Integrable Cosmologies in Supergravity

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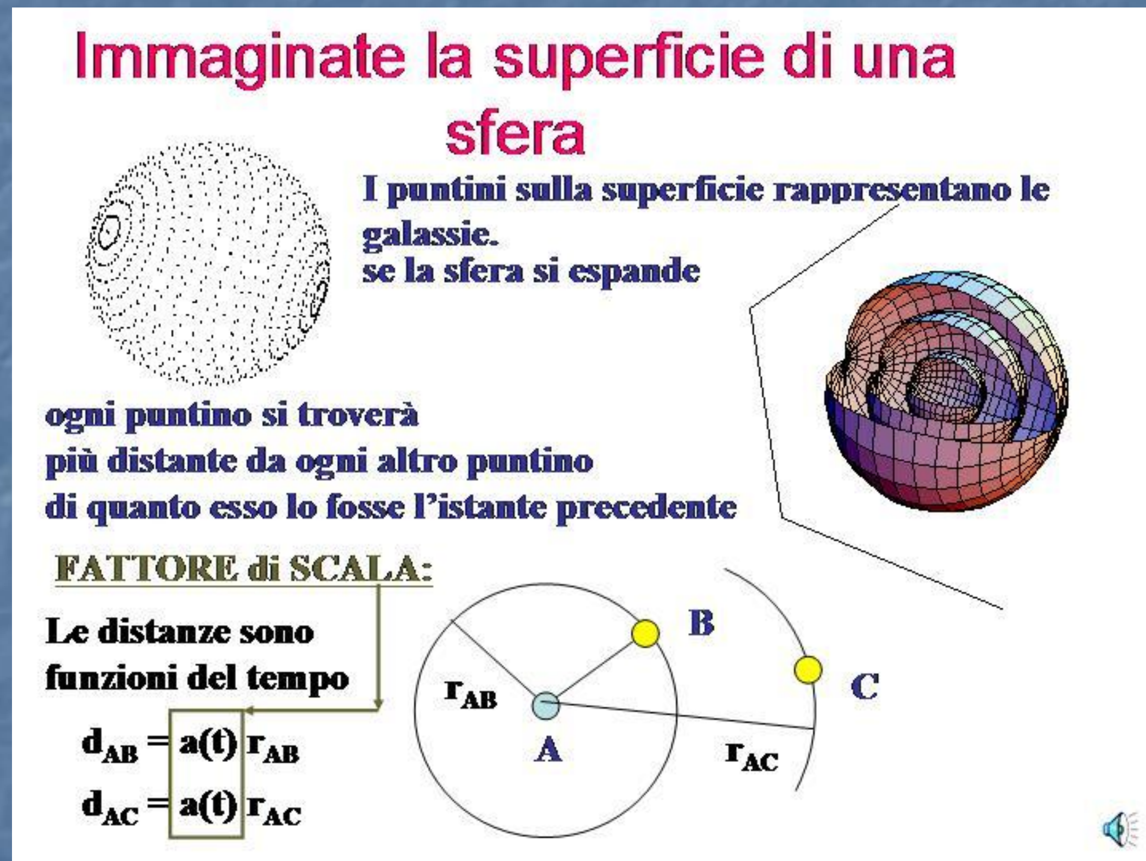
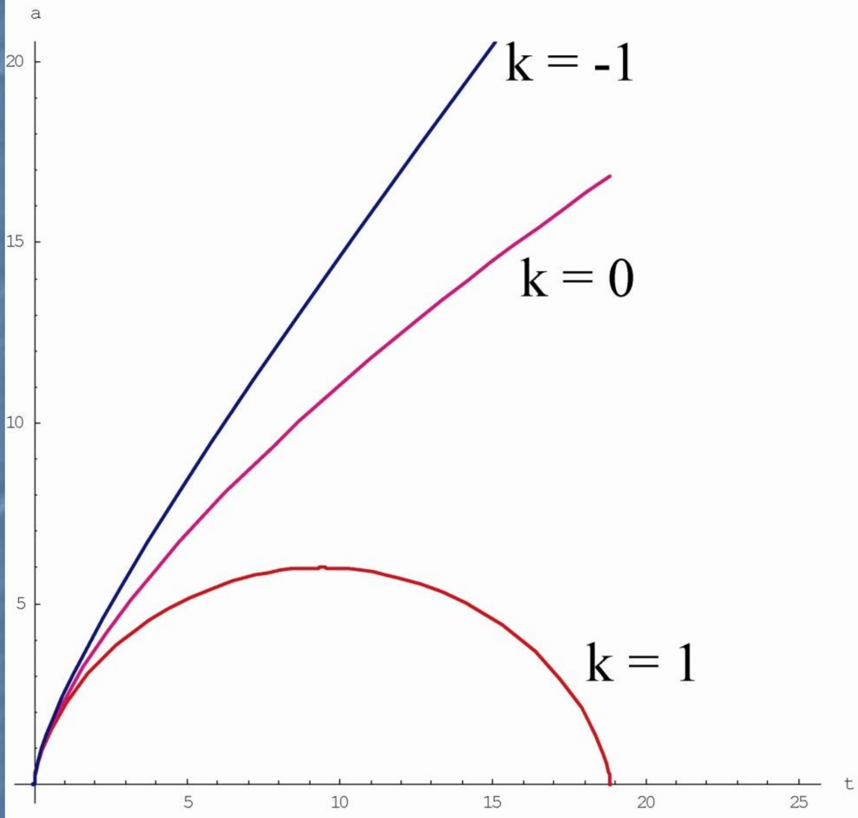
Standard Cosmology

- Standard cosmology is based on the cosmological principle.
- Homogeneity
- Isotropy

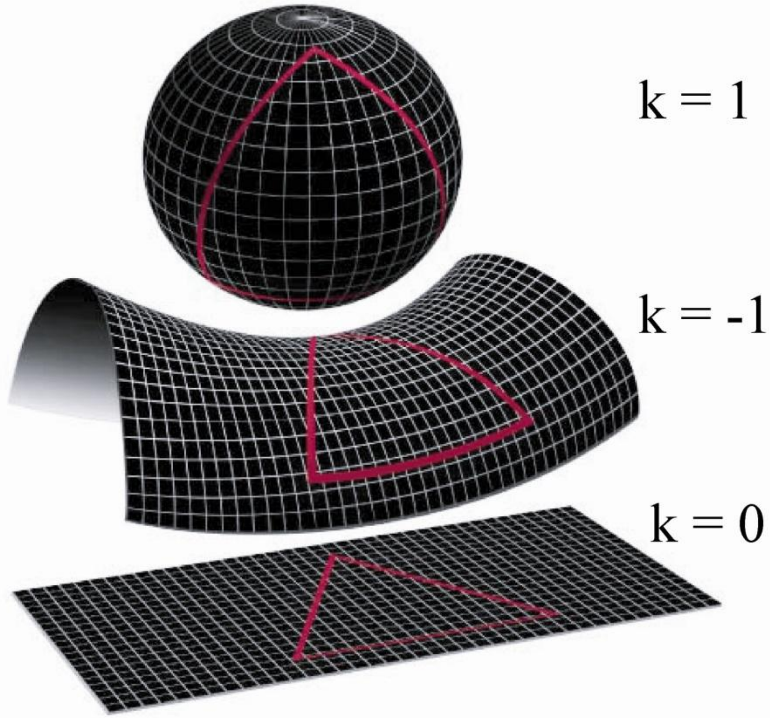


$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - k\,r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$$

Evolution of the scale factor without cosmological constant

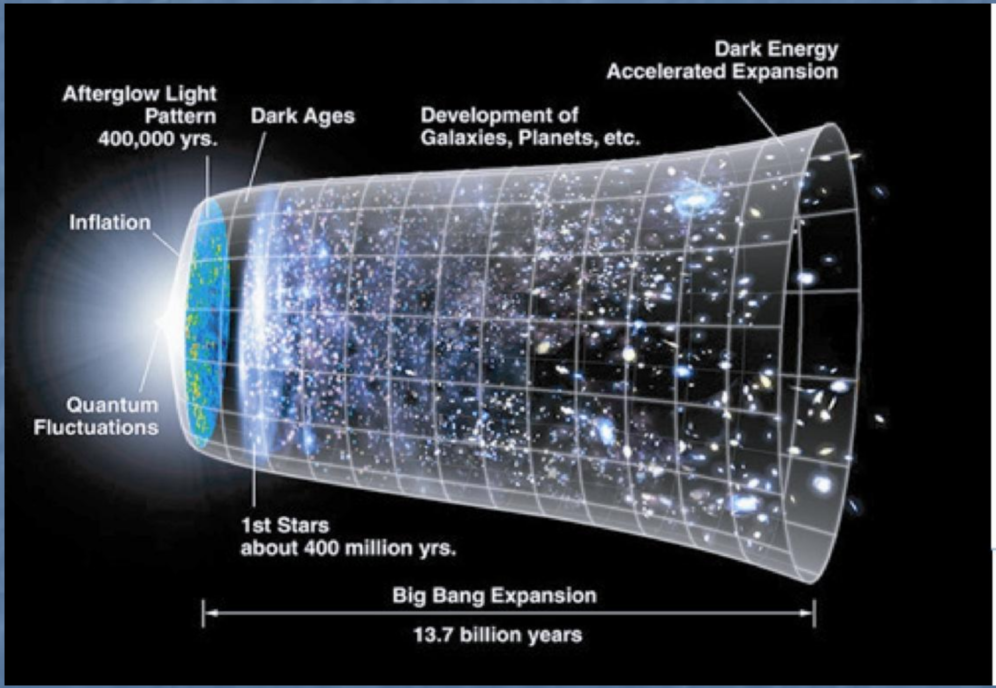


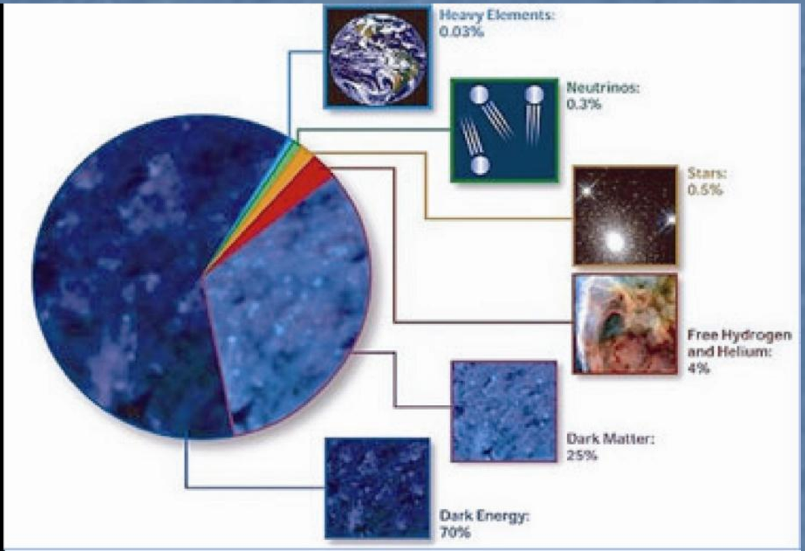
L'evoluzione del fattore di scala predetta dalle equazioni di Einstein nei tre casi di curvatura negativa, nulla e positiva.



I tre possibili tre-spazi massimamente simmetrici.

From 2001 we know that the Universe is spatially flat (k=0) and that it is dominated by dark energy.

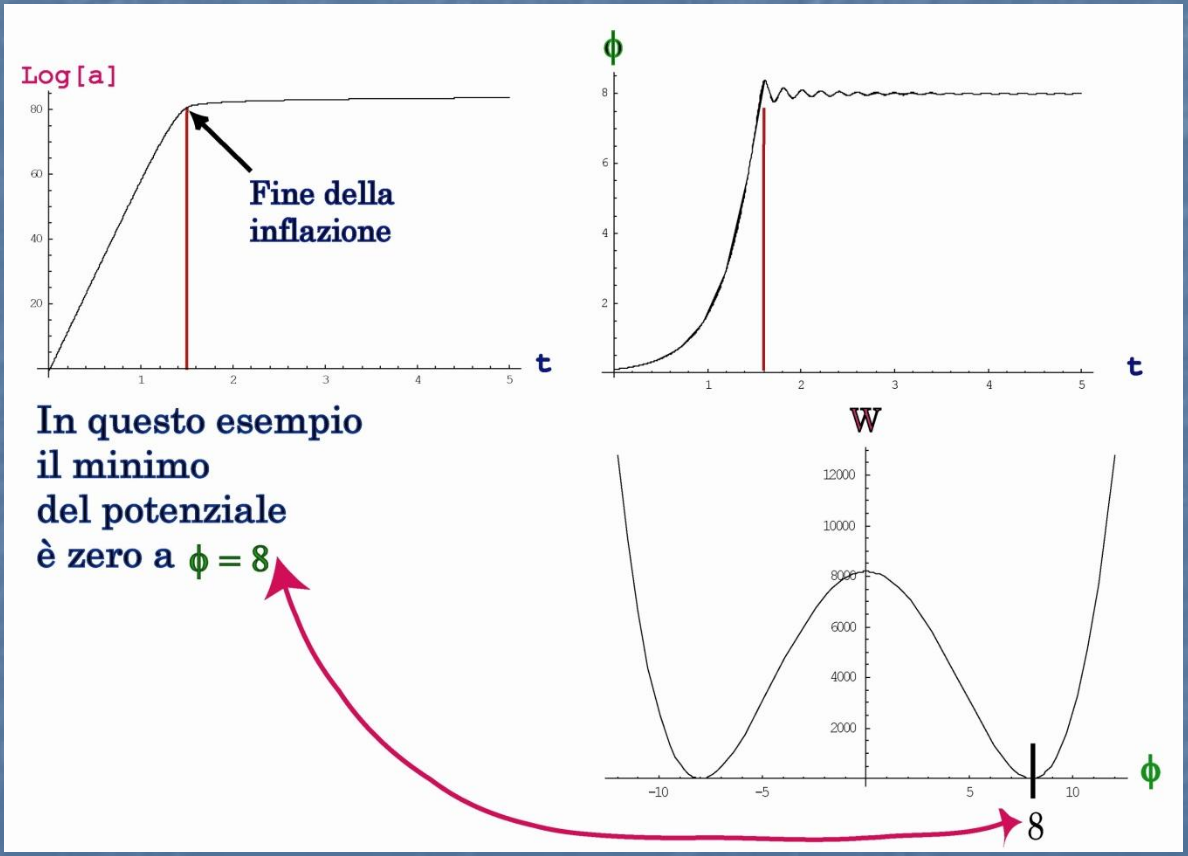


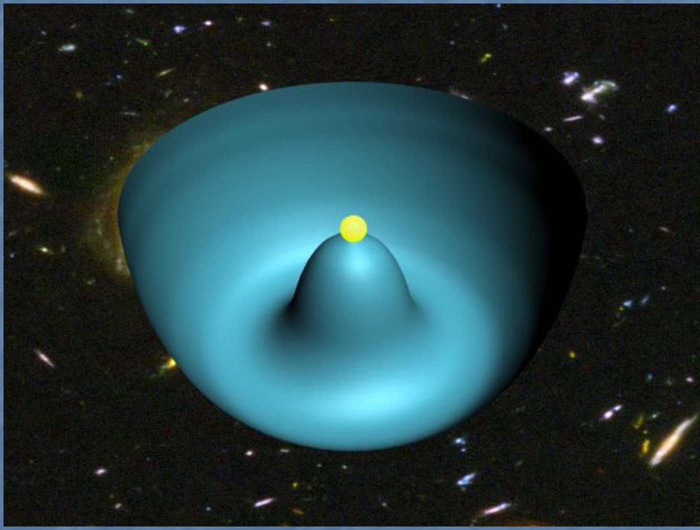


Contenuto dell'Universo al tempo attuale

Most probably there has been inflation

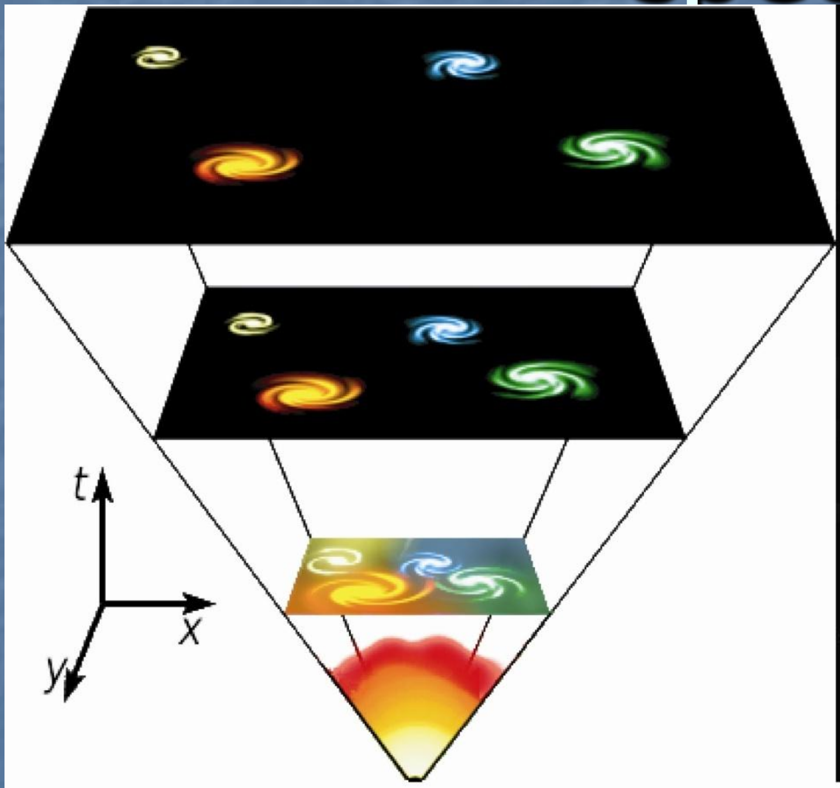
The scalar fields drive inflation while rolling down from a maximum to a minimum

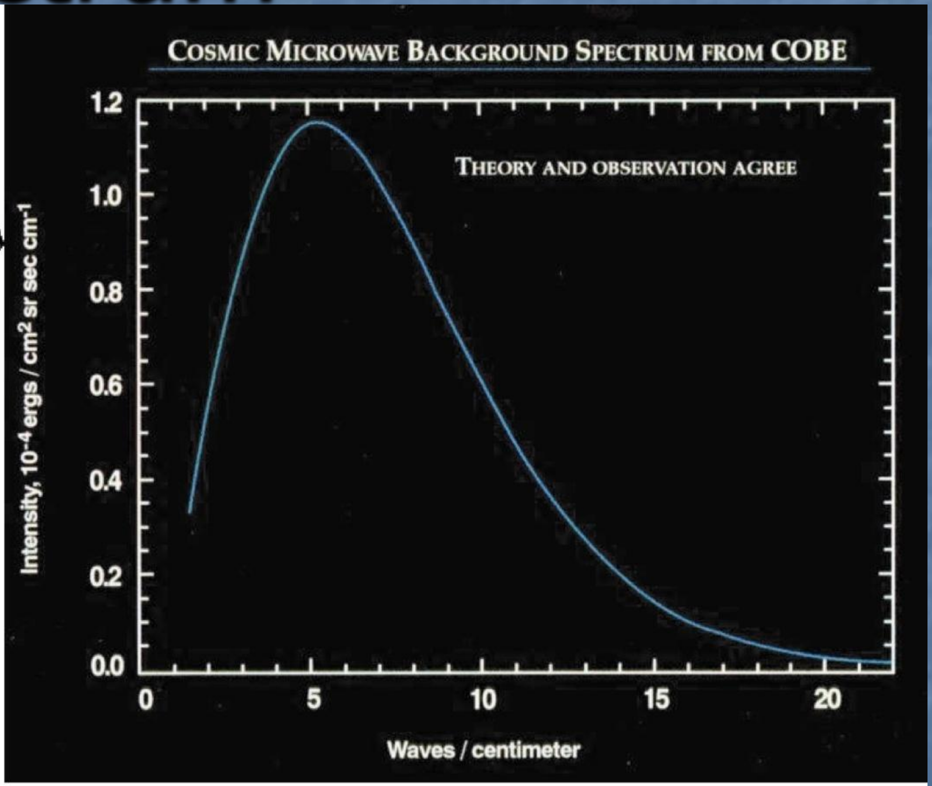




- Exponential expansion during slow rolling
- Fast rolling and exit from inflation
- Oscillations and reheating of the Universe

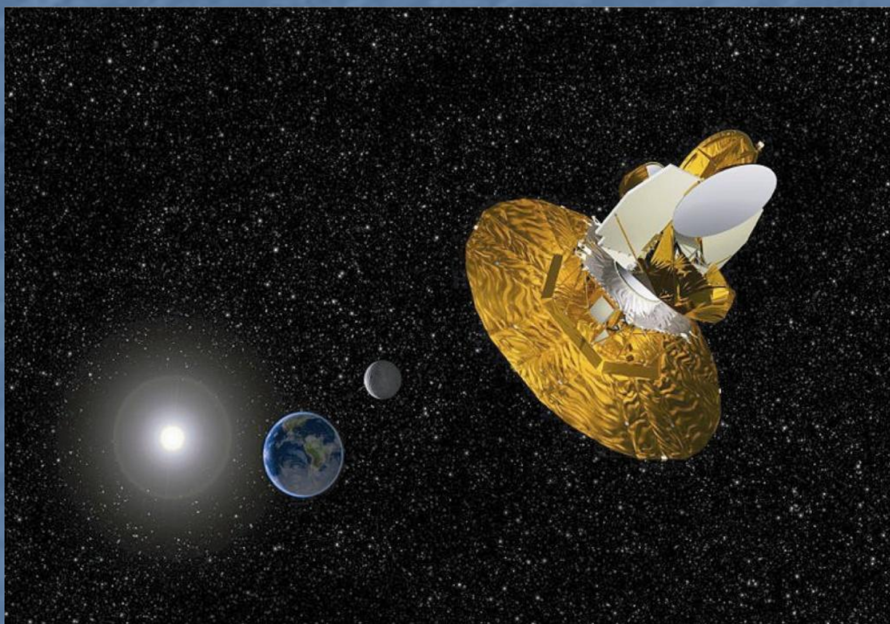
The isotropy and homogeneity are proved by the CMB spectrum

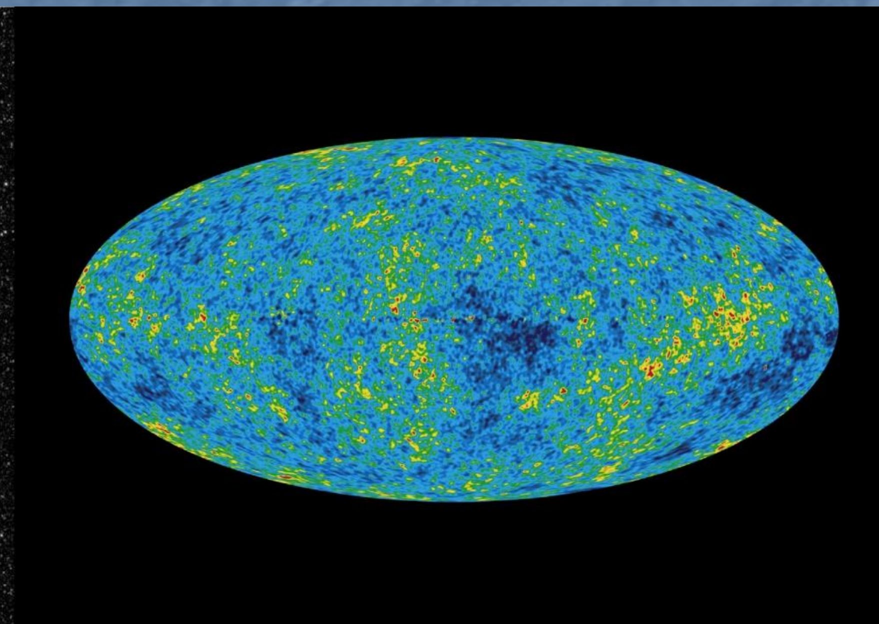




L'Universo si è espanso a partire da uno stato iniziale estremamente denso e caldissimo. Man mano che l'espansione procede l'Universo si raffredda a causa del redshift cosmologico. La radiazione cosmica di fondo è la maggiore evidenza del Big Bang iniziale.

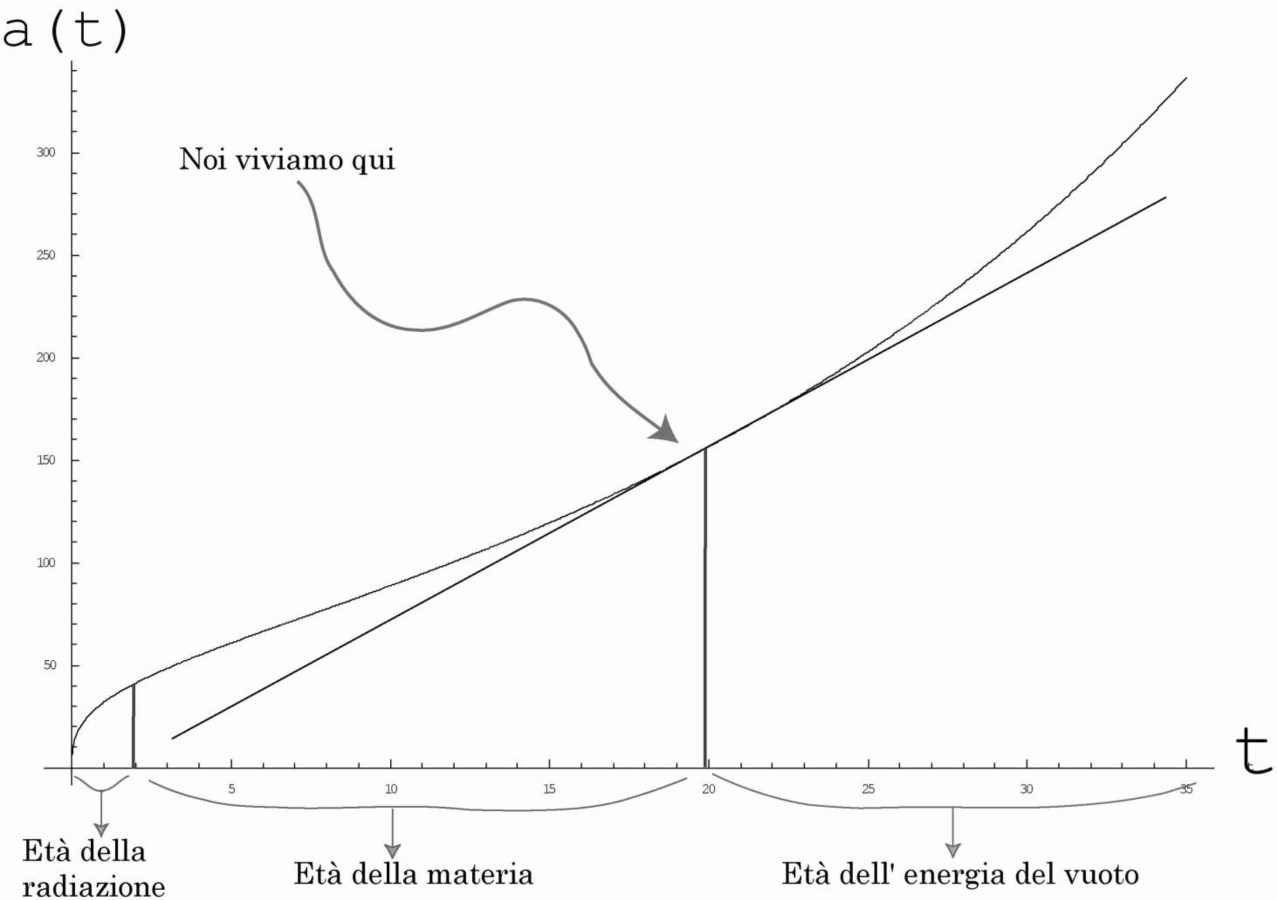
WMAP measured anisotropies of CMB

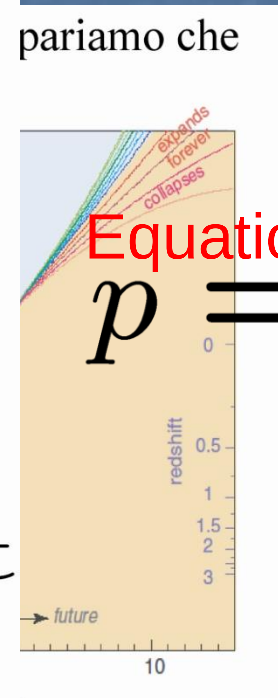




The milliKelvin angular variations of CMB temperature are the inflation blown up image of Quantum fluctuations of the gravitational potential and the seeds of large scale cosmological structures

d by Dark





Equation of State

$$p = w\,\rho$$

Equazione di stato	TIPO DI FLUIDO	ANDAMENTO della densità di energia
$w = \frac{1}{3}$	Radiazione elettromagnetica	$\frac{cost}{a^4(t)}$
$w = 0$	Materia barionica visibile	$\frac{cost}{a^3(t)}$
$w = 1$	Campo scalare fast rolling	$\frac{cost}{a^6(t)}$
$w = -1$	Energia Oscura, cioè energia potenziale del campo scalare slow rolling	$cost$

The Friedman equations govern this evolution

$$
\begin{aligned}
H^2 &= \frac{1}{3}\alpha^2\,\dot{h}^2 + \frac{2}{3}\mathfrak{W}(h) \\
\dot{H} + H^2 &= -\frac{2}{3}\alpha^2\,\dot{h}^2 + \frac{2}{3}\mathfrak{W}(h) \\
0 &= \ddot{h} + 3\,H\,\dot{h} + \frac{1}{\alpha^2}\partial_h\mathfrak{W}(h)
\end{aligned}
$$

where

$$H(t) = \frac{\dot{a}}{a}$$

and

$$\text{kin. term} = \frac{\alpha^2}{2}\dot{h}^2$$

is the Hubble function

In general, also for simple power like potentials the Friedman equations are not integrable. Solutions are known only numerically. Yet some new results are now obtained in gauged supergravity…….!

Gauged and Ungauged Supergravity

$$\mathcal{L}^{(4)} = \sqrt{|\det g|}\left[\frac{R[g]}{2} - \frac{1}{4}\nabla_\mu\phi^a\nabla^\mu\phi^b h_{ab}(\phi) + g^2 V(\phi)\right.$$
$$\left. + \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}(\phi)\, F^\Lambda_{\mu\nu}F^{\Sigma|\mu\nu}\right]$$
$$+\frac{1}{2}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}(\phi)\, F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}$$

$$\nabla_\mu\,\phi^I = \partial_\mu\,\phi^I + g\,A^\Lambda_\mu\,k_\Lambda(\phi)^I$$

g = 0 UNGAUGED SUPERGRAVITY

Gauging of isometries of the scalar manifold

Non isotropic Universes in UNGAUGED SUPERGRAVITY

- We saw what happens if there is isotropy !
- Relaxing isotropy an entire new world of phenomena opens up
- In a multidimensional world, as string theory predicts, there is no isotropy among all dimensions!

Cosmic Billiards before 2003

A challenging phenomenon, was proposed, at the *beginning of this millenium*, by a number of authors under the name of **cosmic billiards**. This proposal was a development of the pioneering ideas of Belinskij, Lifshits and Khalatnikov, based on the Kasner solution of Einstein equations. The Kasner solution corresponds to a regime, where the scale factors of a *D*-dimensional universe have an exponential behaviour . Einstein equations are simply solved by imposing quadratic algebraic constraints on the coefficients . An inspiring mechanical analogy is at the root of the name billiards.

Some general considerations on roots and gravity.......

String Theory implies D=10 space-time dimensions.

Hence a generalization of the standard cosmological metric is of the type:

$$ds^2_{10D} = -e^{2a\tau} d\tau^2 + \sum_{i=1}^{9} e^{2p_i\tau}\, dx_i^2$$

In the absence of matter the conditions for this metric to be Einstein are:

$$\overset{9}{\underset{i=1}{}} h_i(\tau) = \overset{9}{\underset{i=1}{}} p_i\,\tau = 0 \qquad a^2 = \sum_{i=1}^{9} p_i^2$$

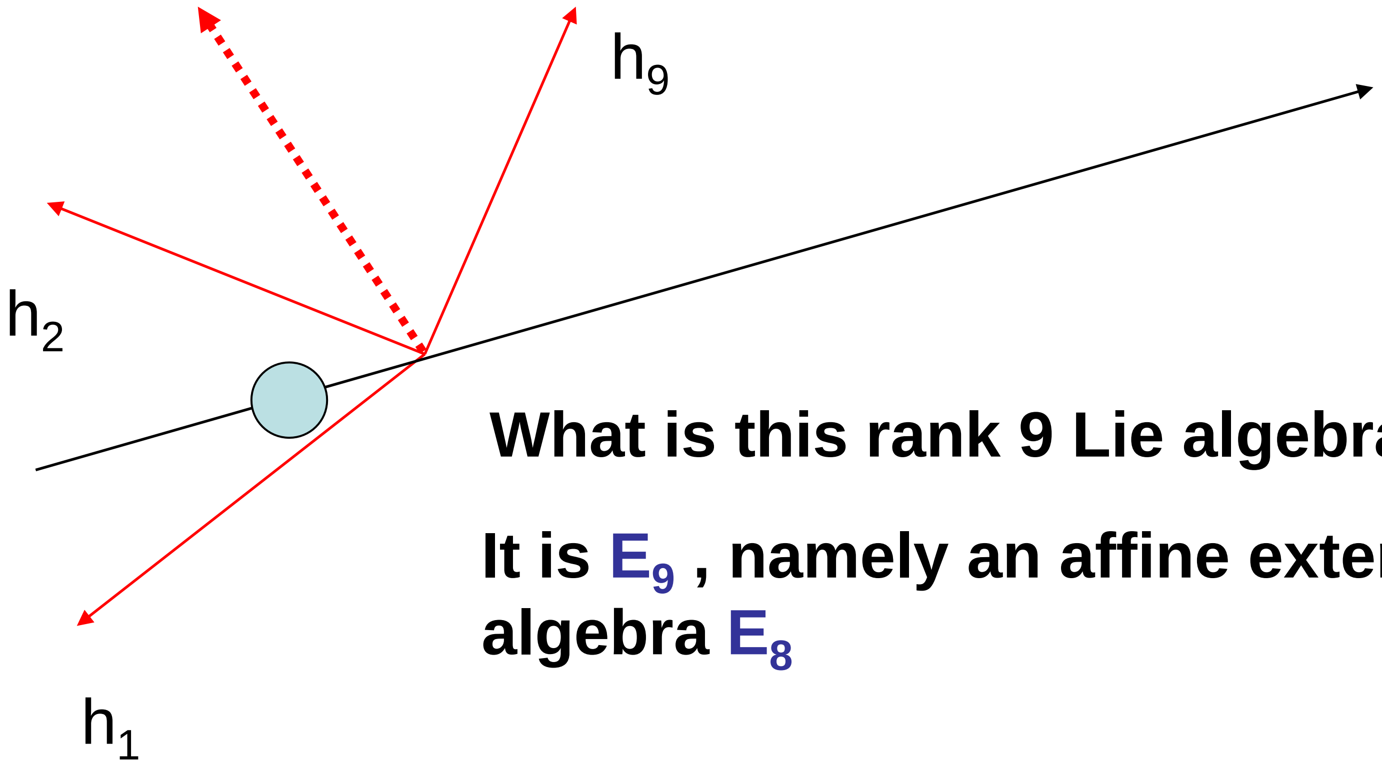
are the coordinates of a ball moving linearly with constant velocity

$\Rightarrow p_i$

What is the space where this fictitious ball moves

ANSWER:

The Cartan subalgebra of a rank 9 Lie algebra.



What is this rank 9 Lie algebra?

It is E_9 , namely an affine extension of the Lie algebra E_8

Lie algebras and root systems

$$[H_i , H_j] = 0$$

$$[H_i , E^{\alpha}] = \alpha^i E^{\alpha}$$

$$[E^{\alpha} , E^{\beta}] = E^{\alpha+\beta} \quad \text{if } \alpha + \beta \text{ is a root}$$

$$[E^{\alpha} , E^{\beta}] = 0 \quad \text{if } \alpha + \beta \text{ is not a root}$$

$$[E^{\alpha} , E^{-\alpha}] = \alpha \cdot H$$

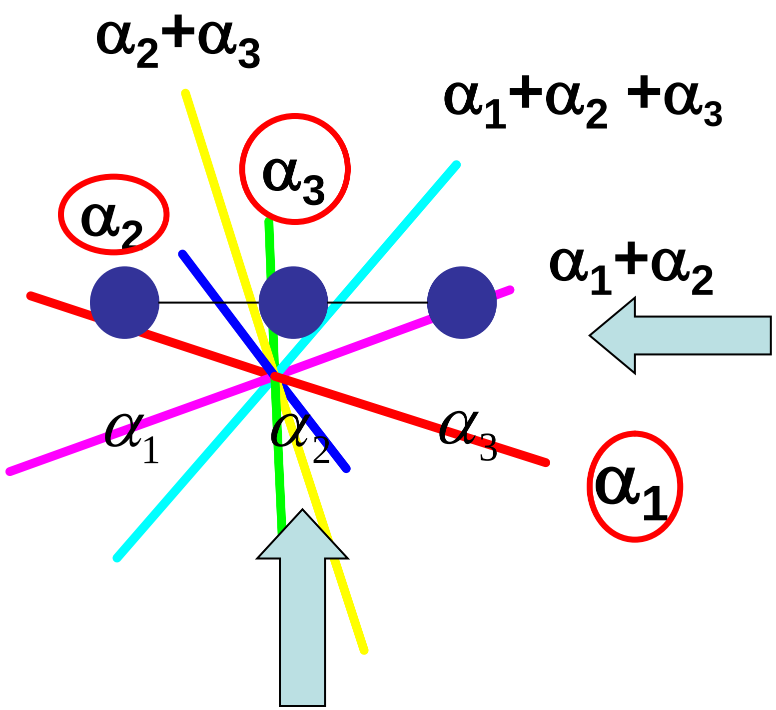
$$2\frac{(\alpha , \beta)}{(\beta \;\; \beta)} \in \mathbf{Z} \qquad \alpha - 2\frac{(\alpha , \beta)}{(\beta , \beta)}\beta \quad \text{is a root}$$

Lie algebras are classified.......

by the properties of simple roots. For instance for A_3 we have $\alpha_1, \alpha_2, \alpha_3$ such that..........

It suffices to specify the scalar products of simple roots

For instance for A_3



$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

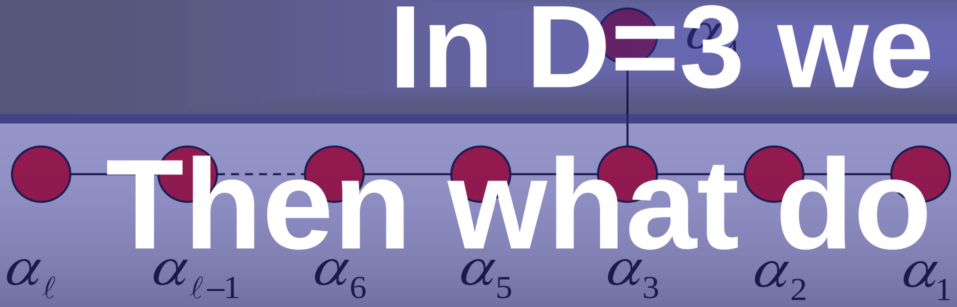
And all the roots are given

There is a simple way of representing these scalar products:Dynkin diagrams

Algebras of the type

exist for any ℓ

Algebras of the type



exist only for

$\ell = 1, 2, \ldots, 7, 8$

E series (exceptional)

In an euclidean space we cannot fit more than 8 linear independent vectors with angles of 120 degrees !!

The group E_r is the duality group of String Theory in dimension D = 10 – r + 1

In D=3 we have E_8

Then what do we have for D=2 ?

We have E_9 !

How come? More than 8 vectors cannot be fitted in an euclidean space at the prescribed angles !

Yes! Euclidean!! Yet in a non euclidean space we can do that !!

Do you remember the condition on the exponent p_i = (*velocity of the little ball*)

$$0 = \sum_{i=1}^{9} p_i^2 - \left(\sum_{i=1}^{9} p_i\right)^2 = p_i K_{ij} p_j$$

where

$$K_{ij} = \begin{pmatrix} 0 & 2 & 2 & \dots & 2 \\ 2 & 0 & 2 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & \dots & 2 & 2 & 0 \end{pmatrix}$$

If we diagonalize the matrix K_{ij} we find the eigenvalues

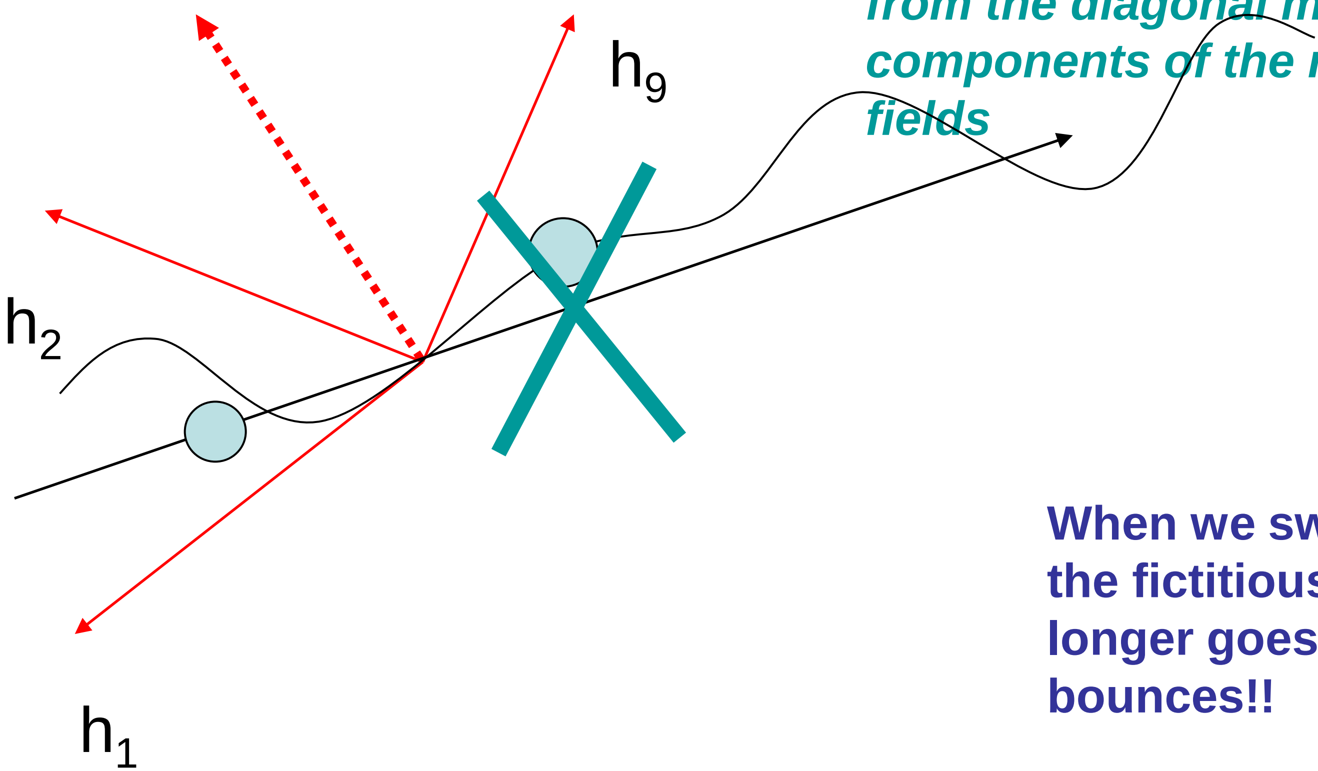
$$(-8, 1, 1, 1, 1, 1, 1, 1)$$

Here is the non-euclidean signature in the Cartan subalgebra of E_9. It is an infinite dimensional algebra (= infinite number of roots!!)

Now let us introduce also the roots......

There are infinitely many, but the time-like ones are in finite number. There are 120 of them as in E_8. All the others are light-like

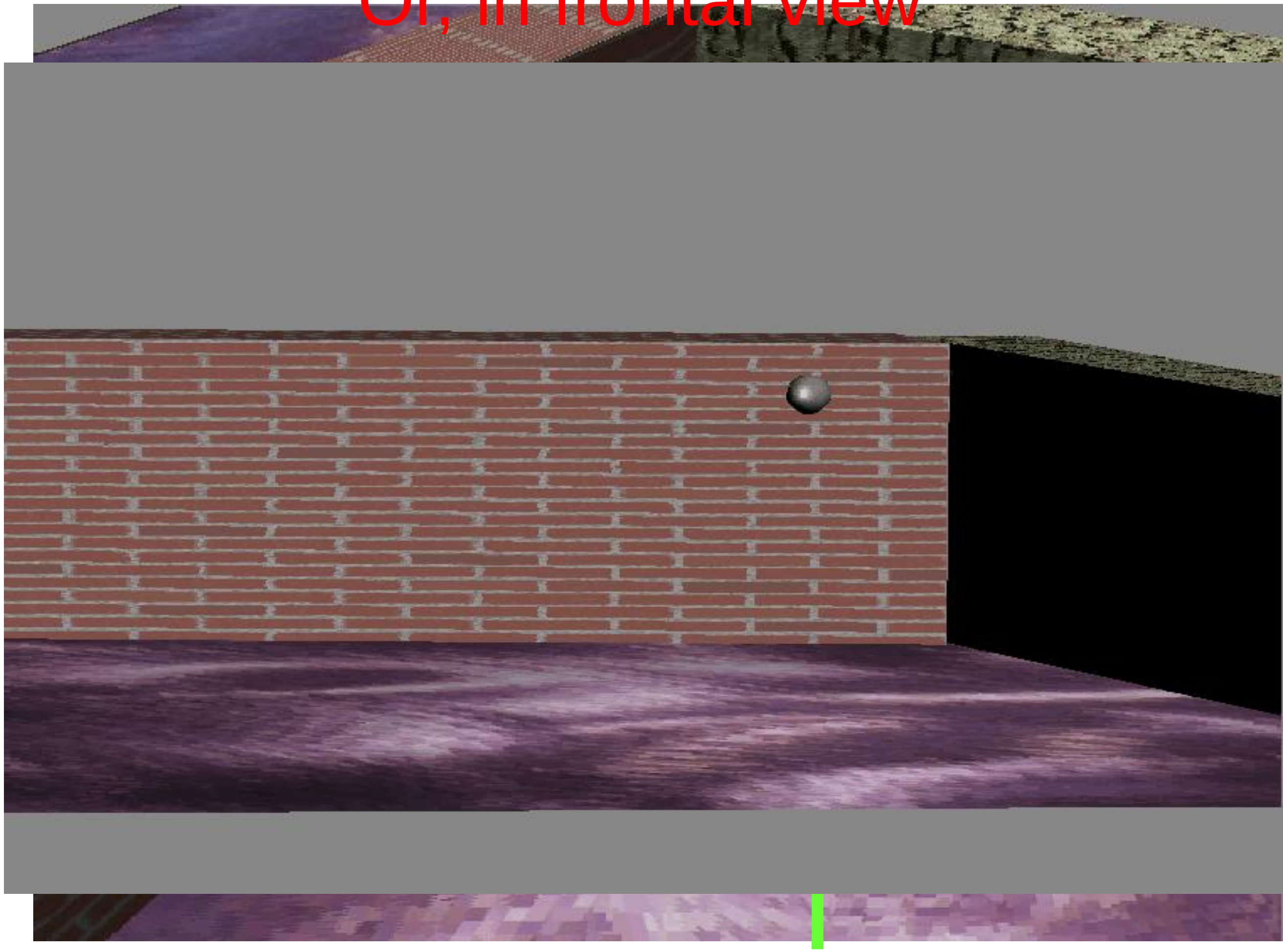
Time like roots, correspond to the light fields of Superstring Theory different from the diagonal metric: off-diagonal components of the metric and p-form fields



When we switch on the roots, the fictitious cosmic ball no longer goes on straight lines. It bounces!!

The cosmic Billiard

Or, in frontal view



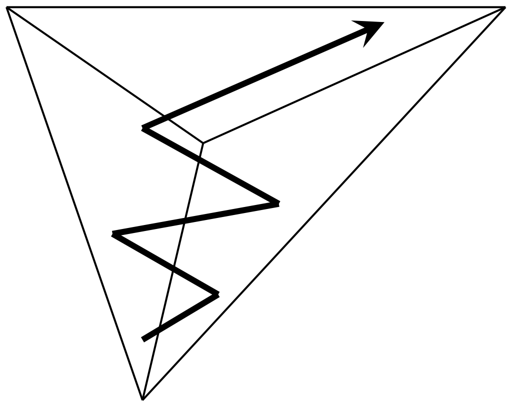
The Lie algebra roots correspond to off-diagonal elements of the metric, or to matter fields (the p+1 forms which couple to p-branes)

Switching a root β we raise a wall on which the cosmic ball bounces

Before 2003: Rigid Billiards

Asymptotically any time—dependent solution defines a ***zigzag*** in **ln** $\mathbf{a_i}$ space

The Supergravity billiard is completely determined by U-duality group



Damour, Henneaux, Nicolai 2002 --

h-space	⟺	***CSA of the U algebra***
walls	⟺	***hyperplanes orthogonal to positive roots*** $\alpha(h_i)$
bounces	⟺	***Weyl reflections***
billiard region	⟺	***Weyl chamber***

Smooth billiards: *Exact cosmological solutions can be constructed using U-duality (**in fact billiards are exactly integrable**)*

Frè, Sorin,

and collaborators,

2003-2008

series of papers

bounces	⟺	***Smooth Weyl reflections***
walls	⟺	***Dynamical hyperplanes***

What is the meaning of the smooth cosmic billiard ?

- *The number of effective dimensions varies dynamically in time!*
- *Some dimensions are suppressed for some cosmic time and then enflate, while others contract.*
- *The walls are also dynamical. First they do not exist then they raise for a certain time and finally decay once again.*
- *The walls are euclidean p-branes! (Space-branes)*
- *When there is the brane its parallel dimensions are big and dominant, while the transverse ones contract.*
- *When the brane decays the opposite occurs*

Cosmic Billiards after 2008

Results established by P.Frè and A.Sorin

- The billiard phenomenon is the generic feature of all exact solutions **of ungauged supergravity** restricted to time dependence.
- We know all solutions where two scale factors are equal. In this case one-dimensional σ–model on the coset U/H. We proved complete integrability.
- We established an integration algorithm which provides the general integral.
- We discovered new properties of the moduli space of the general integral. This is the compact coset H/G_{paint} , further modded by the relevant Weyl group. This is the Weyl group W_{TS} of the Tits Satake subalgebra U_{TS} ½ U.
- There exist both trapped and (super)critical surfaces. Asymptotic states of the universe are in one-to-one correspondence with elements of W_{TS}.
- Classification of integrable supergravity billiards into a short list of universality classes.
- Arrow of time. The time flow is in the direction of increasing the disorder:
  - Disorder is measured by the number of elementary transpositions in a Weyl group element.
  - Glimpses of a new cosmological entropy to be possibly interpreted in terms of superstring microstates, as it happens for the Bekenstein-Hawking entropy of black holes.

Main Points

Definition

<< A supergravity billiard is a one-dimensional σ-model whose target space is a non-compact coset manifold U/H, *metrically equivalent, in force of a general theorem, to a solvable group manifold* exp $[Solv\,(\mathrm{U/H})]$. *>>*

Because ***t-dependent*** supergravity field equations are equivalent to the ***geodesic equations*** for a manifold

U/H

Statement

Supergravity billiards are exactly integrable by means of a general algorithm constructing the Toda-like flow

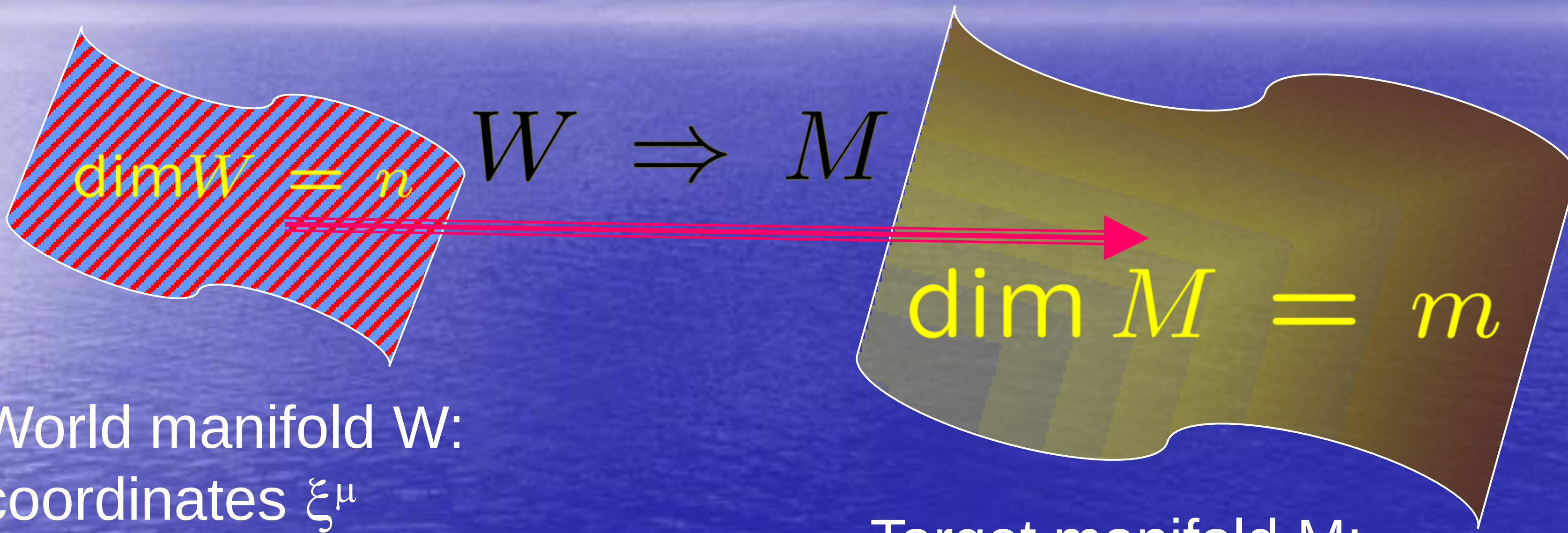
Because **U/H** is always metrically equivalent to a solvable group manifold **exp[Solv(U/H)]** and this defines a canonical embedding

$$\mathbb{U} \hookrightarrow \mathfrak{sl}(N,\mathbb{R}) \; ,$$

$$\mathbb{U} \supset \mathbb{H} \hookrightarrow \mathfrak{so}(N) \subset \mathfrak{sl}(N,\mathbb{R}) \; .$$

What is a σ - model ?

It is a theory of maps from one manifold to another one:



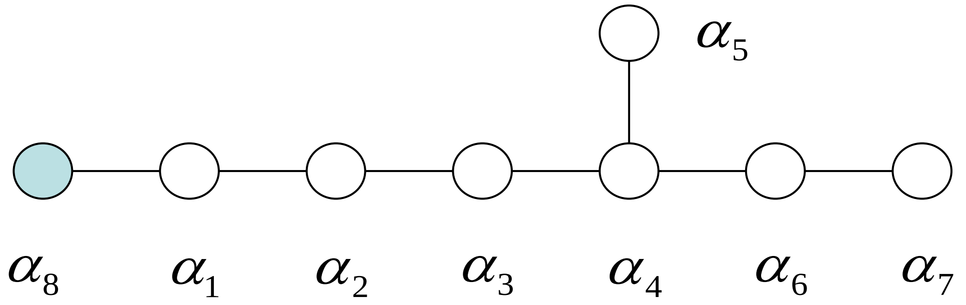
World manifold W: coordinates ξ^μ

Target manifold M:

coordinates ϕ^I

$$\mathcal{A} = \int g^{\mu\nu}(\xi)\, \partial_\mu \phi^I\, \partial_\nu \phi^J\, h_{IJ}(\phi)\, \sqrt{-\det g}\, d^n\xi$$

Starting from D=3 (*D=2 and D=1, also*) all the (bosonic) degrees of freedom are scalars



The bosonic Lagrangian of both Type IIA and Type IIB reduces, upon *toroidal dimensional reduction* from D=10 to D=3, to the gravity coupled sigma model

$$\mathcal{L}^{\sigma-model} = \sqrt{-\det g}\left[2\,R[g] + \frac{1}{2} h_{IJ}(\phi)\,\partial_\mu\phi^I\partial_\nu\phi^J\,g^{\mu\nu}\right]$$

With the target manifold being the **maximally non-compact** coset space

$$M_{target} = \frac{E_{8(8)}}{SO(16)}$$

The discovered Principle

$<<$ The asymptotic states at $t = \pm\infty$ are in one-to-one correspondence with the elements $w_i \in$ Weyl($\mathbb{U}$). The Weyl group admits a natural ordering in terms of $\ell_T(w)$, i.e. the number of transpositions of the corresponding permutation when Weyl(U) is embedded in the symmetric group. Time flows goes always in the direction of increasing ℓ_T which, therefore, plays the role of entropy. $>>$

The relevant **Weyl group** is that of the **Tits Satake** projection. *It is a property of a **universality class** of theories.*

$$\Pi_{TS} \quad : \quad \frac{U}{H} \rightarrow \frac{U_{TS}}{H_{TS}}$$

There is an interesting topology of parameter space for the LAX EQUATION

The Weyl group of a Lie algebra

- Is the discrete group generated by all reflections with respect to all roots
- Weyl(L) is a discrete subgroup of the orthogonal group O(r) where r is the rank of L.

$$\forall \alpha \in \Delta \quad : \quad \sigma_\alpha(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha$$

Full Integrability

Lax pair representation

and the integration algorithm

Lax Representation and Integration Algorithm

- $$\mathbb{L}(\phi) = \prod_{I=m}^{I=1} \exp\left[\varphi_I\, E^{\alpha_I}\right] \exp\left[h_i \mathcal{H}^i\right]$$ Solvable coset representative

- $$L(t) = \sum_i \mathrm{Tr}\left(\mathbb{L}^{-1} \frac{d}{dt}\mathbb{L}\, \mathsf{K}_i\right) \mathsf{K}_i \ ,$$ Lax operator (symm.)

- $$W(t) = \sum_\ell \mathrm{Tr}\left(\mathbb{L}^{-1} \frac{d}{dt}\mathbb{L}\, \mathsf{H}_\ell\right) \mathsf{H}_\ell$$ Connection (antisymm.)

$$\frac{d}{dt} L = [W\, ,\, L]$$ Lax Equation

$$W = \Pi(L) : = L_{>0} - L_{<0}$$

Parameters of the time flows

From initial data we obtain the time flow (complete integral)

$$\mathcal{I}_K \quad : \quad L_0 \mapsto L\left(t, L_0\right)$$

Initial data are specified by a pair: an element of the non-compact Cartan Subalgebra and an element of maximal compact group:

$$\mathcal{C}_0 \in \mathsf{CSA} \bigcap \mathbb{K} \quad ; \quad \mathcal{O} \in \mathsf{H} \, .$$

$$L_0 = \mathcal{O}^T \mathcal{C}_0 \, \mathcal{O}$$

Properties of the flows

The flow is isospectral

$$\forall\, t \in \mathbb{R} \quad : \quad \text{Eigenvalues}\,[L(t)] =$$

$$= \{\lambda_1 \ldots \lambda_N\} = \text{const}$$

$$\boxed{L(t) = \mathcal{O}^T(t)\, \mathcal{C}_0\, \mathcal{O}(t)}$$

The asymptotic values of the Lax operator are diagonal (Kasner epochs)

$$\lim_{t \to \pm\infty} L(t) = L_{\pm\infty} \in \text{CSA}$$

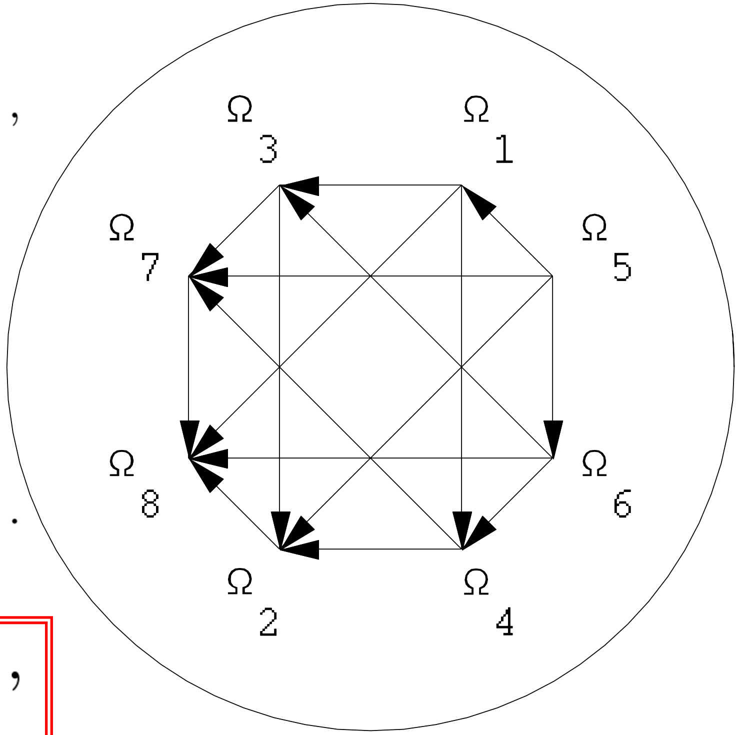
$$\lim_{t \to \pm\infty} \mathcal{O}(t) \in \text{Weyl}\,(\mathbb{U}_{\text{TS}})$$

Parameter space	$\mathcal{P} = \frac{\mathsf{H}}{\mathsf{G_{paint}}} / \mathsf{Weyl}(\mathbb{U})$
Proposition	*<<Consider now the* $\mathsf{N}^2 - 1$ *minors of* $\mathcal{O}(t)$ *obtained by intersecting the first* k *columns with any set of* k*-rows, for* $k = 1, \ldots, \mathsf{N} - 1$. *If any of these minors vanishes at any finite time* $t \neq \pm\infty$ *then it is constant and vanishes at all times.>>*
Trapped submanifolds	There are $N^2 - 1$ **trapped hypersurfaces** $\Sigma_i \subset \mathcal{P}$. defined by the vanishing of one of the minors. They can be intersected.
ARROW OF TIME	*On any (trapped) manifold the flow is from the lowest, to the highest accessible Weyl element*

Example. The Weyl group of Sp(4)» SO(2,3)

ℓ_T	Weyl group of $\mathfrak{sp}(4,\mathbb{R})$
0	w_5
1	w_6
2	w_1
3	w_3
3	w_4
4	w_2
5	w_7
6	w_8

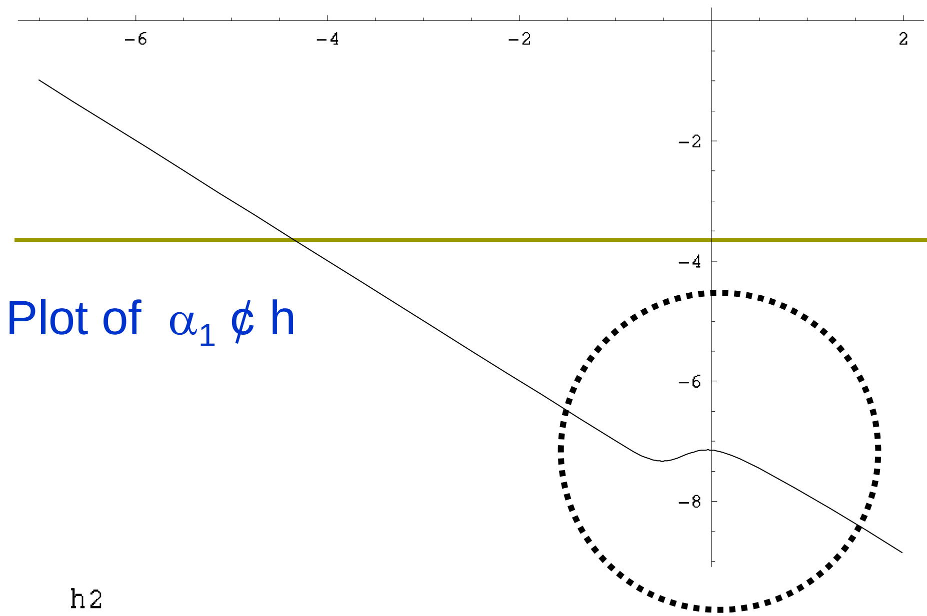
$$\begin{aligned}
w_1 &: (h_1,h_2) \to (h_1,h_2)\,,\\
w_2 &: (h_1,h_2) \to (-h_1,-h_2)\,,\\
w_3 &: (h_1,h_2) \to (-h_1,h_2)\,,\\
w_4 &: (h_1,h_2) \to (h_1,-h_2)\,,\\
w_5 &: (h_1,h_2) \to (h_2,h_1)\,,\\
w_6 &: (h_1,h_2) \to (h_2,-h_1)\,,\\
w_7 &: (h_1,h_2) \to (-h_2,h_1)\,,\\
w_8 &: (h_1,h_2) \to (-h_2,-h_1)\,.
\end{aligned}$$

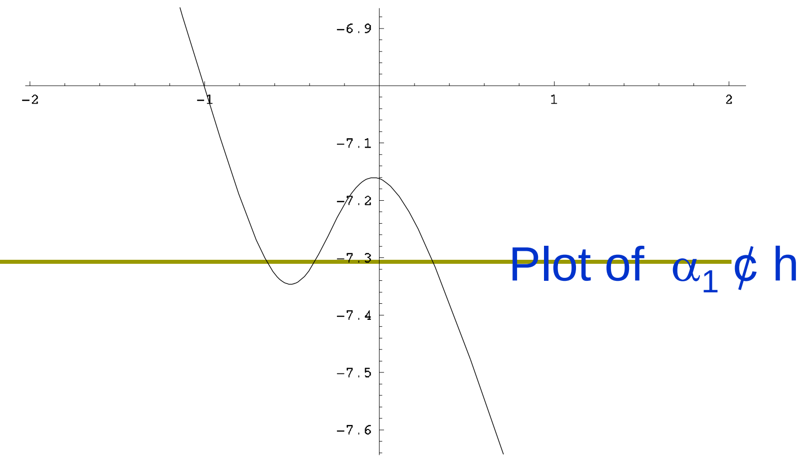


Available flows on 3-dimensional critical surfaces

$$\begin{aligned}
1 &: w_1 \mapsto w_8\,,\\
2 &: w_5 \mapsto w_2\,,\\
3 &: w_5 \mapsto w_7\,,\\
4 &: w_5 \mapsto w_8\,,\\
5 &: w_6 \mapsto w_8\,.
\end{aligned}$$

Available flows on edges, i.e. 1-dimensional critical surfaces

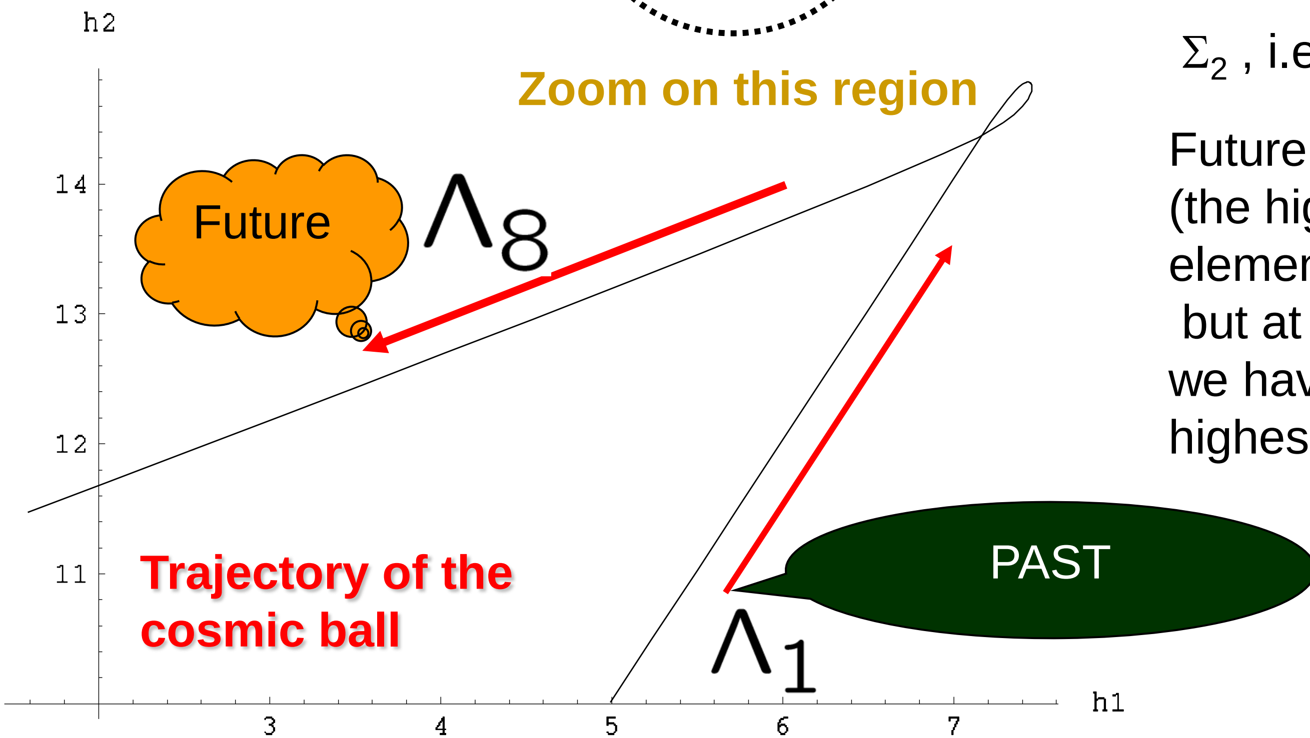


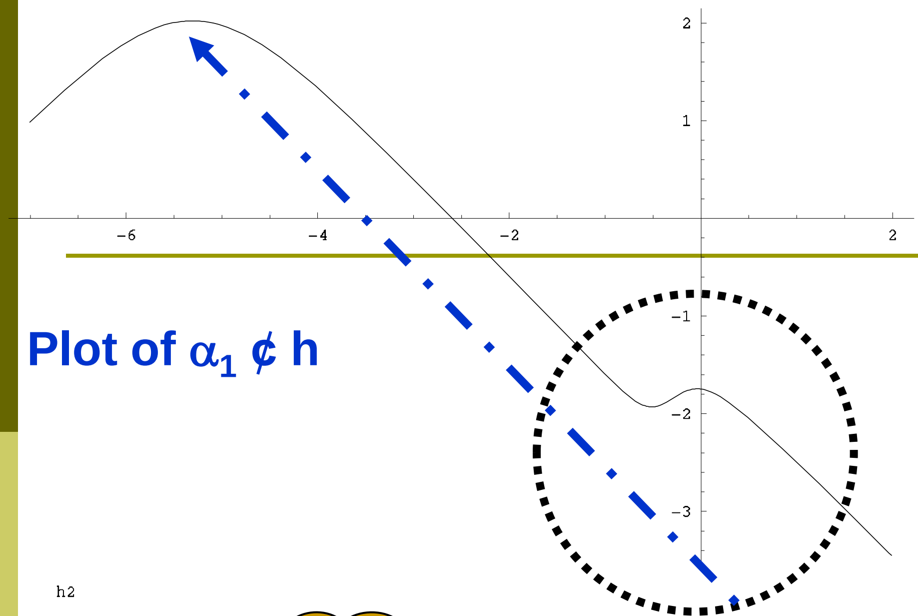


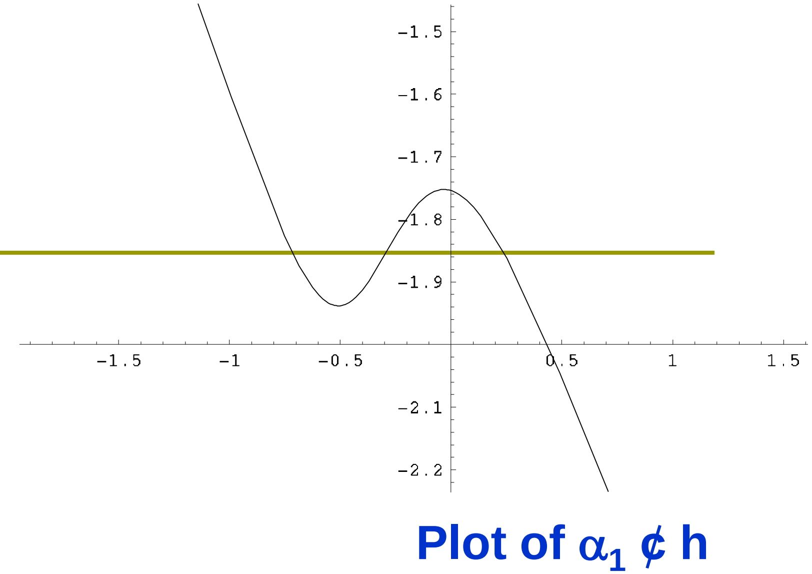
An example of flow on a critical surface for SO(2,4).

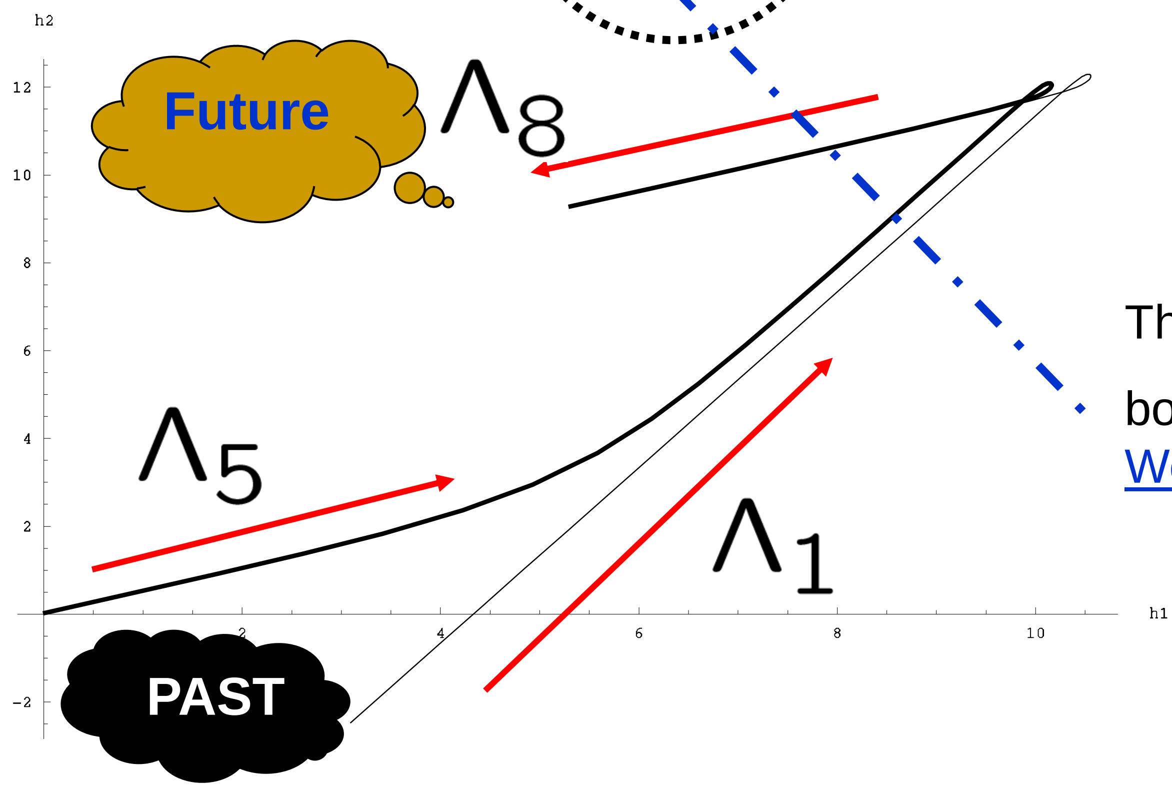
Σ_2 , i.e. $\mathbf{O_{2,1} = 0}$

Future infinity is Λ_8 (the highest Weyl group element),
but at past infinity we have Λ_1 (not the highest) = **criticality**









$O_{2,1}$ ' 0.01 (Perturbation of critical surface)

There is an extra primordial bounce and we have the lowest Weyl group element Λ_5 at

t = -1

Let us turn to Gauged Supergravity

Inflation & CMB spectrum require the presence of a potential for the scalar fields

One scalar flat cosmologies

$$\mathcal{L}(g,\phi) = \left[R[g] + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right]\sqrt{-g}$$

Generalized ansatz for spatially flat metric

$$ds^2 = -\,e^{2\mathcal{B}(t)}\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}\,,$$

Friedman equations

when B(t) = 0

$$H^2 = \frac{1}{3}\,\dot{\phi}^2 + \frac{2}{3}\,V(\phi)\,,$$

$$\dot{H} = -\dot{\phi}^2\,,$$

$$\ddot{\phi} + 3\,H\,\dot{\phi} + V' = 0\,,$$

In a recent paper by P.F. , Sagnotti & Sorin

It has been derived a BESTIARY of potentials that lead to integrable models of cosmology. There we also described the explicit integration for the scalar field and the scale factor for each of the potentials in the list.

The question is: Can any of these cosmologies be embedded into a Gauged Supergravity model?

This is a priori possible and natural within a subclass of the mentioned Bestiary

The issue of the scalar potential in N=1 supergravity and extended supergravity

The Hodge Kahler geometry of the scalar sector

The hermitian Kahler metric that defines the kinetic terms is determined in terms of the Kahler potential:

$$g_{ij^\star} = \partial_i \partial_{j^\star} \mathcal{K} \quad \Rightarrow \quad ds^2_K = \partial_i \partial_{j^\star} \mathcal{K} \, dz^i \otimes dz^{j^\star}$$

$$\Gamma^i{}_k = \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} dz^j = g^{i\ell^\star} \partial_j \, g_{\ell^\star k} \, dz^j$$

$$\Gamma^{i^\star}{}_{k^\star} = \text{Complex Conjugate of} \, \Gamma^i{}_k$$

The Levi Civita connection

The Kahler 2-form is the curvature of the Hodge line bundle

$$K = \mathrm{i} \, g_{ij^\star} \, dz^i \wedge dz^{j^\star}$$

The holomorphic sections W(z) of the Hodge bundle are the possible superpotentials

$$||W||^2 = \exp[\mathcal{K}] \, W(z) \, \overline{W}(\bar{z})$$

The function G and the Momentum Map

$$G(z,\bar{z}) = \log ||W||^2 = \mathcal{K} + \log W + \log \overline{W}$$

$$z^i \to z^i + \epsilon^\Lambda k^i_\Lambda(z)$$

An infinitesimal isometry defines a Killing vector

The holomorphic Killing vectors can be derived from a real prepotential called the momentum map:

$$k^i_\Lambda(z) = \mathrm{i}\, g^{ij^\star}\, \partial_{j^\star} \mathcal{P}_\Lambda \quad ; \quad k^{j^\star}_\Lambda(\bar{z}) = \mathrm{i}\, g^{ij^\star}\, \partial_i \mathcal{P}_\Lambda$$

The momentum map is constructed as follows in terms of the G function

$$\mathcal{P}_\Lambda = -\frac{1}{2}\left(k^i_\Lambda\, \partial_i G - k^{i^\star}_\Lambda\, \partial_{i^\star} G\right)$$

The structure of the scalar potential

$$V = \underbrace{\frac{1}{4} g_{ij^\star} \mathcal{H}^i \mathcal{H}^{j^\star}}_{\text{chiralinos contr.}} - \underbrace{3\, S\, S^\star}_{\text{gravitino contr.}} + \underbrace{\frac{1}{3} \left(\mathrm{Im}\, \mathcal{N}^{-1}\right)_{\Lambda\Sigma} D^\Lambda D^\Sigma}_{\text{gaugino contr.}}$$

$$= e^2 \exp\left[\mathcal{K}\right] \left(g^{ij^\star} \mathcal{D}_i W\, D_{j^\star} \overline{W} - 3\,|W|^2\right) + \frac{1}{3} g^2\, \mathrm{Im}\, \mathcal{N}_{\Lambda\Sigma}\, \mathcal{P}^\Lambda \mathcal{P}^\Sigma$$

The scalar potential is a quadratic form in the auxiliary fields of the various multiplets:

1) **The auxiliary fields of the chiral multiplets H^i**
2) **The auxiliary field of the graviton multiplet S**
3) **The auxiliary fields of the vector multiplets P^Λ**

The auxiliary fields on shell become functions of the scalar fields with a definite geometric interpretation.

Embedding Inflaton Models

$$\mathcal{M}_{\text{Kähler}} = \mathcal{M}_J \otimes \mathcal{M}_\mathcal{K}$$

Direct product of manifolds

$$\Omega = \mathrm{i}C + B$$

Distinguished complex scalar

$$\widehat{\mathcal{K}} = J\left(\mathrm{Im}\Omega\right) + \mathcal{K}(z, z)$$

Translational symmetry. Does not depend on B !

$$ds^2_{\text{Kähler}} = \frac{1}{4} J''\left(\mathrm{Im}\Omega\right)|d\Omega|^2 + g_{ij^\star}\, dz^i\, dz^{j^\star}$$

$$G = J\left(\mathrm{Im}\Omega\right) + \mathcal{K}(z, z) + \log\left[W(\Omega, z)\,\bar{W}(\bar{\Omega}, \bar{z})\right]$$

Final structure of the potential

$$V = V_{YM} + V_{ZM}$$

$$V_{WZ} = \exp[J] \left[\underbrace{\exp[\mathcal{K}] \left(g^{ij^\star} \mathcal{D}_i W \, D_{j^\star} \overline{W} - 3 |W|^2 \right)}_{\text{potential of the } n \text{ multiplets}} + \exp[\mathcal{K}] \frac{(J')^2}{J''} |W|^2 \right]$$

$$\mathcal{P}_0 = -\frac{1}{2} \frac{\mathrm{d}J}{\mathrm{d}C} \equiv -\frac{1}{2} J'$$

$$V_{YM} = \frac{1}{12} g^2 \left(J'(C) \right)^2$$

The complete potential can be reduced to a function of the single field C if the other moduli fields z^i can be stabilized in a C-independent way

D-type inflaton embedding

Critical point of the superpotential

$$\mathcal{D}_i W\big|_{z^i = z^i_0} = 0 \quad ; \quad W(z_0) = 0$$

$$V(C) = \text{const}^2 \times \left(J'(C)\right)^2$$

$$\mathcal{L}_{scalar} = 2\left(\frac{1}{4} J''(C)\, \partial^\mu C\, \partial_\mu C - \text{const}^2 \times \left(J'(C)\right)^2\right)$$

F-type Embedding

$$J' = 0$$

$$V = \exp[\mathcal{K}] \left(g^{zz^\star} \mathcal{D}_z W \, D_{z^\star} \overline{W} - 3 |W|^2 \right)$$

If $\mathcal{K} = I(C)$ where $z = \mathrm{i}\, C + B$

and $\partial_B V|_{B=0} = 0$

we have a consistent truncation to a single inflaton model

The F-type Embedding of some integrable cosmological models

We begin by considering the issue of the F-type embedding of the integrable potentials in the Bestiary compiled by Sagnotti, Sorin and P.F.

Later we will consider the issue of D-type embedding of the same

The integrable potentials candidate in SUGRA via F-type

From Friedman equations to

$$\phi = \sqrt{q}\,\mathfrak{h}$$

Conversion formulae

$$\begin{aligned} \varphi &= \sqrt{3q}\,\mathfrak{h} \\ \mathcal{V}(\varphi) &= 3\,V(\mathfrak{h}) \\ &= 3\,V\left(\frac{\varphi}{\sqrt{3q}}\right) \end{aligned}$$

Effective dynamical model

$$\mathcal{L}_{eff} = \text{const} \times e^{3A - \mathcal{B}} \left[-\frac{3}{2}\dot{A}^2 + \frac{q}{2}\dot{\mathfrak{h}}^2 - e^{2\mathcal{B}}\,V(\mathfrak{h}) \right]$$

	Potential function
[1]	$C_{11}\,e^{\varphi} + 2\,C_{12} + C_{22}\,e^{-\varphi}$
[2]	$C_1\,e^{2\gamma\varphi} + C_2 e^{(\gamma+1)\varphi} \quad (\gamma^2 \neq 1)$
[3]	$C_1\,e^{2\varphi} + C_2$
[7]	$C_1\,(\cosh\gamma\varphi)^{\frac{2}{\gamma}-2} + C_2(\sinh\gamma\varphi)^{\frac{2}{\gamma}-2}$
[8]	$\Im[C\,(2(\mathrm{i} + \sinh 2\gamma\varphi))^{\frac{1}{\gamma}-1}]$
[9]	$C_1\,e^{2\gamma\varphi} + C_2\,e^{\frac{2}{\gamma}\varphi} \quad (\gamma^2 \neq 1)$

Sporadic Integrable Potentials

$$\mathcal{V}_{Ia}(\varphi) = \frac{\lambda}{4}\left[(a+b)\cosh\left(\tfrac{6}{5}\varphi\right) + (3a-b)\cosh\left(\tfrac{2}{5}\varphi\right)\right]$$

$$\mathcal{V}_{Ib}(\varphi) = \frac{\lambda}{4}\left[(a+b)\sinh\left(\tfrac{6}{5}\varphi\right) - (3a-b)\sinh\left(\tfrac{2}{5}\varphi\right)\right]$$

where
$$\{a,b\} = \left\{\begin{array}{cc} 1 & -3 \\ 1 & -\frac{1}{2} \\ 1 & -\frac{3}{16} \end{array}\right\}$$

$$\mathcal{V}_{II}(\varphi) = \frac{\lambda}{8}\left[3a+3b-c+4(a-b)\cosh\left(\tfrac{2}{3}\varphi\right) + (a+b+c)\cosh\left(\tfrac{4}{3}\varphi\right)\right] ,$$

where
$$\{a,b,c\} = \left\{\begin{array}{ccc} 1 & 1 & -2 \\ 1 & 1 & -6 \\ 1 & 8 & -6 \\ 1 & 16 & -12 \\ 1 & \frac{1}{8} & -\frac{3}{4} \\ 1 & \frac{1}{16} & -\frac{3}{4} \end{array}\right\}$$

$$\mathcal{V}_{IIIa}(\varphi) = \frac{\lambda}{16}\left[\left(1-\frac{1}{3\sqrt{3}}\right)e^{-6\varphi/5} + \left(7+\frac{1}{\sqrt{3}}\right)e^{-2\varphi/5} + \left(7-\frac{1}{\sqrt{3}}\right)e^{2\varphi/5} + \left(1+\frac{1}{3\sqrt{3}}\right)e^{6\varphi/5}\right] .$$

$$\mathcal{V}_{IIIb}(\varphi) = \frac{\lambda}{16}\left[(2-18\sqrt{3})\,e^{-6\varphi/5} + (6+30\sqrt{3})\,e^{-2\varphi/5} + (6-30\sqrt{3})\,e^{2\varphi/5} + (2+18\sqrt{3})\,e^{6\varphi/5}\right]$$

There are additional integrable sporadic potentials in the class that might be fit into supergravity

Connection with Gauged SUGRA

In all integrable models the potential $V(\phi)$ is a polynomial or rational function of exponentials $\exp[\beta\,\phi]$ of a field ϕ with canonical kinetic term. The polynomial cases naturally connect to Gauged Supergravity with scalar fields belonging to *non compact, symmetric coset manifolds* G/H. There one can resort to a *solvable parameterization*. The scalar fields fall into two classes:

1. The *Cartan fields* $\mathfrak{h}^i$ associated with the Cartan generators of the Lie algebra $\mathbb{G}$, whose number equals the rank r of G/H.
2. The *axion fields* b^I associated with the roots of the Lie algebra $\mathbb{G}$.

From the gauging procedure the potential emerges as a polynomial function of the coset representative and hence as a polynomial function in the exponentials of the Cartan fields h_i

coset D=4	coset D=3	susy
$\frac{SU(1,1)}{U(1)}$	$\frac{G_{2(2)}}{SL(2,R)\times SL(2,R)}$	$\mathcal{N}=2$ n=1
$\frac{Sp(6,R)}{SU(3)\times U(1)}$	$\frac{F_{4(4)}}{Sp(6,R)\times SL(2,R)}$	$\mathcal{N}=2$ $n=6$
$\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$	$\frac{E_{6(2)}}{SU(3,3)\times SL(2,R)}$	$\mathcal{N}=2$ $n=9$
$\frac{SO^*(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO^*(12)\times SL(2,R)}$	$\mathcal{N}=2$ n=15
$\frac{E_{7(-25)}}{E_{6(-78)}\times U(1)}$	$\frac{E_{8(-24)}}{E_{7(-25)}\times SL(2,R)}$	$\mathcal{N}=2$ $n=27$
$\frac{SL(2,\mathbb{R})}{O(2)}\times\frac{SO(2,2+p)}{SO(2)\times SO(2+p)}$	$\frac{SO(4,4+p)}{SO(2,2)\times SO(2,2+p)}$	$\mathcal{N}=2$ n=3+p
$\frac{SU(p+1,1)}{SU(p+1)\times U(1)}$	$\frac{SU(p+2,2)}{SU(p+1,1)\times SL(2,R)_{h^*}}$	$\mathcal{N}=2$

The N=2 playing ground

In N=2 or more extended gauged SUGRA we have found no integrable submodel, so far. The full set of gaugings has been onstructed only for the STU model

p=0

The classification of other gaugings has to be done and explored

Some results from a new paper by P.F.,Sagnotti, Sorin & Trigiante (*to appear*)

- We have classified all the gaugings of the STU model, excluding integrable truncations
- We have found two integrable truncations of gauged N=1 Supergravity. In short, suitable superpotentials that lead to potentials with consistent integrable truncations
- Analysing in depth the solutions of one of the supersymmetric integrable models we have discovered some new mechanisms with potentially important cosmological implicactions….

N=1 SUGRA potentials

N=1 SUGRA coupled to n Wess Zumino multiplets

$$\mathcal{L}^{\mathcal{N}=1}_{SUGRA} = \sqrt{-g}\left[\mathcal{R}[g] + 2\, g^{HK}_{ij^\star}\, \partial_\mu z^i\, \partial^\mu \bar{z}^{j^\star} - 2\, V(z,\bar{z})\right] ,$$

where

$$g_{ij^\star} = \partial_i \partial_{j^\star} \mathcal{K}$$
$$\mathcal{K} = \overline{\mathcal{K}} = \text{Kähler potential}$$

$$V = 4\, e^2 \exp[\mathcal{K}] \left(g^{ij^\star} \mathcal{D}_i W_h(z)\, \mathcal{D}_{j^\star} \overline{W_h}(\bar{z}) - 3\, |W_h(z)|^2\right) ,$$

and

$$\mathcal{D}_i W = \partial_i W + \partial_i \mathcal{K}\, W$$
$$\mathcal{D}_{j^\star} \overline{W} = \partial_{j^\star} \overline{W} + \partial_{j^\star} \mathcal{K}\, \overline{W}$$

If one multiplet, for instance

$$\mathcal{M}_K = \frac{SU(1,1)}{U(1)} \Rightarrow \mathcal{K} = \log\left[-(z-\bar{z})^3\right]$$

Integrable SUGRA model N=1

If in supergravity coupled to one Wess Zumino multiplet spanning the SU(1,1) / U(1) Kaehler manifold we introduce the following superpotential

$$W_{int} = \lambda z^4 + \mathrm{i}\,\kappa z^3 \,,$$

$$\lambda = \frac{6}{\sqrt{5}} \quad ; \quad \kappa = \frac{2\omega}{\sqrt{5}}$$

we obtain a scalar potential

$$V_{int}(z,\bar{z}) = \frac{12 z^2 \bar{z}^2 \left((4\mathrm{i}\bar{z} + \omega) z^2 - 4\mathrm{i}\bar{z}^2 z + \bar{z}^2 \omega \right)}{5(z-\bar{z})^2}$$

where

$$z = \mathrm{i}\exp[h] + b$$

Truncation to zero axion b=0 is consistent

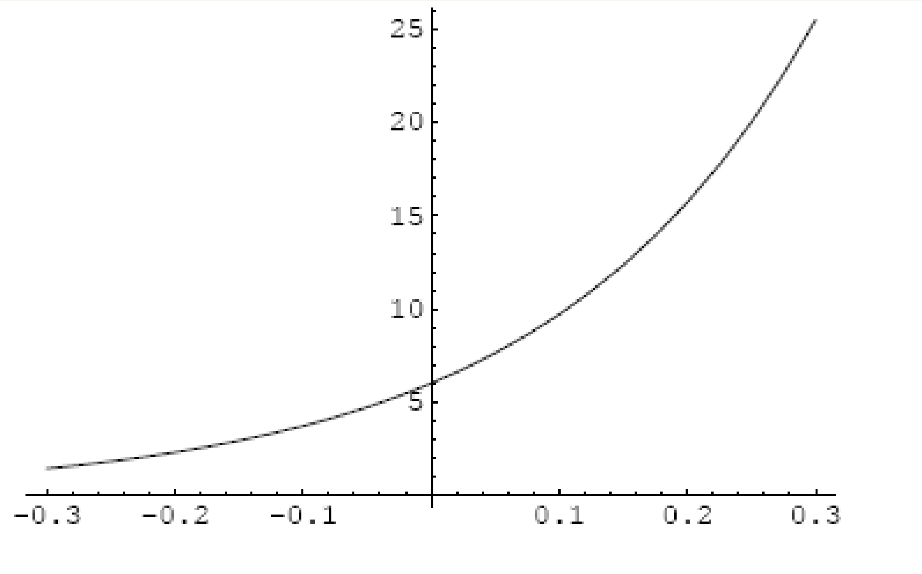
$$V_{Int} = \frac{6}{5} e^{4\mathfrak{h}} \left(\omega + 4 e^{\mathfrak{h}} \right)$$

THIS IS AN INTEGRABLE MODEL

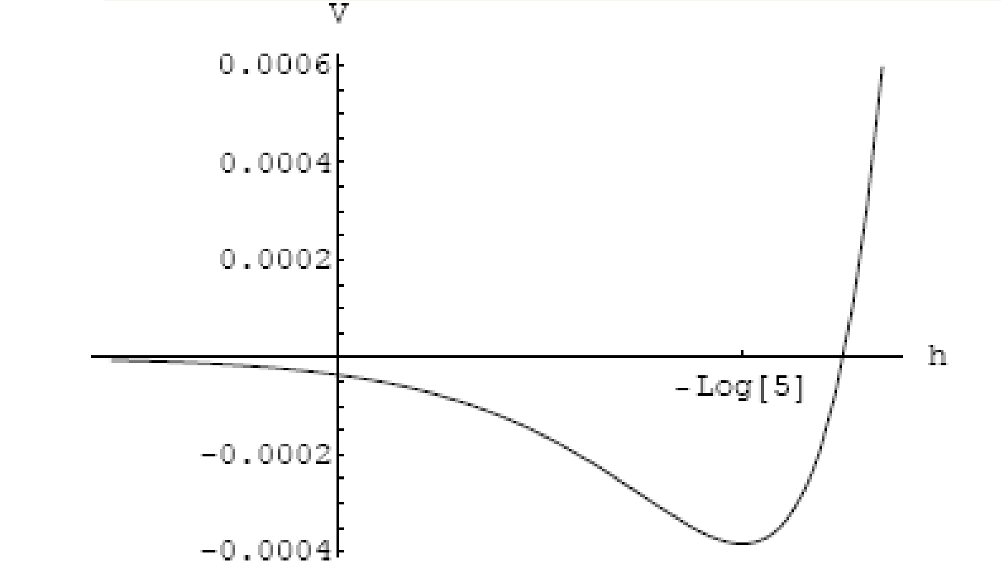
$$\mathfrak{h} = \frac{\varphi}{3}$$

$$\begin{aligned} \mathcal{V}(\varphi) &= C_1 \exp\left[2\gamma\,\varphi\right] + C_2 \exp\left[(\gamma+1)\,\varphi\right] \\ \gamma &= \frac{2}{3} \end{aligned}$$

The form of the potential



Hyperbolic: $\omega > 0$
Runaway potential



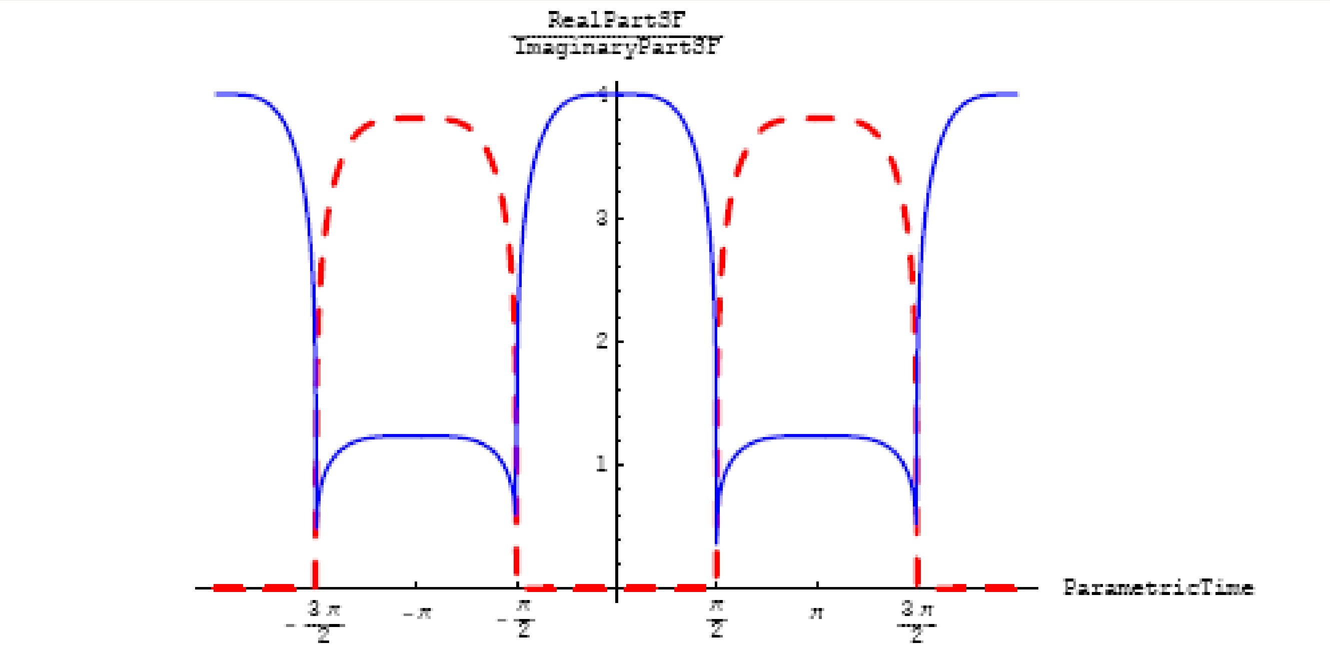
Trigonometric $\omega < 0$
Potential with a negative extremum: stable AdS vacuum

The General Integral in the trigonometric case

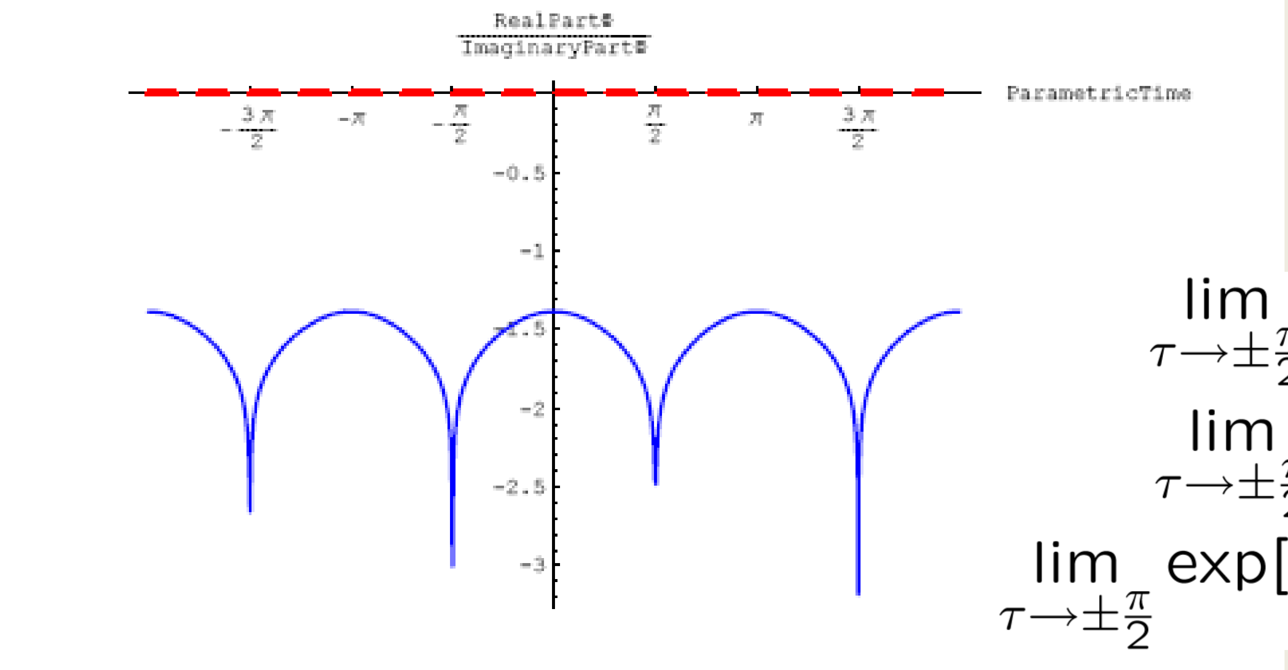
$$
\begin{aligned}
a(\tau,\Lambda,Y) = \Lambda\,\mathfrak{a}(\tau,Y) &= \Lambda\left[\frac{4}{5}\cos^{\frac{2}{5}}(\tau)\left((\cos\tau)^{9/5}\,{}_2F_1\left(\frac{1}{2},\frac{9}{10};\frac{3}{2};\sin^2(\tau)\right)\tan^2(\tau)+5\right)\right.\\
&\quad\left.-Y\cos^{\frac{1}{5}}(\tau)\sin(\tau)\right]\\
\exp\left[\mathcal{B}(\tau,Y)\right] &= \frac{1}{25}\left(4\cos^2(\tau)^{9/10}\,{}_2F_1\left(\frac{1}{2},\frac{9}{10};\frac{3}{2};\sin^2(\tau)\right)\tan^2(\tau)-\frac{5Y\sin(\tau)}{\cos^{\frac{1}{5}}(\tau)}+20\right)^2\\
\mathfrak{h}(\tau,Y) &= -\log\left(\frac{4}{5}\cos^2(\tau)^{9/10}\,{}_2F_1\left(\frac{1}{2},\frac{9}{10};\frac{3}{2};\sin^2(\tau)\right)\tan^2(\tau)-\frac{Y\sin(\tau)}{\cos^{\frac{1}{5}}(\tau)}+4\right)
\end{aligned}
$$

The scalar field tries to set down at the negative extremum but it cannot since there are no spatial flat sections of AdS space!

The result is a BIG CRUNCH. General Mechanism whenever there is a negative extremum of the potential



The simplest solution Y=0

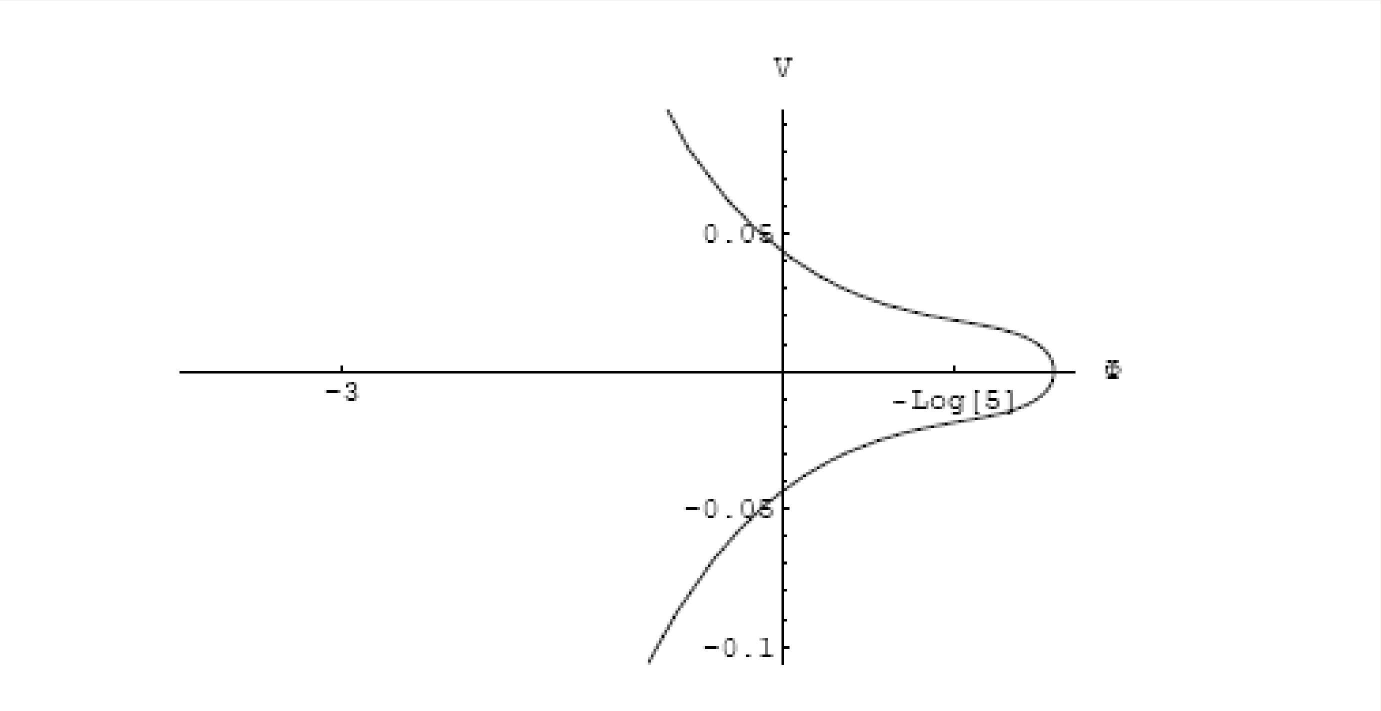


$$\lim_{\tau \to \pm\frac{\pi}{2}} \mathfrak{a}(\tau;0) = 0$$

$$\lim_{\tau \to \pm\frac{\pi}{2}} \mathfrak{h}(\tau,0) = -\infty$$

$$\lim_{\tau \to \pm\frac{\pi}{2}} \exp[\mathcal{B}(\tau,0)] = +\infty$$

Phase portrait of the simplest solution

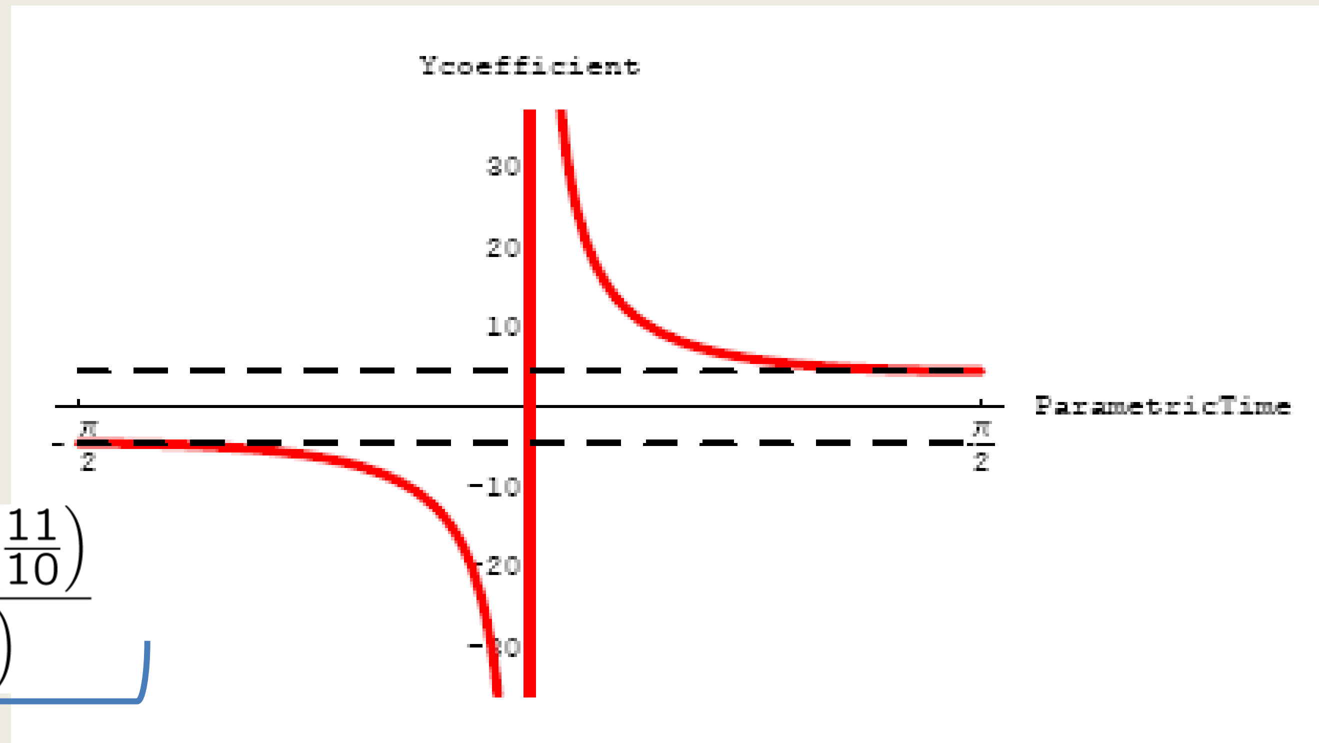


The extremum of the potential is at $\Phi_0 = -\log[5]$. It is reached by the solution however with a non vanishing velocity. There is no fixed point and the trajectory is from infinity to infinity.

Y-deformed solutions

An additional zero of the scale factor occurs for τ_0 such that

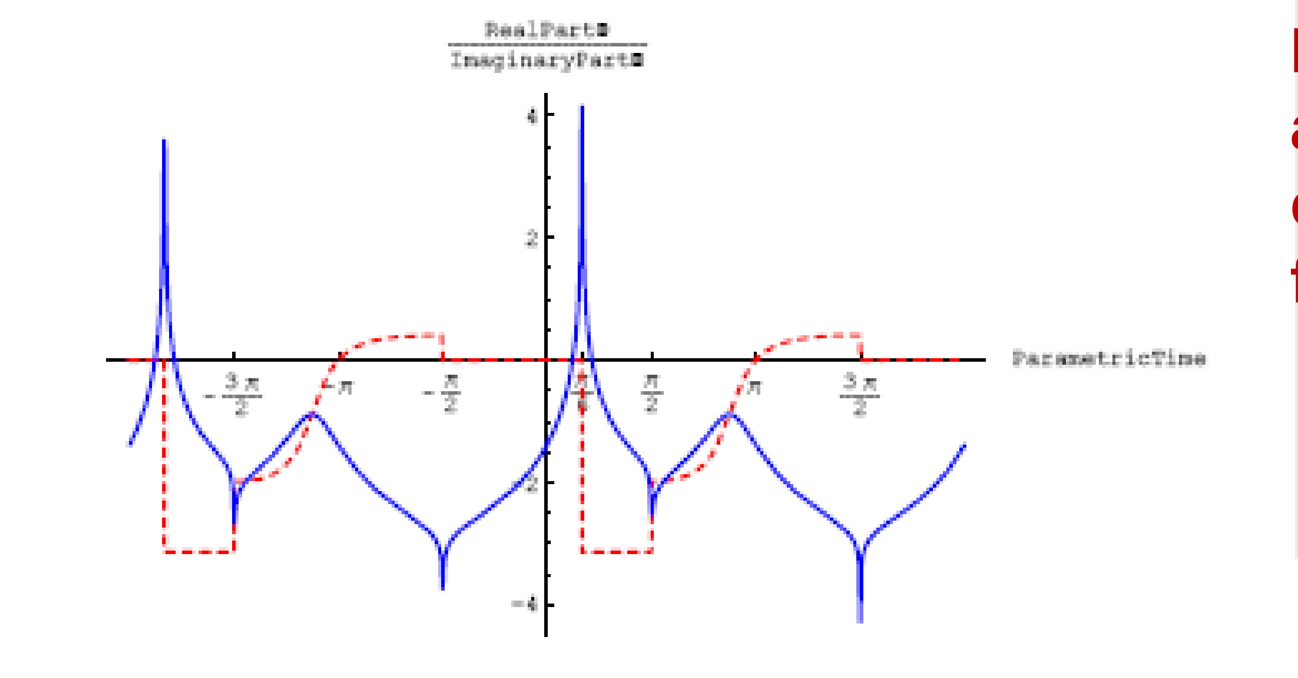
$$Y = \frac{4}{5}\cos^{\frac{1}{5}}(\tau_0)\csc(\tau_0)\left(\cos^2(\tau_0)^{9/10}\,{}_2F_1\left(\frac{1}{2},\frac{9}{10};\frac{3}{2};\sin^2(\tau_0)\right)\tan^2(\tau_0)+5\right) \equiv \mathfrak{f}(\tau_0)$$



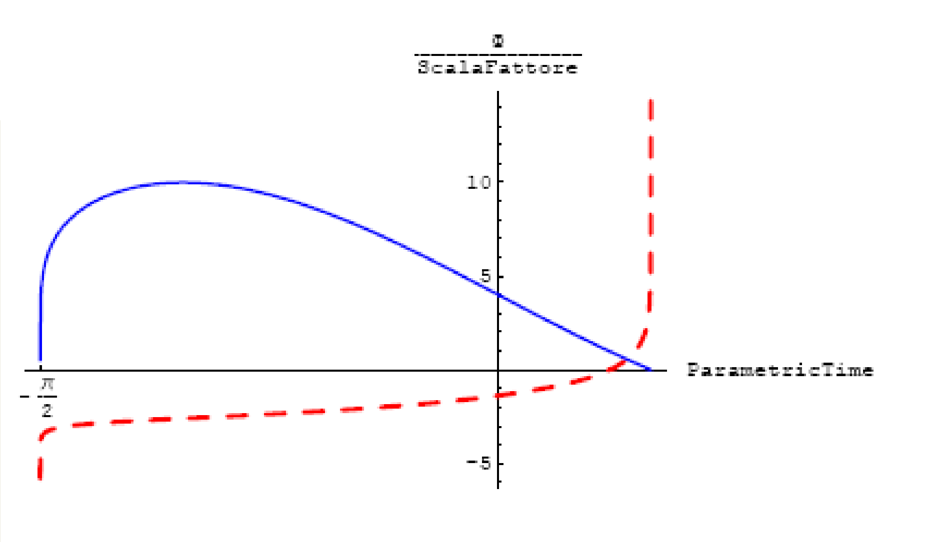
$$|Y| \leq Y_0 \equiv \frac{4\sqrt{\pi}\,\Gamma\left(\frac{11}{10}\right)}{\Gamma\left(\frac{3}{5}\right)}$$

Region of moduli space
without early Big Crunch

What new happens for $Y > Y_0$?



Early Big Bang and climbing scalar from -1 to + 1



Particle and Event Horizons

Radial light-like geodesics

$$0 = \exp\left[2\mathcal{B}(\tau)\right] d\tau^2 - \mathfrak{a}^2(\tau)\, dr^2$$

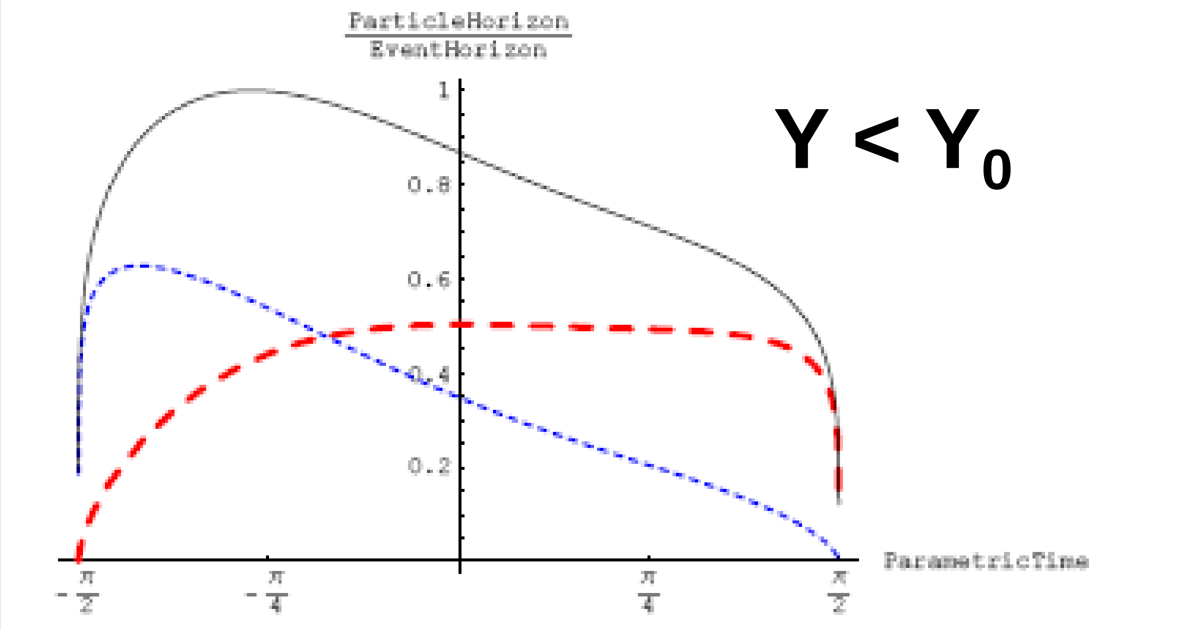
Particle horizon: boundary of the visible universe at time T

$$\mathcal{P}(T) = \frac{\mathfrak{a}(T)}{a_{max}\, r_{max}} \int_{-\frac{\pi}{2}}^{T} d\tau \frac{\exp\left[\mathcal{B}(\tau)\right]}{\mathfrak{a}(\tau)}$$

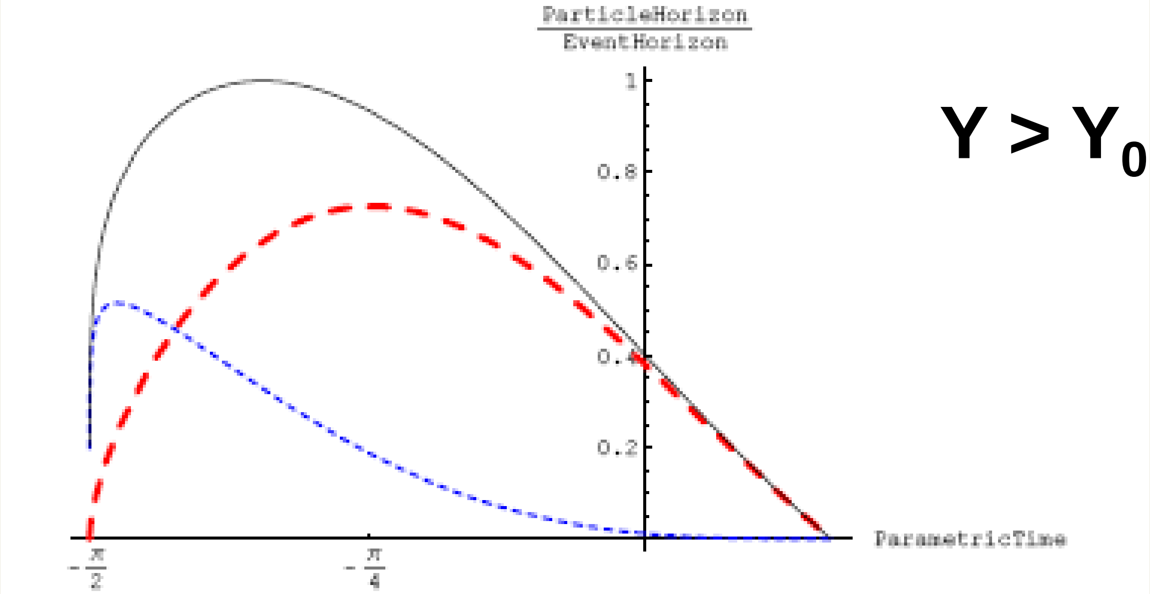
Event Horizon: Boundary of the Universe part from which no signal will ever reach an observer living at time T

$$\mathcal{E}(T) = \frac{\mathfrak{a}(T)}{a_{max}\, r_{max}} \int_{T}^{T_{max}} d\tau \frac{\exp\left[\mathcal{B}(\tau)\right]}{\mathfrak{a}(\tau)}$$

Particle and Event Horizons do not coincide!

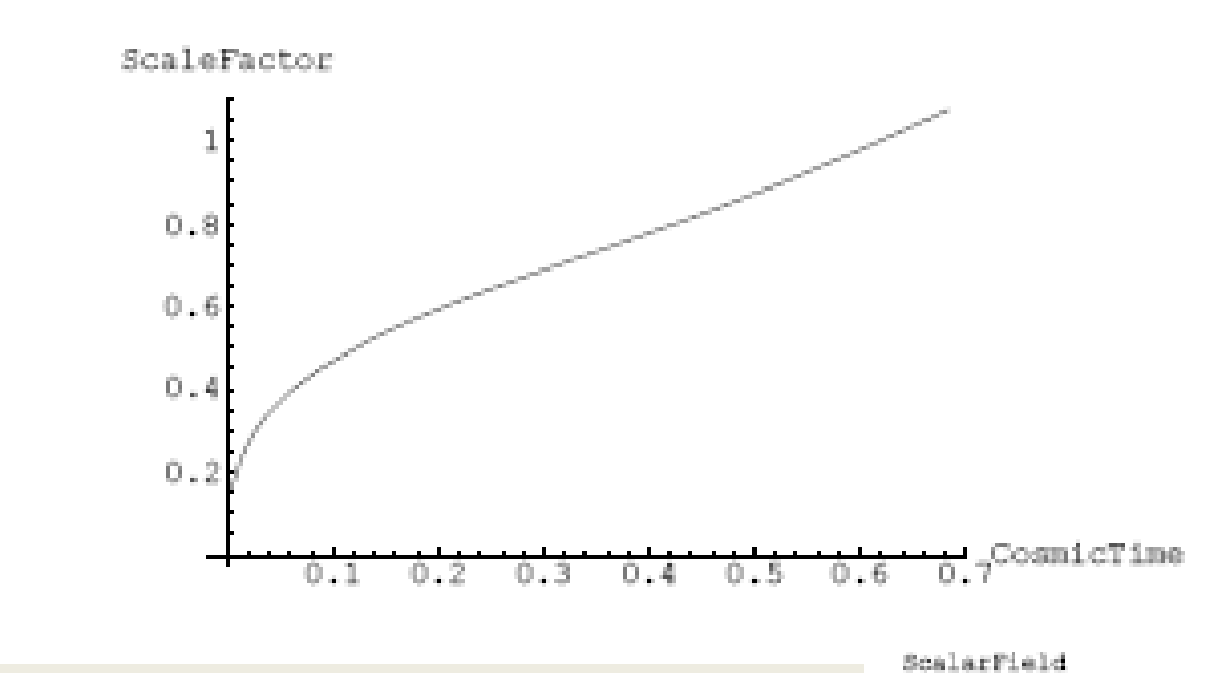


$Y < Y_0$



$Y > Y_0$

Hyperbolic solutions



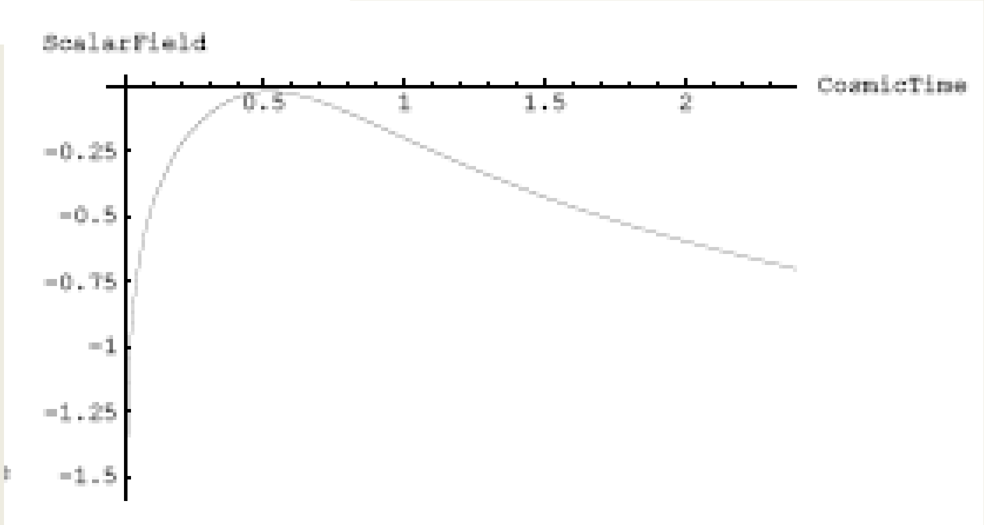
We do not write the analytic form. It is also given in terms of hypergeometric functions of exponentials

$$a(T_c) \sim T_c^{\frac{1}{3}}$$

at Big Bang

$$a(T_c) \sim (T_c - T_{max})^{\frac{3}{4}}$$

at Big Crunch



FLUX compactifications and another integrable model

In string compactifications on T^6 / Z_2 £ Z_2

one arrives at 7 complex moduli fields

$$S,\, T_1,\, T_2,\, T_3,\, U_1,\, U_2,\, U_3$$

imposing a global SO(3) symmetry one can reduce the game to three fields

$$S \quad ; \quad T = T_1 = T_2 = T_3 \quad ; \quad U = U_1 = U_2 = U_3$$

with Kahler potential

$$\mathcal{K} = -\log[-i\,(S - \bar{S})] - \log[i\,(T - \bar{T})^3] - \log[i\,(U - \bar{U})^3]$$

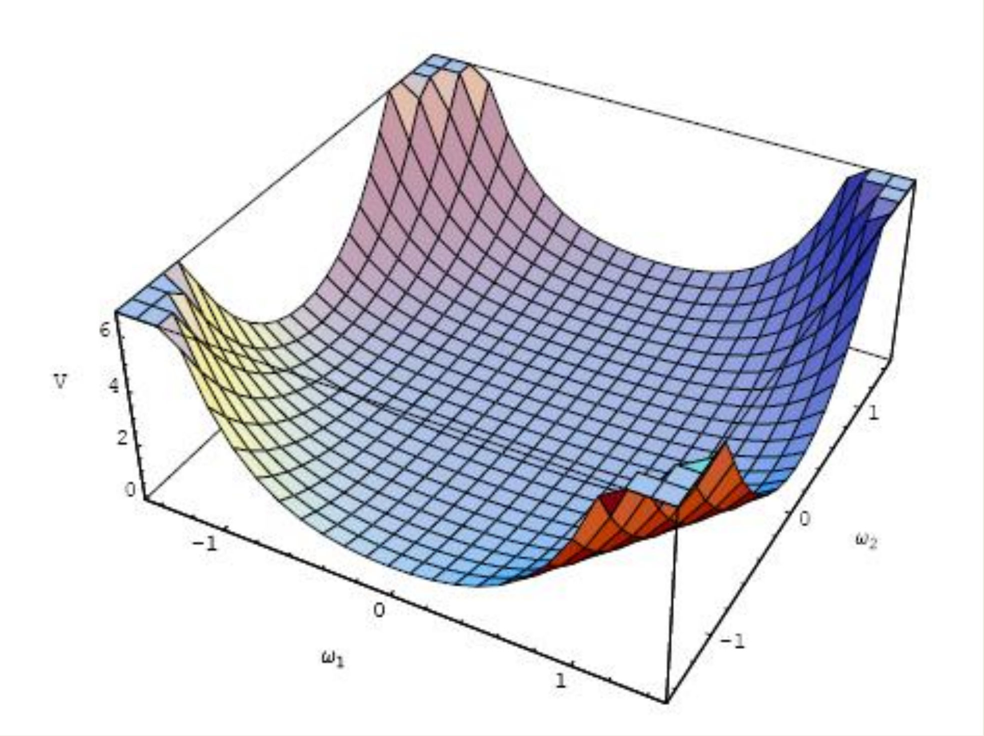
Switching on Fluxes introduces a superpotential W polynomial in S,T,U and breaks SUSY N=4 into N=1

A special case

$$W_{integ} = \left(iT^3 + 1\right)\left(SU^3 - 1\right)$$

induces a potential depending on three dilatons and three axions.
The axions can be consistently truncated and one has a potential in three dilatons with an extremum at $h1=h_2=h_3 = 0$ that is a ***STABLE dS VACUUM***

There are two massive and one massless eigenstates. The potential depends only on the two massive eigenstates ω_1 and ω_2



The truncation to either one of the mass eigenstates is consistent

one obtains:

$$3\,V_{dil}(\vec{\omega})\big|_{\omega_1=0\,;\,\omega_2=\frac{\varphi}{\sqrt{3}}} = \frac{3}{8}(\cosh(\varphi)+1)$$

$$3\,V_{dil}(\vec{\omega})\big|_{\omega_1=\frac{2\varphi}{3\sqrt{3}}\,;\,\omega_2=0} = \frac{3}{32}\left(\cosh\left(\frac{2\varphi}{\sqrt{3}}\right)+7\right)$$

THIS MODEL is INTEGRABLE. Number 1) in the list

Hence we can derive exact cosmological solutions in this supergravity from flux compactifications

Conclusion on F-type

The study of integrable cosmologies within superstring and supergravity scenarios has just only begun.
Integrable cases in the F-type approach are rare but do exist and can provide a lot of unexpected information that illuminates also the Physics behind the non integrable cases.

Via D-type all positive potentials can be embedded into N=1 SUGRA

New challenging interpretation problems !

Let us go back to where we were.....

Imposing:

$$J''(C)\,(\mathrm{d}C)^2 = J''(C)\left(\frac{\mathrm{d}C}{\mathrm{d}\phi}\right)^2 (\mathrm{d}\phi)^2 = (\mathrm{d}\phi)^2$$

$$D(\phi) = -2\,\mathcal{P}_0(C) = J'(C)$$

momentum map

$$\left(\frac{\mathrm{d}C}{\mathrm{d}\phi}\right) = \left(\frac{\mathrm{d}D(\phi)}{\mathrm{d}\phi}\right)^{-1}$$

$$\mathcal{L}_{scalar} = 2\left(\frac{1}{4}\partial^\mu\phi\,\partial_\mu\phi - \mathrm{const}^2 \times (D(\phi))^2\right)$$

The square root of the potential is interpreted as the momentum map of the translational symmetry

$$\mathcal{P} = -\frac{1}{2}\sqrt{V(\phi)}$$

The Kahler curvature from the scalar Potential

$$ds^2_{\text{Kähler}} = \frac{1}{4}\mathrm{d}\phi^2 + \left(\mathcal{P}'(\phi)\right)^2 dB^2$$

$$\left.\begin{aligned} E^1 &= \frac{1}{2}\mathrm{d}\phi \\ E^2 &= \mathcal{P}'(\phi)\,\mathrm{d}B \end{aligned}\right\} \text{zweibein}$$

$$\begin{aligned} \mathrm{d}E^1 + \omega \wedge E^2 &= 0 \\ \mathrm{d}E^2 - \omega \wedge E^1 &= 0 \end{aligned}$$

$$\omega = -2\,\mathcal{P}''(\phi)\,\mathrm{d}B$$

$$\mathfrak{R} \equiv \mathrm{d}\omega \equiv R(\phi)\,E^1 \wedge E^2$$

$$R(\phi) = -4\left(\frac{V'''}{V'} - \frac{3}{2}\frac{V''}{V} - \frac{3}{4}\left(\frac{V'}{V}\right)^2\right)(\phi)$$

Example:
the best fit model in the γ series

INTEGRABLE SERIES

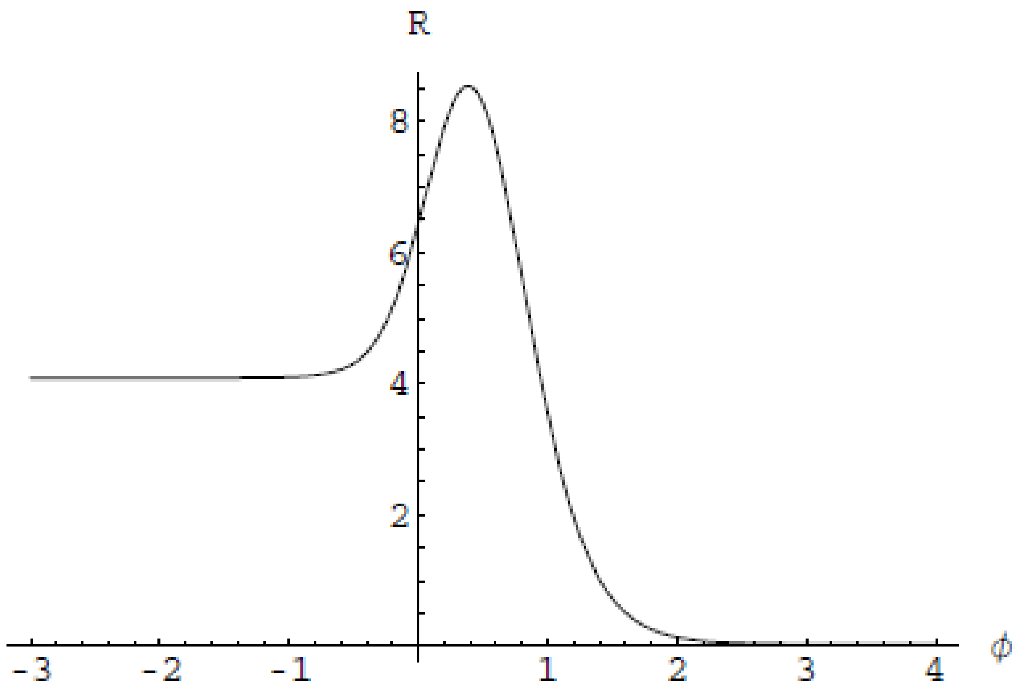
$$V(\phi) \;=\; a\,\exp\left[2\sqrt{3}\,\gamma\,\phi\right] + b\,\exp\left[\sqrt{3}\,(\gamma+1)\,\phi\right]$$

Best fit for CMB (Sagnotti et al) $\gamma = -\frac{7}{6}$

$$R_{-\frac{7}{6}}(\phi) \;=\; \frac{1}{48}\left(39\lambda\left(\frac{280}{14\lambda + e^{\frac{13\phi}{2\sqrt{3}}}} - \frac{15\lambda + 28e^{\frac{13\phi}{2\sqrt{3}}}}{\left(\lambda + e^{\frac{13\phi}{2\sqrt{3}}}\right)^2}\right) + 1\right)$$

$$\lambda = \frac{b}{a}$$

Interpolating kink between two Poincaré spaces



$$R_{-\frac{7}{6}}(-\infty) = \frac{49}{12} \quad ; \quad R_{-\frac{7}{6}}(\infty) = \frac{1}{48}$$

This is just the beginnig.....

We should find the geometric interpretation of the Kahler manifolds associated with integrable potentials and their string origin.........

THANK YOU FOR YOUR ATTENTION