Quantum Integrable Systems and Yang-Baxter Equations.

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Plan

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2. **Yang-Baxter Equations.**
    - Integrable systems
    - Factorizable scattering and Yang-Baxter Equations.
    - Zamolodchikov algebra
    - Quantum Integrable Models.

1. Introduction

- In modern theoretical physics, the ideas of symmetry and invariance play a very important role. Symmetry transformations form groups, and therefore the most natural language for describing symmetries in physics is the group theory language.
- About 30 years ago, in the study of quantum integrable systems (*R. Baxter, A.B.Zamolodchikov and Al.B.Zamolodchikov, A.A.Belavin, E.K.Sklyanin, L.A.Takhtajan, L.D.Faddeev, M.Jimbo, P.P.Kulish, N.Reshetikhin, and many others*), in particular in the framework of the quantum inverse scattering method (*E.K.Sklyanin, L.A.Takhtajan, L.D.Faddeev*), new algebraic structures arose, the generalizations of which were later called quantum groups (*V.G.Drinfeld*). Yang-Baxter equations became a unifying basis of all these investigations.

1. Introduction

- Although quantum groups are deformations of the usual groups, they nevertheless still possess several properties that make it possible to speak of them as ”symmetry groups”. Moreover, one can claim that the quantum groups serve as symmetries and provide integrability in exactly solvable quantum systems (Yangian symmetries are the symmetries of that type).
- Quantum Group structures and in particular Yang-Baxter equations appear in 1D and 2D quantum integrable systems (spin chains, 2D quantum (conformal) field theories, multy-particle systems like Toda chains and Calodgero-Moser systems, ...).

1. Introduction

- Quantum Groups (Yangians) and integrable structures were observed in 4D quantum (conformal) field theories.
  1. Analytical evaluations of Feynman diagrams and related statistical models (*A.B. Zamolodchikov (1980)*)
  2. Evaluations of the anomalous dimensions in 4D supersymmetric Yang-Mills theories use the methods developed for investigations of quantum integrable systems (*L.N. Lipatov; L.D. Faddeev and G.P. Korchemsky; V.Kazakov; J. Minahan and K. Zarembo; N. Beisert and M. Staudacher; a.o.*)
  3. Alday-Gaiotto-Tachikawa correspondence between 4D supersymmetric Yang-Mills theories and 2D conformal field theories (A.Belavin and collaborators, A.Morozov, A.Mironov a.o.).

2. Integrable systems

As a rule, for integrable dynamical systems the equations of motion are written as zero-curvature condition

$$\left[\frac{\partial}{\partial t} - M(\theta)\,,\; L(\theta)\right] = 0\,, \qquad (1)$$

where t - time, θ - spectral parameter, and Lax pair (L, M) are operators (matrices in general) which depend on phase space coordinates of the system. For integrable field theories in d=(1+1) space-time with coordinates (x,t), operators L and M are also differential operators which depend on $\frac{\partial}{\partial x} \equiv \partial_x$.

Example. KdV equation

$$\partial_t V(x,t) = \frac{1}{4}\partial_x^3 V(x,t) + \frac{3}{2} V(x,t)\, V'(x,t)\,,$$

where $V'(x,t) = \partial_x V(x,t)$, is written as (1) if we take Lax pair:

$$M = \partial_x^3 + \frac{3}{4} V' + \frac{3}{2} V\, \partial_x\,, \quad L(\theta) = \partial_x^2 + V + \theta\,.$$

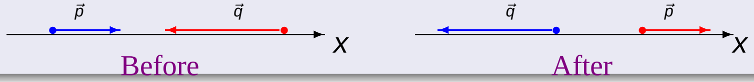
Zero-curvature condition can be written as

$$\partial_t L(\theta) = [M(\theta)\,,\; L(\theta)] \quad \Rightarrow$$

$$\partial_t \mathrm{Tr}\left(L^k\right) = \mathrm{Tr}\left(\left[M\,, L^k\right]\right) = 0 \quad \Rightarrow$$

$\mathbf{I}_k = \mathrm{Tr}\left(L^k(\theta)\right)$ are integrals of motion of the system. In other words the integrals of motion (IM) are eigenvalues of the operator $L(\theta)$.

For d=(1+1) integrable quantum relativistic field theories one obtains infinite number of IM $\Rightarrow$ all particles save their momenta after scattering $\Rightarrow$ factorizing scattering



$$p_0^2 - p_1^2 = m^2 \;\Rightarrow\; (p_0 + p_1) = e^{\theta}\, m\,,\;\; (p_0 - p_1) = e^{-\theta}\, m \;\Rightarrow$$

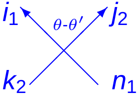
$$p_0 = m\,\mathrm{ch}(\theta)\;, \quad p_1 = m\,\mathrm{sh}(\theta)$$

The scattering of two particles with 2-momenta $\vec{p}$ and $\vec{q}$ is described by the two-particle S-matrix which depends on the invariant

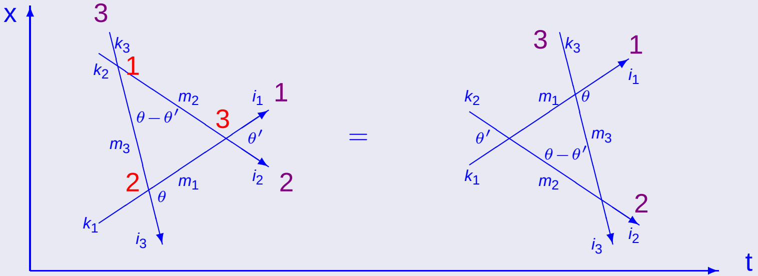
$$(\vec{p}, \vec{q}) = m^2(\mathrm{ch}(\theta)\mathrm{ch}(\theta') - \mathrm{sh}(\theta)\mathrm{sh}(\theta')) = m^2\mathrm{ch}(\theta - \theta')\,.$$

2. Yang-Baxter Equations

For two-particle S matrix (a single act of scattering) we have

$$S^{n_1 k_2}_{i_1 j_2}(\theta - \theta') =$$ 

arrowed lines show trajectories of particles; $\theta_{1,2}$- rapidities; $i_1, n_1, \ldots = 1, \ldots, N$ - colors. For 3-particle scattering and corresponding S matrix we have:



(2)

This is a graphical representation of the Yang-Baxter eqs. (triangle eqs.):

$$S^{k_2 k_3}_{m_2 m_3}(\theta - \theta')\, S^{k_1 m_3}_{m_1 i_3}(\theta)\, S^{m_1 m_2}_{i_1 i_2}(\theta') = S^{k_1 k_2}_{m_1 m_2}(\theta')\, S^{m_1 k_3}_{i_1 m_3}(\theta)\, S^{m_2 m_3}_{i_2 i_3}(\theta - \theta')$$

2. Yang-Baxter Equations

In concise matrix notations YBE is written as

$$S_{23}(\theta-\theta')\,S_{13}(\theta)\,S_{12}(\theta') = S_{12}(\theta')\,S_{13}(\theta)\,S_{23}(\theta-\theta')\ .$$

With the additional conditions of unitarity

$$S_{12}(\theta)\,S_{21}(-\theta) = I_{12} \quad \Leftrightarrow \quad S^{i_1 i_2}_{k_1 k_2}(\theta)\,S^{k_2 k_1}_{\ell_2 \ell_1}(-\theta) = \delta^{i_1}_{\ell_1}\,\delta^{i_2}_{\ell_2}\ ,$$

and crossing symmetry

$$S_{12}(\theta) = (S_{21}(i\pi-\theta))^{t_1}\ ,$$

the YB equations uniquely determine factorizable S matrices (with a minimal set of poles) describing the scattering of particle-like excitations in $(1+1)$-dimensional integrable relativistic models. The matrix $S^{i_1 i_2}_{j_1 j_2}(\theta)$ is the S matrix which describes the scattering of two neutral particles with isotopic spins i_1 and i_2 into two particles with spins j_1 and j_2. The spectral parameter θ is nothing but the difference of the rapidities of these particles.

Zamolodchikov algebra

Factorizable scattering on a line can be described by means of a Zamolodchikov algebra with generators $\{A^i(\theta)\}$ $(i = 1, \ldots, N)$ and defining relations

$$A^{i_1}(\theta)A^{i_2}(\theta') = S^{i_1\,i_2}_{k_1k_2}(\theta-\theta')A^{k_2}(\theta')A^{k_1}(\theta) \quad \Rightarrow$$
$$A^{1\rangle}(\theta)\,A^{2\rangle}(\theta') = S_{12}(\theta-\theta')\,A^{2\rangle}(\theta')\,A^{1\rangle}(\theta)\,.$$

For example: unitarity condition is obtained as

$$A^{1\rangle}(\theta)A^{2\rangle}(\theta') = S_{12}(\theta-\theta')A^{2\rangle}(\theta')A^{1\rangle}(\theta) =$$
$$= \underline{S_{12}(\theta-\theta')\,S_{21}(\theta'-\theta)}\,A^{1\rangle}(\theta)A^{2\rangle}(\theta')\,.$$

Zamolodchikov algebra

Yang-Baxter equations appears as associativity condition

$$\left(A^{1\rangle}(\theta)\,A^{2\rangle}(\theta')\right)A^{3\rangle}(\theta'') = A^{1\rangle}(\theta)\left(A^{2\rangle}(\theta')\,A^{3\rangle}(\theta'')\right),$$

for permutation:

$$A^{1\rangle}(\theta)\,A^{2\rangle}(\theta')\,A^{3\rangle}(\theta'') \;\to\; A^{3\rangle}(\theta'')\,A^{2\rangle}(\theta')\,A^{1\rangle}(\theta)$$

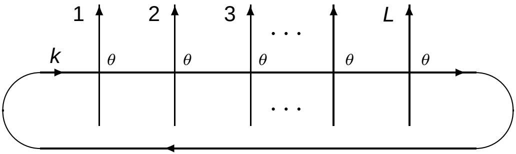
$$\left(A^{1\rangle}(\theta)\,A^{2\rangle}(\theta')\right)A^{3\rangle}(\theta'') = S_{12}(\theta-\theta')\,A^{2\rangle}(\theta')\left(A^{1\rangle}(\theta)A^{3\rangle}(\theta'')\right) =$$
$$= S_{12}(\theta-\theta')\,S_{13}(\theta-\theta'')\left(A^{2\rangle}(\theta')\,A^{3\rangle}(\theta'')\right)A^{1\rangle}(\theta) =$$
$$= \underline{S_{12}(\theta-\theta')\,S_{13}(\theta-\theta'')\,S_{13}(\theta'-\theta'')}\,A^{3\rangle}(\theta'')\,A^{2\rangle}(\theta')\,A^{1\rangle}(\theta)\,.$$

Any solution of the Yang-Baxter equations define quantum integrable system!

Introduce the transfer matrix (L - length of the chain)

$$t(\theta) = \left(\underbrace{S^{i_k i_1}_{j_k j_1}(\theta) S^{j_k i_2}_{r_k j_2}(\theta) \cdots S^{s_k i_L}_{i_k j_L}(\theta)}_{T_{12\ldots L}(\theta)} \right) =$$

$$= \mathrm{Tr}_k \left(\underbrace{S_{k1}(\theta) S_{k2}(\theta) \cdots S_{kL}(\theta)}_{T_{k;\,12\ldots L}(\theta)} \right) \in \mathrm{End}(\underbrace{V_N \otimes \cdots \otimes V_N}_{L}) \,.$$



3. Quantum Integrable Models

Transfermatrix $t(\theta)$ gives a commuting family of operators ("quantum integrals of motion"):

$$[t(\theta), t(\theta')] = 0 \ .$$

Proof is based on key relation (RTT-relation):

$$R_{km}(\theta - \theta')\, T_k(\theta)\, T_m(\theta') = T_m(\theta')\, T_k(\theta)\, R_{km}(\theta - \theta') \ .$$

Here $R_{km}(\theta) \sim S_{km}(\theta)$; we call rapidity θ – spectral parameter.
Using the operators $t(\theta)$ a set of integrals of motion can be constructed

$$\mathcal{I}_n = \frac{d^n}{d\theta^n} \ln\left(t(\theta)\, t(0)^{-1}\right) |_{\theta=0} \ .$$

and we identify the local Hamiltonian with

$$H \equiv \mathcal{I}_1 = \frac{d}{d\theta} \ln\left(t(\theta)\, t(0)^{-1}\right) |_{\theta=0} \ .$$

Example: $GL(N)$- (and $GL(N|M)$-) invariant R (or S) matrix.
Introduce permutation operator $P \in \mathrm{End}(V_N \otimes V_N)$:

$$P \cdot (v_1 \otimes v_2) = v_2 \otimes v_1 \ , \quad \forall v_1, v_2 \in V_N \ .$$

Let $(e_1, \dots, e_N)$ be basis vectors in V_N. Then

$$P \cdot e_i \otimes e_r = e_k \otimes e_\ell \, P_{ir}^{k\ell} = e_r \otimes e_i \quad \Rightarrow \quad P_{ir}^{k\ell} = \delta_r^k \ \delta_i^\ell \ .$$

In supersymmetric case we have super-permutation matrix

$$\mathbf{P} \cdot e_i \otimes e_r = e_k \otimes e_\ell \, \mathbf{P}_{ir}^{k\ell} = (-1)^{(r)(i)} e_r \otimes e_i \quad \Rightarrow \quad \mathbf{P}_{ir}^{k\ell} = (-1)^{(r)(i)} \delta_r^k \ \delta_i^\ell \ .$$

For comparison we discuss the unit operator I which acts as following

$$I \cdot (v_1 \otimes v_2) = v_1 \otimes v_2 \ , \quad \forall v_1, v_2 \in V_N \ .$$

$$I \cdot e_i \otimes e_r = e_k \otimes e_\ell \, I_{ir}^{k\ell} = e_i \otimes e_r \quad \Rightarrow \quad I_{ir}^{k\ell} = \delta_i^k \ \delta_r^\ell \ .$$

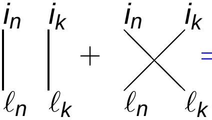
Any operator $R(\theta) \in \mathrm{End}(V_N \otimes V_N)$ which is invariant under $GL(N)$ transformations

$$(T \otimes T)\, R(\theta) = R(\theta)\, (T \otimes T) \;\; \forall T \in GL(N) \;\; \Rightarrow$$

$$T_1\, T_2 \;\, R_{12}(\theta) \;\, (T_1\, T_2)^{-1} = R_{12}(\theta) \quad ,$$

is represented as $R(\theta) = a(\theta) I + b(\theta) P$. Such operator solves YB equations only if it is proporional to the Yangian R-matrix $R = \theta\, I + P$, or in matrix representation

$R^{i_n i_k}_{\ell_n \ell_k}(\theta) = \theta$



$= (\theta\, \delta^{i_n}_{\ell_n} \delta^{i_k}_{\ell_k} + \delta^{i_n}_{\ell_k} \delta^{i_k}_{\ell_n}) \Rightarrow$

$$\Rightarrow \; R_{nk}(\theta) = (\theta\, I_{n,k} + P_{n,k}) \; .$$

For $GL(N|M)$ invariant R- matrix we have

$$R_{nk}(\theta) = (\theta\, I_{n,k} + \mathbf{P}_{n,k}) \; .$$

$$t(\theta) = \mathrm{Tr}_k \left(R_{k1}(\theta) R_{k2}(\theta) \cdots R_{kL}(\theta) \right) .$$

$$t(0) = \mathrm{Tr}_k \left(P_{k1} \cdots P_{kL} \right) = \mathrm{Tr}_k \left(P_{12} \cdots P_{1L} P_{k1} \right) = P_{12} \cdots P_{1L} .$$

$$H = \partial_\theta t(\theta) t(0)^{-1} \Big|_{\theta=0} = \sum_{r=1}^{L} \mathrm{Tr}_k \left(P_{k1} \cdots P_{k,r-1} P_{k,r+1} \cdots P_{kL} \right) t(0)^{-1} =$$

$$= \sum_{r=1}^{L} \mathrm{Tr}_k \left(P_{k1} \cdots P_{k,r-1} P_{k,r+1} \cdots P_{kL} \right) t(0)^{-1} =$$

$$= \sum_{r=1}^{L} \mathrm{Tr}_k \left(P_{12} \cdots P_{1,r-1} P_{1,r+1} \cdots P_{1L} P_{k1} \right) t(0)^{-1} =$$

$$= \mathrm{Tr}_k (P_{k2} \cdots P_{k,L}) P_{1,L} \cdots P_{12} + \sum_{r=2}^{L} P_{12} \cdots P_{1,r-1} P_{1,r} P_{1,r-1} \cdots P_{12} =$$

$$= P_{1,2} + P_{2,3} + \cdots + P_{L-1,L} + P_{L,1} = H .$$

is the Hamiltonian for $GL(N)$ periodic spin chain of length L. For $N = 2$ this is the Hamiltonian for the XXX Heisenberg spin chain.

A.P. Isaev,
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