Lattice QCD in magnetic fields

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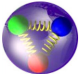
Outline

- introduction
  - QCD phase diagram
  - strongly interacting matter exposed to magnetic fields
  - the lattice approach
- magnetic field setup: continuum, torus, lattice
- energy levels of charged particle: continuum, torus, lattice
- two recent results about magnetic fields on the lattice
  - phase diagram
  - magnetic susceptibility, equation of state

Introduction

QCD and quark-gluon plasma

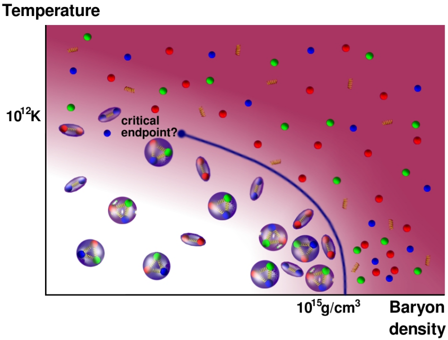
- elementary particle interactions:
  gravitational, $\underbrace{\text{electromagnetic, weak, strong}}_{\text{Standard Model}}$
- strong sector: Quantum Chromodynamics
- elementary particles: quarks ($\sim$ electrons) and gluons ($\sim$ photons)
  but: they cannot be observed directly
  $\Rightarrow$ *confinement* at low temperatures



- asymptotic freedom [Gross, Politzer, Wilczek '04]
  $\Rightarrow$ heating or compressing the system leads to *deconfinement*: quark-gluon plasma is formed
- transition between the two phases
  characteristics: order (1st/2nd/crossover)
  critical temperature T_c
  equation of state

QCD phase diagram

- why is the physics of the quark-gluon plasma interesting?
  - large T: early Universe, cosmological models
  - large ρ: neutron stars
  - large T and/or ρ: heavy-ion collisions, experiment design



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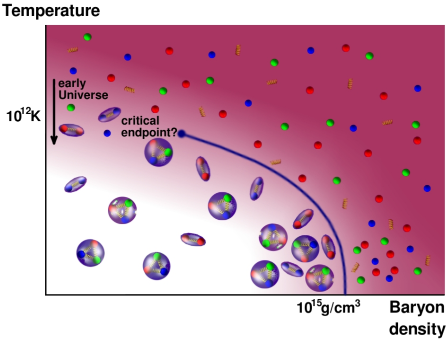

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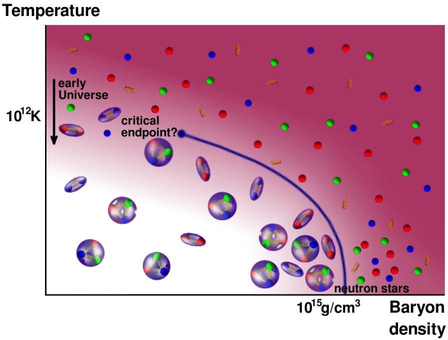

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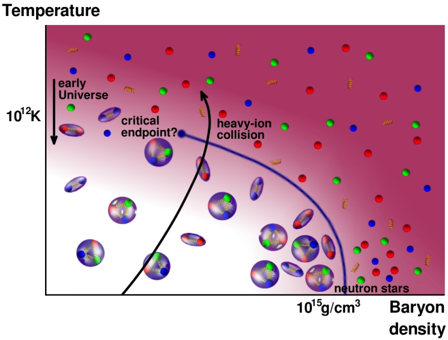

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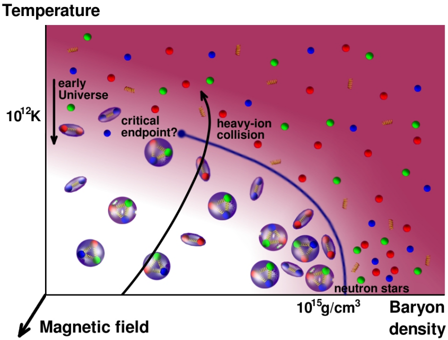
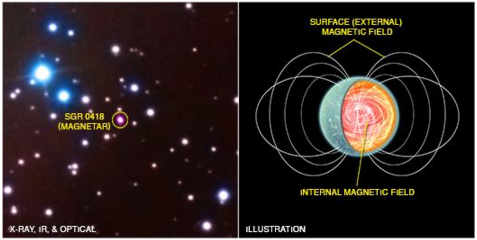


- additional, relevant parameter:
  - background magnetic field B

Example 1: neutron star



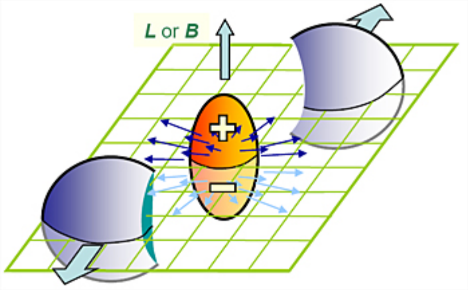
[Rea et al. '13]

- possible quark core at center with high density, low temperature
- magnetars: extreme strong magnetic fields are measured at the surface
- need models to describe magnetic field configuration (field strength in the center)

Typical magnetic fields

• magnetic field of Earth	10^{-5} T
• common magnet	10^{-3} T
• strongest human-made field in lab	10^{2} T
• magnetar surface	10^{10} T
• magnetar core	10^{14} T?

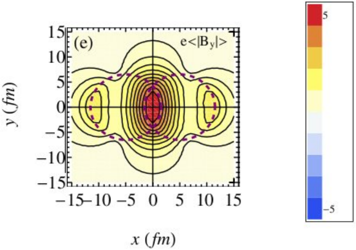
Example 2: heavy-ion collision



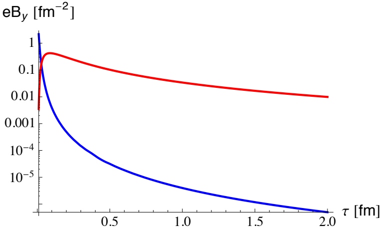
[STAR collaboration, '10]

- off-central collisions generate magnetic fields:
  strength controlled by $\sqrt{s}$ and impact parameter (centrality)
- strong (but very uncertain) time-dependence

Example 2: heavy-ion collision



[Deng et al '12]



[Gursoy et al '13]

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• LHC Pb-Pb at 2.7 TeV, $b = 10$ fm [Skokov '09]	10^{15} T

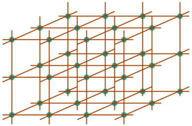
convert: $e \cdot 10^{15}\ \mathrm{T} \approx 3m_\pi^2 \approx \Lambda_{\mathrm{QCD}}^2$

$\Rightarrow$ electromagnetic and strong interactions compete

Approaches to study QCD

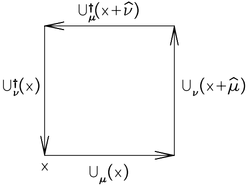
- various methods in various regimes:
  - high T/B: perturbation theory
  - low T/B: chiral perturbation theory, hadronic models
  - transition region: non-perturbative methods, lattice gauge theory [Wilson, '74]

- discretize quark and gluon fields ψ and A_μ on a 4D space-time lattice with spacing a



  - use $U_\mu = e^{iaA_\mu}$ instead of A_μ
  - U_μ: links, ψ: sites

- example: gauge action $F_{\mu\nu}F_{\mu\nu}(x) \sim$



(remember [Müller-Preussker Lect.2 + tutorial])

Lattice simulations

- functional integral

$$\mathcal{Z} = \int \mathcal{D}U_\mu \, \mathcal{D}\bar{\psi} \, \mathcal{D}\psi \; \mathsf{exp}\left(- \int \mathsf{d}^4 x \, \mathcal{L}_{\mathrm{QCD}} \right)$$

Lattice simulations

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$$\mathcal{Z} = \int \mathcal{D}U_\mu \; \exp\left(- \int d^4x \, \frac{1}{2} \mathrm{Tr} F_{\mu\nu} F_{\mu\nu} \right) \cdot \prod_f \det\left(\not{D} + m_f^{\mathrm{lat}} \right)$$

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- after discretization, $\mathcal{Z}$ becomes a $\sim 10^9$ dimensional integral
  - importance sampling, Monte-Carlo methods
- biggest challenges are to
  - extrapolate $a \to 0$ 'continuum limit'
    and keep physical size fixed: $\#$ of lattice points $\to \infty$
  - fix bare parameters of $\mathcal{L}$: quark masses
    tune m_f^{lat} such that the measured $m_\pi, m_p, m_\rho, \ldots$ are the same as in nature

Computational requirements

- typical computational requirement $\mathcal{O}(10\ \text{Tflop/s} \times \text{year})$

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$\mathcal{O}(40$ mio. core hours$)$

$\mathcal{O}(100 \text{ GPU} \times \text{year})$

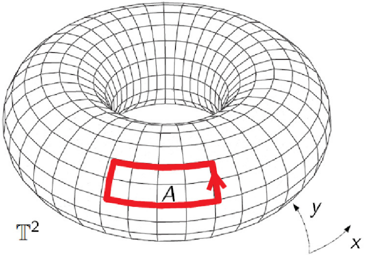
Magnetic field - setup

Magnetic field

- constant and uniform electromagnetic field
- choose coordinates such that $\mathbf{B} = (0, 0, B)$
- represented by an electromagnetic vector potential $A_\mu = (0, \mathbf{A})$ for which $\nabla \times \mathbf{A} = \mathbf{B}$
  - a possible gauge: $A_x = A_z = 0$, $A_y = Bx$
- interaction with charged particles via minimal coupling

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$$

Magnetic field on the torus



torus $\mathbb{T}^2$ with surface area $L_x L_y$

picture from [D'Elia et al '11]

- phase factor for a charged particle transported along path $\mathcal{C}$: $\exp(iq \oint_{\mathcal{C}} \mathrm{d}x_\mu A_\mu)$
- Stokes theorem: $\oint_{\mathcal{C}} \mathrm{d}x_\mu A_\mu = \iint_A \mathrm{d}\sigma B = B \cdot A$
  but also $= -\iint_{\mathbb{T}^2 - A} \mathrm{d}\sigma B = -B \cdot (L_x L_y - A)$
- equality of phase factors gives quantization condition [Hashimi, Wiese '09]

$$\exp(iqBL_xL_y) = 1 \quad \rightarrow \quad qBL_xL_y = 2\pi \cdot N_b, \quad N_b \in \mathbb{Z}$$

Magnetic field on the torus

- qB cannot be arbitrary
- what if there are several charged particles in the system?
  - e.g. fermion flavors ψ_f with charge q_f $(f = u, d, s, \ldots)$
  - all flavors have to obey the quantization condition:

$$q_f B L_x L_y = 2\pi N_b^{(f)}, \quad N_b^{(f)} \in \mathbb{Z}$$

  - option A: charges are incommensurable $q_{f1}/q_{f2} \in \mathbb{R}$
    $\Rightarrow$ bad luck
  - option B (nature): charges are commensurable
    $q_u = 2e/3$, $q_d = q_s = -e/3$
    $\Rightarrow$ need to set magnetic field according to *lowest* charge

$$q_d B L_x L_y = 2\pi N_b^{(d)},$$

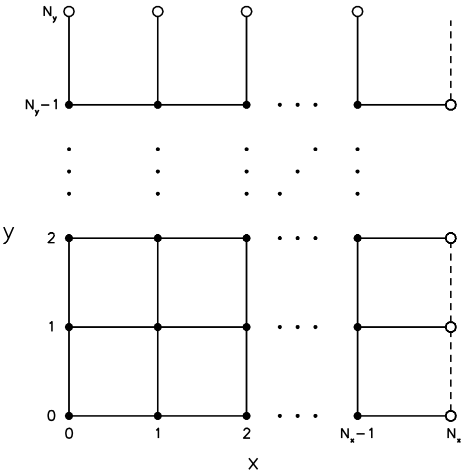
$$N_b^{(d)} \in \mathbb{Z}, \quad N_b^{(s)} = N_b^{(d)}, \quad N_b^{(u)} = -2N_b^{(d)}$$

Magnetic field on the lattice

- how to discretize A_μ on the lattice?
- as usual, work with group elements $u_\mu = \exp(iaA_\mu)$
- Dirac operator at nonzero B, schematically:

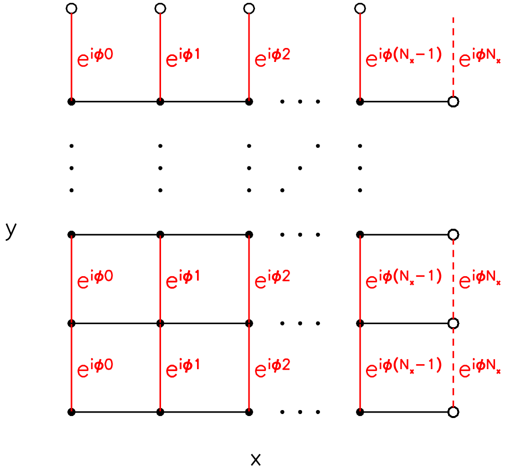
$$\not{D} = \frac{1}{2} \sum_\mu \left[\gamma_\mu U_\mu u_\mu - \gamma_\mu U_\mu^\dagger u_\mu^* \right]$$

Magnetic field on the lattice



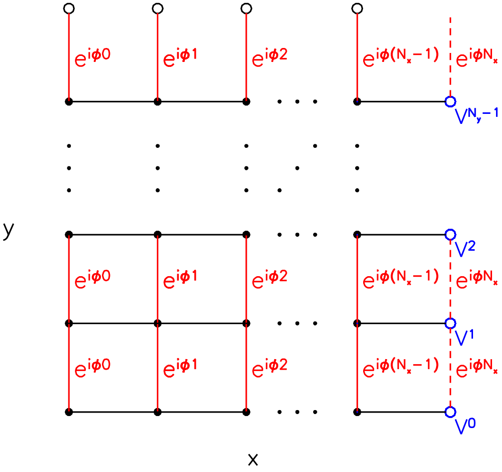
- 2D slice of the lattice: sites (n_x, n_y) with $n_\mu = 0 \ldots N_\mu - 1$
- $A_y = Bx = Bn_x a$

Magnetic field on the lattice



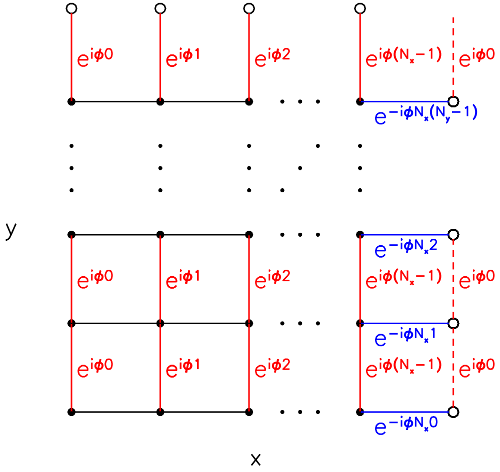
- simplest choice $u_y = \exp(iaqA_y) = \exp(i\phi n_x)$, with the flux unit $\phi = a^2qB$
- periodic b.c. in the x-direction is violated $\rightarrow$ inconvenient

Magnetic field on the lattice



- do a local $\mathrm{U}(1)$ gauge transformation
  $\psi(N_x, n_y) \rightarrow \psi(N_x, n_y) \cdot V^{n_y}$ with $V = \exp(i\phi N_x)$

Magnetic field on the lattice



- restores periodicity in the x-direction
- changes last x-links to $u_x(N_x - 1, n_y) = \exp(-i\phi N_x n_y)$

Magnetic field on the lattice

- flux quantization on the lattice (finite volume)

$$q_d B \cdot a^2 = \frac{2\pi N_b}{N_x N_y}, \qquad N_b \in \mathbb{Z}$$

$\Rightarrow$ smallest flux is $N_b = 1$

- phase factor along a single plaquette (finite lattice spacing)

$$\exp(ia^2 q_d B) = \exp\left(i \frac{2\pi N_b}{N_x N_y} \right)$$

$\Rightarrow$ largest flux is $N_b = N_x N_y$

- remark: $\det(\not{D}(B) + m_f^{\mathrm{lat}}) > 0$ so no sign problem

Energy levels of a free charged particle in magnetic field

Energy levels in the continuum

- relativistic particle with electric charge q (but no color charge), subject to the Dirac equation

$$(\not{D} + m)\psi = 0, \qquad D_y = \partial_y + iqBx, \quad D_\nu = \partial_\nu \quad (\nu \neq y)$$

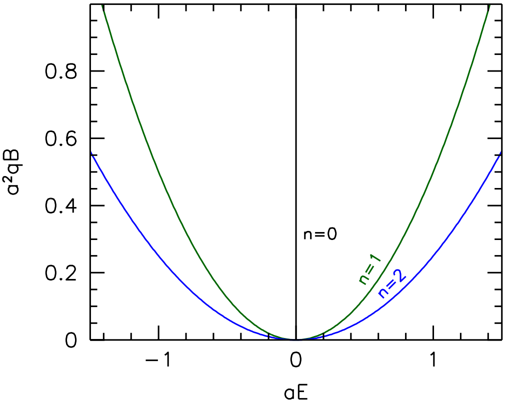
- solutions (assuming $qB > 0$)

$$E_n^2 = p_z^2 + m^2 + 2qB(n + 1/2 - \sigma_z)$$

  in terms of the quantum numbers $n \in \mathbb{Z}_0^+$, $\sigma_z = \pm 1/2$ and p_z
- degeneracy is ∞ in an infinite volume and is $qB \cdot L_x L_y/(2\pi)$ in a finite box
- for the massless case in 2D ($p_z = 0$) the lowest Landau levels:

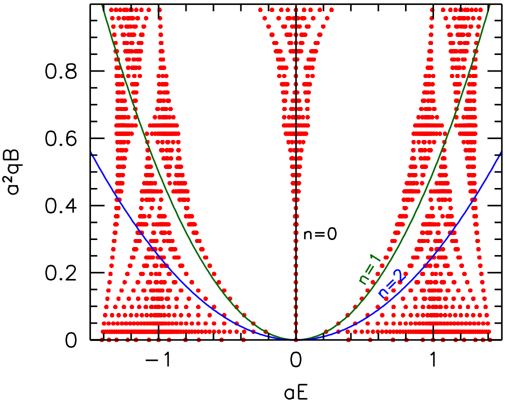
$$E_0^2 = 0, \qquad E_1^2 = 2qB, \qquad E_2^2 = 4qB$$

Landau levels in 2D



- lowest Landau levels in the continuum $E_n^2 = 2n \cdot qB$
- now solve eigenvalue problem on the lattice:
  $\not{D}\psi_n = E_n\psi_n$ for each N_b

Landau levels in 2D



- lowest Landau levels in the continuum $E_n^2 = 2n \cdot qB$
- now solve eigenvalue problem on the lattice:
  $\not{D}\psi_n = E_n\psi_n$ for each N_b
  - continuum levels dissolve into discrete bands

Lattice distorted Landau levels

- zoom out to view all $N_b = 0 \ldots N_x N_y$ (with $N_x = N_y = 16$)

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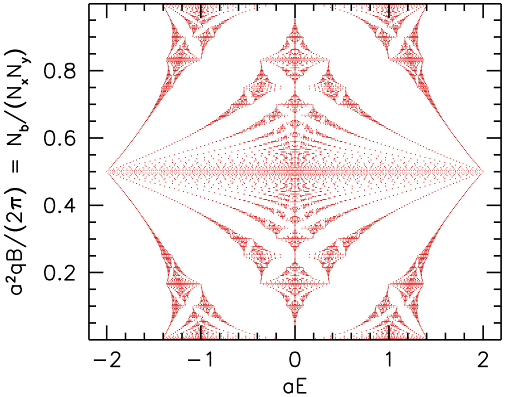

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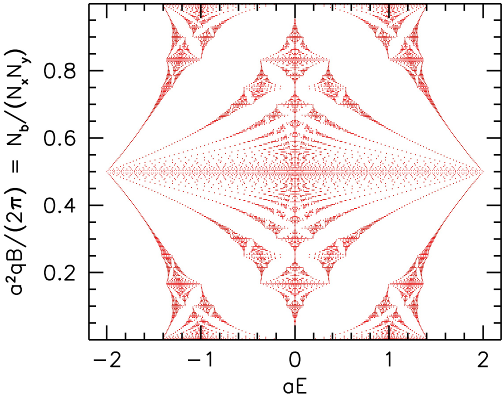

  - recursive pattern: fractality
  - twofold mirror symmetry:

    $N_b \leftrightarrow N_x N_y - N_b$ (flux periodicity + parity)

    $E \leftrightarrow -E$ (charge conjugation)

Lattices and butterflies

- charged particle on a lattice in magnetic field:
  familiar setup in solid state physics
- almost the same eigenvalue problem in [Hofstadter '76]
  (non-relativistic case)

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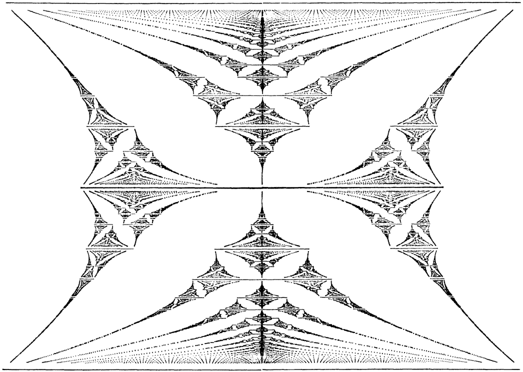
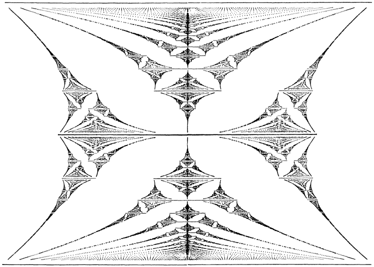


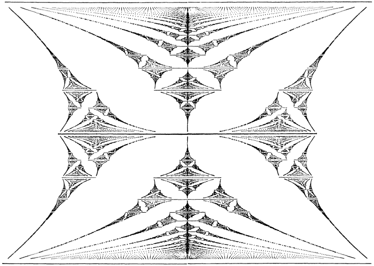
'Bloch electron in magnetic field' ⇒ Hofstadter's butterfly

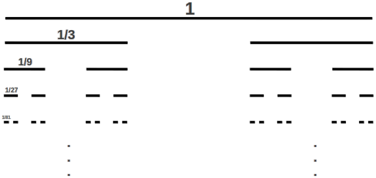
Hofstadter's butterfly [Hofstadter '76]



- true fractal structure if lattice size is infinite
- energy levels form finite bands if $a^2 qB/2\pi \in \mathbb{Q}$
- energy levels isomorphic to Cantor's set if $a^2 qB/2\pi \notin \mathbb{Q}$

Hofstadter's butterfly [Hofstadter '76]





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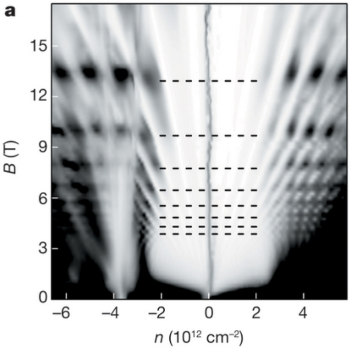
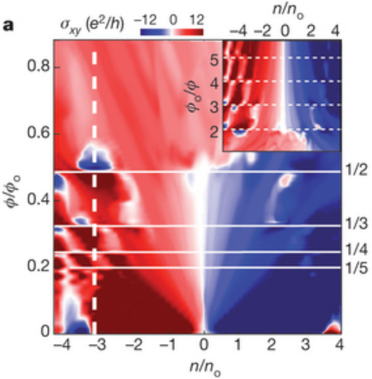
Hofstadter's butterfly: experiments

- "catching the butterfly"
  - challenge: get $a^2 \cdot qB$ of order 1 (typical a too small)
  - solution: overlay sheets of graphene to effectively increase a



[Ponomarenko et al '13]



[Dean et al '13]

Hofstadter's butterfly: impact on QCD

- Hoftadter's butterfly (solid state physics)

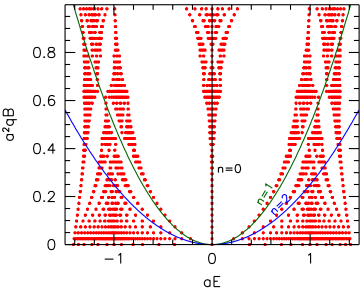
  $=$

  $\not{D}$ eigenvalues for free quarks (quantum field theory)

- lattice is a crystal in solid state physics

  $\neq$

  lattice is a regulator in QFT ($a \to 0$ continuum limit)



- still, the low-N_b lattice spectrum contains continuum information

Hofstadter's butterfly: impact on QCD

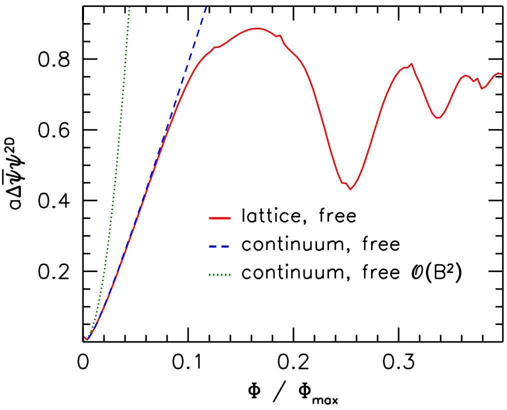
- the butterfly disappears in the continuum limit, but its wings around $a^2 qB \approx 0$ contain physical information
- in contrast to electron energies, Dirac eigenvalues cannot be measured
- physical observable composed of the eigenvalues: condensate of quarks with mass m

$$\bar{\psi}\psi^{\rm 2D} = \sum_n \frac{m}{E_n^2 + m^2}$$

- nonzero quark mass washes out the fractal structure up to $qB \propto m^2$

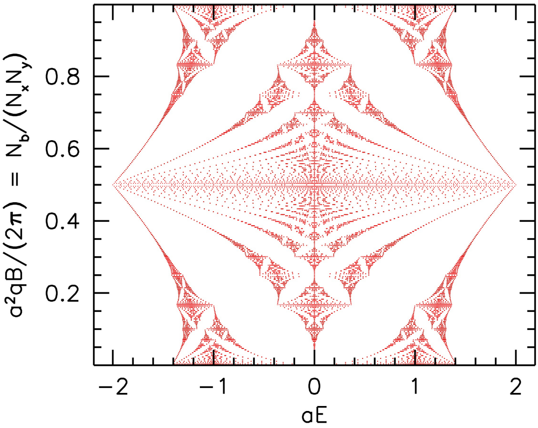
  $\rightarrow$ animation

Hofstadter's butterfly and the condensate



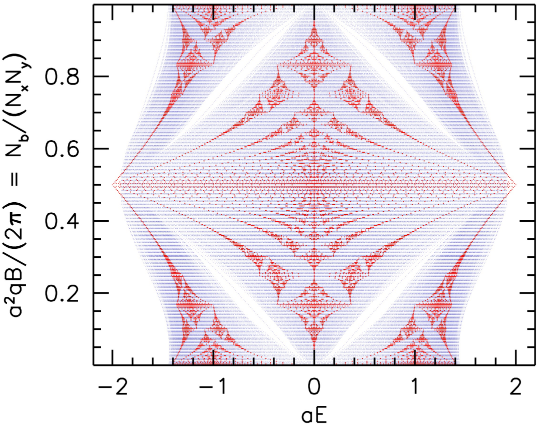
- the lattice condensate matches the continuum curve at small a^2qB
  - 'butterfly carries continuum information in its low-B wings'

Hofstadter's butterfly: impact on QCD



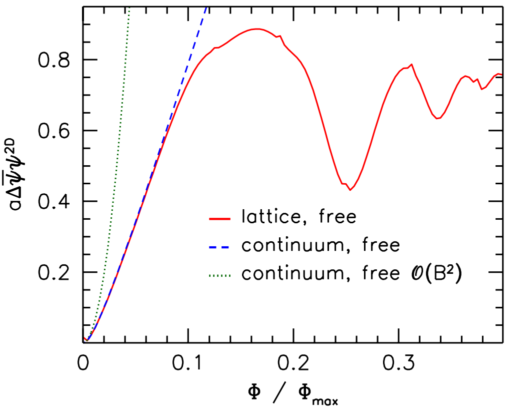
- Dirac eigenvalues for quarks in a magnetic field in 2D

Hofstadter's butterfly: impact on QCD



- Dirac eigenvalues for quarks in a magnetic field in 2D with QCD interactions switched on (perturbation lifts degeneracy)

Hofstadter's butterfly and the condensate



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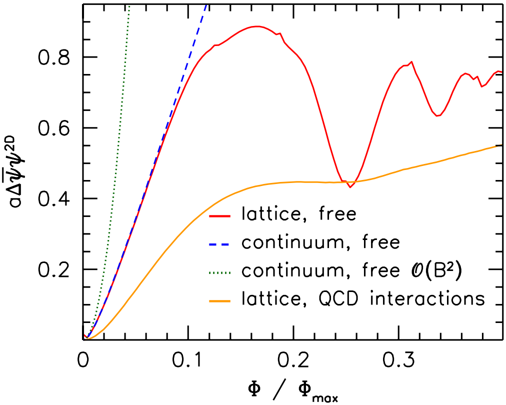

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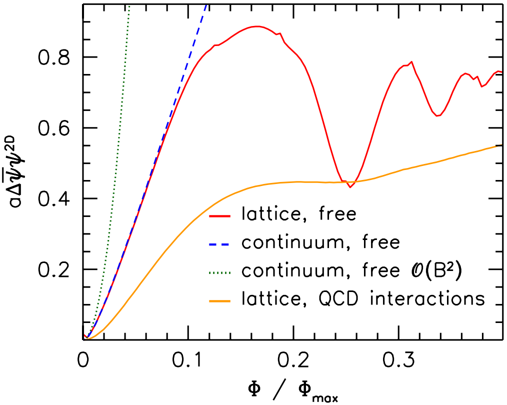
Hofstadter's butterfly and the condensate



- QCD interactions wash out the fractal structure, but qualitative tendency remains
- more on this in
  [GE 1301.1307] and [Bali, Bruckmann, GE, Katz, Schäfer 1406.0269]

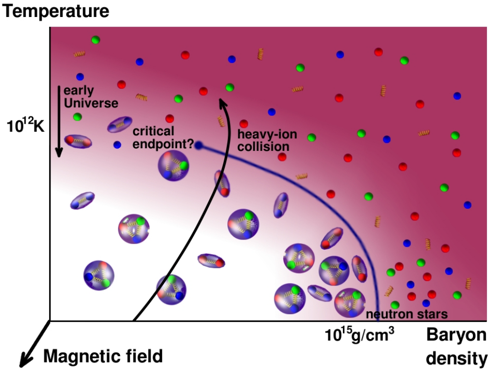
Magnetic fields in full 4D QCD

Magnetic field-induced effects

- QCD phase diagram
- magnetic susceptibility and equation of state in QCD

QCD phase diagram

QCD phase diagram



- keep density=0 and explore $B - T$ plane
  - what is the transition temperature $T_c(B)$?
  - what is the nature of the transition at $B > 0$?

Approximate order parameters

- symmetries of $\mathcal{L}_{\mathrm{QCD}}$ (remember [Philipsen Lect.2])
  - chiral symmetry at $m_f^{\mathrm{lat}} = 0$
    order parameter: chiral condensate
  - center symmetry at $m_f^{\mathrm{lat}} = \infty$
    order parameter: Polyakov loop
- at the physical masses no exact symmetries
  $\rightarrow$ no exact order parameters
- still, there are approximate order parameters: observables that are sensitive to the transition

Observables sensitive to the transition

- chiral condensate
  $\rightarrow$ chiral symmetry breaking

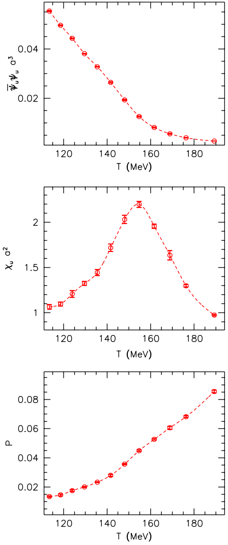
$$\bar{\psi}_f \psi_f = \frac{\partial \log \mathcal{Z}}{\partial m_f}$$

- chiral susceptibility
  $\rightarrow$ chiral symmetry breaking

$$\chi_f = \frac{\partial^2 \log \mathcal{Z}}{\partial m_f^2}$$

- Polyakov loop
  $\rightarrow$ deconfinement

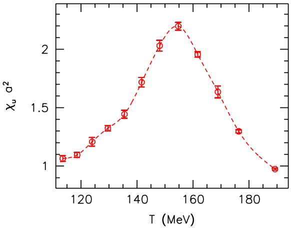
$$P = \frac{1}{V} \sum_{\mathbf{x}} \mathrm{Tr} \prod_{x_4} U_4(\mathbf{x}, x_4)$$



Transition characteristics

- chiral susceptibility
  ($\sim$ specific heat)

$$\chi = \frac{\partial^2 \log \mathcal{Z}}{\partial m^2}$$



- ▸ transition temperature: peak maximum
- ▸ order of transition: volume-dependence of height $h(V) \propto V^\alpha$
  1st ($\alpha = 1$), 2nd ($0 < \alpha < 1$) or crossover ($\alpha = 0$)

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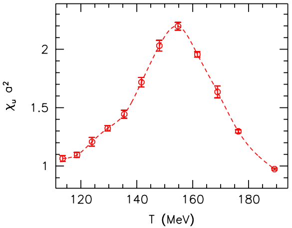
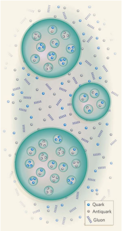
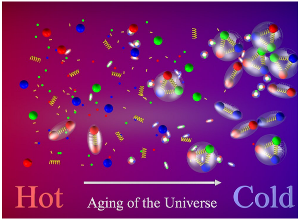


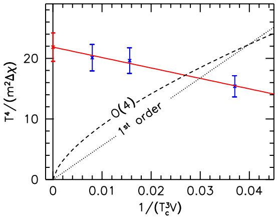
  - transition temperature: peak maximum
  - order of transition: volume-dependence of height $h(V) \propto V^\alpha$
    1st ($\alpha = 1$), 2nd ($0 < \alpha < 1$) or crossover ($\alpha = 0$)
- bubble nucleation versus smooth transition





Transition characteristics at $B = 0$

- simulations with physical $m_f^{\rm lat}$, continuum extrapolation
- no singular behavior as $V \to \infty$
  $\Rightarrow$ transition is analytic crossover
  [Aoki, GE, Fodor, Katz, Szabó '06]



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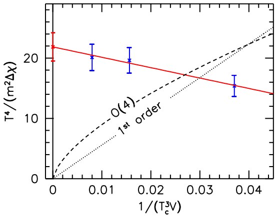

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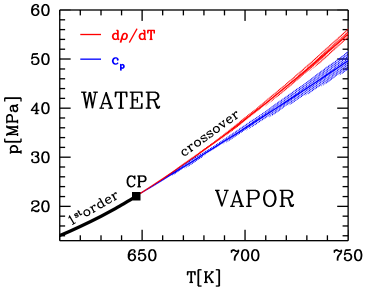

- compare to the water-vapor transition at high pressures

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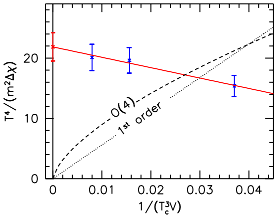

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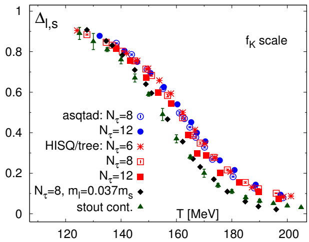
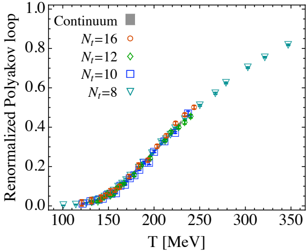




- $T_c^{\bar\psi\psi} \approx 150$ MeV, $T_c^P \approx 175$ MeV [BWc '06,'09,'10]

Condensate at $B > 0$: 'magnetic catalysis'

- what happens to $\bar\psi\psi$ ($\langle +q\uparrow, -q\downarrow\rangle$) in magnetic field?
  $\Rightarrow$ magnetic moments parallel, energetically favored state
  (cf. Cooper-pairs in superconductors: Meissner effect)
- dimensional reduction $3+1 \to 1+1$ in the lowest Landau level

$$E_0(B=0) = \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}$$

$$E_0(B>0) = \sqrt{p_z^2 + m^2}, \qquad \#_0 = \frac{|qB| \cdot L_x L_y}{2\pi}$$

- chiral condensate $\leftrightarrow$ spectral density around 0 [Banks, Casher '80]

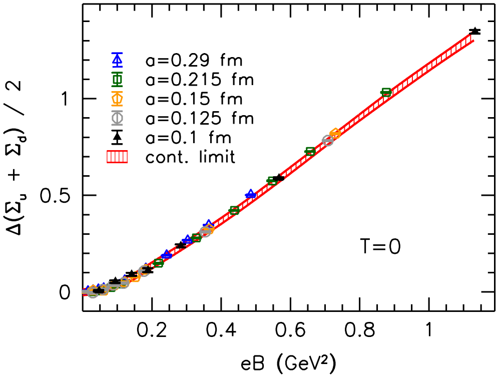
$$\bar\psi\psi \propto \rho(0)$$

- in the chiral limit, to maintain $\bar\psi\psi > 0$ (NJL [Gusynin et al '96])

$B = 0$	$\rho(p)\mathrm{d}p \sim p^2\mathrm{d}p$	"we need a strong interaction"
$B \gg m^2$	$\rho(p)\mathrm{d}p \sim qB\mathrm{d}p$	"the weakest interaction suffices"

Magnetic catalysis – zero temperature

- magnetic catalysis at zero temperature is a robust concept: χPT, NJL model, AdS-CFT, linear σ model, ...
  lattice QCD at physical/unphysical m_π



lattice QCD, physical m_π, continuum limit [Bali et al '12]

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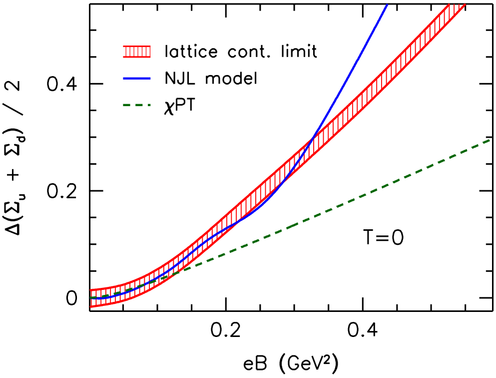

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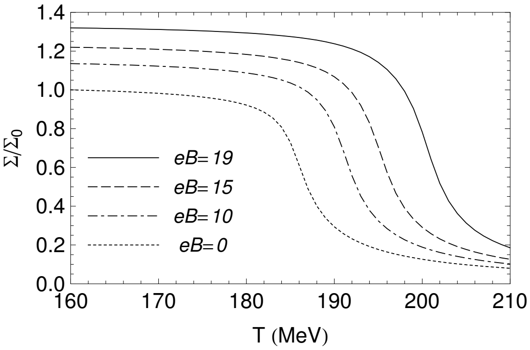

PNJL model [Gatto, Ruggieri '11]

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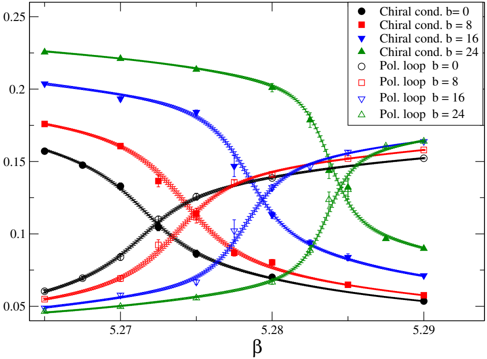
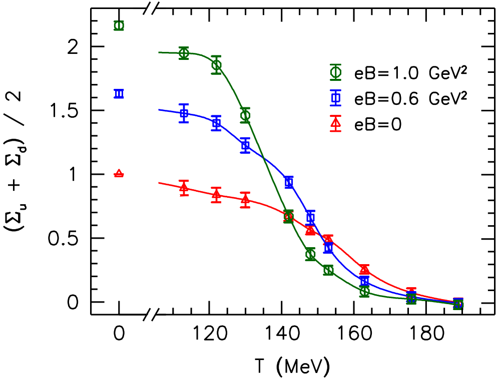


lattice QCD, unphysical m_π, coarse lattice [D'Elia et al '10]

Inverse magnetic catalysis

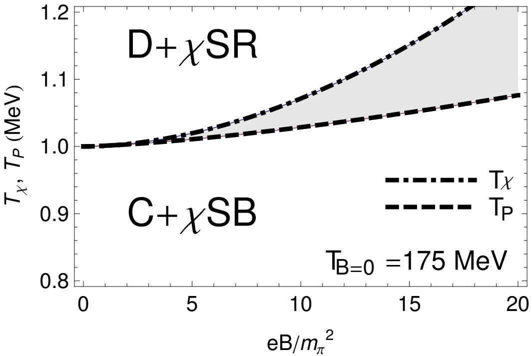
- lattice QCD, physical m_π, continuum limit [Bali et al '11, '12]



- at $T \approx 150$ MeV the condensate is reduced by B dubbed 'inverse magnetic catalysis'

Phase diagram

- inflection point of $\bar{\psi}\psi(T)$ defines T_c
- sinificant difference whether IMC is exhibited or not:



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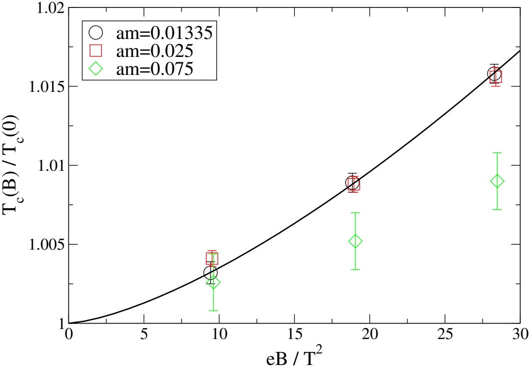

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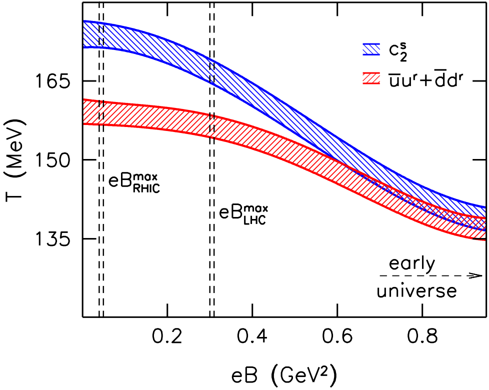

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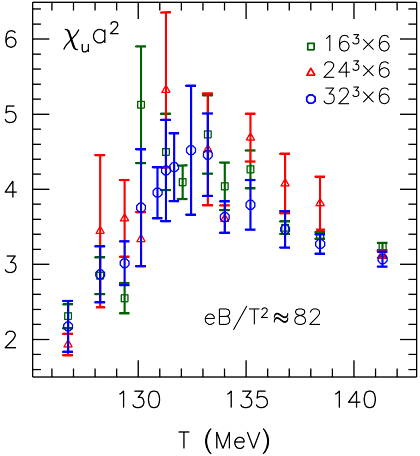
Order of transition at $B > 0$

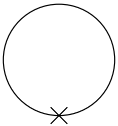


- at the largest B, no volume-dependence is visible
  $\Rightarrow$ crossover persists up to $eB \approx 1$ GeV^2

Mechanism behind IMC

- two competing mechanisms at finite B
  [D'Elia et al '11, Bruckmann, GE, Kovács '13]
  - direct (valence) effect $B \leftrightarrow q_f$
  - indirect (sea) effect $B \leftrightarrow q_f \leftrightarrow g$

$$\left\langle \bar{\psi}\psi(B) \right\rangle \propto \int \mathcal{D}U \, e^{-S_g} \underbrace{\det(\not{D}(B,U)+m)}_{\text{sea}} \underbrace{\mathrm{Tr}\left[(\not{D}(B,U)+m)^{-1}\right]}_{\text{valence}}$$



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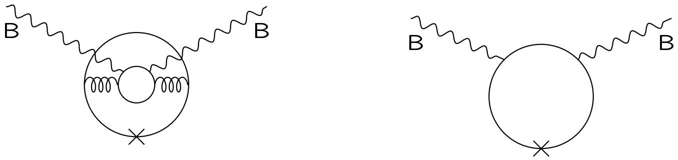

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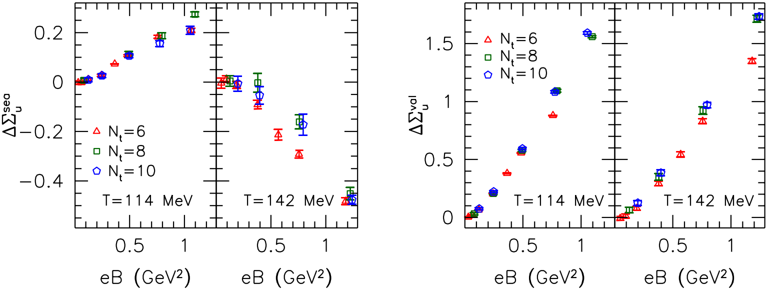
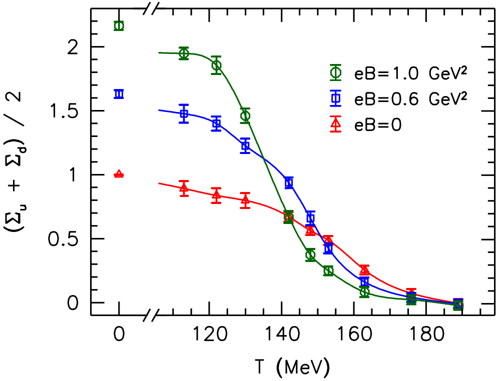


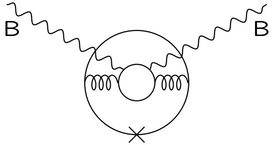
Phase diagram – conclusions

- valence and sea effects compete and around T_c the sea wins



Phase diagram – conclusions

- valence and sea effects compete and around T_c the sea wins
- lessons learned:
  - LL-picture not applicable to non-perturbative QCD in the transition region
  - inclusion of dynamical quarks necessary in the models to reproduce the real phase diagram



  - important to improve effective theories/models
  - already many attempts to reproduce IMC, e.g. [Fraga et al., Farias et al., Ferreira et al., Ayala et al. '14]

Magnetic susceptibility and equation of state

Matter in magnetic fields (linear response)

- paramagnets: attracted by magnetic field
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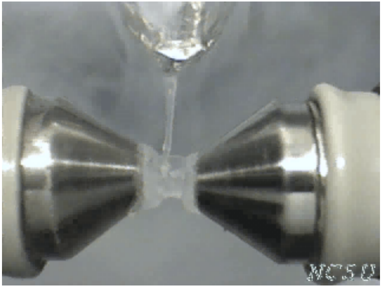

paramagnet: liquid oxygen
[NCSU physics demonstrations]

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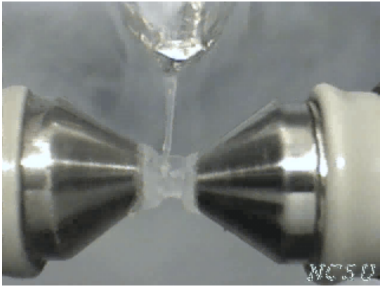

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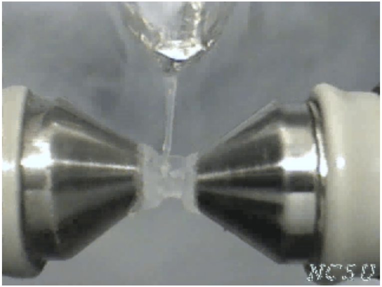

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[NCSU physics demonstrations]

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- ▶ is thermal QCD as a medium para- or diamagnetic?

Magnetic susceptibility

- free energy density in background magnetic field

$$f(B) = -\frac{T}{V}\log \mathcal{Z}(B)$$

- magnetization

$$\mathcal{M} = -\frac{\partial f}{\partial (eB)}, \qquad \mathcal{M}\big|_{B=0} = 0$$

- susceptibility

$$\chi = \frac{\partial \mathcal{M}}{\partial (eB)}\bigg|_{B=0} = -\left.\frac{\partial^2 f}{\partial (eB)^2}\right|_{B=0}$$

- sign distinguishes between
  - paramagnets ($\chi > 0$) enjoy magnetic field
  - diamagnets ($\chi < 0$) repel magnetic field

Magnetic susceptibility at high T

- consider a free quark with charge q
  - propagator in B can be calculated exactly [Schwinger '51]
- using Schwinger proper time regularization to obtain $f(B)$, the susceptibility reads [Bali, Bruckmann, GE et al 1406.0269]

$$\chi_r^{\rm free} = -N_c \cdot \beta_1 \cdot (q/e)^2 \int \frac{\mathrm{d}s}{s} e^{-m^2 s} \cdot \left\{ \Theta_4\left[0, e^{-1/(4sT^2)}\right] - 1 \right\}$$

$$\xrightarrow{T\to\infty} N_c \cdot \beta_1 \cdot (q/e)^2 \log\left(\frac{T^2}{m^2}\right)$$

- QED is not asymptotically free ($\beta_1 = 1/12\pi^2 > 0$)
  $\Rightarrow$ free quarks at high T are paramagnetic
  (see also [Elmfors et al. '94])
  - remark: additive renormalization to set $\chi_r(T=0)=0$

Magnetic susceptibility at low T

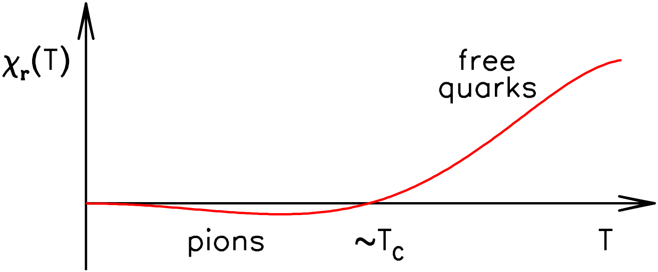
- low-energy regime of QCD: dominant degrees of freedom are pions (χPT)
  - so consider a free pion (charge $\pm e$)
- similarly as before: proper-time regularization
  [Bali, Bruckmann, GE et al 1406.0269]

$$\chi_r^{\text{pion}}(T) = -\beta_1^{\text{scalar}} \underbrace{\int \frac{\mathrm{d}s}{s} e^{-m^2 s} \cdot \left\{ \Theta_3 \left[0, e^{-1/(4sT^2)}\right] - 1 \right\}}_{\text{finite and positive}}$$

  - scalar QED β-function $\beta_1^{\text{scalar}} = 1/48\pi^2 > 0$
    $\Rightarrow$ free pions are diamagnetic
    (see also [Elmfors et al. '94])

Expectation for the susceptibility

- $\chi_r(T = 0) = 0$ due to renormalization prescription
- asymptotic freedom in QCD + no asymptotic freedom in QED
  $\Rightarrow \chi_r > 0$ for high temperatures
- expectation: pions are relevant at low energies
  $\Rightarrow \chi_r < 0$ for low temperatures



Magnetic susceptibility on the lattice

- remember magnetic flux quantization

$$a^2 qB = \frac{2\pi N_b}{N_x N_y}, \qquad N_b = 0, 1, \ldots, N_x N_y$$

  - χ as derivative is not directly accessible

- various methods to circumvent this problem
[Bali et al 1303.1328] [DeTar et al 1309.1142]
[Bonati et al 1307.8063] [Bali et al 1406.0269]

  - here: calculate $f(B)$ and differentiate it numerically

- lattice setup: stout smeared staggered quarks + Symanzik gauge action, physical pion mass, continuum estimate based on $N_t = 6, 8, 10$ [Bali et al 1406.0269]

Lattice method

- with conventional Monte-Carlo techniques, derivatives of $\log \mathcal{Z}$ can be calculated, but not $\log \mathcal{Z} \propto f$ itself (compare [Philipsen Lect.2])
  - rewrite $\log \mathcal{Z}$ as the integral of its derivatives at constant N_b

$$\log \mathcal{Z}(\infty) - \log \mathcal{Z}(m_f^{\mathrm{ph}}) = \int_{m_f^{\mathrm{ph}}}^{\infty} \mathrm{d}m_f \, \frac{\partial \log \mathcal{Z}}{\partial m_f}$$

  - take difference $\Delta \log \mathcal{Z} = \log \mathcal{Z}(N_b) - \log \mathcal{Z}(0)$

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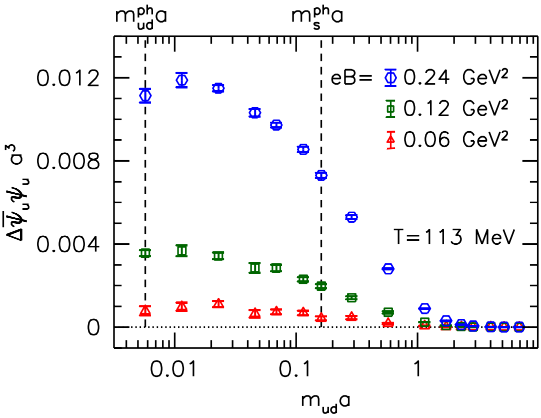
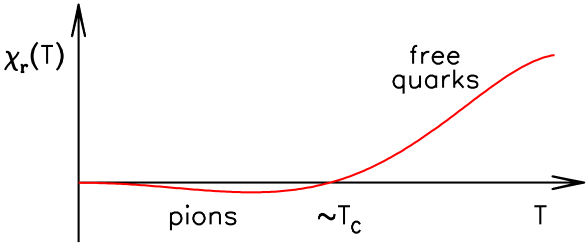
- $\Delta \log \mathcal{Z}$ obtained as integral of condensates

Lattice method



- obtain $\Delta \log \mathcal{Z}$ as an integral for each B
- interpolate $\Delta \log \mathcal{Z}$ as function of B
- differentiate to obtain $\chi \propto \Delta \log \mathcal{Z}''$

Susceptibility from the lattice



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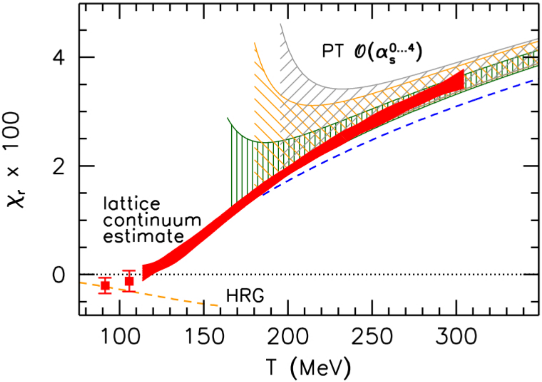

- confirms free-case expectation qualitatively transition from diamagnetism to paramagnetism slightly below T_c [Bali, Bruckmann, GE et al 1406.0269]
- comparison to Hadron Resonance Gas model (low T) and to perturbation theory (high T)

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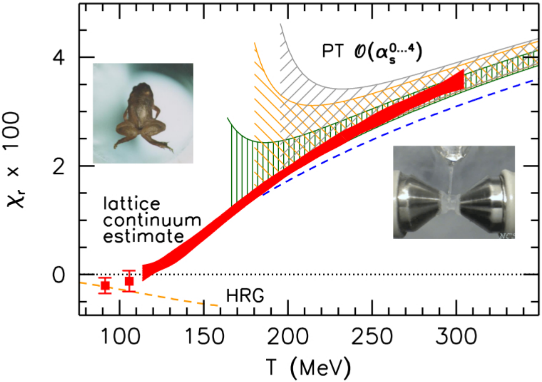

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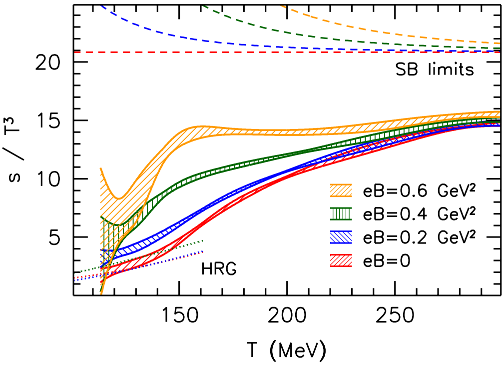
Equation of state

- using $\log \mathcal{Z}(B)$, all thermodynamic observables can be calculated $\Rightarrow$ equation of state
  - energy density ϵ
  - pressures $p_x = p_y, p_z$
  - entropy density s
  - interaction measure $\epsilon - 3p_z$



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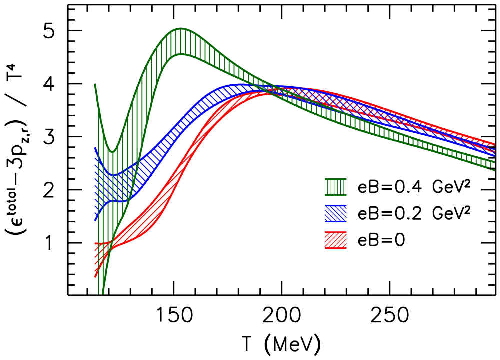
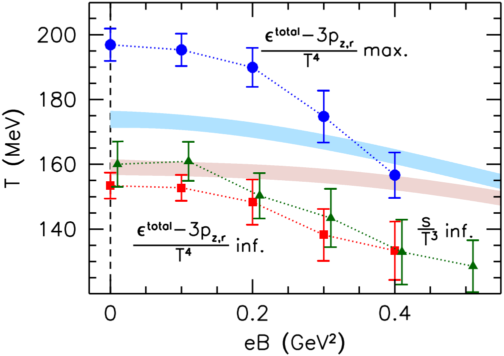


Phase diagram

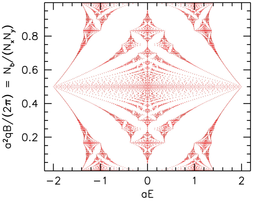
- inflection points of s and $\epsilon - 3p_z$ are measures for T_c
- maximum of $\epsilon - 3p_z$ is just a characteristic point



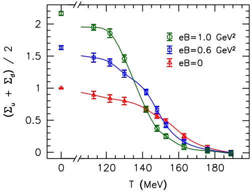
- results again show that T_c is reduced by B

Summary

- $\not{D}$ eigenvalues give Hofstadter's butterfly: solid state physics $\leftrightarrow$ QFT



- magnetic catalysis versus inverse catalysis $\Rightarrow$ phase diagram



- magnetic susceptibility: QCD matter can either be dia- or paramagnetic