BPS branes in 10 and 11 dimensional supergravity


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Introduction

1. Introduction
   - $10D$ and $11D$ Supergravity
   - $P$-brane Solution
   - The Supersymmetry Algebra and BPS Bound

Preserved Supersymmetries

2. Preserved Supersymmetries
   - Generalization to curved manifolds
   - Multiple Configurations
   - Configurations on Ricci-flat factor spaces
   - Killing Spinor Equations
   - The supersymmetry conditions

How does it work?

3. How does it work?
   - Pure electric background
   - Pure magnetic background
   - The intersection of two electric and one magnetic branes
   - The intersection of one electric and two magnetic branes

Conclusions
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Duality and reduction
**10D and 11D Supergravity**

**11 dimensional $\mathcal{N} = 1$ SUGRA**

$$S_{11D} = \int d^{11}z \sqrt{|g|} \left\{ R[g] - \frac{1}{2(4!)} \hat{F}^2 \right\} - \frac{1}{6} \int \hat{A} \wedge \hat{F} \wedge \hat{F},$$

where

$$\hat{F} = d\hat{A} = \frac{1}{4!} \hat{F}_{NPQR} dz^N \wedge dz^P \wedge dz^Q \wedge dz^R$$

is the 4-form field strength of the 3-form potential $A$. 
10D and 11D Supergravity

11 dimensional $\mathcal{N} = 1$ SUGRA

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\]

is the 4-form field strength of the 3-form potential $A$.

10 dimensional $\mathcal{N} = 2$ SUGRA

\[
S_{IIA, SF} = \int d^{10}x \sqrt{|g|} \left\{ e^{-2\varphi} \left[ R[g] + 4\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2(3!)} |F(3)|^2 \right] - \frac{1}{2(2!)} |F(2)|^2 - \frac{1}{2(4!)} |\tilde{F}(4)|^2 \right\} - \frac{1}{2} \int A_2 \wedge F(4) \wedge F(4),
\]

where $\varphi$ is the dilaton, $F(3) = dA_2$ is the field strength of the NS-NS two form, $F(2) = dA_1$ is the field strength of the R-R 1-form, $F(4) = dA_3$, $\tilde{F}(4) = dA_3 + F(3) \wedge A_1$ are the Ramond-Ramond field strengths.
P-brane Solution

Ansatz for the metric

\[ M_0 \times M_1 \]

the transverse space \hspace{1cm} the worldvolume
An ansatz for the metric is given by:

\[
M_0 \times M_1
\]

where \( M_0 \) is the transverse space and \( M_1 \) is the worldvolume. The metric can be written as:

\[
g = e^{2\phi(x)} g_0 + e^{2\gamma(x)} g_1.
\]

The harmonic function \( H \) satisfies the Laplace-Beltrami equation:

\[
\nabla^2 H = 0,
\]

where \( \nabla^2 \) is the Laplacian on the transverse space.
**P-brane Solution**

**Ansatz for the metric**

\[
\begin{align*}
\underbrace{M_0}_{\text{the transverse space}} & \times \underbrace{M_1}_{\text{the worldvolume}} \\
SO(D - p - 1) & \times (\text{Poincaré}_{(p+1)})
\end{align*}
\]

\[
g = e^{2\phi(x)}g^0 + e^{2\gamma(x)}g^1.
\]

**The harmonic function**

A \( p \)-brane solution depends on a harmonic function \( H \)

\[
\nabla^2 H = 0,
\]

where \( (\nabla^2) \) is the Laplacian on the transverse space.
The SUSY algebra

\[ \frac{1}{p!} (C \Gamma_{\mu_1 \ldots \mu_p})_{\alpha \beta} Z^{\mu_1 \ldots \mu_p}, \]

\[ Z^{\mu_1 \ldots \mu_p} = Q(p) \int dX^{\mu_1} \wedge dX^{\mu_2} \wedge \ldots \wedge dX^{\mu_p} \]

is the topological charge, \( C \) is the charge conjugation matrix, \( X^\mu \) are spacetime coordinates and \( \Gamma \) is an antisymmetric combination of Gamma matrices.

\[ \{ \hat{Q}_\alpha, \hat{Q}_\beta \} = Z_{\alpha \beta} \]

The BPS bound

\[ T = Q(p) \]
The SUSY algebra

\[
\frac{1}{p!}(CT_{\mu_1...\mu_p})_{\alpha\beta} Z^{\mu_1...\mu_p},
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11D SUGRA: \( M2 \)-brane, \( M5 \)-brane, KK monopole, \( M9 \)-brane
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11D SUGRA: \( M2 \)-brane, \( M5 \)-brane, KK monopole, \( M9 \)-brane
10D IIA SUGRA: \( D0 \)-brane, \( D2 \)-brane, \( D4 \)-brane, \( D6 \)-brane, \( D8 \)-brane, \( F \)-string, NS5, NS9, KK monopole
The SUSY algebra

\[
\frac{1}{p!} (CT_{\mu_1...\mu_p})_{\alpha\beta} Z^{\mu_1...\mu_p},
\]

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10D IIA SUGRA: \(D0\)-brane, \(D2\)-brane, \(D4\)-brane, \(D6\)-brane, \(D8\)-brane, \(F\)-string, NS5, NS9, KK monopole
10D IIB SUGRA: \(D1\)-brane, \(D3\)-brane, \(D5\)-brane, \(D7\)-brane, \(D9\)-brane, \(F\)-string, NS5, NS9, KK monopole
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4 Conclusions
The amount of preserved supersymmetries

\[ \mathcal{N} = n/\nu, \]

where \( \nu \) — a number of maximal supersymmetries of the system.
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11D SUGRA

- \( n = 0, 1, 2, 4, 5, 6, 8, 16, \nu = 32. \)
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- \( n = 0, 1, 2, 4, 5, 6, 8, 16, \nu = 32. \)
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- M-brane configurations on flat factor spaces

\[ M_0 \]
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### 11D SUGRA

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\[ M_0 \times M_1 \times \ldots \times M_n, \quad M_i = \mathbb{R}^{k_i} \]
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\[ \mathcal{N} = 2^{-k}, \quad k = 1, 2, 3, 4, 5 \]

The amount of preserved supersymmetries

\[ \mathcal{N} = \frac{n}{\nu}, \]

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- \( \Rightarrow \) P-branes on Ricci-flat manifolds, \( \mathcal{N} = ? \)
The generalization of worldvolume manifold to Ricci-flat manifold admitting Killing spinors

Generalization to curved manifolds

- The generalization of worldvolume manifold to Ricci-flat manifold admitting Killing spinors
- The solutions with indecomposable Ricci-flat brane worldvolumes
Orthogonally intersecting branes

The three branes have the \((p + 1)\)-dimensional common worldvolume space, the relative transverse \((q_1 + q_2 + q_3)\)-dimensional space, and the totally transverse \(\tilde{D}\)-dimensional space.
Orthogonally intersecting branes

\[(p + q_1) - \text{brane} :\]
\[(p + q_2) - \text{brane} :\]
\[(p + q_3) - \text{brane} :\]
Orthogonally intersecting branes

\[ (p + q_1) - \text{brane: } X \cdots X \]
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\[ (p + q_3) - \text{brane: } X \cdots X \]
Orthogonally intersecting branes

\[(p + q_1) \text{-- brane:} \quad \overbrace{X \ldots X}^{p + 1} \quad \overbrace{X \ldots X}^{q_1} \quad \overbrace{- \ldots -}^{q_2} \quad \overbrace{- \ldots -}^{q_3}\]

\[(p + q_2) \text{-- brane:} \quad \overbrace{X \ldots X}^{p + q_1} \quad \overbrace{- \ldots -}^{q_1} \quad \overbrace{X \ldots X}^{q_2} \quad \overbrace{- \ldots -}^{q_3}\]

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Orthogonally intersecting branes

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<tr>
<th></th>
<th>(p + 1)</th>
<th>(q_1)</th>
<th>(q_2)</th>
<th>(q_3)</th>
<th>(\tilde{D})</th>
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<td>(X \cdots X)</td>
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Orthogonally intersecting branes

The three branes have

- the \((p + 1)\)-dimensional common worldvolume space
- the relative transverse \((q_1 + q_2 + q_3)\)-dimensional space
- the totally transverse \(\tilde{D}\)-dimensional space.
Configurations on Ricci-flat factor spaces

The product of manifolds

\[ M = M_0 \times M_1 \times \ldots \times M_n \]
\[ g = e^{2\gamma(x)} g^0 + \sum_{i=1}^{n} e^{2\phi_i(x)} g^i. \]

The diagonalizing D-beins

\[ g_{MN} = \eta_{AB} e^A_M e^B_N, \]
with \( \eta_{AB} = \eta^{AB} = \eta_A \delta_{AB}, \quad e^A = e^A_M dx^M \)

The field strength for composite intersecting branes

\[ F = \sum_{s=1}^{m} c_s \mathcal{F}_s, \]
where \( \mathcal{F}_s \) is an elementary 4-form corresponding to s-th p-brane, \( c_s = \pm 1 \) is the sign factor of s-th p-brane, which defines the orientation of the worldvolume.
SUSY equations

\[ \delta \psi = (D_M + B_M) \epsilon = 0, \quad \text{where} \quad D_M = \partial_M + \frac{1}{4} w_{ABM} \hat{\Gamma}^A \hat{\Gamma}^B, \]

\( \epsilon \) is a SUSY transformation parameter, \( w_{ABM} \) is a spin connection.

\[ \hat{\Gamma}^A \hat{\Gamma}^B + \hat{\Gamma}^B \hat{\Gamma}^A = 2 \eta^{AB} 1_{32}, \]

\[ B_M = \frac{1}{288} \left( \Gamma_M \Gamma^N \Gamma^P \Gamma^Q \Gamma^R - 12 \delta^N_M \Gamma^P \Gamma^Q \Gamma^R \right) F_{NPQR} \]

and \( \Gamma_M \) are world gamma matrices satisfying Clifford algebra relations

\[ \Gamma_M = e^A_M \Gamma_A, \quad \Gamma_M \Gamma_N + \Gamma_N \Gamma_M = 2 g_{MN} 1_{32} \]
SUSY equations

### Killing Spinor Equations

\[ \delta\psi = (D_M + B_M)\varepsilon = 0, \quad \text{where} \quad D_M = \partial_M + \frac{1}{4} w_{ABM} \hat{\Gamma}^A \hat{\Gamma}^B, \]

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and \(\Gamma_M\) are world gamma matrices satisfying Clifford algebra relations

\[ \Gamma_M = e^A_M \Gamma_A, \quad \Gamma_M \Gamma_N + \Gamma_N \Gamma_M = 2g_{MN} 1_{32} \]

### The number of unbroken SUSY

\[ \mathcal{N} = N/32, \]

where \(N\) is the dimension of the linear space of solutions to differential equation \((D_M + B_M)\varepsilon = 0\).
The chirality conditions

The solutions admit spinors in the form

\[ \varepsilon = \left( \prod_{s \in S_e} H_s^{-1/6} \right) \left( \prod_{s \in S_m} H_s^{-1/12} \right) \eta \]

with the parallel spinor \( \eta \)

\[ \bar{D}_m^l \eta = 0, \quad \bar{D}_m^l = \partial_m^l + w_{a(l)b m l}, \quad l = 0, \ldots, n, \quad (1) \]

satisfying brane chirality conditions

\[ \hat{\Gamma}_{[s]} \eta = c_s \eta. \]

The chirality operators

\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}^{A_1} \hat{\Gamma}^{A_2} \hat{\Gamma}^{A_3}, \quad \text{for} \quad s \in S_e \]

\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}^{B_1} \hat{\Gamma}^{B_2} \hat{\Gamma}^{B_3} \hat{\Gamma}^{B_4} \hat{\Gamma}^{B_5}, \quad \text{for} \quad s \in S_m \]

\[ \left( \hat{\Gamma}_{[s]} \right)^2 = 1 \]
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The product of manifolds

\[ M_0 \times M_1 \times M_2 \times M_3 \times M_4 \]

\[ d_0 = 4, \quad d_1 = d_2 = d_3 = 2 \quad \text{and} \quad d_4 = 1 \]

The solution

\[ g = H_1^{1/3} H_2^{1/3} H_3^{1/3} \{ \tilde{g}^0 + H_1^{-1} \tilde{g}^1 + H_2^{-1} \tilde{g}^2 + H_3^{-1} \tilde{g}^3 + H_1^{-1} H_2^{-1} H_3^{-1} \tilde{g}^4 \} \]

\[ F = c_1 dH_1^{-1} \wedge \hat{\tau}_1 \wedge \hat{\tau}_4 + c_2 dH_2^{-1} \wedge \hat{\tau}_2 \wedge \hat{\tau}_4 + c_3 dH_3^{-1} \wedge \hat{\tau}_3 \wedge \hat{\tau}_4, \]

where \( c_1^2 = c_2^2 = c_3^2 = 1 \). The metrics \( g^i, \ i = 0, 1, 2, 3 \), have Euclidean signatures and \( g^4 = -dt \otimes dt \).
The set of gamma matrices

\[
(\hat{\Gamma}^A) = \\
\begin{align*}
(\hat{\Gamma}^0_0) & \otimes 1_2 \otimes 1_2 \otimes 1_2 \otimes 1, \\
\hat{\Gamma}_0 & \otimes \hat{\Gamma}^1_0 \otimes 1_2 \otimes 1_2 \otimes 1, \\
i\hat{\Gamma}_0 & \otimes \hat{\Gamma}_1 \otimes \hat{\Gamma}^2_0 \otimes 1_2 \otimes 1, \\
\hat{\Gamma}_0 & \otimes \hat{\Gamma}_1 \otimes \hat{\Gamma}_2 \otimes \hat{\Gamma}^3_0 \otimes 1, \\
\hat{\Gamma}_0 & \otimes \hat{\Gamma}_1 \otimes \hat{\Gamma}_2 \otimes \hat{\Gamma}_3 \otimes 1,
\end{align*}
\]

where

\[
\hat{\Gamma}_0 = \hat{\Gamma}^{10}_0 \ldots \hat{\Gamma}^{40}_0, \quad \hat{\Gamma}_i = \hat{\Gamma}^{1i}_i \hat{\Gamma}^{2i}_i,
\]

obey

\[
(\hat{\Gamma}^0_0)^2 = 1_4, \quad (\hat{\Gamma}^i_0)^2 = -1_2, \quad i = 1, 2, 3.
\]

The monomial spinor reads

\[
\eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4),
\]

where \(\eta_0 = \eta_0(x)\) is 4-component spinor on \(M_0\), \(\eta_i = \eta_i(y_i)\) is 2-component spinor on \(M_i\), \(i = 1, 2, 3\), and \(\eta_4 = \eta_4(y_4)\) is 1-component spinor on \(M_4\).
\[ \tilde{D}^{(0)}_{m_0} \eta = (D^{(0)}_{m_0} \eta_0) \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4, \]
\[ \tilde{D}^{(1)}_{m_1} \eta = \eta_0 \otimes (D^{(1)}_{m_1} \eta_1) \otimes \eta_2 \otimes \eta_3 \otimes \eta_4, \]
\[ \tilde{D}^{(2)}_{m_2} \eta = \eta_0 \otimes \eta_1 \otimes (D^{(2)}_{m_2} \eta_2) \otimes \eta_3 \otimes \eta_4, \]
\[ \tilde{D}^{(3)}_{m_3} \eta = \eta_0 \otimes \eta_1 \otimes \eta_2 \otimes (D^{(3)}_{m_3} \eta_3) \otimes \eta_4, \]
\[ \tilde{D}^{(4)}_{m_4} \eta = \eta_0 \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes (D^{(4)}_{m_4} \eta_4), \]

where \( D^{(i)}_{m_i} \) correspond to \( M_i, \ i = 0, 1, 2, 3 \). Here \( D^{(4)}_{m_4} = \partial_{m_4} \).

\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}^{11}_{(0)} \hat{\Gamma}^{21}_{(1)} \hat{\Gamma}^{14}_{(3)} = -\hat{\Gamma}_{(0)} \otimes 1_2 \otimes \hat{\Gamma}_{(2)} \otimes \hat{\Gamma}_{(3)} \otimes 1, \quad \text{for} \quad s = I_1, \]
\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}^{12}_{(0)} \hat{\Gamma}^{22}_{(1)} \hat{14}_{(3)} = -\hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(1)} \otimes 1_2 \otimes \hat{\Gamma}_{(3)} \otimes 1, \quad \text{for} \quad s = I_2, \]
\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}^{13}_{(0)} \hat{\Gamma}^{23}_{(1)} \hat{14}_{(3)} = -\hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(1)} \otimes \hat{\Gamma}_{(2)} \otimes 1_2 \otimes 1, \quad \text{for} \quad s = I_3. \]

The chirality restrictions are satisfied if

\[ \hat{\Gamma}_{(0)} \eta_0 = c_{(0)} \eta_0, \quad c_{(0)}^2 = 1, \]
\[ \hat{\Gamma}_{(j)} \eta_j = c_{(j)} \eta_j, \quad c_{(j)}^2 = -1, \]

\[ j = 1, 2, 3 \text{ with } \]
\[ c_{(0)} = c_1 c_2 c_3, \quad c_{(j)} = \pm ic_j. \]
The following solution to SUSY equations corresponding to the field configuration from

\[
\varepsilon = H_1^{-1/6} H_2^{-1/6} H_3^{-1/6} \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4.
\]

Here \(\eta_i, i = 0, 1, 2, 3\), are chiral parallel spinors defined on \(M_i\), respectively \((D_{m_i}^{(i)}\eta_i = 0)\), \(\eta_4\) is constant.

The number of linear independent solutions to \((D_M + B_m)\varepsilon = 0\)

\[
N = 32N = n_0(c_1c_2c_3) \sum_{c=\pm 1} n_1(icc_1)n_2(icc_2)n_3(icc_3),
\]

where \(n_j(c_{(j)})\) is the number of chiral parallel spinors on \(M_j, j = 0, 1, 2, 3\).

<table>
<thead>
<tr>
<th>(M)</th>
<th>(M_0)</th>
<th>(M_1)</th>
<th>(M_2)</th>
<th>(M_3)</th>
<th>(N = 1/16n_0(c_1c_2c_3))</th>
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</thead>
<tbody>
<tr>
<td>(\mathbb{R}^4)</td>
<td>(\mathbb{R}^2)</td>
<td>(\mathbb{R}^2)</td>
<td>(\mathbb{R}^2)</td>
<td>1/8</td>
<td></td>
</tr>
<tr>
<td>(K3 = CY_2)</td>
<td>(\mathbb{R}^2)</td>
<td>(\mathbb{R}^2)</td>
<td>(\mathbb{R}^2)</td>
<td>1/8 for (c_1c_2c_3 = 1) / 0 for (c_1c_2c_3 = -1)</td>
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<tr>
<td>(\mathbb{C}^2*/\mathbb{Z}_2)</td>
<td>(\mathbb{R}^2)</td>
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<td>1/8 for (c_1c_2c_3 = 1) / 0 for (c_1c_2c_3 = -1)</td>
<td></td>
</tr>
</tbody>
</table>
The product manifold

\[ M_0 \times M_1 \times M_2 \times M_3 \times M_4, \]

where \( d_0 = 3, \ d_1 = d_2 = d_3 = d_4 = 2. \)

The solutions for the metric and field strengths

\[ g = H_1^{2/3} H_2^{2/3} H_3^{2/3} \left\{ g^0 + H_2^{-1} H_3^{-1} g^1 + H_1^{-1} H_3^{-1} g^2 + H_1^{-1} H_2^{-1} g^3 + H_1^{-1} H_2^{-1} H_3^{-1} g^4 \right\}, \]

\[ F = c_1 (\star_0 dH_1) \wedge \tau_1 + c_2 (\star_0 dH_2) \wedge \tau_2 + c_3 (\star_0 dH_3) \wedge \tau_3, \]

where \( c_1^2 = c_2^2 = c_3^2 = 1. \)
The set of gamma matrices

\[
(\hat{\Gamma}^A) =
\begin{pmatrix}
1 & \hat{\Gamma}^{(1)} & \hat{\Gamma}^{(2)} & \hat{\Gamma}^{(3)} & \hat{\Gamma}^{(4)} \\
1 & i\hat{\Gamma}^{a_1(1)} & \hat{\Gamma}^{(2)} & \hat{\Gamma}^{(3)} & \hat{\Gamma}^{(4)} \\
1 & \hat{1}_2 & \hat{\Gamma}^{a_2(2)} & \hat{\Gamma}^{(3)} & \hat{\Gamma}^{(4)} \\
1 & \hat{1}_2 & \hat{1}_2 & i\hat{\Gamma}^{a_3(3)} & \hat{\Gamma}^{(4)} \\
1 & \hat{1}_2 & \hat{1}_2 & \hat{1}_2 & \hat{\Gamma}^{a_4(4)}
\end{pmatrix}
\]

Here the operators \(\hat{\Gamma}^{(i)}\), \(i = 1, 2, 3, 4\), are given by

\[
\hat{\Gamma}^{(1)} = \hat{\Gamma}^{1_1(1)}\hat{\Gamma}^{2_1(1)}, \quad \hat{\Gamma}^{(2)} = \hat{\Gamma}^{1_2(2)}\hat{\Gamma}^{2_2(2)}, \quad \hat{\Gamma}^{(3)} = \hat{\Gamma}^{1_3(3)}\hat{\Gamma}^{2_3(3)}, \quad \hat{\Gamma}^{(4)} = \hat{\Gamma}^{1_4(4)}\hat{\Gamma}^{2_4(4)}\hat{\Gamma}^{3_4(4)}\hat{\Gamma}^{4_4(4)}
\]

obey \((\hat{\Gamma}^{(i)})^2 = -1_2\), \((\hat{\Gamma}^{(4)})^2 = -1_4\), with \(i = 1, 2, 3\).

\[
\eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4),
\]

where \(\eta_0(x)\) is a 1-component spinor on \(M_0\), \(\eta_i = \eta_i(y_i)\) is a 2-component spinor on \(M_i\), \(i = 1, 2, 3\), \(\eta_4 = \eta_4(y_4)\) is a 4-component spinor on \(M_4\).
The covariant derivatives can be written down as

\[
\bar{D}_{m_1} = \partial_{m_1} + \frac{1}{4} \omega^{(1)}_{a_1 b_1 m_1} \left( 1 \otimes \hat{\Gamma}^{a_1}_{(1)} \hat{\Gamma}^{b_1}_{(1)} \otimes 1_2 \otimes 1_2 \otimes 1_4 \right),
\]

\[
\bar{D}_{m_2} = \partial_{m_2} + \frac{1}{4} \omega^{(2)}_{a_2 b_2 m_2} \left( 1 \otimes 1_2 \otimes \hat{\Gamma}^{a_2}_{(2)} \hat{\Gamma}^{b_2}_{(2)} \otimes 1_2 \otimes 1_4 \right),
\]

\[
\bar{D}_{m_3} = \partial_{m_3} + \frac{1}{4} \omega^{(3)}_{a_3 b_3 m_3} \left( 1 \otimes 1_2 \otimes 1_2 \otimes \hat{\Gamma}^{a_3}_{(3)} \hat{\Gamma}^{b_3}_{(3)} \otimes 1_4 \right),
\]

\[
\bar{D}_{m_4} = \partial_{m_4} + \frac{1}{4} \omega^{(4)}_{a_4 b_4 m_4} \left( 1 \otimes 1_2 \otimes 1_2 \otimes 1_2 \otimes \hat{\Gamma}^{a_4}_{(4)} \hat{\Gamma}^{b_4}_{(4)} \right),
\]

where \( \omega^{(i)}_{a_i b_i c_i} \) are components of the spin connection corresponding to the manifold \( M_i \), \( D^{(i)}_{m_i} \) is a covariant derivatives corresponding to \( M_i \), \( i = 1, 2, 3, 4 \), \( \bar{D}_{m_0} = \partial_{m_0} \) and \( D^{(0)}_{m_0} = \partial_{m_0} \).

\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{10} \hat{\Gamma}^{12} \hat{\Gamma}^{22} \hat{\Gamma}^{13} \hat{\Gamma}^{23} = 1 \otimes \hat{\Gamma}^{(1)} \otimes 1_2 \otimes 1_2 \otimes \hat{\Gamma}^{(4)} \quad \text{for } s = I_1,
\]

\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{10} \hat{\Gamma}^{11} \hat{\Gamma}^{21} \hat{\Gamma}^{13} \hat{\Gamma}^{23} = 1 \otimes 1_2 \otimes \hat{\Gamma}^{(2)} \otimes 1_2 \otimes \hat{\Gamma}^{(4)} \quad \text{for } s = I_2,
\]

\[
\hat{\Gamma}_{[s]} = \hat{\Gamma}^{10} \hat{\Gamma}^{11} \hat{\Gamma}^{21} \hat{\Gamma}^{12} \hat{\Gamma}^{22} = 1 \otimes 1_2 \otimes 1_2 \otimes \hat{\Gamma}^{(3)} \otimes \hat{\Gamma}^{(4)} \quad \text{for } s = I_3.
\]
The supersymmetry constraints

\[ \hat{\Gamma}(j)\eta_j = c(j)\eta_j, \quad c(j)^2 = -1, \quad j = 1, 2, 3, 4, \]

and

\[ c(1)c(4) = c_1, \quad c(2)c(4) = c_2, \quad c(3)c(4) = c_3. \]

\[ \varepsilon = \prod_{s=1}^{3} H_s^{-\frac{1}{12}} \eta_0 \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_2(y_3) \otimes \eta_4(y_4), \]

where \( \eta_i, \ i = 1, 2, 3, 4 \) are parallel spinors defined on \( M_i \), respectively.

\[ c(1) = -ic_1, \quad c(2) = -ic_2, \quad c(3) = -ic_3, \quad c(4) = i, \]
\[ c(1) = ic_1, \quad c(2) = ic_2, \quad c(3) = ic_3, \quad c(4) = -i. \]

The number of preserved supersymmetries

\[ N = 32\mathcal{N} = n_1(-ic_1)n_2(-ic_2)n_3(-ic_3)n_4(i) + n_1(ic_1)n_2(ic_2)n_3(ic_3)n_4(-i), \]

where \( n_j(c_j) \) is the number of chiral parallel spinors on \( M_j, \ j = 1, 2, 3, 4. \)
Examples

Let $M_0 = \mathbb{R}$ and $M_1 = M_2 = M_3 = \mathbb{R}^2$. Then all $n_j(c) = 1$, $j = 1, 2, 3$, with $c = \pm i$, and hence

$$N = 32 \mathcal{N} = n_4(i) + n_4(-i).$$

(2)

<table>
<thead>
<tr>
<th>$M$</th>
<th>$M_0$</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
<th>$M_4$</th>
<th>$\mathcal{N}$</th>
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<td>$\mathbb{R}^{1,3}$</td>
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<td>$(\mathbb{R}^{1,1}/Z_2) \times \mathbb{R}^2$</td>
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<td>a 4d pp-wave manifold</td>
<td>1/16</td>
</tr>
</tbody>
</table>
The product manifold

\[ M_0 \times M_1 \times M_2 \times M_3 \times M_4 \times M_5 \times M_6, \]

\[ d_0 = 3, \quad d_1 = d_2 = d_4 = d_5 = d_6 = 1 \text{ and } d_3 = 3. \]

The solution reads

\[
g = H_1^{1/3} H_2^{1/3} H_3^{2/3} \left\{ g^0 + H_1^{-1} g^1 + H_2^{-1} g^2 + H_3^{-1} g^3 + H_2^{-1} H_3^{-1} g^4 + H_1^{-1} H_3^{-1} g^5 + H_1^{-1} H_2^{-1} H_3^{-1} g^6 \right\},
\]

\[ F = c_1 dH_1^{-1} \wedge \tau_1 \wedge \tau_5 \wedge \tau_6 + c_2 dH_2^{-1} \wedge \tau_2 \wedge \tau_4 \wedge \tau_6 + c_3 (\ast_0 dH_3) \wedge \tau_1 \wedge \tau_2, \]
The set of $\Gamma$-matrices

\[
(\hat{\Gamma}^A) = (\hat{\Gamma}^{a_0}_{(0)} \otimes 1 \otimes 1 \otimes \mathbf{1}_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_3 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2,
\mathbf{1}_2 \otimes 1 \otimes 1 \otimes \mathbf{1}_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_1 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2,
\mathbf{1}_2 \otimes 1 \otimes 1 \otimes \mathbf{1}_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_3 \otimes \mathbf{1}_2,
\mathbf{1}_2 \otimes 1 \otimes 1 \otimes \hat{\Gamma}^{a_3}_{(3)} \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_1 \otimes \mathbf{1}_2,
\mathbf{1}_2 \otimes 1 \otimes 1 \otimes \mathbf{1}_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_3,
\mathbf{1}_2 \otimes 1 \otimes 1 \otimes \mathbf{1}_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_1,
\mathbf{1}_2 \otimes 1 \otimes 1 \otimes \mathbf{1}_2 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes i \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2).\]

\[
\hat{\Gamma}_{(0)} = \hat{\Gamma}^{10}_{(0)} \hat{\Gamma}^{20}_{(0)} \hat{\Gamma}^{30}_{(0)}, \quad \hat{\Gamma}_{(3)} = \hat{\Gamma}^{13}_{(3)} \hat{\Gamma}^{23}_{(3)} \hat{\Gamma}^{33}_{(3)}
\]

\[
(\hat{\Gamma}^{a_i}_{(i)}) = (\sigma_1, \sigma_2, \sigma_3), \quad \hat{\Gamma}_{(i)} = i\mathbf{1}_2, \quad i = 0, 6
\]
\[ \eta = \eta_0(x) \otimes \eta_1(y_1) \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4) \otimes \eta_5(y_5) \otimes \eta_6(y_6) \otimes \chi, \]

where \( \eta_i = \eta_i(y_i) \) is a 1-component spinor on \( M_i, i = 1, 2, 4, 5, 6 \), \( \eta_0 = \eta_0(x) \) is a 2-component spinor on \( M_0 \), \( \eta_3 = \eta_3(y_3) \) is a 2-component spinor on \( M_3 \) and \( \chi \) belongs to \( V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \). The covariant derivatives \( \bar{D}_{m_i} \) act on \( \eta \) as follows

\[
\bar{D}_{m_0} \eta = \left( D^{(0)}_{m_0} \eta_0 \right) \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4 \otimes \eta_5 \otimes \eta_6 \otimes \chi, \\
\bar{D}_{m_3} \eta = \eta_0 \otimes \eta_1 \otimes \eta_2 \otimes \left( D^{(3)}_{m_3} \eta_3 \right) \otimes \eta_4 \otimes \eta_5 \otimes \eta_6 \otimes \chi, \\
\bar{D}_{m_6} \eta = \eta_0 \otimes \eta_1 \otimes \eta_2 \otimes \eta_3 \otimes \eta_4 \otimes \eta_5 \otimes \left( D^{(6)}_{m_6} \eta_6 \right) \otimes \chi,
\]

where

\[
\bar{D}_{m_0} = \partial_{m_0} + \frac{1}{4} \omega^{(0)}_{a_0 b_0 m_0} \left( \hat{\Gamma}^{a_0}_{0} \hat{\Gamma}^{b_0}_{0} \otimes 1 \otimes 1 \otimes 1_2 \otimes 1 \otimes 1 \otimes 1_2 \otimes 1_2 \otimes 1_2 \right), \\
\bar{D}_{m_3} = \partial_{m_3} + \frac{1}{4} \omega^{(3)}_{a_3 b_3 m_3} \left( 1_2 \otimes 1 \otimes 1 \otimes \hat{\Gamma}^{a_3}_{3} \hat{\Gamma}^{b_3}_{3} \otimes 1 \otimes 1 \otimes 1 \otimes 1_2 \otimes 1_2 \otimes 1_2 \right).
\]
The operators corresponding to the $M2$-branes and the $M5$-brane

\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}^{11} \hat{\Gamma}^{15} \hat{\Gamma}^{16} = -\mathbf{1}_2 \otimes 1 \otimes 1 \otimes \mathbf{1}_2 \otimes 1 \otimes 1 \otimes 1 \otimes B_1, \]

for $s = I_1$,

\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}^{12} \hat{\Gamma}^{14} \hat{16} = -\mathbf{1}_2 \otimes 1 \otimes 1 \otimes \mathbf{1}_2 \otimes 1 \otimes 1 \otimes 1 \otimes B_2, \]

for $s = I_2$,

\[ \hat{\Gamma}_{[s]} = \hat{\Gamma}^{10} \hat{\Gamma}^{20} \hat{\Gamma}^{30} \hat{\Gamma}^{11} \hat{12} = i\hat{\Gamma}(0) \otimes 1 \otimes 1 \otimes \mathbf{1}_2 \otimes 1 \otimes 1 \otimes 1 \otimes B_3, \]

for $s = I_3$.

$B_s$ are self-adjoint commuting idempotent operators acting on

\[ V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \]

\[ B_1 = \sigma_1 \otimes \mathbf{1}_2 \otimes \sigma_3, \quad B_2 = \sigma_2 \otimes \sigma_3 \otimes \sigma_1, \quad B_3 = \mathbf{1}_2 \otimes \sigma_3 \otimes \mathbf{1}_2. \]

\[ \varepsilon = H_1^{-1/6} H_2^{-1/6} H_3^{-1/12} \eta_0(x) \otimes \eta_1 \otimes \eta_2 \otimes \eta_3(y_3) \otimes \eta_4 \otimes \eta_5 \otimes \eta_6 \otimes \psi_{\varepsilon_1,\varepsilon_2,\varepsilon_3}, \]

where $\eta_0(x)$ and $\eta_3(y_3)$ are parallel spinors defined on $M_0$ and $M_3$, respectively, $\eta_i$ is a constant 1-dimensional spinor on $M_i$, $i = 1, 2, 4, 5, 6$. 

\[ \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_10. \]
The number of preserved SUSY

\[ \mathcal{N} = n_0 n_3 / 32 \]

where \( n_j \) is the number of parallel spinors on the 3-dimensional manifolds \( M_j \), \( j = 0, 3 \).

<table>
<thead>
<tr>
<th>( M )</th>
<th>( M_0 )</th>
<th>( M_1 )</th>
<th>( M_2 )</th>
<th>( M_3 )</th>
<th>( M_4 )</th>
<th>( M_5 )</th>
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<th>( \mathcal{N} )</th>
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<td>( \mathbb{R} )</td>
<td>( 1/8 )</td>
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</tbody>
</table>
The product of manifolds

\[ M_0 \times M_1 \times M_2 \times M_3 \times M_4 \times M_5, \]

\[ d_0 = d_2 = d_3 = d_4 = d_5 = 2 \text{ and } d_1 = 1. \]

The solution

\[ g = \left\{ \begin{array}{ccccccc}
\hat{g}^0 + H_1^{-1} \hat{g}^1 + H_2^{-1} \hat{g}^2 + H_3^{-1} \hat{g}^3 + H_1^{-1} H_2^{-1} H_3^{-1} \hat{g}^4 + \\
H_1^{-1} H_2^{-1} H_3^{-1} \hat{g}^5 \end{array} \right\}. \]

\[ F = c_1 dH_1^{-1} \wedge \hat{\tau}_1 \wedge \hat{\tau}_5 + c_2 (\ast_0 dH_2) \wedge \hat{\tau}_1 \wedge \hat{\tau}_3 + c_3 (\ast_0 dH_3) \wedge \hat{\tau}_1 \wedge \hat{\tau}_2, \]

\[ c_1^2 = c_2^2 = c_3^2 = 1. \]

\[ g^i \ (i = 0, 1, 2, 3, 4) \] have Euclidean signatures \( g^5 \) has the signature \((-+,+).\)
\( M_2 \cap M_5 \cap M_5 \)

**Γ-matrices**

\[
(\hat{\Gamma}^A) =
\begin{align*}
(\hat{\Gamma}^0_0) & \otimes 1 \otimes 1_2 \otimes 1_2 \otimes 1_2 \otimes 1_2, \\
\hat{\Gamma}_0 & \otimes 1 \otimes \hat{\Gamma}^{a_2}_{(2)} \otimes \hat{\Gamma}^{a_3}_{(3)} \otimes \hat{\Gamma}^{a_4}_{(4)} \otimes \hat{\Gamma}^{a_5}_{(5)}, \\
i\hat{\Gamma}_0 & \otimes 1 \otimes \hat{\Gamma}^{a_2}_{(2)} \otimes \hat{\Gamma}^{a_3}_{(3)} \otimes \hat{\Gamma}^{a_4}_{(4)} \otimes \hat{\Gamma}^{a_5}_{(5)}, \\
\hat{\Gamma}_0 & \otimes 1 \otimes \hat{\Gamma}^{a_2}_{(2)} \otimes \hat{\Gamma}^{a_3}_{(3)} \otimes \hat{\Gamma}^{a_4}_{(4)} \otimes \hat{\Gamma}^{a_5}_{(5)}, \\
\hat{\Gamma}_0 & \otimes 1 \otimes \hat{\Gamma}^{a_2}_{(2)} \otimes \hat{\Gamma}^{a_3}_{(3)} \otimes \hat{\Gamma}^{a_4}_{(4)} \otimes \hat{\Gamma}^{a_5}_{(5)}, \\
\hat{\Gamma}_0 & \otimes 1 \otimes \hat{\Gamma}^{a_2}_{(2)} \otimes \hat{\Gamma}^{a_3}_{(3)} \otimes \hat{\Gamma}^{a_4}_{(4)} \otimes \hat{\Gamma}^{a_5}_{(5)}.
\end{align*}
\]

\[
\hat{\Gamma}^{(5)} = \hat{\Gamma}^{15}_{(5)} \hat{\Gamma}^{25}_{(5)}, \quad \hat{\Gamma}^{(i)} = \hat{\Gamma}^{1i}_{(i)} \hat{\Gamma}^{2i}_{(i)},
\]
satisfy

\[
(\hat{\Gamma}^{(i)})^2 = -1_2, \quad (\hat{\Gamma}^{(5)})^2 = 1_2, \quad i = 0, 2, 3, 4.
\]

\[
\eta = \eta_0(x) \otimes \eta_1 \otimes \eta_2(y_2) \otimes \eta_3(y_3) \otimes \eta_4(y_4) \otimes \eta_5(y_5)
\]

where \( \eta_i = \eta_i(y_i) \) is a 2-component spinor on \( M_i, i = 0, 2, 3, 4, 5 \), \( \eta_1 \) is a 1-component spinor on \( M_1 \).
The operator $\bar{D}_{m_i}^{(i)}$ acts on $\eta$ as

$$\bar{D}_{m_i}^{(i)} \eta = \ldots \otimes \eta_{i-1} \otimes \left(D_{m_i}^{(i)} \eta_i\right) \otimes \eta_{i+1} \otimes \ldots$$

The chirality operators

$$\hat{\Gamma}_{[s]} = \hat{\Gamma}^{11} \hat{\Gamma}^{13} \hat{\Gamma}^{23} = \hat{\Gamma}(0) \otimes 1 \otimes \hat{\Gamma}(2) \otimes \hat{\Gamma}(3) \otimes \hat{\Gamma}(4) \otimes 1_2, \quad \text{for } s = I_1,$$

$$\hat{\Gamma}_{[s]} = \hat{\Gamma}^{10} \hat{\Gamma}^{20} \hat{\Gamma}^{11} \hat{\Gamma}^{13} \hat{\Gamma}^{23} = 1_2 \otimes 1 \otimes \hat{\Gamma}(2) \otimes 1_2 \otimes \hat{\Gamma}(4) \otimes \hat{\Gamma}(5), \quad \text{for } s = I_2,$$

$$\hat{\Gamma}_{[s]} = \hat{\Gamma}^{10} \hat{\Gamma}^{20} \hat{\Gamma}^{11} \hat{\Gamma}^{12} \hat{\Gamma}^{22} = 1_2 \otimes 1 \otimes 1_2 \otimes \hat{\Gamma}(3) \otimes \hat{\Gamma}(4) \otimes \hat{\Gamma}(5), \quad \text{for } s = I_3.$$

The restrictions are satisfied if

$$\hat{\Gamma}(3) \eta_3 = c(3) \eta_3, \quad c^2(3) = 1,$$

$$\hat{\Gamma}(j) \eta_j = c(j) \eta_j, \quad c^2(j) = -1, \quad j = 0, 2, 4, 5$$

$$c(0)c(2)c(3)c(4) = c_1, \quad c(2)c(4)c(5) = c_2, \quad c(3)c(4)c(5) = c_3.$$
The solution to SUSY equations

\[ \varepsilon = H^{-1/6} H_2^{-1/12} H_3^{-1/12} \eta_0(x) \otimes \eta_1 \otimes \eta_2(y_2) \otimes \eta_2(y_3) \otimes \eta_4(y_4) \otimes \eta_5(y_5), \]

where \( \eta_i, i = 0, 2, 3, 4, 5 \) are chiral parallel spinors defined on \( M_i \), \( \eta_1 \) is constant.

The number of linear independent solutions to \( (D_M + B_m)\varepsilon = 0 \)

\[ N = 32N = \sum_{\varepsilon_2 = \pm 1, \varepsilon_4 = \pm 1} n_0(i\varepsilon_4 c_1 c_2 c_3)n_2(i\varepsilon_2)n_3(i\varepsilon_2 c_2 c_3)n_4(i\varepsilon_4)n_5(-\varepsilon_2 \varepsilon_4 c_2), \]

where \( n_j(c_j) \) is the number of chiral parallel spinors on \( M_j \), \( j = 0, 2, 3, 4, 5 \), \( \varepsilon_2 = \pm 1, \varepsilon_4 = \pm 1 \).

<table>
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   - $P$-brane Solution
   - The Supersymmetry Algebra and BPS Bound

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   - Configurations on Ricci-flat factor spaces
   - Killing Spinor Equations
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3 How does it work?
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   - Pure magnetic background
   - The intersection of two electric and one magnetic branes
   - The intersection of one electric and two magnetic branes

4 Conclusions
Conclusions

- We have obtained relations for computing the amount of preserved SUSY $M^2$/$M^5$-branes defined on product spaces including Ricci-flat manifolds and flat spaces with non-trivial topology.
- All possible orthogonal intersections of two M-branes: $M^2 \cap M^2$, $M^2 \cap M^5$, $M^5 \cap M^5$.
- All possible triple intersections: $M^2 \cap M^2 \cap M^2$, $M^5 \cap M^5 \cap M^5$ (three configurations), $M^2 \cap M^5 \cap M^5$ (two configurations), $M^2 \cap M^2 \cap M^5$. 
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The amount of preserved supersymmetries for the model defined on the product of factor spaces including Ricci-flat manifolds or manifolds with non-trivial topology is less than for the case of the product of flat spaces.
Conclusions

- We have obtained relations for computing the amount of preserved SUSY
  - $M2$-$M5$-branes defined on product spaces including Ricci-flat manifolds and flat spaces with non-trivial topology.
  - All possible orthogonal intersections of two M-branes: $M2 \cap M2$, $M2 \cap M5$, $M5 \cap M5$.
  - All possible triple intersections: $M2 \cap M2 \cap M2$, $M5 \cap M5 \cap M5$ (three configurations), $M2 \cap M5 \cap M5$ (two configurations), $M2 \cap M2 \cap M5$.
- The amount of preserved supersymmetries for the model defined on the product of factor spaces including Ricci-flat manifolds or manifolds with non trivial topology is less than for the case of the product of flat spaces.
- For the intersections $M2 \cap M2 \cap M2$, $M5 \cap M5 \cap M5$ we have presented examples where $\mathcal{N}$ depend on upon brane sign factors $c_s = \pm 1$.


THANK YOU FOR YOUR ATTENTION!