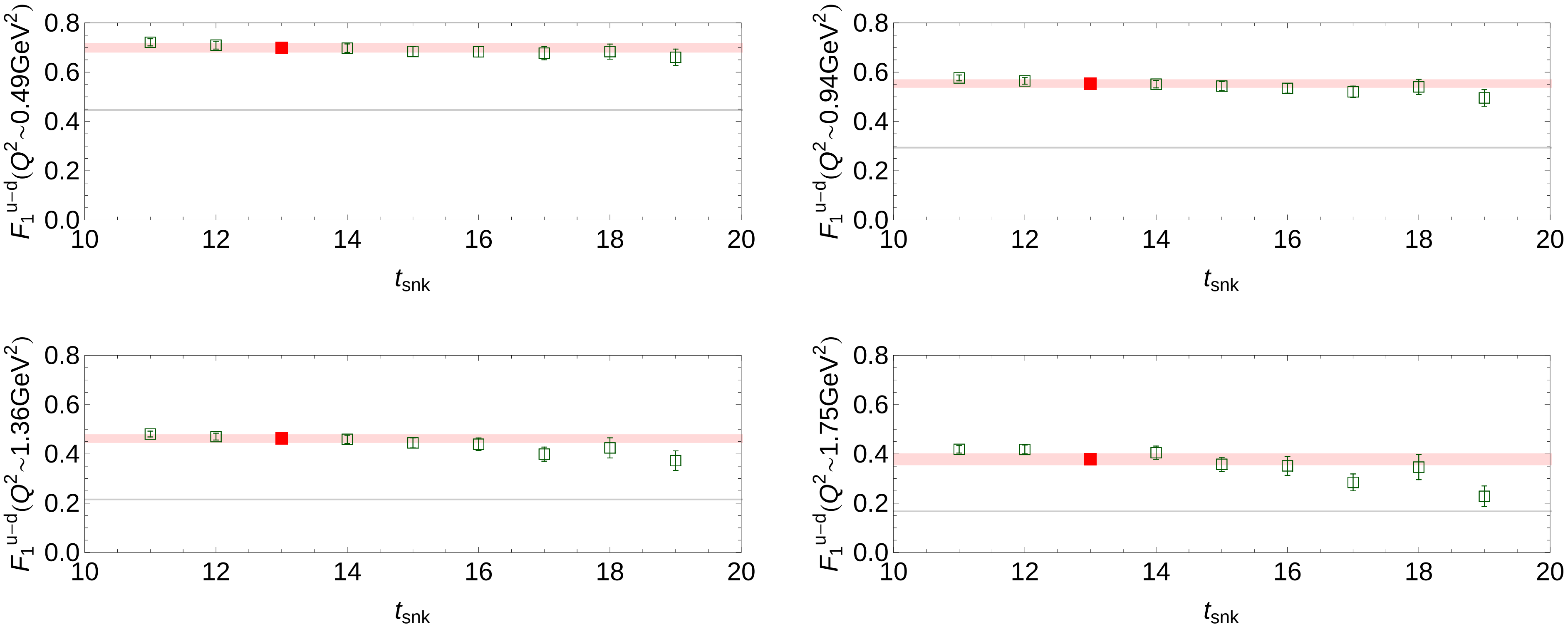


isovector Pauli form factor (QCDSF, $n_f = 2$)

darker colours $\leftrightarrow$ smaller pion masses

gray shaded band:
parametrisation of the experimental data

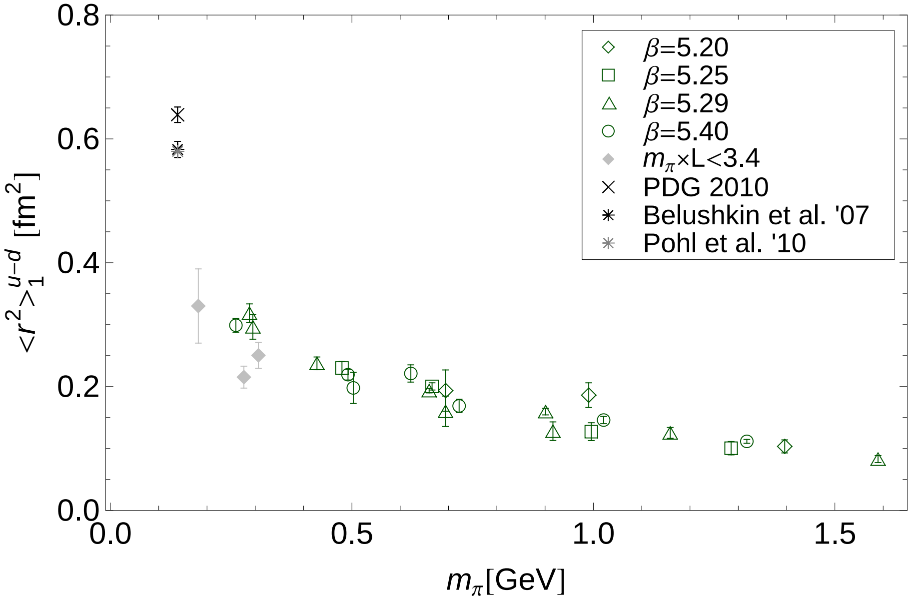
dependence of $F_1^{u-d}(Q^2)$ (Q^2 fixed) on the sink time $t_{\rm snk} \equiv t$ in the three-point function

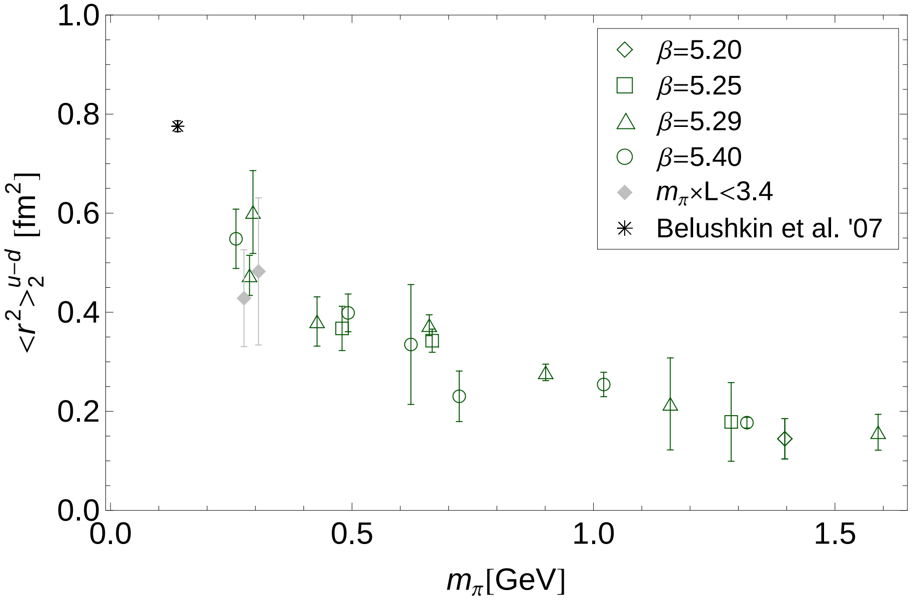


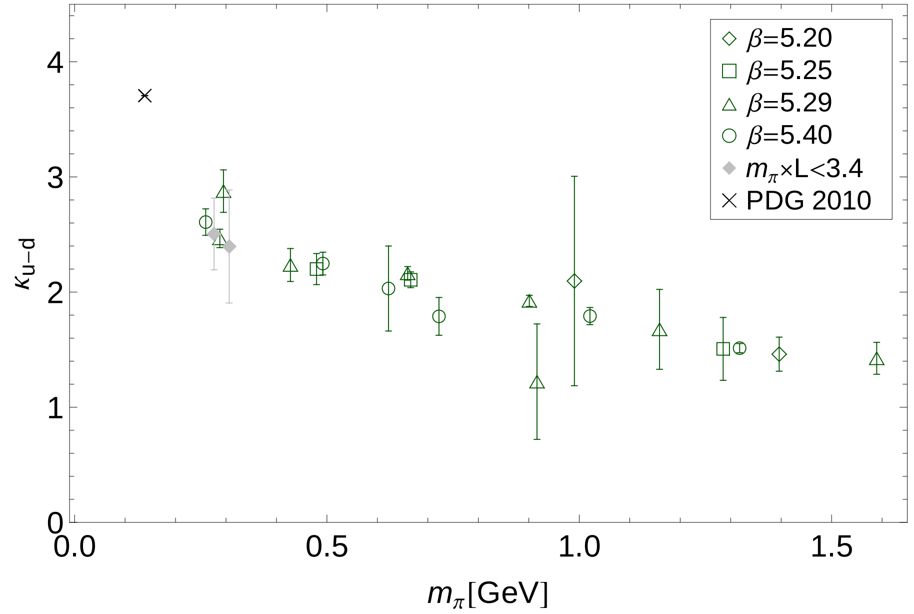
$\beta = 5.29$, $\kappa = 0.13590$ ($a = 0.071\,\mathrm{fm}$, $m_\pi = 660\,\mathrm{MeV}$)

standard choice: $t_{\rm snk}/a = 13$

thin gray shaded band: parametrisation of the experimental data







QCDSF, $n_f = 2$:

isovector Dirac radius (squared) $\langle r^2 \rangle_1^{u-d}$

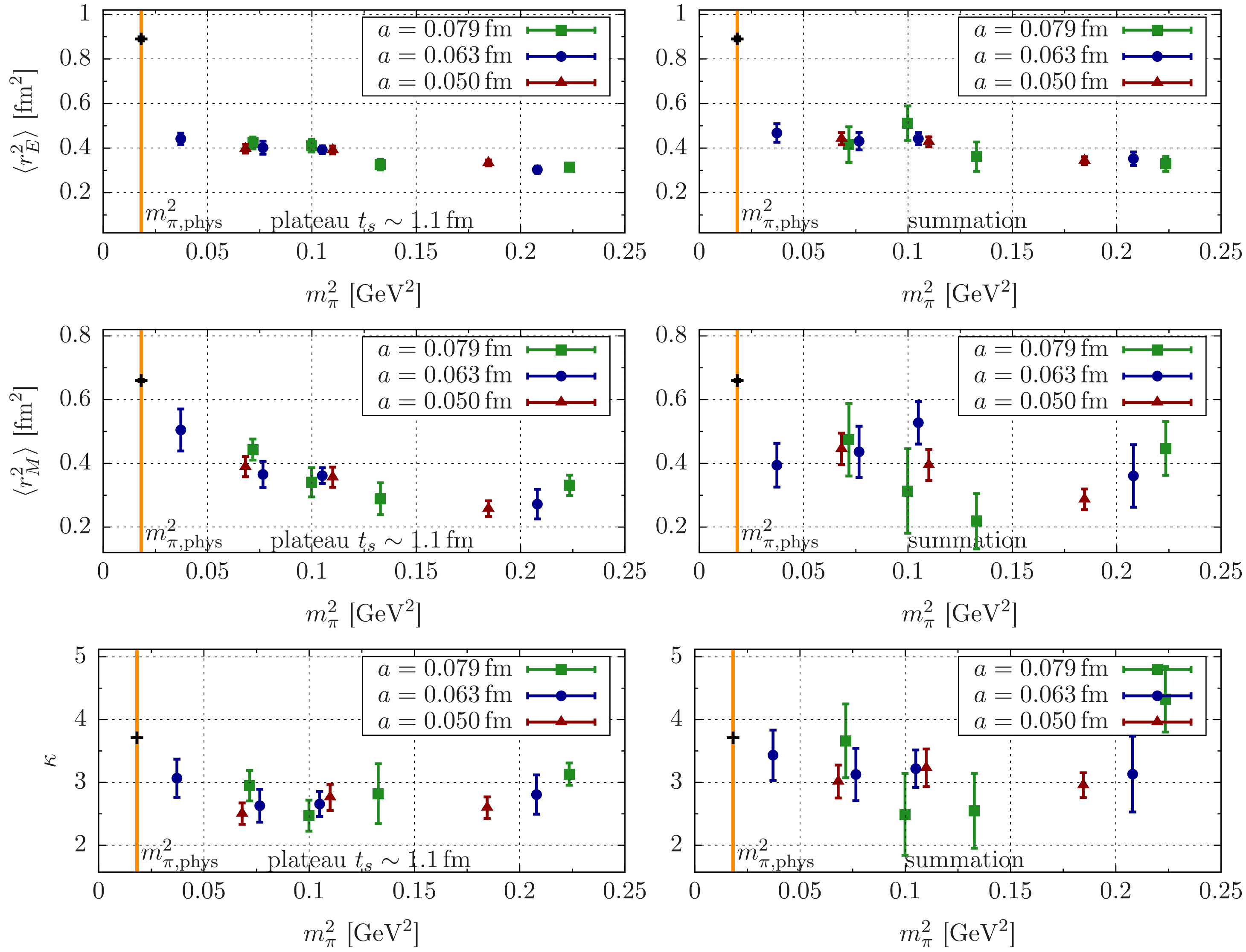
isovector Pauli radius (squared) $\langle r^2 \rangle_2^{u-d}$

isovector anomalous magnetic moment κ_{u-d}

Dirac radius: different experimental values!

results of the Mainz group comparing different methods for the suppression of excited states

B. Jäger et al., arXiv:1311.5804 (Lattice 2013)

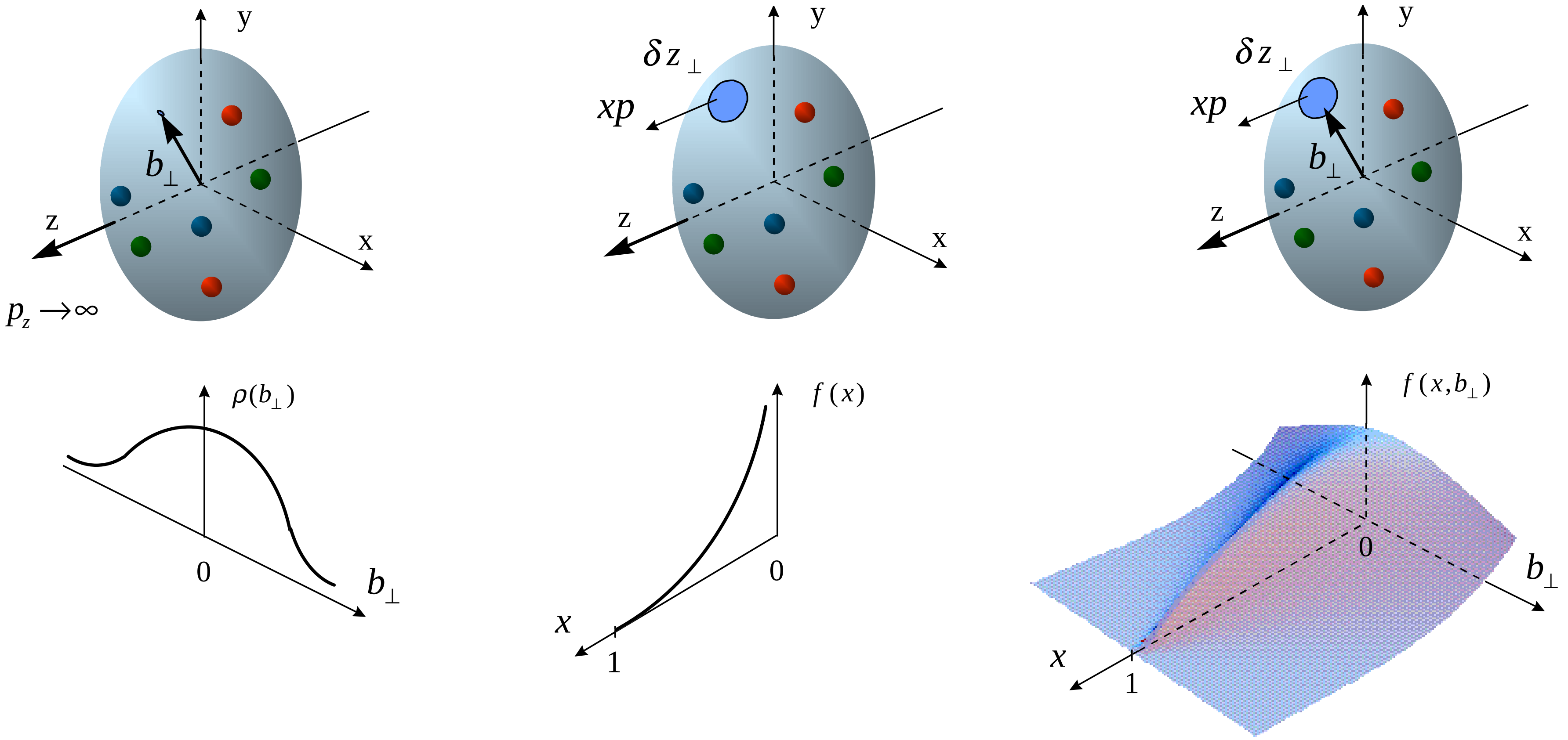


Generalised parton distributions (GPDs)

form factor

PDF

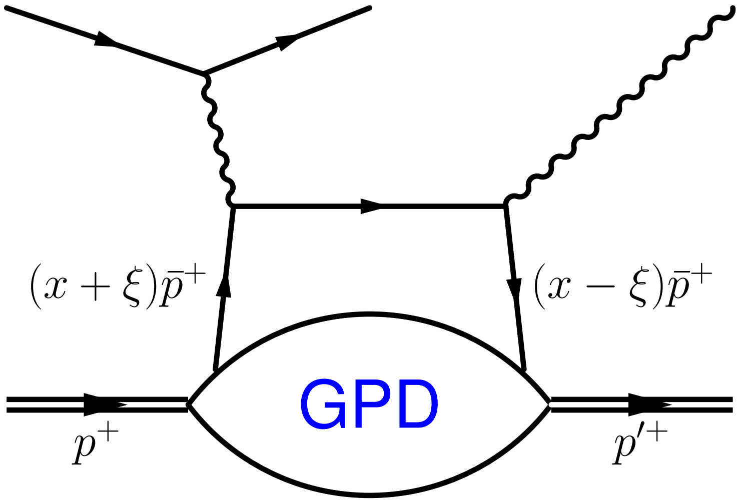
GPD at $\xi = 0$



pictures by Dieter Müller

for $\xi = 0$: probabilistic interpretation in impact parameter space (M. Burkardt)

Formal definition of GPDs



$$\int \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{\mathrm{i}\lambda x} {}_{\mathrm{c}}\langle p', s' | \bar{q}(-\tfrac{1}{2}\lambda n)\, \not{n}\, \overset{\text{Wilson line}}{\overset{\downarrow}{\mathcal{U}}} q(\tfrac{1}{2}\lambda n) | p, s\rangle_{\mathrm{c}}$$

$$= H_q(x,\xi,t)\bar{U}(p',s')\,\not{n}U(p,s)$$

$$+\, E_q(x,\xi,t)\bar{U}(p',s')\frac{\mathrm{i}\sigma^{\mu\nu}n_\mu\Delta_\nu}{2m_N}U(p,s)$$

$\bar{p} = \frac{1}{2}(p'+p)$, $\Delta = p' - p$, n: light-like vector with $\bar{p}\cdot n = 1$, $\xi = -n\cdot\Delta/2$, $t = \Delta^2$
(Minkowski space, dependence on renormalisation scale suppressed)

ξ: called skewness

special cases:

ordinary parton distributions

$$H_q(x,0,0) = \begin{cases} q(x) & \text{for } x > 0 \\ -\bar{q}(-x) & \text{for } x < 0 \end{cases}$$

electromagnetic form factors

$$\int_{-1}^{1} \mathrm{d}x\, H_q(x,\xi,t) = F_1^q(t)$$

$$\int_{-1}^{1} \mathrm{d}x\, E_q(x,\xi,t) = F_2^q(t)$$

probabilistic interpretation in impact parameter space (M. Burkardt)

$\xi = 0$: momentum transfer purely transverse $\Delta = \Delta_\perp$

$$q(x, \mathbf{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}\mathbf{b}_\perp \cdot \boldsymbol{\Delta}_\perp} H_q(x, 0, -\boldsymbol{\Delta}_\perp^2)$$

with $\int \mathrm{d}^2 b_\perp q(x, \mathbf{b}_\perp) = q(x)$ expect: $q(x, \mathbf{b}_\perp) \overset{x\to 1}{\to} \delta(\mathbf{b}_\perp)$

Note: momentum fraction of the quarks fixed

→ longitudinal position undetermined (Heisenberg)

→ distribution in impact parameter space meaningful

experimental access: DVCS (deeply virtual Compton scattering): $ep \to ep\,\gamma$

meson electroproduction: $ep \to ep\,\pi, \rho, \omega, \ldots$

however: direct (model-independent) extraction from experimental data difficult (impossible?)

→ additional input highly welcome, e.g., from the lattice moments w.r.t. x in terms of generalised form factors (GFFs) A, B, C:

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} H_q(x,\xi,t) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} A^q_{n,2i}(t)(-2\xi)^{2i} + \mathrm{Mod}(n+1,2) C^q_n(t)(-2\xi)^n$$

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} E_q(x,\xi,t) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} B^q_{n,2i}(t)(-2\xi)^{2i} - \mathrm{Mod}(n+1,2) C^q_n(t)(-2\xi)^n$$

GFFs from matrix elements of local (twist 2) operators (momentum transfer $\Delta = p' - p \neq 0$)

$$\begin{aligned}
{}_{\mathrm{c}}\langle p', s'|\mathcal{O}^q_{(\mu_1\cdots\mu_n)}|p,s\rangle_{\mathrm{c}} &= \bar{U}(p',s')\gamma_{(\mu_1} U(p,s) \sum_{i=0}^{\left[\frac{n-1}{2}\right]} A^q_{n,2i}(t)\Delta_{\mu_2}\cdots\Delta_{\mu_{2i+1}}\bar{p}_{\mu_{2i+2}}\cdots\bar{p}_{\mu_n)} \\
&- \frac{\bar{U}(p',s')\mathrm{i}\Delta^\alpha\sigma_{\alpha(\mu_1}U(p,s)}{2m_N} \sum_{i=0}^{\left[\frac{n-1}{2}\right]} B^q_{n,2i}(t)\Delta_{\mu_2}\cdots\Delta_{\mu_{2i+1}}\bar{p}_{\mu_{2i+2}}\cdots\bar{p}_{\mu_n)} \\
&+ C^q_n(t)\mathrm{Mod}(n+1,2)\frac{1}{m_N}\bar{U}(p',s')U(p,s)\Delta_{(\mu_1}\cdots\Delta_{\mu_n)}
\end{aligned}$$

with $\mathcal{O}^q_{\mu_1\cdots\mu_n} = (\mathrm{i}/2)^{n-1}\ \bar{q}\gamma_{\mu_1}\overset{\leftrightarrow}{D}_{\mu_2}\cdots\overset{\leftrightarrow}{D}_{\mu_n} q$

analogous equations in the polarised case:

$$\mathcal{O}^{q,5}_{\mu_1\cdots\mu_n} = (\mathrm{i}/2)^{n-1}\ \bar{q}\gamma_{\mu_1}\overset{\leftrightarrow}{D}_{\mu_2}\cdots\overset{\leftrightarrow}{D}_{\mu_n}\gamma_5 q$$

$$\begin{aligned}{}_{\mathrm{c}}\langle p',s'|\mathcal{O}^{q,5}_{(\mu_1\cdots\mu_n)}|p,s\rangle_{\mathrm{c}} &= -\bar{U}(p',s')\gamma_5\gamma_{(\mu_1}U(p,s)\sum_{i=0}^{\left[\frac{n-1}{2}\right]}\tilde{A}^q_{n,2i}(t)\Delta_{\mu_2}\cdots\Delta_{\mu_{2i+1}}\overline{p}_{\mu_{2i+2}}\cdots\overline{p}_{\mu_n)}\\ &-\frac{\bar{U}(p',s')\gamma_5 U(p,s)}{2m_N}\sum_{i=0}^{\left[\frac{n-1}{2}\right]}\tilde{B}^q_{n,2i}(t)\Delta_{(\mu_1}\cdots\Delta_{\mu_{2i+1}}\overline{p}_{\mu_{2i+2}}\cdots\overline{p}_{\mu_n)}\end{aligned}$$

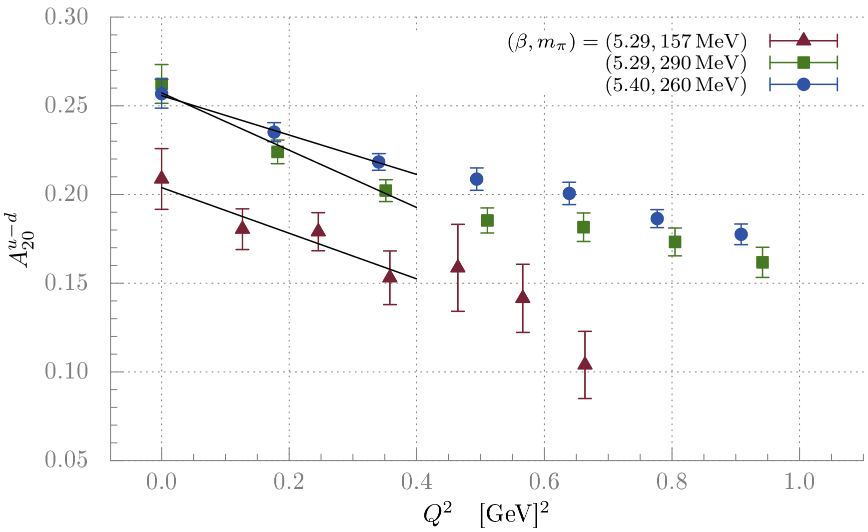
$$\int_{-1}^{1}\mathrm{d}x\,x^{n-1}\tilde{H}_q(x,\xi,t) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]}\tilde{A}^q_{n,2i}(t)(-2\xi)^{2i} \qquad \int_{-1}^{1}\mathrm{d}x\,x^{n-1}\tilde{E}_q(x,\xi,t) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]}\tilde{B}^q_{n,2i}(t)(-2\xi)^{2i}$$

similarly for $(\mathrm{i}/2)^n\ \bar{q}\mathrm{i}\sigma_{\mu\nu}\overset{\leftrightarrow}{D}_{\mu_1}\cdots\overset{\leftrightarrow}{D}_{\mu_n}q$

similar towers of gluon operators

particularly easy to manage: flavour non-singlet operators
(quark-line) disconnected contributions and gluonic operators drop out

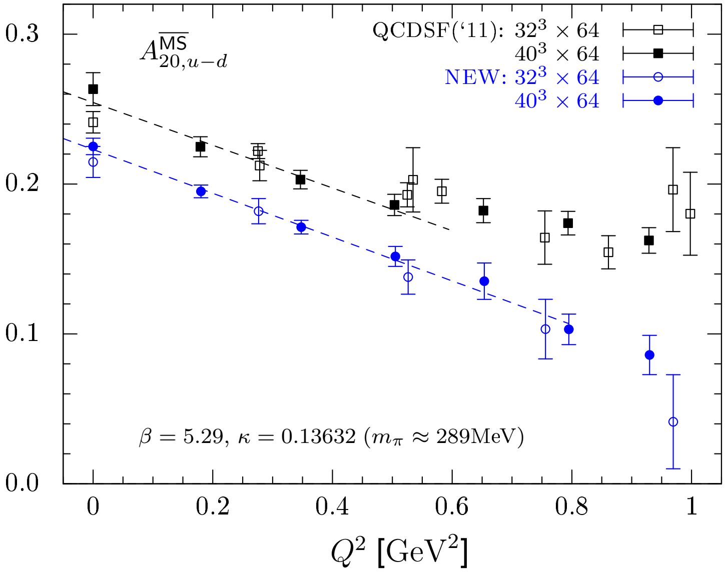
Lattice results for GPDs: distributions in impact parameter space



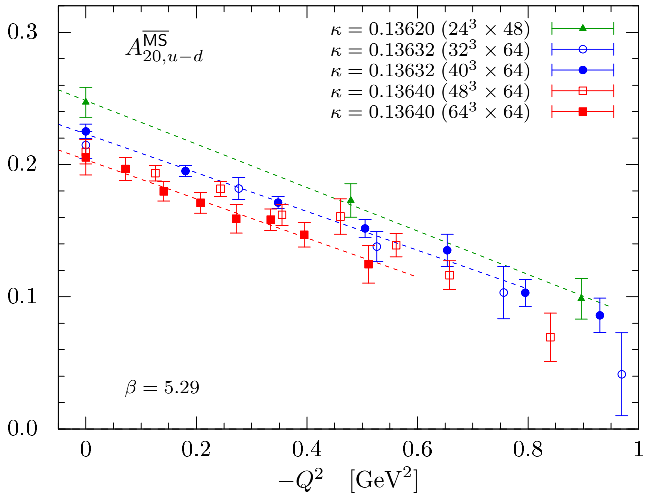
extrapolations of $A_{20}^{u-d}(Q^2)$ to $Q = 0$

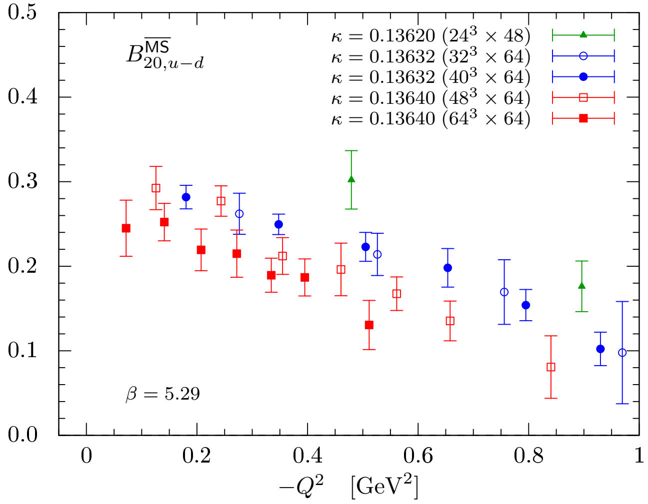
points at $Q = 0$ determined directly from the forward matrix element

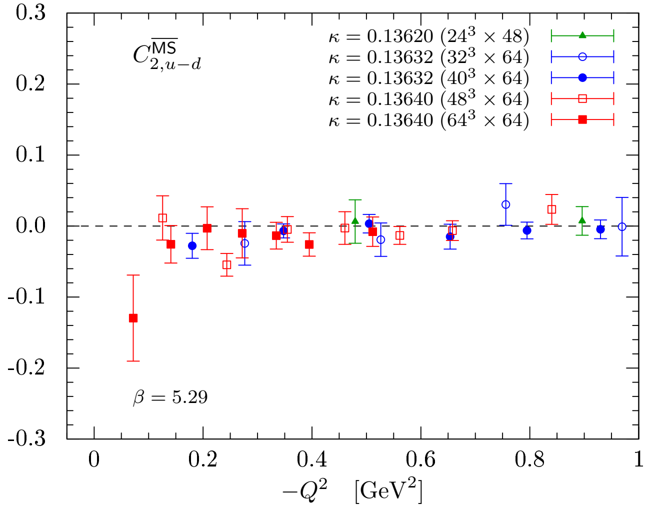
G.S. Bali et al.,
Phys. Rev. D86 (2012) 054504



black squares: Jacobi smearing (old data)
blue circles: improved smearing
(preliminary data)
A. Sternbeck, Lattice 2013 (arXiv:1312.0828)







GFFs (preliminary) (improved smearing)

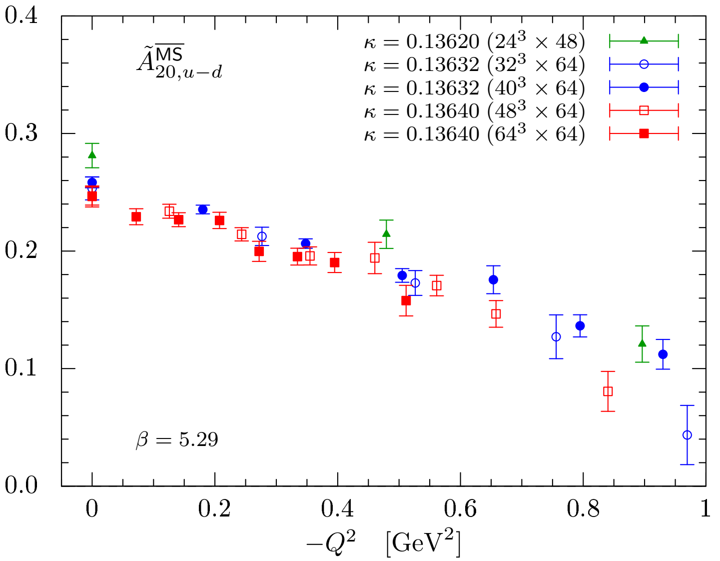
red squares: $m_\pi = 151\,\text{MeV}$
blue circles: $m_\pi = 289\,\text{MeV}$
green diamonds: $m_\pi = 429\,\text{MeV}$

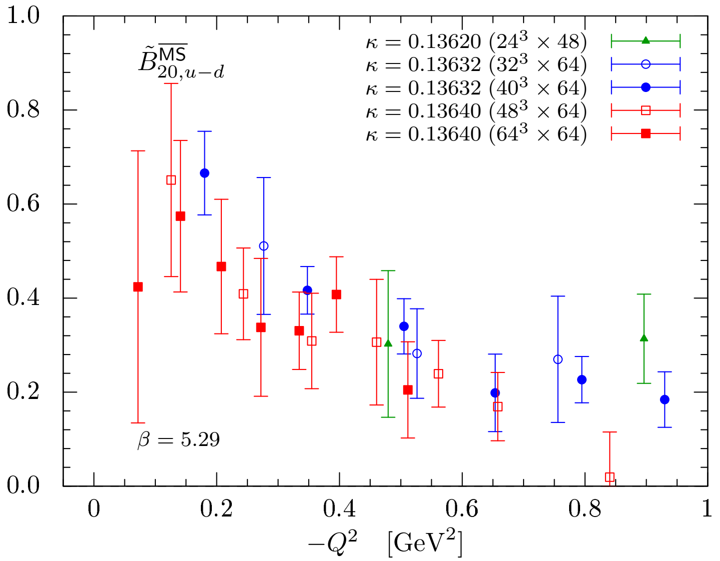
A. Sternbeck, Lattice 2013 (arXiv:1312.0828)

$$\int_{-1}^{1} \mathrm{d}x\, x H_q(x,\xi,t) = A^q_{20}(t) + 4\xi^2 C^q_2(t)$$

$$\int_{-1}^{1} \mathrm{d}x\, x E_q(x,\xi,t) = B^q_{20}(t) - 4\xi^2 C^q_2(t)$$

A. Sternbeck, Lattice 2013 (arXiv:1312.0828)





$$
{}_{\mathrm{c}}\langle p',s'|\mathcal{O}^{q,5}_{(\mu_1\cdots\mu_n)}|p,s\rangle_{\mathrm{c}} = -\bar{U}(p',s')\gamma_5\gamma_{(\mu_1}U(p,s)\sum_{i=0}^{\left[\frac{n-1}{2}\right]}\tilde{A}^q_{n,2i}(t)\Delta_{\mu_2}\cdots\Delta_{\mu_{2i+1}}\bar{p}_{\mu_{2i+2}}\cdots\bar{p}_{\mu_n)}
$$

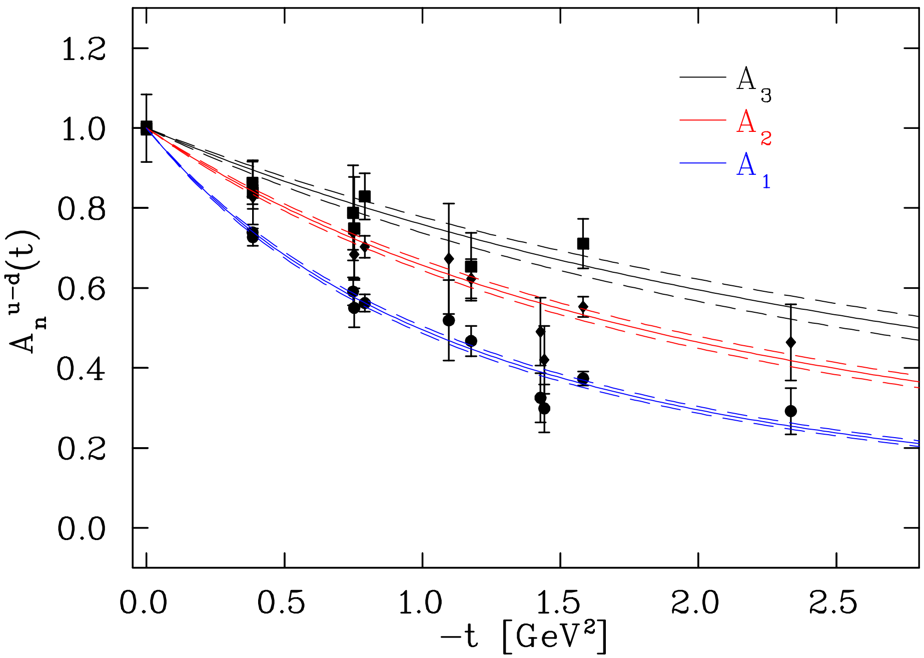
$$
-\frac{\bar{U}(p',s')\gamma_5 U(p,s)}{2m_N}\sum_{i=0}^{\left[\frac{n-1}{2}\right]}\tilde{B}^q_{n,2i}(t)\Delta_{(\mu_1}\cdots\Delta_{\mu_{2i+1}}\bar{p}_{\mu_{2i+2}}\cdots\bar{p}_{\mu_n)}
$$

with $\mathcal{O}^{q,5}_{\mu_1\cdots\mu_n} = (\mathrm{i}/2)^{n-1}\ \bar{q}\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}\gamma_5 q$

$$
\int_{-1}^{1}\mathrm{d}x\, x^{n-1}\tilde{H}_q(x,\xi,t) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]}\tilde{A}^q_{n,2i}(t)(-2\xi)^{2i}
$$

$$
\int_{-1}^{1}\mathrm{d}x\, x^{n-1}\tilde{E}_q(x,\xi,t) = \sum_{i=0}^{\left[\frac{n-1}{2}\right]}\tilde{B}^q_{n,2i}(t)(-2\xi)^{2i}
$$

GFFs $A_{10}^{u-d} = F_1^{u-d}(t)$, A_{20}^{u-d}, A_{30}^{u-d} (non-singlet), normalised to unity at $t = 0$



$\beta = 5.4$, $\kappa = 0.1350$, $24^3 \times 48$ lattice
$a = 0.060\,\mathrm{fm}$, $m_\pi = 1320\,\mathrm{MeV}$

dipole fit:

$$A_{n0}(t) = \frac{A_{n0}(0)}{(1 - t/M_n^2)^2} = \frac{\langle x^{n-1} \rangle}{(1 - t/M_n^2)^2}$$

form factor $A_{n0}(t)$ flattens as n grows
↔ dipole mass M_n grows with n

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} H_q(x, \xi = 0, t) = A_{n0}^q(t)$$

H_q (as a function of t) becomes wider as x grows

$$q(x, \mathbf{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}\mathbf{b}_\perp \cdot \boldsymbol{\Delta}_\perp} H_q(x, 0, -\boldsymbol{\Delta}_\perp^2)$$

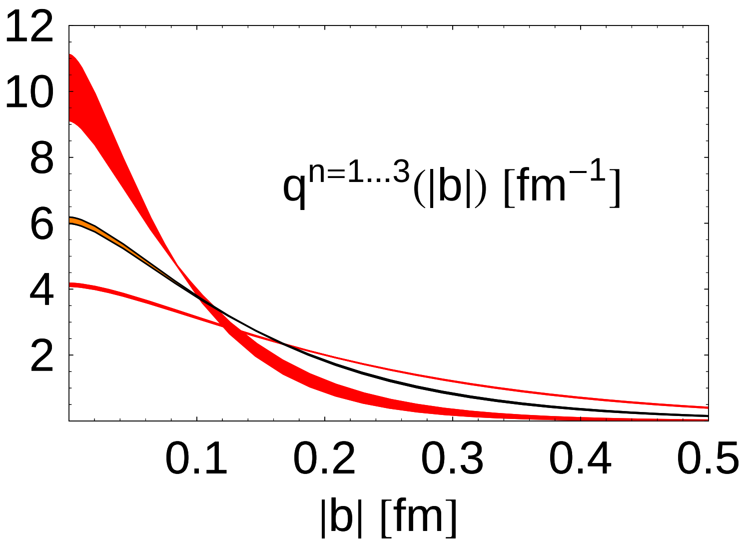
q (as a function of $\mathbf{b}_\perp$) becomes narrower as x grows (as expected) lowest three moments of $H(x, \xi = 0, t)$ and $\tilde{H}(x, \xi = 0, t)$

Fourier transform to impact parameter space

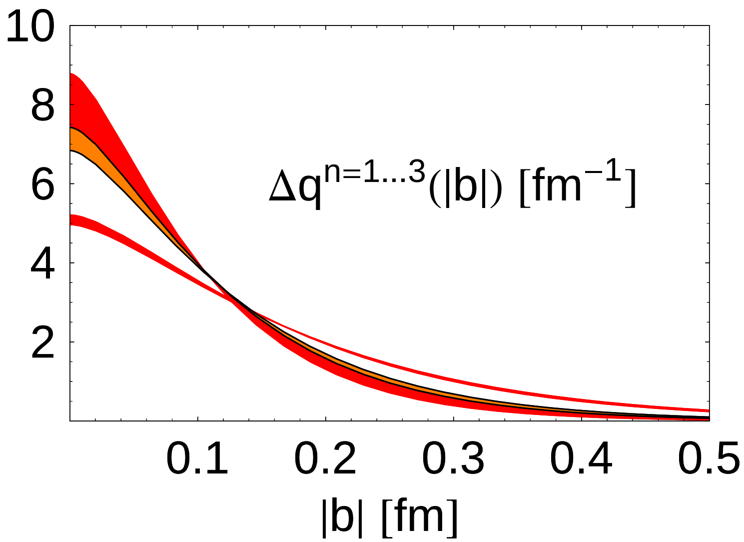
with the help of the dipole ansatz extrapolated linearly to the chiral limit:

$$\int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}\mathbf{b}_\perp\cdot\boldsymbol{\Delta}_\perp} \int_{-1}^{1} \mathrm{d}x\, x^{n-1} H_q(x, 0, -\boldsymbol{\Delta}_\perp^2)$$

$$= \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}\mathbf{b}_\perp\cdot\boldsymbol{\Delta}_\perp} \quad \frac{A^q_{n0}(0)}{(1 + \boldsymbol{\Delta}_\perp^2/M_n^2)^2} \quad = \quad \int_{-1}^{1} \mathrm{d}x\, x^{n-1} q(x, \mathbf{b}_\perp)$$



H



$\tilde{H}$

larger n corresponds to a narrower distribution

flavour $u - d$

M. G. et al., Eur. Phys. J. A32 (2007) 445 [hep-lat/0609001]

Lattice results for GPDs: transverse spin structure

what about the GPDs (GFFs) connected with the tensor operators $(\mathrm{i}/2)^{n-1}\ \bar{q}\mathrm{i}\sigma_{\lambda\mu_1}\overset{\leftrightarrow}{D}_{\mu_2}\cdots\overset{\leftrightarrow}{D}_{\mu_n}q$?

together with the vector operators $(\mathrm{i}/2)^{n-1}\ \bar{q}\gamma_{\mu_1}\overset{\leftrightarrow}{D}_{\mu_2}\cdots\overset{\leftrightarrow}{D}_{\mu_n}q$

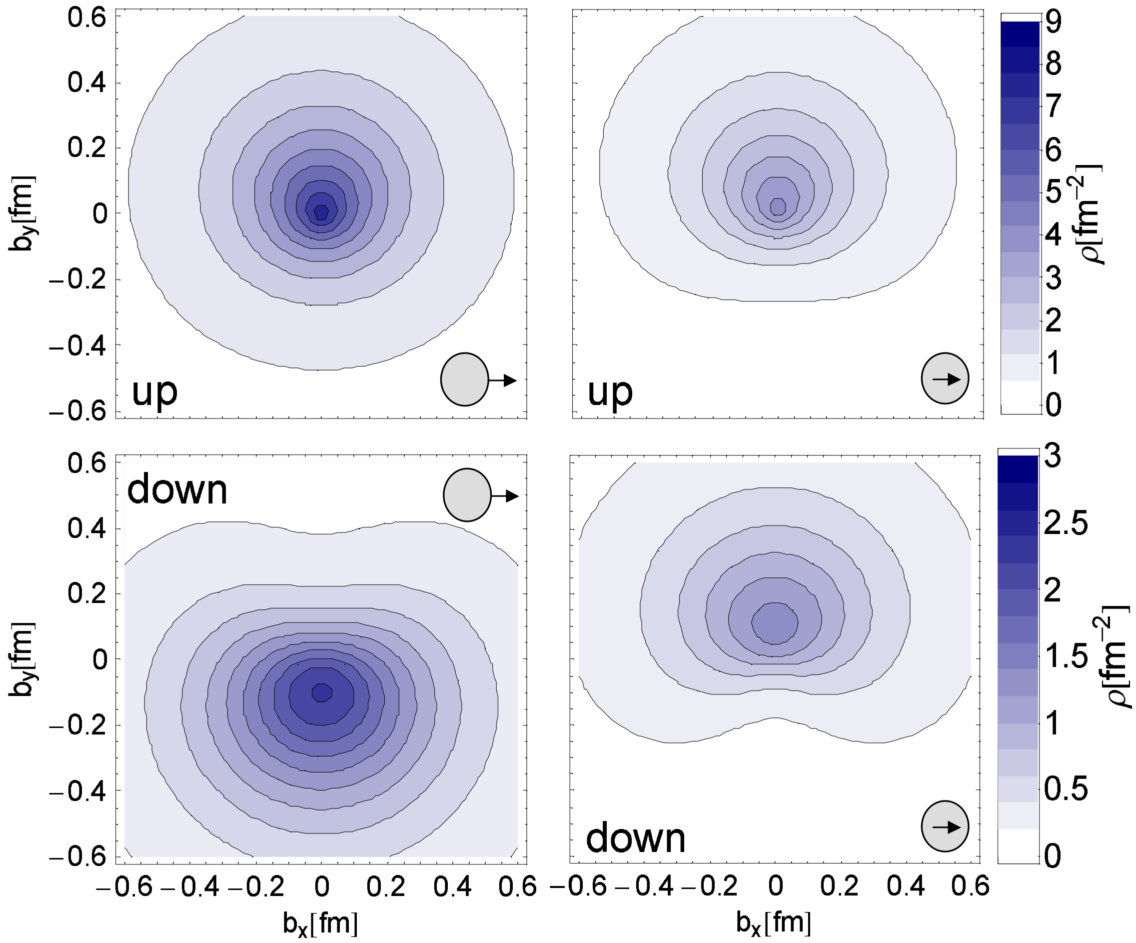
→ (moments of) the density of transversely polarised quarks in a transversely polarised nucleon in impact parameter space

M. Diehl, Ph. Hägler, Eur. Phys. J. C44 (2005) 87

$$\begin{aligned}&\int_{-1}^{1}\mathrm{d}x\,x^{n-1}\rho(x,\mathbf{b}_\perp,\mathbf{s}_\perp,\mathbf{S}_\perp)\\&=\frac{1}{2}\left\{A_{n0}(b_\perp^2)+s_\perp^iS_\perp^i\left(A_{Tn0}(b_\perp^2)-\frac{1}{4m_N^2}\Delta_{b_\perp}\widetilde{A}_{Tn0}(b_\perp^2)\right)\right.\\&\left.+\frac{b_\perp^j\epsilon^{ji}}{m_N}\left(S_\perp^iB'_{n0}(b_\perp^2)+s_\perp^i\overline{B}'_{Tn0}(b_\perp^2)\right)+s_\perp^i(2b_\perp^ib_\perp^j-b_\perp^2\delta^{ij})S_\perp^j\frac{1}{m_N^2}\widetilde{A}''_{Tn0}(b_\perp^2)\right\}\end{aligned}$$

$\mathbf{s}_\perp$: transverse spin of the quark $\mathbf{S}_\perp$: transverse spin of the nucleon

unpolarised quark in a ⊥ polarised nucleon:	only contributions from vector operators
⊥ polarised quark in an unpolarised nucleon:	also contributions from tensor operators



QCDSF/UKQCD,
PRL 98 (2007) 222001

(gen.) dipole parametrisation
+ linear chiral extrapolation

x^0 moment $(q - \bar{q})$
quark spins ↔ inner arrows
nucleon spins ↔ outer arrows

transversely polarised quarks
in an unpolarised nucleon:
distortion in positive y-direction
for u and d quarks

$\downarrow ?$

unpolarised quark
in a polarised nucleon:
distortion $\overset{?}{\to}$ Sivers effect

sizable negative Boer-Mulders function for u and d quarks
(correlation of quark $\perp$ momentum and the $\perp$ quark spin)
M. Burkardt, Phys. Rev. D72 (2005) 094020

Lattice results for GPDs: quark angular momentum in the nucleon

Ji’s sum rule for the total angular momentum of quarks of flavour q in the nucleon:

$$J_q = \frac{1}{2}\int_{-1}^{1} \mathrm{d}x\, x\left(H_q(x,\xi,0) + E_q(x,\xi,0)\right) = \frac{1}{2}\left(A^q_{20}(t=0) + B^q_{20}(t=0)\right)$$

quark spin contribution to the nucleon spin:

$$S_q = \frac{1}{2}\int_{-1}^{1} \mathrm{d}x\, \tilde{H}_q(x,\xi,0) = \frac{1}{2}\tilde{A}^q_{10}(t=0) = \frac{1}{2}\Delta q$$

quark orbital angular momentum: $L_q = J_q - S_q$

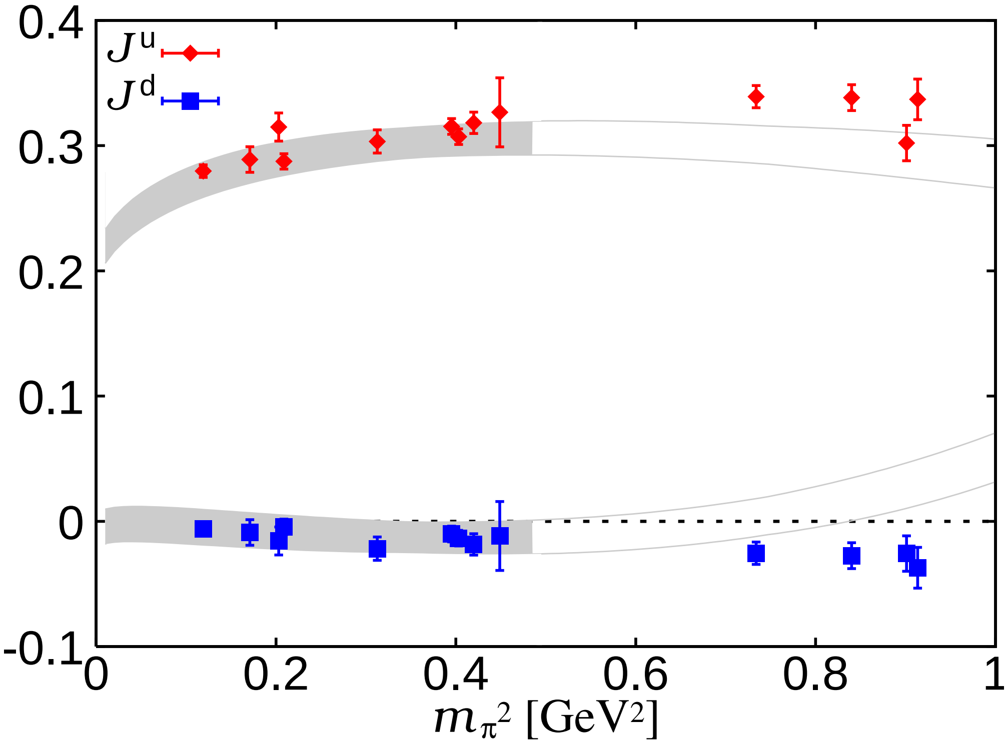
decomposition controversial!

difficult problem:

- disconnected contributions (not yet included)
- $B^q_{20}(t=0)$ requires an extrapolation from $t \neq 0$ to the forward limit
- chiral extrapolation and finite size corrections

for GFFs at vanishing momentum transfer t

- heavy-baryon chiral perturbation theory
  e.g., M. Diehl, A. Manashov, A. Schäfer, Eur. Phys. J. A31 (2007) 335
- covariant chiral perturbation theory in the baryon sector
  e.g., M. Dorati, T.A. Gail, T.R. Hemmert, Nucl. Phys. A798 (2008) 96

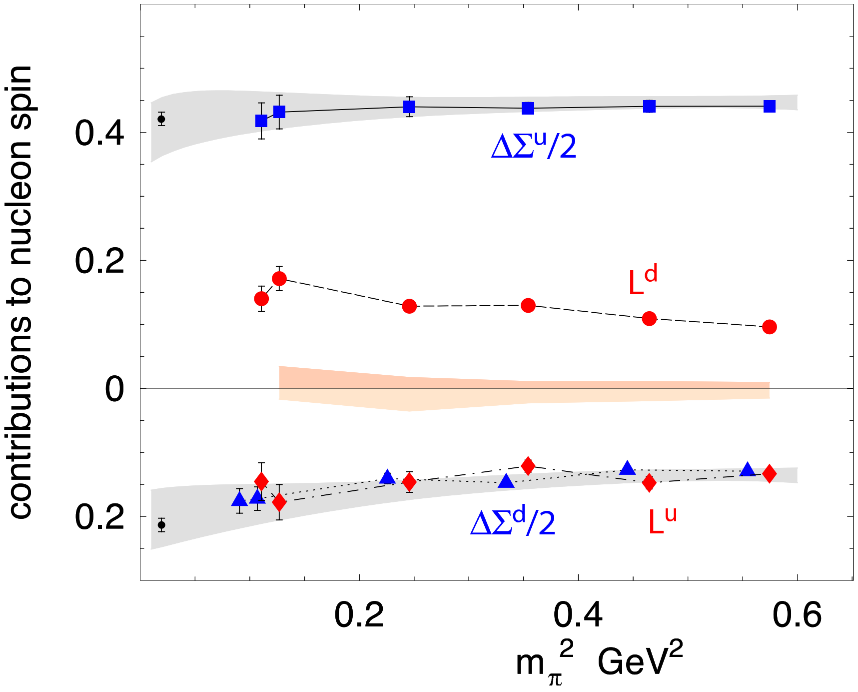


total angular momentum of quarks in the nucleon with χPT fit

QCDSF-UKQCD, arXiv:0710.1534 (Lattice 2007)

note: $J_d \approx 0$

LHPC: Ph. Hägler et al., Phys. Rev. D77 (2008) 094502



grey bands:
(preliminary) chiral extrapolations

brown bands:
errors for L_q from the extrapolation in t

stars:
experimental results from HERMES 2007

similar findings as QCDSF: signs of $\frac{1}{2}\Delta\Sigma^q = S_q$ and L_q opposite

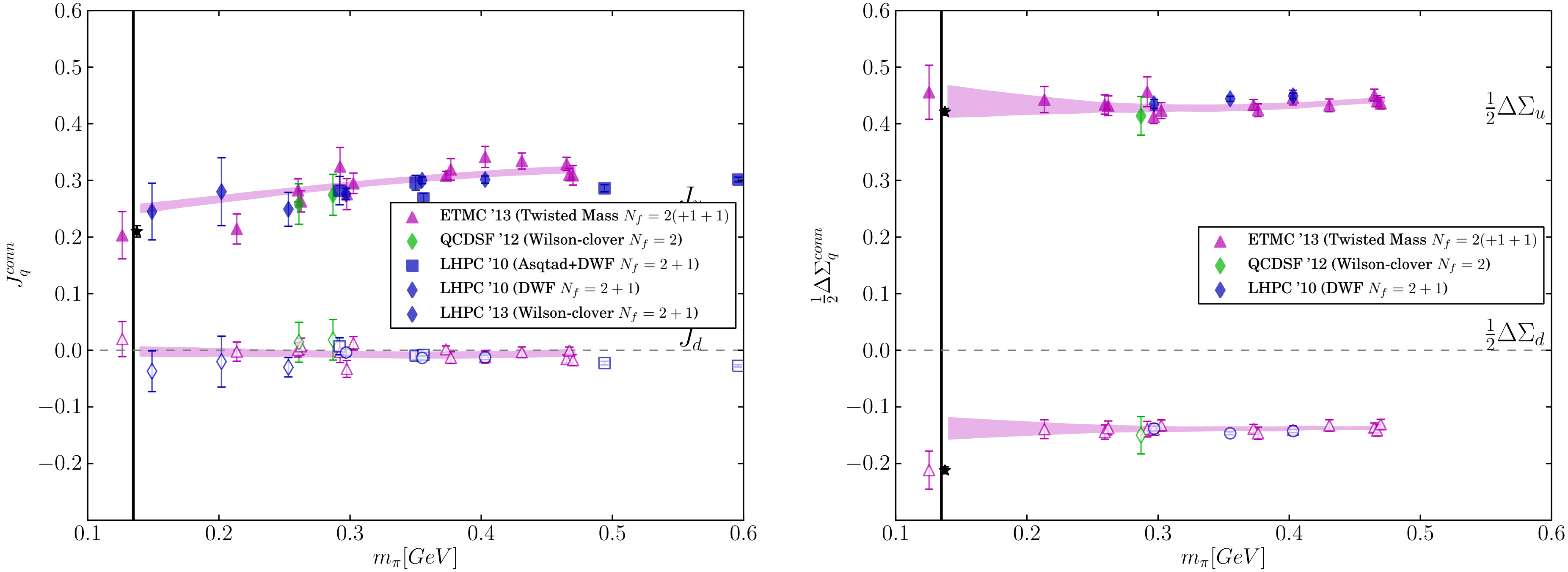
$$J_d = L_d + S_d \approx 0$$

$L_u + L_d \approx 0$ in strong disagreement with relativistic quark models

strong scale dependence? lattice data at a scale of $4\,$GeV^2!

some more recent results compiled by S. Syritsyn

arXiv:1403.4686 (Lattice 2013)



Renormalisation of composite operators

renormalisation: let bare parameters and renormalisation factors Z depend on a in such a way that the limit $a \to 0$ (of correlation functions) is finite

$q_{\mathrm{R}}(x) = Z_q^{1/2} q(x)$ quark field

$\mathcal{O}_{\mathrm{R}}(x) = Z_{\mathcal{O}} \mathcal{O}(x)$ composite operator like $\bar{q}q$

direct calculation of physical observables (e.g. hadron masses):
Z factors unnecessary (cancel in physical quantities)
↑
scheme and renormalisation scale dependent

Why then worry about Z factors?

It is not always possible to calculate the physical observables directly!

example: deep-inelastic lepton-nucleon scattering

OPE: structure function = Wilson coefficient $\otimes$ hadronic matrix element of a (local) composite operator

observable — short distance, perturbative — long distance, non-perturbative

Need: renormalisation of (local) composite operators
conversion to the $\overline{\mathrm{MS}}$ scheme (Wilson coefficients!)

bare lattice operator $\mathcal{O}(a)$

$$\to \text{renormalised continuum operator } \mathcal{O}_{\mathrm{R}}(\mu) = Z_{\mathcal{O}}(a, \underset{\substack{\uparrow \\ \text{renormalisation scale}}}{\mu})\mathcal{O}(a)$$

μ dependence should cancel between the operator and the Wilson coefficient

calculation of the Z factors:

- lattice perturbation theory (+ tadpole improvement) poor convergence!
- nonperturbative renormalisation (Monte Carlo simulation)
  in a scheme which can be implemented on the lattice (unlike $\overline{\mathrm{MS}}$) and in the continuum
  (RI-MOM scheme, Schrödinger functional, . . .)

in general, all operators of the same symmetry can contribute!

“they mix with each other”

renormalisation of an operator of dimension d:

$$\mathcal{O}_{\mathrm{R}}^{(d)} = Z\mathcal{O}^{(d)} + \sum_i Z_i \mathcal{O}_i^{(d)} + \underset{\uparrow}{\frac{1}{a^2}} \sum_i Z_i' \mathcal{O}_i^{(d-2)} + \cdots$$

large subtraction in the continuum limit

Perturbative calculation of the mixing with lower-dimensional operators (coefficients Z_i') unreliable:

$$Z' = b_1 g^2 + b_2 g^4 + \cdots + \underbrace{A\mathrm{e}^{-c/g^2}}_{\propto \Lambda_{\mathrm{QCD}}^2 a^2}$$

b_i: logarithmic a dependence

avoid mixing with lower-dimensional operators whenever possible!

DIS: we have to deal with operators in the Euclidean continuum like

$$\bar{q}\gamma_{\mu_1}\overset{\leftrightarrow}{D}_{\mu_2}\cdots\overset{\leftrightarrow}{D}_{\mu_n}q \qquad , \qquad \bar{q}\gamma_{\mu_1}\gamma_5\overset{\leftrightarrow}{D}_{\mu_2}\cdots\overset{\leftrightarrow}{D}_{\mu_n}q$$

or rather O(4) irreducible multiplets with definite C-parity.

All operators in one multiplet have the same renormalisation factor.

twist-2 operators: symmetrise the indices and subtract traces

(representation $D^{(n/2,n/2)}$ of SO(4))

flavour-nonsinglet case: no mixing

corresponding lattice operators: continuum $D \to$ lattice D

O(4) → H(4) (the hypercubic group with 384 elements)

∞ many irreducible representations → 20 irreducible representations

irreducible O(4) multiplet of operators → several irreducible H(4) multiplets

→ more possibilities for mixing

example: operator $\mathcal{O}_{\mu\nu} = \bar{q}\gamma_\mu \overset{\leftrightarrow}{D}_\nu q$ → twist 2: $\frac{1}{2}\left(\mathcal{O}_{\mu\nu} + \mathcal{O}_{\nu\mu}\right) - \frac{1}{4}\delta_{\mu\nu}\sum_\lambda \mathcal{O}_{\lambda\lambda}$

9-dimensional irreducible O(4) multiplet with a common Z

decomposes under H(4) into two irreducible multipletts:

$\frac{1}{2}\left(\mathcal{O}_{\mu\nu} + \mathcal{O}_{\nu\mu}\right) \quad (1 \leq \mu < \nu \leq 4)$	$\mathcal{O}_{11} - \mathcal{O}_{22}\ ,\ \mathcal{O}_{33} - \mathcal{O}_{44}\ ,\ \mathcal{O}_{11} - \mathcal{O}_{44}$
6-dimensional	3-dimensional
Z_6	Z_3

mixing pattern on the lattice for $\bar{q}\gamma_{(\mu_1}\overset{\leftrightarrow}{D}_{\mu_2}\cdots\overset{\leftrightarrow}{D}_{\mu_n)}q$

n	d	SO(4)	H(4)
0	3	$D^{(0,0)}$	$\tau_1^{(1)}$
1	3	$D^{(\frac{1}{2},\frac{1}{2})}$	$\tau_1^{(4)}$
2	4	$D^{(1,1)}$	$\tau_1^{(3)} \oplus \tau_3^{(6)}$
3	5	$D^{(\frac{3}{2},\frac{3}{2})}$	$\tau_1^{(4)} \oplus \tau_2^{(4)} \oplus \tau_1^{(8)}$
4	6	$D^{(2,2)}$	$\tau_1^{(1)} \oplus \tau_2^{(1)} \oplus \tau_1^{(2)} \oplus \tau_1^{(3)} \oplus \tau_1^{(6)} \oplus \tau_2^{(6)} \oplus \tau_3^{(6)}$

Try to avoid mixing by a suitable choice of indices (representations).

simplest case: a single operator without any mixing

$$\mathcal{O}_{\mathrm{R}}(\mu) = Z(a\mu, g_{\mathrm{R}}(\mu))\mathcal{O}(a)$$

The renormalised operator $\mathcal{O}_{\mathrm{R}}$ is scale and scheme dependent.
anomalous dimension (scaling violations neglected):

$$\gamma(g_{\mathrm{R}}) = -\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln Z \bigg|_a = \gamma_0 \frac{g_{\mathrm{R}}^2}{16\pi^2} + \gamma_1 \left(\frac{g_{\mathrm{R}}^2}{16\pi^2} \right)^2 + \cdots$$

$$\beta(g_{\mathrm{R}}) = \mu \frac{\mathrm{d}g_{\mathrm{R}}}{\mathrm{d}\mu} \bigg|_a = -\beta_0 \frac{g_{\mathrm{R}}^3}{16\pi^2} - \beta_1 \frac{g_{\mathrm{R}}^5}{(16\pi^2)^2} + \cdots$$

(cutoff a and bare quantities fixed)
→ scale dependence of Z governed by the renormalisation group:

$$\frac{Z(a\mu, g_{\mathrm{R}}(\mu))}{Z(a\mu_0, g_{\mathrm{R}}(\mu_0))} = \exp\left\{ - \int_{g_{\mathrm{R}}(\mu_0)}^{g_{\mathrm{R}}(\mu)} \mathrm{d}g \, \frac{\gamma(g)}{\beta(g)} \right\}$$

scale and scheme independent:

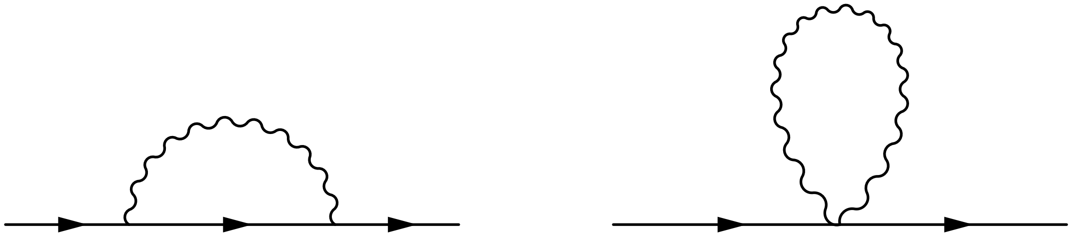
$$\mathcal{O}_{\mathrm{RGI}} = \left(2\beta_0 \frac{g_{\mathrm{R}}(\mu)^2}{16\pi^2} \right)^{-\gamma_0/(2\beta_0)} \exp\left\{ \int_0^{g_{\mathrm{R}}(\mu)} \mathrm{d}g \, \left(\frac{\gamma(g)}{\beta(g)} + \frac{\gamma_0}{\beta_0 g} \right) \right\} \mathcal{O}_{\mathrm{R}}(\mu)$$

Perturbative renormalisation on the lattice

1 loop: $Z = 1 - \frac{g^2}{16\pi^2}\left(\gamma_0 \ln(a\mu) + \Delta\right) + O(g^4)$

lattice perturbation theory often has poor convergence properties
one reason: tadpoles (lattice artefacts)

1-loop contributions to the quark propagator:



tadpoles originate from $U(x,\mu) = \mathrm{e}^{\mathrm{i}agA_\mu(x)} = 1 + \mathrm{i}agA_\mu(x) - \frac{1}{2}a^2g^2A_\mu(x)^2 + \cdots$

→ tadpole improvement (G.P. Lepage, P.B. Mackenzie, Phys. Rev. D48 (1993) 2250)

“gauge invariant link variable”

$$u_0 = \langle \tfrac{1}{3}\mathrm{tr}U_\Box \rangle^{1/4} = 1 - \frac{g^2}{16\pi^2}\frac{4}{3}\pi^2 + O(g^4) \qquad \text{SU(3)}$$

recipe for an operator with n_D covariant derivatives:

$$Z = \left(\frac{u_0}{u_0}\right)^{n_D-1} Z = u_0^{1-n_D}\left[1 - \frac{g_*^2}{16\pi^2}\left(\gamma_0 \ln(a\mu) + \Delta + (n_D-1)\frac{4}{3}\pi^2\right) + O(g^4)\right]$$

u_0: value from simulations

$g_*^2 = g^2 + O(g^4)$: “physical” coupling constant, e.g. $g_*^2 = g_\Box^2 \equiv \dfrac{g^2}{u_0^4}$

“boosted perturbation theory”

motivated by the appearance of

n_D operator tadpole diagrams and 1 leg tadpole diagram

contributing with opposite sign

tadpole improvement does what it is expected to do:

operator	$3\Delta/4$	$3\bar{\Delta}/4$	operator	$3\Delta/4$	$3\bar{\Delta}/4$
	no derivatives			2 derivatives	
l	12.95240	3.08280	v3	-12.12740	-2.25779
vls	20.61780	10.74820	r3	-12.86094	-2.99133
vas	15.79630	5.92670	a2	-12.11715	-2.24754
ts	17.01810	7.14850	h2a	-11.54826	-1.67866
zpp	16.64440	6.77480	h2b	-11.86877	-1.99917
	1 derivative		h2c	-11.74773	-1.87813
v2a	1.27958	1.27958	h2d	-12.92681	-3.05721
v2b	2.56185	2.56185		3 derivatives	
r2a	0.34512	0.34512	v4	-25.50303	-5.76382

for unimproved Wilson fermions

$\bar{\Delta} = \Delta + (n_D - 1)\frac{4}{3}\pi^2$: tadpole improved counterpart of Δ

tadpole improvement does improve convergence,
but uncertainty remains, in particular if one is restricted to 1 loop order

lattice perturbation theory very complicated for more than one loop

way out: numerical stochastic perturbation theory (NSPT)?

Nonperturbative renormalisation (RI-MOM)

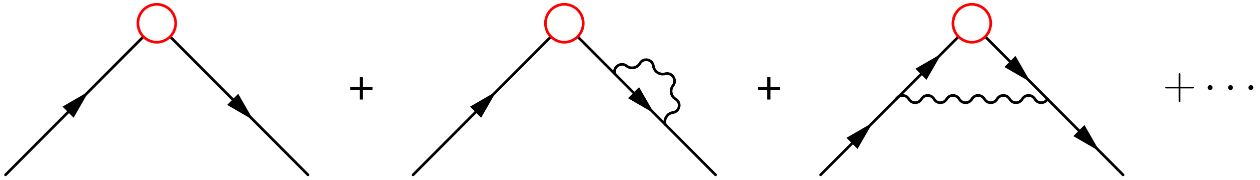
G. Martinelli, C. Pittori, C.T. Sachrajda, M. Testa, A. Vladikas, Nucl. Phys. B445 (1995) 81

idea: mimic the continuum definition

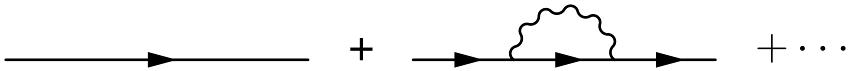
three-point function of a quark-antiquark operator ($\mathcal{O} = \bar{q} \cdots q$) in Landau gauge

$$G^{ij}_{\alpha\beta}(p) = \frac{a^{12}}{V} \sum_{x,y,z} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \langle q^i_\alpha(x) \mathcal{O}(z) \bar{q}^j_\beta(y) \rangle$$

$V = L_s^3 L_t =$ lattice volume



quark propagator: $S^{ij}_{\alpha\beta}(p) = \frac{a^8}{V} \sum_{x,y} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \langle q^i_\alpha(x) \bar{q}^j_\beta(y) \rangle$



vertex function: $\Gamma(p) = S^{-1}(p) G(p) S^{-1}(p)$

renormalised: $\Gamma_{\mathrm{R}}(p) = Z_q^{-1} Z_{\mathcal{O}} \Gamma(p)$

renormalisation condition: $\frac{1}{12}\mathrm{tr}_{\mathrm{DC}}\left(\Gamma_{\mathrm{R}}(p)\Gamma_{\mathrm{Born}}(p)^{-1}\right)\big|_{p^2=\mu^2} = 1$ in the chiral limit

MOM-like: RI-MOM

renormalisation of the quark fields: $Z_q(p) = \left.\frac{\mathrm{tr}_{\mathrm{DC}}\left(-\mathrm{i}\sum_\lambda \gamma_\lambda \sin(ap_\lambda) a S^{-1}(p)\right)}{12\sum_\lambda \sin^2(ap_\lambda)}\right|_{p^2=\mu^2}$

($Z_q(p) = 1$ for the free Wilson propagator)

corresponding to the continuum renormalisation condition

$$Z_q(p) = \left.\frac{\mathrm{tr}_{\mathrm{DC}}\left(-\mathrm{i}\sum_\lambda \gamma_\lambda p_\lambda S^{-1}(p)\right)}{12p^2}\right|_{p^2=\mu^2}$$

ideally: $1/L^2 \ll \Lambda^2_{\mathrm{QCD}} \ll \mu^2 \ll 1/a^2$ (scale dependence as in continuum perturbation theory)

MOM → $\overline{\text{MS}}$: perturbation theory in the continuum

$$Z_{\mathrm{RGI}} = \left(2\beta_0 \frac{g_{\mathrm{R}}(\mu)^2}{16\pi^2}\right)^{-\gamma_0/(2\beta_0)} \exp\left\{\int_0^{g_{\mathrm{R}}(\mu)} \mathrm{d}g \left(\frac{\gamma(g)}{\beta(g)} + \frac{\gamma_0}{\beta_0 g}\right)\right\} Z(\mu)$$

such that $\mathcal{O}_{\mathrm{RGI}} = Z_{\mathrm{RGI}}\mathcal{O}(a)$

independent of μ for sufficiently large scales μ (contact with perturbation theory)

Nonperturbative renormalisation (RI-SMOM)

RI-MOM: vanishing momentum at the operator (exceptional momentum configuration)

RI-SMOM scheme (s = symmetric) uses a non-exceptional momentum configuration:
three momenta of the same size at the external legs and at the operator

→ three-point function $G^{ij}_{\alpha\beta}(p,q) = \dfrac{a^{12}}{V} \sum_{x,y,z} \mathrm{e}^{-\mathrm{i}p\cdot x - \mathrm{i}(q-p)\cdot z + \mathrm{i}q\cdot y} \langle q^i_\alpha(x) \mathcal{O}(z) \bar{q}^j_\beta(y) \rangle$

→ vertex function $\Gamma(p,q) = S^{-1}(p) G(p,q) S^{-1}(q)$

impose suitable renormalisation condition (in the chiral limit) at $p^2 = q^2 = (p-q)^2 = \mu^2$
(twisted boundary conditions for exact realisation)

remember: moments of GPDs and meson DAs ↔ nonforward matrix elements

→ more operators contribute and mix under renormalisation

example: $\bar{q}\gamma_\mu \overset{\leftrightarrow}{D}_\nu \overset{\leftrightarrow}{D}_\lambda q$ mixes with $\partial_\nu \partial_\lambda \bar{q} \gamma_\mu q$

→ RI-SMOM scheme required for a nonperturbative evaluation of the mixing coefficient

Numerical implementation of RI-MOM

3-point function $G(p) = \frac{a^{12}}{V} \sum_{x,y,z} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \langle q(x)\mathcal{O}(z)\bar{q}(y)\rangle$ (quark-line connected)

calculated as gauge field average of $\hat{G}(U|p) = \frac{a^{12}}{V} \sum_{x,y,z,z'} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \hat{S}(U|x,z)J(U|z,z')\hat{S}(U|z',y)$

with the operator $\mathcal{O}$ represented as $\sum_{z} \mathcal{O}(z) = \sum_{z,z'} \bar{q}(z)J(U|z,z')q(z')$

$\hat{S}(U|x,y) =$ quark propagator in the gauge field configuration U

either place the sources for the quark propagator at the operator or rewrite

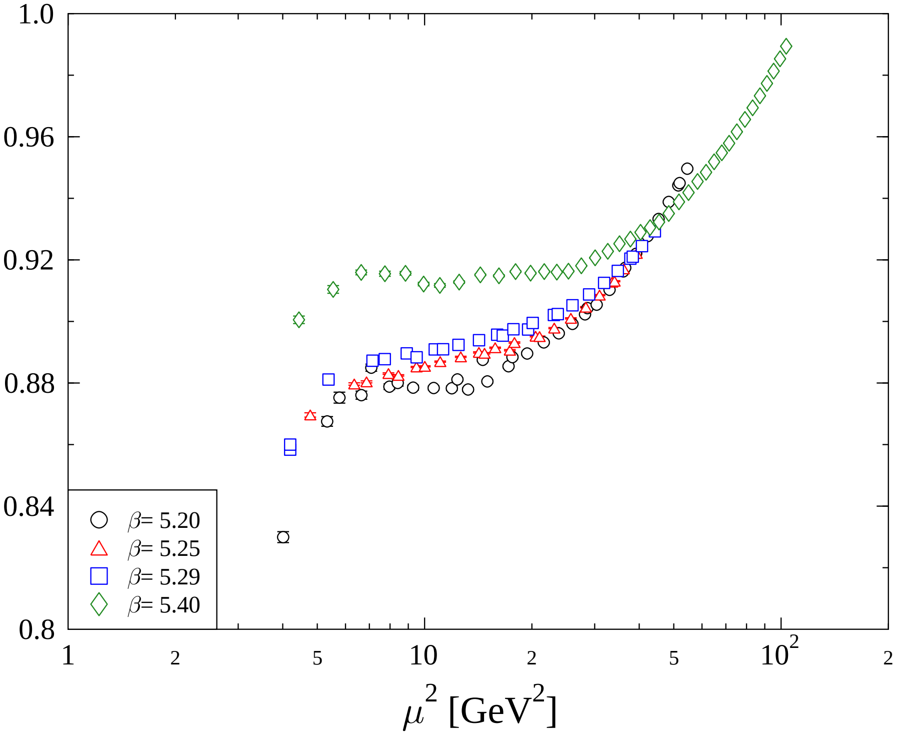
$$\hat{G}(U|p) = \frac{a^4}{V} \sum_{z,z'} \gamma_5 \left(a^4 \sum_{x} \hat{S}(U|z,x)\mathrm{e}^{\mathrm{i}p\cdot x} \right)^{+} \gamma_5 J(U|z,z') \left(a^4 \sum_{y} \hat{S}(U|z',y)\mathrm{e}^{\mathrm{i}p\cdot y} \right)$$

and solve the lattice Dirac equation with a momentum source:

$$a^4 \sum_{z} M(U|y,z) \left(a^4 \sum_{x} \hat{S}(U|z,x)\mathrm{e}^{\mathrm{i}p\cdot x} \right) = \mathrm{e}^{\mathrm{i}p\cdot y}$$

↑ reduced statistical fluctuations, arbitrary operators

↓ number of inversions $\propto$ number of momenta look for a plateau (values independent of μ) in $\quad Z^{\mathrm{RGI}} = \Delta Z^{\mathcal{S}}(\mu) Z^{\mathcal{S}}_{\mathrm{RI'-MOM}}(\mu) Z^{\mathrm{RI'-MOM}}_{\mathrm{bare}}(\mu)$

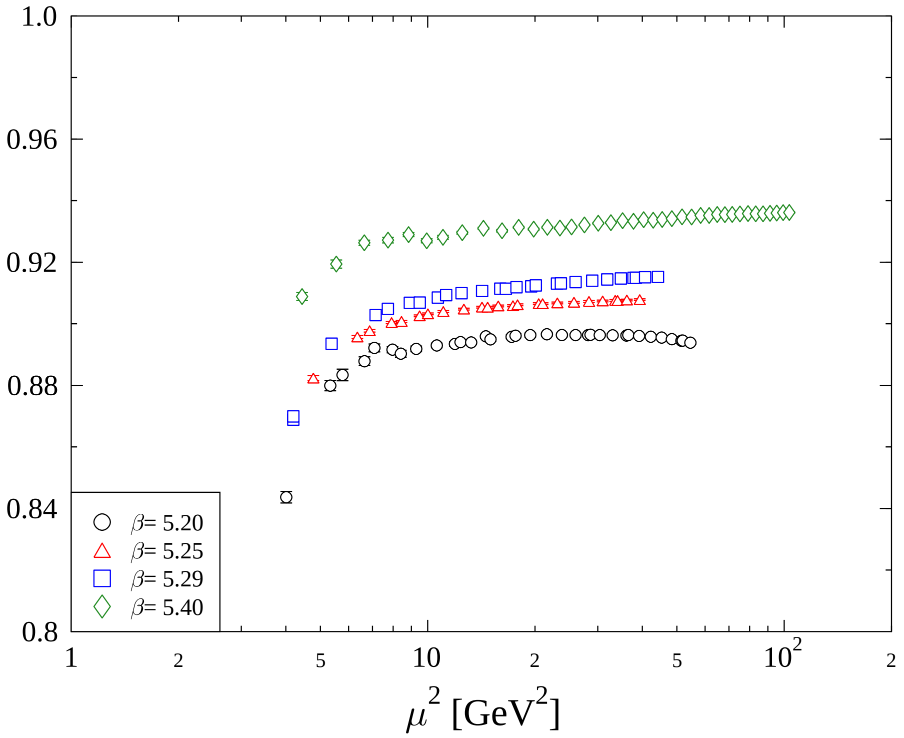


example: $\bar{\psi}\sigma_{\mu\nu}\psi$

plateau jeopardised by

- truncation of the perturbative expansions in $\Delta Z^{\mathcal{S}}$ and $Z^{\mathcal{S}}_{\mathrm{RI'-MOM}}$ (and finite size effects?) at small values of μ
- lattice artefacts vanishing like powers (up to logarithms) of a for $a \to 0$ at large values of μ

therefore: try to subtract lattice artefacts (perturbatively)



Subtraction of lattice artefacts

calculation of Z in lattice perturbation theory neglects lattice artefacts: $a^2p^2 \ll 1$
our momenta usually do not satisfy this condition

(1-loop) lattice perturbative results for arbitrary a^2p^2:
evaluate the loop integrals numerically (for each p separately) perhaps using NSPT

write the MOM scheme Z as $Z(p,a) = 1 + \frac{g^2 C_F}{16\pi^2} F(p,a) + O(g^4)$

$$\text{drop } O(a^2) \text{ terms} \quad \downarrow$$

$$\tilde{F}(p,a)$$

for the scalar density $\bar{q}q$ in the Landau gauge and in the chiral limit:

$$\tilde{F}(p,a) = 3\ln(a^2p^2) - 16.952410 - 7.737917 c_{\mathrm{SW}} + 1.380381 c_{\mathrm{SW}}^2$$

use the calculated difference between F and $\tilde{F}$ to correct for the perturbative discretisation errors in the Monte Carlo data: $D(p,a) \equiv F(p,a) - \tilde{F}(p,a)$
subtracted renormalisation constant:

$$Z_{\mathrm{MC}}(p,a) - \frac{g_\Box^2 C_F}{16\pi^2} D(p,a) \quad \text{with} \quad g_\Box^2 = \frac{g^2}{u_0^4}$$

Disconnected contributions

up to now: disconnected contributions neglected → systematic error unless flavour-nonsinglet quantity considered

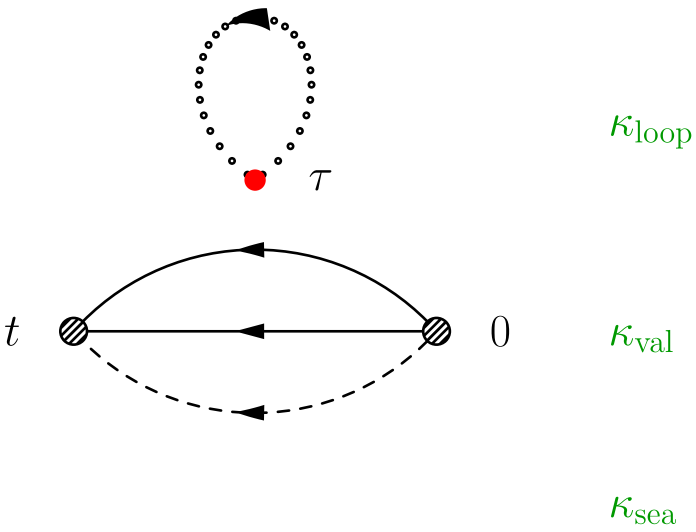
e.g., proton three-point function for an operator of the form $J^{(q)}(x) = \bar{q}(x)\Gamma q(x)$:

$$C^{(q)}_{\alpha\beta}(t,\tau;\mathbf{p},\mathbf{q})^{\mathrm{dis}} = -L_s^3 a^6 \sum_{\substack{x \\ x_4=\tau}} \sum_{\substack{y \\ y_4=t}} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{y}+\mathrm{i}\mathbf{q}\cdot\mathbf{x}} \epsilon_{ijk}\epsilon_{i'j'k'}(C^{-1}\gamma_5)_{\gamma\delta}(\gamma_5 C)_{\gamma'\delta'}$$

$$\times \left\langle \mathrm{tr}_{\mathrm{DC}}\left(\Gamma G_q(x,x)\right) G_d(y,0)^{ki'}_{\delta\gamma'} \left(G_u(y,0)^{ij'}_{\alpha\delta'}\, G_u(y,0)^{jk'}_{\gamma\beta} - G_u(y,0)^{jj'}_{\gamma\delta'}\, G_u(y,0)^{ik'}_{\alpha\beta} \right) \right\rangle_g$$

closed quark loop $\sum_{x,x_4=\tau} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\, \mathrm{tr}_{\mathrm{DC}}\left(\Gamma G_q(x,x)\right)$

correlated with proton propagator



how to compute the closed quark loop $\sum_{x,x_4=\tau} \mathrm{tr}_{\mathrm{DC}}\left(\Gamma G_q(x,x)\right)$?

one source for every x is hardly practical (even for fixed τ)

use a stochastic estimator (noisy estimator)!

choose a vector of random numbers $\omega_\alpha^i(x)$ with

$$\langle \omega_\alpha^i(x)^* \omega_\beta^j(y)\rangle_\omega = \delta_{ij}\delta_{\alpha\beta}\delta_{xy}\,, \quad \langle \omega_\alpha^i(x)\rangle_\omega = 0$$

$\langle\cdots\rangle_\omega$: average over the ωs

several possibilities: Gaussian random numbers, ± 1 (Z$_2$ noise), …

write $$\mathrm{tr}_{DC}\Gamma G(x,x) = \sum_{i\alpha\beta} \Gamma_{\beta\alpha} G(x,x)^{ii}_{\alpha\beta} = \sum_{i\alpha\beta} \Gamma_{\beta\alpha} \langle \omega_\beta^i(x)^* \sum_{zk\gamma} G(x,z)^{ik}_{\alpha\gamma}\, \omega_\gamma^k(z)\rangle_\omega$$

solve the Dirac equation with source ω for every random vector ω:

$$\sum_{x'i'\alpha'} M(x,x')^{ii'}_{\alpha\alpha'} \cdot \sum_{zk\gamma} G(x',z)^{i'k}_{\alpha'\gamma}\, \omega_\gamma^k(z) = \omega_\alpha^i(x)$$

then multiply by the appropriate elements of ω and sum over x to obtain a stochastic estimator of $\sum_{x,x_4=\tau} \mathrm{tr}_{\mathrm{DC}}\left(\Gamma G_q(x,x)\right)$

$$\sum_{x,x_4=\tau} \mathrm{tr}_{DC}\Gamma G(x,x) = \sum_{x,x_4=\tau} \sum_{i\alpha\beta} \Gamma_{\beta\alpha} G(x,x)^{ii}_{\alpha\beta} = \sum_{x,x_4=\tau} \sum_{i\alpha\beta} \Gamma_{\beta\alpha} \langle \omega^i_\beta(x)^* \sum_{zk\gamma} G(x,z)^{ik}_{\alpha\gamma}\, \omega^k_\gamma(z) \rangle_\omega$$

"stochastic" noise in addition to the usual "gauge" noise

"noise reduction" techniques aim at reducing the statistical error for given CPU time, e.g., by dilution:

divide the noise vector into subsets and estimate the trace on each subset separately

example: colour dilution

one random vector of size $3 \cdot 4 \cdot N_s^3$ (for fixed τ) → 3 random vectors of size $4 \cdot N_s^3$

$$\sum_{x,x_4=\tau} \mathrm{tr}_{DC}\Gamma G(x,x) = \sum_{i=1}^{3} \left\{ \sum_{x,x_4=\tau} \sum_{\alpha\beta} \Gamma_{\beta\alpha} \langle \omega_\beta(x)^* \sum_{z\gamma} G(x,z)^{ii}_{\alpha\gamma}\, \omega_\gamma(z) \rangle_\omega \right\}$$

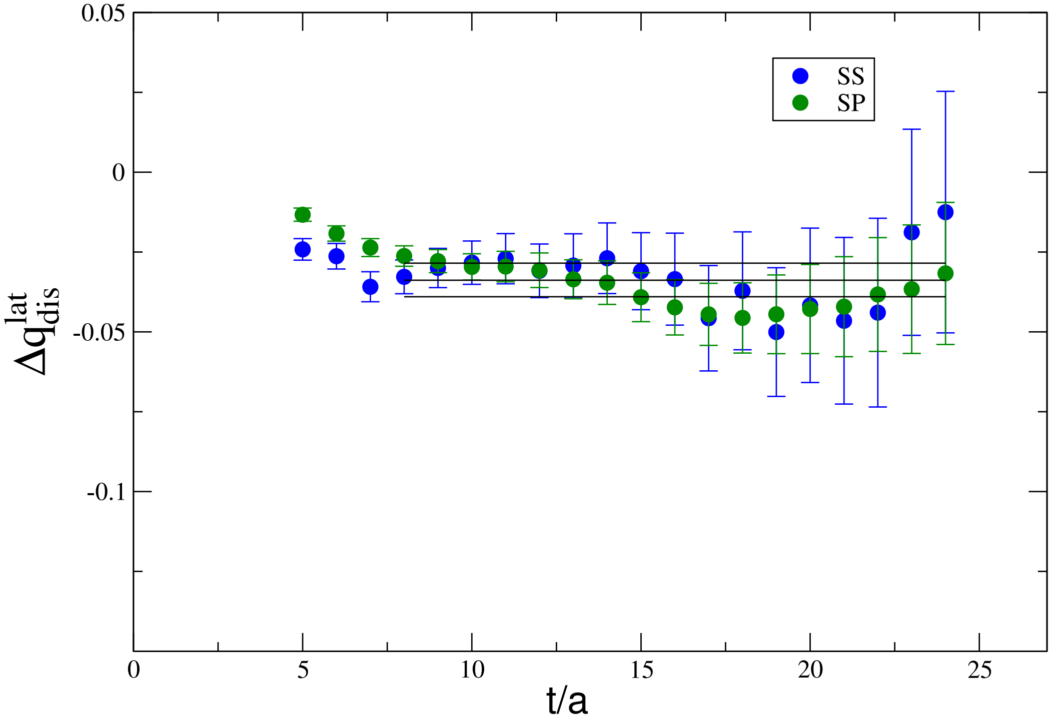
Dirac equation to be solved: $\sum_{x'i'\alpha'} M(x,x')^{ii'}_{\alpha\alpha'} \cdot \sum_{z\gamma} G(x',z)^{i'k}_{\alpha'\gamma}\, \omega_\gamma(z) = \omega_\alpha(x)\delta_{ik}$

additional challenge: renormalisation and (possibly) mixing with gluonic operators, . . .

example 1: Δq in the nucleon

G.S. Bali et al. (QCDSF), Phys. Rev. Lett. 108 (2012) 222001 [arXiv:1112.3354]

disconnected ratio of three-point function (axial current insertion) over two-point function



unrenormalised

SS: smeared-smeared
SP: smeared-point
source-sink combinations

strange quark quenched

$n_f = 2$, $\beta = 5.29$, $\kappa_{\text{sea}} = 0.13632$
$m_\pi^{\text{sea}} = 290\,\text{MeV}$

$\kappa_{\text{val}} = \kappa_{\text{loop}} = 0.1355 (= \kappa_{\text{s}})$

current insertion fixed at $\tau = 4a$

saturation into a plateau (within the statistical errors) at $t \leq 2\tau$
convergence of SS and SP ratios towards the same value

$\tau = 4a$ reasonable

constant fit of the SS ratios for $t \geq 8a$

some results from this study:

q	$\Delta q^{\mathrm{lat}}_{\mathrm{con}}$	$\Delta q^{\mathrm{lat}}_{\mathrm{dis}}$	$\Delta q^{\overline{\mathrm{MS}}}(\mu)$
u	1.071(15)	-0.049(17)	0.787(18)(2)
d	-0.369(9)	-0.049(17)	-0.319(15)(1)
s	0	-0.027(12)	-0.020(10)(1)
Σ	0.702(18)	-0.124(44)	0.448(37)(2)

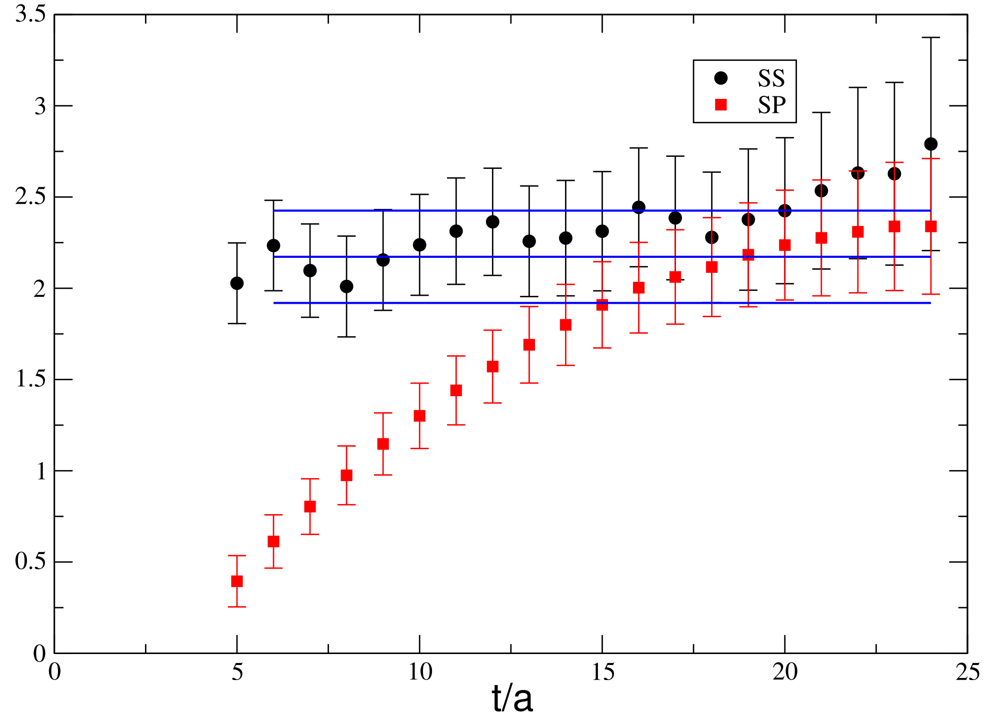
$\frac{1}{2}\Delta\Sigma = \frac{1}{2}(\Delta u + \Delta d + \Delta s)$: quark spin contribution to the nucleon spin

$\Delta q^{\overline{\mathrm{MS}}}(\mu)$: renormalisation scale $\mu^2 = 7.4\,\mathrm{GeV}^2 \approx a^{-2}$
first error statistical, second error from renormalisation
estimated systematic error: 20%

example 2: $\bar{s}s$ in the nucleon

G.S. Bali et al. (QCDSF), Phys. Rev. D85 (2012) 054502 [arXiv:1111.1600]

disconnected part of the ratio of three-point function (insertion of $\bar{q}q$) over two-point function



SS: smeared-smeared
SP: smeared-point
source-sink combinations

strange quark quenched

$n_f = 2$, $\beta = 5.29$, $\kappa_{\text{sea}} = 0.13632$
$m_\pi^{\text{sea}} = 290\,\text{MeV}$

$\kappa_{\text{val}} = \kappa_{\text{loop}} = 0.1355 (= \kappa_{\text{s}})$

current insertion fixed at $\tau = 4a$

saturation into a plateau (within the statistical errors) at $t \leq 2\tau$
convergence of SS and SP ratios towards the same value

$\tau = 4a$ reasonable

constant fit of the SS ratios for $t \geq 6a$

renormalisation not quite straightforward due to explicit breaking of chiral symmetry
(for Wilson fermions)

phenomenologically interesting:

(dimensionless) strange quark contribution to the nucleon mass

$$f_{T_s} = \frac{[m_s \langle N|\bar{s}s|N\rangle]_{\rm ren}}{m_N} = 0.012(14)^{+10}_{-3} \qquad \text{(rather small!)}$$

f_{T_s} relevant for dark matter searches:

models → dark matter candidates scattering from nuclei through Higgs exchange

Higgs expected to couple predominantly to s quarks in the nucleon

(coupling to light quarks too small, heavier quarks too rare)

→ cross section sensitive to matrix element of $\bar{s}s$ entering through f_{T_s}

if small value confirmed: predicted cross sections smaller than expected

Concluding remarks

- lattice QCD can provide results on hadron structure that are difficult (impossible) to obtain by other means
- lattice results important input for dark matter searches, determinations of CKM matrix elements, physics beyond the standard model, . . .
- systematic uncertainties to be reduced:
  continuum extrapolation, chiral extrapolation, finite size corrections, contamination by excited states, . . .
- more interesting results to be expected