Introduction to the Supersymmetry

Sergey Fedoruk

BLTP JINR, Dubna, Russia

Helmholtz International Summer School
“Cosmology, Strings and New Physics”
BLTP JINR, Dubna, Russia
September 2-14, 2013

Plan

- Lecture 1: Grounds for supersymmetry.
  - Unification in elementary particle physics. Symmetries.
  - Haag-Lopushanski-Sohnius theorem: Graded symmetry algebra.
  - Brief sketch on supermatrix and supergroups.
  - Super–Poincare algebra, conformal and dS (AdS) supersymmetry.
  - Wess–Zumino model.
- Lecture 2: 1*D* SUSY in component formulation and in superspace.
  - 1*D* super-Poincare and superconformal symmetries.
  - 1*D* field theory with global SUSY: Hamiltonian analysis and supercharges.
  - 1*D* $\mathcal{N} = 1$ supergravity in component formulation: spin $1/2$ particle model.
  - Superspace formulation. 1*D* supergravity in superspace.
  - Extended SUSY in 1*D* superspace: $\mathcal{N}{=}2$ real and chiral superfields.
  - Superconformal mechanics.
- Lecture 3: 4*D* SUSY in superspace
  - $\mathcal{N} = 1$ 4*D* superspace.
  - $\mathcal{N} = 1$ 4*D* superfields action.
  - $\mathcal{N}$-extended SUSY in superspace.
  - $\mathcal{N} = 2$ 4*D* in harmonic superspace formulation.
  - Superstring action.

Plan

- Lectures 4,5: The elements of twistor theory.
  - Symmetries of massless particle action.
  - Twistor space and its geometry.
  - Twistor transform.
  - Supertwistors.
  - (Super)twistors in HS theory and superstring theory.

Lecture 1: Grounds for supersymmetry

- Unification in elementary particle physics. Symmetries.
- Haag-Lopushanski-Sohnius theorem: Graded symmetry algebra.
- Brief sketch on supermatrix and supergroups.
- Super–Poincare algebra, conformal and dS (AdS) supersymmetry.
- Wess–Zumino model.

In elementary particle physics, the hope is that we will eventually achieve a unified scheme which combines all particles and all their interactions into one consistent theory.

The currently known particles:

$$\text{Bosons}: \quad \underbrace{\overbrace{A_\mu \sim (\vec{E}, \vec{B})}^{\text{Maxwell Theory}, U(1)} \;\oplus\; W^{\pm}_{\mu}, W^{0}_{\mu}}_{\text{Electroweek Theory}, SU(2)\times U(1)} \;\oplus\; \underbrace{G^{r}_{\mu},\ {\scriptstyle r=1,\ldots,8}}_{\text{Strong Interaction}, SU(3)} \;\oplus\; \underbrace{g_{\mu\nu}}_{\text{Gravity}} \;\oplus\; \underbrace{H}_{\text{Higgs}}$$

$$\text{Fermions}: \quad \psi^{i}_{\alpha}\,, \quad \bar{\psi}_{\dot{\alpha} i}$$

Here $\mu, \nu = 0, 1, 2, 3$, $\alpha = 1, 2$, $\dot{\alpha} = 1, 2$ are the Lorentz indices; i is internal symmetry index. Bosons are the carriers of interactions (except Higgs). Fermions describe particles of matter.

Symmetries play fundamental role in the formulation of modern theories, which actually specify the theories. Symmetries are defined by concrete groups and corresponding algebras.

$$\psi \;\rightarrow\; g(\lambda)\,\psi\,, \text{ etc.}, \quad \text{where} \quad g(\lambda) = \exp\{\lambda^{A} B_{A}\}\,, \quad [B_A, B_B] = i c^{C}_{AB} B_C\,.$$

Parameters are functions of the space-time coordinates, $\lambda^{A} = \lambda^{A}(x_\mu)$ – local (gauge) group:

- Maxwell theory: gauge group $U(1)$;
- Electro-week theory: gauge group $U(1) \times SU(2)$;
- Standard model: gauge group $U(1) \times SU(2) \times SU(3)$;
- Gravity: local diffeomorphism group of four-dimensional space-time;
- String theory: local diffeomorphism group of the worldsheet (two-dimensional space-time).

An important role is played by the space-time (relativistic) symmetry.

Space-time symmetries

$$\Big\{\text{Lorentz algebra } SO(3,1) \cong SL(2,C)\Big\} \subset \Big\{\text{Poincare algebra } T^4 \subset\!\!\!\!+ SL(2,C)\Big\} \subset$$
$$\subset \Big\{\text{conformal algebra } SO(4,2) \cong SU(2,2)\Big\}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} [L_{\mu\nu}, L_{\rho\lambda}] = i\,(\eta_{\nu\rho}L_{\mu\lambda} + \eta_{\mu\lambda}L_{\nu\rho} - (\mu\leftrightarrow\nu)) \quad \mathrm{SL(2,C)} \\ [P_\mu, P_\nu] = 0\,, \quad [L_{\mu\nu}, P_\lambda] = i\,(\eta_{\nu\lambda}P_\mu - \eta_{\mu\lambda}P_\nu) \end{array}\right\}\text{Poincare algebra} \\ [D, P_\mu] = i\,P_\mu\,, \quad [D, K_\mu] = -i\,K_\mu\,, \quad [D, L_{\mu\nu}] = 0 \\ [K_\mu, K_\nu] = 0\,, \quad [P_\mu, K_\nu] = -2i\,(\eta_{\mu\nu}D + L_{\mu\nu})\,, \quad [L_{\mu\nu}, K_\lambda] = i\,(\eta_{\nu\lambda}K_\mu - \eta_{\mu\lambda}K_\nu) \end{array}\right\}\text{conformal algebra}$$

$$\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1), \;\; \mu = 0,1,2,3$$

All modern realistic field theories are relativistic and possess Poincare symmetry.

Conformal symmetry is invariance of light-cone

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0$$

and is symmetry in the theories of massless fields (CFT).

$SO(d,1)$ – Lorentz symmetry in $(d+1)$-dimensonal space-time.
$SO(d+1,2)$ – conformal symmetry in $(d+1)$-dimensonal space-time.

Space-time symmetries

$\Big\{$de Sitter algebra $SO(4,1)\Big\}$, $\Big\{$anti de Sitter algebra $SO(3,2) \cong Sp(4)\Big\}$

$$[M_{MN}, M_{KL}] = i\left(\eta_{NL}M_{MK} + \eta_{MK}M_{NL} - (M \leftrightarrow N)\right),$$

$$\eta_{MN} = \mathrm{diag}(+1,-1,-1,-1,\mp 1), \quad M = 0,1,2,3,4 = (\mu, 4)$$

Using notations $M_{\mu\nu} = L_{\mu\nu}$, $M_{\mu 4} = \rho P_\mu$ we obtain

$$\left.\begin{array}{l} [L_{\mu\nu}, L_{\rho\lambda}] = i\left(\eta_{\nu\rho}L_{\mu\lambda} + \eta_{\mu\lambda}L_{\nu\rho} - (\mu \leftrightarrow \nu)\right) \\ [P_\mu, P_\nu] = \pm i\rho^{-2}L_{\mu\nu}, \quad [L_{\mu\nu}, P_\lambda] = i\left(\eta_{\nu\lambda}P_\mu - \eta_{\mu\lambda}P_\nu\right) \end{array}\right\} \xrightarrow{\rho\to\infty} \text{Poincare algebra}$$

de Sitter and anti de Sitter groups are symmetries of maximally symmetric manifold with constant scalar curvature (*dS* or *AdS* spaces)

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \mp (x^4)^2 = \mp\rho^2 .$$

ρ is the radius of curvature of *dS* or *AdS* spaces.
dS and *AdS* spaces are maximally symmetric vacuum solution of Einstein's field equation with cosmological constant.

$SO(d+1,1)$ – de Sitter group in $(d+1)$-dimensonal space-time.
$SO(d,2)$ – anti de Sitter group in $(d+1)$-dimensonal space-time.

(anti de Sitter group in $(d+1)$-dimensonal space-time) $\cong$
(conformal group in d-dimensonal space-time) – present in AdS/CFT correspondence

Internal symmetries

SU(n) algebra
$$[T^i_j, T^k_l] = i\,(\delta^i_l T^k_j - \delta^k_j T^i_l)\,, \quad (T^i_j)^+ = -T^j_i\,, \quad T^i_i = 0$$
$$i = 1, \ldots, n$$

O(n) algebra
$$[J_{ab}, J_{cd}] = i\,(\delta_{bc} J_{ad} + \delta_{ad} J_{bc} - (a \leftrightarrow b))\,, \quad (J_{ab})^+ = J_{ab} = -J_{ba}$$
$$a = 1, \ldots, n$$

Internal symmetries commute with space-time ones.

Example:
$$O(3) \cong SU(2)$$
$$[J_a, J_b] = i\epsilon_{abc} J_c\,, \qquad J_a = \tfrac{1}{2}\sigma_a\,, \qquad a = 1, 2, 3$$
$$J_{ab} = -\epsilon_{abc} J_c$$
$$T^j_i = J_a (\sigma_a)_i{}^j\,, \qquad i = 1, 2$$

Coleman, Mandula, 1967: it is impossible to unify space–time symmetry with internal symmetries in frame of local relativistic field theory in four dimension with finite number of massive particles.

$$W^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}P_\mu L_{\lambda\rho}\,, \qquad W^2 = -m^2\vec{J}^2 \;\; (P^2 \neq 0)\,, \qquad W_\mu = \Lambda P_\mu \;\; (P^2 = 0)$$

$[T^i_j, P^2] = [T^i_j, W^2] = [T^i_j, \Lambda] = 0$ – all particles of an irreducible multiplet must have the same mass and the same spin (helicity)

Bypass of the Coleman–Mandula theorem:

Haag, Lopushanski, Sohnius, 1975 proved that in the context of relativistic field theory the only models which lead to the unification problem are supersymmetric theories.

In supersymmetric theories, the symmetry described by the Lie superalgebras and Lie supergroups.

Symmetry algebras of the supersymmetric models are graded Lie algebras or Lie superalgebras

$$[B_A, B_B] = i\, c_{AB}^{C} B_C\,, \qquad [B_A, Q_K] = i\, g_{AK}^{M} Q_M\,, \qquad \{Q_K\,, Q_M\} = i\, f_{KM}^{A} B_A$$

B_A are even (bosonic) elements; Q_K are odd (fermionic) elements

Graded Jacobi identities

$$[[G_1, G_2\}, G_3\} + \text{graded cyclic} = 0$$

(there is additional minus sign if two fermionic operators are interchanged)

Bosonic subalgebra B_A are defined by Coleman–Mandula theorem.

On the fermionic operators Q_M it is realized the representation of the bosonic subalgebra.

Q_M generate supersymmetric tansformations

$$Q\,|\text{boson} > = |\text{fermion} >, \qquad Q\,|\text{fermion} > = |\text{boson} >$$

Parity: $q(B) = 0$, $q(Q) = 1$, $q(|\text{boson} >) = 0$, $q(|\text{fermion} >) = 1$

Exponential representation of Lie supergroups are given by

$$X = \exp\left\{ i \left(\lambda^A B_A + \xi^M F_M \right) \right\}$$

where λ^A are c-number parameters whereas ξ^M are Grassmann parameters:

$$\xi^M \xi^N = -\xi^N \xi^M \qquad \Rightarrow \qquad (\xi^1)^2 = 0, \quad (\xi^2)^2 = 0, \quad \text{etc.}$$

$$X=\left(\begin{array}{c|c} B_1 & F_1 \\ \hline F_2 & B_2 \end{array}\right);\qquad \begin{array}{l} B_{1,2}\ \text{are ordinary matrices,} \\ F_{1,2}\ \text{are fermionic matrices} \end{array}$$

$$\mathrm{str}\,X=\mathrm{tr}\,B_1-\mathrm{tr}\,B_2\,,\qquad \mathrm{str}\,XY=\mathrm{str}\,YX$$

$$\mathrm{sdet}\begin{pmatrix} B_1 & F_1 \\ 0 & 1 \end{pmatrix}=\det B_1,\quad \mathrm{sdet}\begin{pmatrix} 1 & F_1 \\ 0 & B_2 \end{pmatrix}=\det B_2^{-1};\qquad \mathrm{sdet}\,XY=\mathrm{sdet}\,X\cdot\mathrm{sdet}\,Y$$

$$\begin{pmatrix} B_1 & F_1 \\ F_2 & B_2 \end{pmatrix}=\begin{pmatrix} 1 & F_1 \\ 0 & B_2 \end{pmatrix}\begin{pmatrix} B_1-F_1B_2^{-1}F_2 & 0 \\ B_2^{-1} & 1 \end{pmatrix},\quad \mathrm{sdet}\,X=\det(B_1-F_1B_2^{-1}F_2)\cdot\det B_2^{-1}$$

$$\mathrm{sdet}\,X=\exp\{\mathrm{str}\,(\ln X)\}$$

$$\mathrm{OSp}(m|n):\qquad G=e^X=\left(\begin{array}{c|c} \mathrm{Sp}(n) & F_1 \\ \hline F_2 & \mathrm{SO}(m) \end{array}\right)$$

$$\mathrm{U}(m,n|p):\qquad G=e^X=\left(\begin{array}{c|c} \mathrm{U}(m,n) & F_1 \\ \hline F_2 & \mathrm{U}(p) \end{array}\right)$$

$$\mathrm{SU}(m,n|p):\qquad G=e^X\,,\quad \mathrm{str}\,X=0$$

For $m+n=p$ the identity matrix obeys $\mathrm{tr}\,B_1=\mathrm{tr}\,B_2$ and generates $\mathrm{U}(1)$ subgroup.
The quotient $\mathrm{PSU}(m,n|p)=\mathrm{SU}(m,n|p)/\mathrm{U}(1)$ is simple and is often denoted just $\mathrm{SU}(m,n|p)$.

Notations used: $$(\gamma_\mu)_a{}^b = \begin{pmatrix} 0 & (\sigma_\mu)_{\alpha\dot\beta} \\ (\tilde\sigma_\mu)^{\dot\alpha\beta} & 0 \end{pmatrix}, \qquad \begin{array}{c} a = 1,..,4 \\ \alpha = 1,2,\ \dot\alpha = 1,2 \end{array}$$

$$\gamma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]$$

$$\Psi_a = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix}, \qquad \bar\chi_{\dot\alpha} = \overline{(\psi_\alpha)} \text{ for a Majorana spinor}$$

$$\sigma_\mu\tilde\sigma_\nu + \sigma_\nu\tilde\sigma_\mu = 2\eta_{\mu\nu}, \quad \sigma_{\mu\nu} = i\sigma_{[\mu}\tilde\sigma_{\nu]}, \quad \tilde\sigma_{\mu\nu} = i\tilde\sigma_{[\mu}\sigma_{\nu]}$$

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \quad \bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}, \quad \bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta},$$

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \quad \epsilon_{\dot\alpha\dot\beta} = -\epsilon_{\dot\beta\dot\alpha}, \quad \epsilon_{12} = -\epsilon^{12} = 1$$

$$(\tilde\sigma_\mu)^{\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}(\sigma_\mu)_{\beta\dot\beta}, \quad (\sigma^\mu)_{\alpha\dot\alpha} = (1, \vec\sigma)_{\alpha\dot\alpha}, \quad (\tilde\sigma^\mu)^{\dot\alpha\alpha} = (1, -\vec\sigma)^{\dot\alpha\alpha}$$

$$x_{\alpha\dot\beta} = \sigma^\mu_{\alpha\dot\beta}x_\mu, \quad x^{\dot\alpha\beta} = \tilde\sigma^{\dot\alpha\alpha}_\mu x^\mu, \quad x^\mu = x^{\dot\alpha\alpha}\sigma^\mu_{\alpha\dot\alpha}$$

4D $\mathcal{N}$-extended Poincare superalgebra

$$P_\mu,\, L_{\mu\nu},\, T^i_j \quad + \quad Q^i_\alpha,\, \bar{Q}_{\dot\alpha i} = (Q^i_\alpha)^+ \quad + \quad Z^{ij}, \bar{Z}_{ij} = (Z^{ij})^+$$

$$\{Q^i_\alpha, \bar{Q}_{\dot\beta j}\} = 2\delta^i_j (\sigma^\mu)_{\alpha\dot\beta} P_\mu, \qquad \{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta} Z^{ij}, \qquad \{\bar{Q}_{\dot\alpha i}, \bar{Q}_{\dot\beta j}\} = \epsilon_{\dot\alpha\dot\beta} \bar{Z}_{ij},$$

$$[P_\mu, Q^i_\alpha] = 0, \quad [P_\mu, \bar{Q}_{\dot\alpha i}] = 0, \qquad [L_{\mu\nu}, Q^i_\alpha] = -\tfrac{1}{2}(\sigma_{\mu\nu})_\alpha^\beta Q^i_\alpha, \quad [L_{\mu\nu}, \bar{Q}_{\dot\alpha i}] = \tfrac{1}{2}(\tilde\sigma_{\mu\nu})^{\dot\beta}_{\dot\alpha} Q_{\dot\alpha i},$$

$$[T^i_j, Q^k_\alpha] = \delta^k_j Q^i_\alpha - \tfrac{1}{\mathcal{N}}\delta^i_j Q^k_\alpha, \qquad [T^i_j, \bar{Q}_{\dot\alpha k}] = -\delta^i_k \bar{Q}_{\dot\alpha j} + \tfrac{1}{\mathcal{N}}\delta^i_j \bar{Q}_{\dot\alpha k}$$

$Z^{ij}, \bar{Z}_{ij} = (Z^{ij})^+$ are central charges, $[Z,P] = [Z,L] = [Z,Q] = [Z,Z] = 0$

In massless case, $P^2 = 0$, : $\quad Z^{ij} = 0,\ \bar{Z}_{ij} = 0$

Basic properties of Poincare supersymmetry

- $[P^2, Q] = 0$ - all particles of any supermultiplet have the same mass
- $0 \le \sum_i \sum_\alpha \{Q^i_\alpha, \bar{Q}_{\dot\alpha i}\} = 4\mathcal{N} P_0$ - the energy is non-negative
- $\{|\text{bosons}>\} \stackrel{Q}{\to} \{|\text{fermions}>\} \stackrel{Q}{\to} \{|\text{bosons}>\}$ translated by P_μ $\Rightarrow$
  There are an equal number of bosons and fermions
  when translations are an invertible operator

4D $\mathcal{N}$-extended conformal superalgebra

$$SU(2,2|\mathcal{N})$$

$$\underbrace{P_\mu, L_{\mu\nu}, \overbrace{T^i_j, \; R,}^{U(\mathcal{N})} \qquad K_\mu, D,}_{even} \qquad \underbrace{Q^i_\alpha, \bar{Q}_{\dot\alpha i} = (Q^i_\alpha)^+, \qquad S_{\alpha i}, \bar{S}^i_{\dot\alpha} = (S_{\alpha i})^+}_{odd}$$

$$\{Q^i_\alpha, \bar{Q}_{\dot\beta j}\} = 2\delta^i_j (\sigma^\mu)_{\alpha\dot\beta} P_\mu, \qquad \{S_{\alpha i}, \bar{S}^j_{\dot\alpha}\} = 2\delta^j_i (\sigma^\mu)_{\alpha\dot\alpha} K_\mu \,,$$

$$\{Q^i_\alpha, S^\beta_j\} = -\delta^i_j (\sigma^{\mu\nu})^\beta_\alpha L_{\mu\nu} - 4i\delta^\beta_\alpha T^i_j - 2i\delta^\beta_\alpha \delta^i_j D + \tfrac{2(4-N)}{N}\, \delta^\beta_\alpha \delta^i_j R \,,$$

$$[K_\mu, Q^i_\alpha] = (\sigma_\mu)_{\alpha\dot\alpha} \, \bar{S}^{\dot\alpha i} \,, \quad [K_\mu, \bar{Q}_{\dot\alpha i}] = -\, (\sigma_\mu)_{\alpha\dot\alpha} \, S^\alpha_i \,,$$

$$[P_\mu, S_{\alpha i}] = (\sigma_\mu)_{\alpha\dot\alpha} \, \bar{Q}^{\dot\alpha}_i \,, \quad [P_\mu, \bar{S}^i_{\dot\alpha}] = -\, (\sigma_\mu)_{\alpha\dot\alpha} \, Q^{\alpha i} \,,$$

$$[D, Q] = \tfrac{i}{2}\, Q \,, \qquad [D, \bar{Q}] = \tfrac{i}{2}\, \bar{Q} \,, \qquad [D, S] = -\tfrac{i}{2}\, S \,, \qquad [D, \bar{S}] = -\tfrac{i}{2}\, \bar{S} \,,$$

$$[R, Q] = -\tfrac{1}{2}\, Q \,, \qquad [R, \bar{Q}] = \tfrac{1}{2}\, \bar{Q} \,, \qquad [R, S] = \tfrac{1}{2}\, S \,, \qquad [R, \bar{S}] = -\tfrac{1}{2}\, \bar{S} \,,$$

4D $\mathcal{N}$-extended AdS superalgebra

$$OSp(4|\mathcal{N})$$

$$\underbrace{P_\mu,\, L_{\mu\nu},\, \overbrace{T_{ij}}^{O(\mathcal{N})},}_{even} \qquad \underbrace{Q_{ai}\,, \qquad (Q_i = C\bar{Q}_i^T)}_{odd}$$

$$\{Q_i, \bar{Q}_j\} = \delta_{ij}\left(2\gamma^\mu P_\mu + \rho^{-1}\gamma^{\mu\nu}L_{\mu\nu}\right)\,,$$

$$[P_\mu, Q_i] = -\frac{1}{2}\rho^{-1}\gamma_\mu Q_i\,,$$

$$[L_{\mu\nu}, Q_i] = -\frac{1}{2}\rho^{-1}\gamma_{\mu\nu} Q_i\,,$$

$$[T_{kl}, Q_i] = i(\delta_{li}Q_k - \delta_{ki}Q_l)$$

Simple 4*D* supersymmetric field theory: Wess–Zumino model

$$\underbrace{\phi(x),\ \bar\phi(x),\ \ F(x),\ \bar F(x),}_{\text{bosons (c}-\text{number)}}\quad \underbrace{\psi_\alpha(x),\ \bar\psi_{\dot\alpha}(x)}_{\text{fermions (Grassmann)}}$$

ε_α, $\bar\varepsilon_{\dot\alpha}$ – Grassmann parameters of SUSY translations , $\{\varepsilon_\alpha,\varepsilon_\beta\}=\{\varepsilon_\alpha,\bar\varepsilon_{\dot\beta}\}=0$

$$\delta\phi=(\varepsilon^\alpha Q_\alpha+\bar\varepsilon_{\dot\alpha}\bar Q^{\dot\alpha})\phi\,,\qquad \ldots\,,\qquad \delta\bar\psi_{\dot\alpha}=(\varepsilon^\alpha Q_\alpha+\bar\varepsilon_{\dot\alpha}\bar Q^{\dot\alpha})\bar\psi_{\dot\alpha}$$

$$(\delta_1\delta_2-\delta_2\delta_1)\phi=2(\varepsilon_1\sigma^\mu\bar\varepsilon_2-\varepsilon_2\sigma^\mu\bar\varepsilon_1)P_\mu\phi,\quad \ldots,\quad (\delta_1\delta_2-\delta_2\delta_1)\bar\psi_{\dot\alpha}=2(\varepsilon_1\sigma^\mu\bar\varepsilon_2-\varepsilon_2\sigma^\mu\bar\varepsilon_1)P_\mu\bar\psi_{\dot\alpha}$$

Off-shell supersymmetry transformation of component fields:

$$\delta\phi=-\varepsilon^\alpha\psi_\alpha\,,\quad \delta\psi_\alpha=-2i\sigma^\mu_{\alpha\dot\alpha}\bar\varepsilon^{\dot\alpha}\partial_\mu\phi-2\varepsilon_\alpha F\,,\quad \delta F=-i\sigma^\mu_{\alpha\dot\alpha}\bar\varepsilon^{\dot\alpha}\partial_\mu\psi^\alpha$$

Supersymmetry invariant action: $S = \int d^4x\, \mathcal{L}\,,$

$$\mathcal{L} = \partial^\mu \bar\phi \partial_\mu \phi - \tfrac{i}{2}\, \psi\sigma^\mu \partial_\mu \bar\psi - \tfrac{1}{4}\, m\,(\psi\psi + \bar\psi\bar\psi) - \tfrac{1}{2}\, g\,(\phi\psi\psi + \bar\phi\bar\psi\bar\psi)$$

$$\underbrace{+\bar F F + m\,(\phi F + \bar\phi \bar F) + g\,(\phi^2 F + \bar\phi^2 \bar F)}_{-(m+g\phi)(m+g\bar\phi)\phi\bar\phi}$$

$F(x)$, $\bar F(x)$ are auxiliary fields; its equations of motion is purely algebraic:

$$F + m\bar\phi + g\bar\phi^2 = 0, \qquad \bar F + m\phi + g\phi^2 = 0$$

$\delta\phi = -\varepsilon^\alpha \psi_\alpha\,, \quad \delta\psi_\alpha = -2i\sigma^\mu_{\alpha\dot\alpha}\bar\varepsilon^{\dot\alpha}\partial_\mu\phi + 2\,\varepsilon_\alpha\bar\phi(m + g\bar\phi)$ - on-shell SUSY transformations (nonlinear in case of the interaction, closed only on field equations of motion)

In massless case $m = 0$ the Wess-Zumino action

$$S = \int d^4x \left[\partial^\mu \bar\phi \partial_\mu \phi - \tfrac{i}{2}\, \psi\sigma^\mu \partial_\mu \bar\psi - g^2(\phi\bar\phi)^2 - \tfrac{1}{2}\, g\,(\phi\psi\psi + \bar\phi\bar\psi\bar\psi)\right]$$

possesses superconformal invariance. Direct generalization

$$S = \int \mathcal{D}x \left[\eta^{\mu\nu} \nabla_\mu \bar\phi \nabla_\mu \phi - \tfrac{i}{2}\, \psi\sigma^\mu \nabla_\mu \bar\psi + m^2\phi\bar\phi - g^2(\phi\bar\phi)^2 - \tfrac{1}{2}\, g\,(\phi\psi\psi + \bar\phi\bar\psi\bar\psi)\right]$$

invariant under *AdS* supersymmetry. Here, ∇_μ is *AdS* covariant derivative, $\rho = m^{-1}$ is *AdS* radius and $\mathcal{D}x$ is *AdS* is invariant volume element.

Lecture 2: 1D SUSY in component formulation and in superspace

- 1D super-Poincare and superconformal symmerties.
- 1D field theory with global SUSY:
  Hamiltonian analysis and supercharges.
- 1D $\mathcal{N}=1$ supergravity in component formulation:
  spin $1/2$ particle model.
- Superspace formulation. 1D supergravity in superspace.
- Extended SUSY in 1D superspace:
  $\mathcal{N}$=2 real and chiral superfields.
- Superconformal mechanics.

1D $\mathcal{N}$-extended super-Poincare algebra

$$\{Q_a, Q_b\} = 2\delta_{ab} H\,, \qquad (Q_a)^+ = Q\,, \qquad a = 1, ..., \mathcal{N}$$

1D $\mathcal{N}$-extended superconformal algebra

1D superconformal algebra $\supset$ 1D conformal symmetry $\mathrm{SO}(1,2) \sim \mathrm{Sp}(2) \sim \mathrm{SU}(1,1)$

$$\sim \left(\begin{array}{c|c} \mathrm{Sp}(2) & Q+S \\ \hline Q-S & \mathrm{SO}(N) \end{array} \right), \qquad \sim \left(\begin{array}{c|c} \mathrm{SU}(1,1) & Q+S \\ \hline Q-S & \mathrm{SU}(M) \end{array} \right)$$

$\{Q,Q\} \sim H$, $\{S,S\} \sim K$, $\{Q,S\} \sim D+J$, $(H,K,D)\subset su(1,1)$, $J\subset o(N)$ or $su(M)$

$\mathcal{N}=1$: $\mathrm{OSp}(1|2)$
$\mathcal{N}=2$: $\mathrm{OSp}(2|2) \sim \mathrm{SU}(1,1|1)$
$\mathcal{N}=4$: $D(2,1;\alpha)$

$\alpha = -1/2$, $\alpha = 1$: $D(2,1;\alpha) \sim \mathrm{OSp}(4|2)$
$\alpha = 0$, $\alpha = -1$: $D(2,1;\alpha) \sim \mathrm{SU}(1,1|2) \oplus_S \mathrm{SU}(2)$

$$D(2,1;\alpha): \quad \{Q^{ai'i}, Q^{bk'k}\} = 2\left(\epsilon^{ik}\epsilon^{i'k'}T^{ab} + \alpha\epsilon^{ab}\epsilon^{i'k'}J^{ik} - (1+\alpha)\epsilon^{ab}\epsilon^{ik}I^{i'k'}\right),$$
$$[T^{ab}, T^{cd}] = i(\epsilon^{ac}T^{bd} + \epsilon^{bd}T^{ac}), \quad ..., \quad [T^{ab}, Q^{ci'i}] = i\epsilon^{c(a}Q^{b)i'i}, ...$$

$$Q^{21'i}{=}-Q^i, \quad Q^{22'i} = -\bar{Q}^i, \quad Q^{11'i}{=}S^i, \quad Q^{12'i} = \bar{S}^i, \qquad T^{22}{=}H, \quad T^{11}{=}K, \quad T^{12}{=}-D.$$

Bosonic generators T^{ab}, J^{ik} and $I^{i'k'}$ form $su(1,1)$, $su(2)$ and $su\,'(2)$ algebras.

t - time coordinate; 1D field theory - mechanics

$$\tilde{S} = \int dt\, \tilde{L}, \qquad \tilde{L} = \tfrac{1}{2}\, \dot{\phi} \cdot \dot{\phi}$$ - 1D massless Klein-Gordon action

Global invariance: $t' = t + a, \;\; \phi'(t') = \phi(t)$

Local invariance: 1D gravity

4D: $g_{MN} = \eta_{AB} e^m_A e^n_B$, e^A_M are vielbein fields

$\Gamma^\lambda_{\mu\nu} = g^{\lambda\rho} \tfrac{1}{2} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$, $R^\rho_{\mu\nu\lambda} = \partial_\mu \Gamma^\rho_{\nu\lambda} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - (\mu \leftrightarrow \nu)$

1D: $g_{\mu\nu} = e^2$, $g^{\mu\nu} = e^{-2}$, $\sqrt{g} = e$, $\Gamma = (\ln e)^{\cdot}$, $R \equiv 0$

$$S = \int dt\, L, \qquad L = \tfrac{1}{2}\, \sqrt{g} g^{\mu\nu} \partial_\mu \phi \cdot \partial_\mu \phi + \tfrac{1}{2}\, \sqrt{g} m^2 = \tfrac{1}{2}\, (e^{-1} \dot{\phi} \cdot \dot{\phi} + e m^2)$$

$$H = \tfrac{1}{2}\, e\, (p \cdot p - m^2), \qquad T_1 = p \cdot p - m^2 \approx 0, \qquad T_2 = p_e \approx 0$$

Quantization:

$$[\hat{A}, \hat{B}] = i \{A, B\}_P : \quad \hat{\phi} = x, \quad \hat{p} = -i\partial_x, \quad \hat{e} = e, \quad \hat{p}_e = -i\partial_e$$

$$\Phi = \Phi(x, e, t), \qquad i\partial_t \Phi = \hat{H}\Phi, \quad \hat{T}_{1,2}\Phi = 0$$

$$\Rightarrow \qquad (\Box + m^2)\Phi(x) = 0, \qquad \text{Klein} - \text{Gordon equation}$$

Quantization 1D matter fields in 1D gravity background $\Rightarrow$ spin 0 target space field

Note: String action $S_{string} = \int d^2\sigma \mathcal{L}_{string}$, $\mathcal{L}_{string} = T\, \sqrt{g}\, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu$

$$S=\int dt\,L,\qquad L=\tfrac{1}{2}\,\dot\phi^{2}+\tfrac{i}{2}\,\psi\dot\psi$$

$$[\phi(t_1),\phi(t_2)]=\phi(t_1)\phi(t_2)-\phi(t_2)\phi(t_1)=0,\qquad \{\psi(t_1),\psi(t_2)\}=\psi(t_1)\psi(t_2)+\psi(t_2)\psi(t_1)=0$$

$$\phi^+=\phi,\qquad \psi^+=\psi;\qquad (AB)^+=B^+A^+$$

$Q:\quad \phi\ \to\ \psi,\quad \psi\ \to\ \phi\qquad\Rightarrow\qquad$ the parameter $\varepsilon=\varepsilon^+$ must be anticommuting

$$[S/\hbar]=0,\quad \hbar=1\qquad\Rightarrow\qquad [L]=+1,\quad [t]=-1\qquad\Rightarrow\qquad [\phi]=-1/2,\quad [\psi]=0$$

$$\delta\phi=i\varepsilon\psi\qquad\Rightarrow\qquad [\varepsilon]=-1/2\qquad\Rightarrow\qquad \delta\psi\sim\varepsilon\dot\phi$$

$$\delta\phi=i\varepsilon\psi,\qquad \delta\psi=-\varepsilon\dot\phi$$

$$\delta L=\tfrac{i}{2}\,(\varepsilon\psi\dot\phi)^{\cdot}+i\dot\varepsilon\psi\dot\phi=0,\qquad \varepsilon=const,\quad \phi|_{t=\pm\infty}=\phi|_{t=\pm\infty}=0$$

$$[\delta_1,\delta_2]\,\phi=2i\varepsilon_1\varepsilon_2\dot\phi,\qquad [\delta_1,\delta_2]\,\psi=2i\varepsilon_1\varepsilon_2\dot\psi$$

Note: In $N>1$ 1*D* and $D>1$ $\qquad [\delta_1,\delta_2]\,\psi=2i\varepsilon_1\varepsilon_2\dot\psi+(\text{eq.of motion})$

$$p = \frac{\partial L}{\partial \dot\phi}, \quad \pi = \frac{\partial^r L}{\partial \dot\psi} \qquad \Rightarrow \qquad p = \dot\phi, \quad \pi = \tfrac{i}{2}\,\psi$$

$$H_0 = p\dot\phi + \pi\dot\psi - L = \tfrac{1}{2}\,p^2$$

$$G \equiv \pi - \tfrac{i}{2}\,\psi \approx 0 \qquad - \quad \text{the constraint}$$

$$H = H_0 + \lambda G$$

$$\{\phi, p\}_P = 1, \qquad \{\psi, \pi\}_P = 1$$

$$\{G, G\}_P = -i \neq 0 \qquad - \quad \text{second class constraint}$$

$$\dot G = \{H, G\}_P = 0 \qquad \Rightarrow \qquad \lambda = 0 \qquad \Rightarrow \qquad H = \tfrac{1}{2}\,p^2$$

$$\{A, B\}_D = \{A, B\}_P - \{A, G\}_P \{G, G\}_P^{-1} \{G, B\}_P$$

$$\{\phi, p\}_D = 1, \quad \{\psi, \psi\}_D = -i \qquad \Rightarrow \qquad [\phi, p] = i, \quad \{\psi, \psi\} = 1$$

$$\delta S = \int dt \dot\Lambda = \int dt \left(\frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot q}\,\delta \dot q\right) \quad \Rightarrow \quad \int dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\,\delta q - \Lambda\right) = \int dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\delta q = 0$$

$$p\,\delta q - \Lambda = const \qquad \text{on} - \text{shell}$$

$$p\,\delta\phi + \pi\,\delta\psi - \Lambda = i\varepsilon p\psi = i\,\varepsilon Q,$$

$$Q = p\psi, \qquad \{Q, Q\}_D = -2i\,H \qquad \Rightarrow \qquad \{Q, Q\} = 2H$$

$$S=\int dt\,(\tfrac{1}{2}\,\dot\phi^2+\tfrac{i}{2}\,\psi\dot\psi), \qquad \delta S=i\int dt\,\dot\varepsilon\psi\dot\phi \qquad !!!\ \varepsilon=\varepsilon(t)$$

Introduce the gauge fermionic field (the gravitino) $\chi=\chi^+$: $\delta\chi=\dot\varepsilon+\ldots$

$$\text{New term}: \quad S'=-i\int dt\,\chi\psi\dot\phi, \qquad \text{additional terms} \quad \delta S'=-i\int dt\,\varepsilon\chi(i\psi\dot\psi+\dot\phi\dot\phi)$$

The first term can be canceled by adding a new term in $\delta\psi$

The last term can only be canceled by introducing a new field h (the graviton) and coupling $\dot\phi^2$

Local SUSY is a theory of gravity!!!

$$\text{New term}: \quad S''=-\int dt\,h\dot\phi\dot\phi$$

$$L_{1D\,SUGRA}=\tfrac{1}{2}\,\dot\phi^2+\tfrac{i}{2}\,\psi\dot\psi-i\,\chi\psi\dot\phi-h\dot\phi\dot\phi$$

$$\text{Local SUSY}: \quad \delta\phi=i\varepsilon\psi, \qquad \delta\psi=-\varepsilon(1-2h)\dot\phi+i\varepsilon\chi\psi,$$

$$\delta h=-i\varepsilon(1-2h)\chi, \qquad \delta\chi=(1-2h)\dot\varepsilon$$

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\phi=2i\varepsilon_1\varepsilon_2(1-2h)\dot\phi-\varepsilon_1\varepsilon_2\chi\psi=\hat\xi\dot\phi+i\hat\varepsilon\psi$$

$$\text{Gravity transformation}: \quad \delta\phi=\xi\dot\phi, \qquad \delta\psi=\xi\dot\psi, \qquad \delta h=\tfrac{1}{2}\,\dot\xi+\xi\dot h-\dot\xi h, \qquad \delta\chi=\xi\dot\chi$$

Note: $\delta g_{\mu\nu}=\xi^\lambda\partial_\lambda g_{\mu\nu}+\partial_\mu\xi^\lambda g_{\lambda\nu}+\partial_\nu\xi^\lambda g_{\nu\lambda}$, $\delta e^m_\mu=\xi^\lambda\partial_\lambda e^m_\mu+\partial_\mu\xi^\lambda e^m_\lambda$,
$\delta e^\mu_m=\xi^\lambda\partial_\lambda e^\mu_m-e^\lambda_m\partial_\lambda\xi^\mu$, $\quad e^\mu_m=1-2h$

$$\phi=(\phi_\mu),\qquad \psi=(\psi_\mu),\qquad \mu=0,1,...,D-1,\qquad e^{-1}=1-2h$$

$$S=\int dt\,L,\qquad L=\tfrac{1}{2}\,e^{-1}\dot\phi^\mu\dot\phi_\mu+\tfrac{i}{2}\,\psi^\mu\dot\psi_\mu-i\,\chi\dot\phi^\mu\psi_\mu$$

$$p_\mu=e^{-1}\dot\phi_\mu-i\chi\psi_\mu,\quad \pi_\mu=\tfrac{i}{2}\,\psi_\mu,\quad p_e=0,\quad \pi_\chi=0$$

$$H_0=p_\mu\dot\phi^\mu+\pi_\mu\dot\psi^\mu+p_e\dot e-L=\tfrac{1}{2}\,e\,p^\mu p_\mu+\tfrac{i}{2}\,\chi\,p^\mu\psi_\mu$$

$$p_e\approx 0,\quad \pi_\chi\approx 0,\quad G_\mu\equiv\pi_\mu-\tfrac{i}{2}\,\psi_\mu\approx 0\qquad -\qquad \text{primary constraints}$$

$$H=H_0+\lambda^\mu G_\mu+\lambda_e p_e+\lambda_\chi\pi_\chi$$

$$\{\phi_\mu,p_\nu\}_P=\eta_{\mu\nu},\qquad \{\psi_\mu,\pi_\nu\}_P=\eta_{\mu\nu}\qquad \{e,p_e\}_P=1\qquad \{\chi,p_\chi\}_P=1$$

$$\{G_\mu,G_\nu\}_P=-i\eta_{\mu\nu}\qquad -\qquad \text{second class constraints}$$

$$\dot G_\mu=\{H,G_\mu\}_P=0\qquad\Rightarrow\qquad \lambda_\mu=0$$

$$\dot p_e=\{H,p_e\}_P=0\qquad\Rightarrow\qquad T\equiv p^\mu p_\mu\approx 0$$

$$\dot\pi_\chi=\{H,\pi_\chi\}_P=0\qquad\Rightarrow\qquad D\equiv p^\mu\psi_\mu\approx 0$$

$$\{A,B\}_D = \{A,B\}_P - \{A,G^\mu\}_P \{G_\mu, G_\nu\}_P^{-1} \{G^\nu, B\}_P$$
$$\{\phi_\mu, p_\nu\}_D = \eta_{\mu\nu}, \qquad \{\psi_\mu, \psi_\nu\}_D = -i\eta_{\mu\nu}$$

Quantization:

$$[\hat{A},\hat{B}] = i\{A,B\}_D : \qquad [\hat{\phi}_\mu, \hat{p}_\nu] = i\,\eta_{\mu\nu}, \quad \{\hat{\psi}_\mu, \hat{\psi}_\nu\} = \eta_{\mu\nu}$$
$$\hat{\phi}_\mu = x_\mu, \quad \hat{p}_\mu = -i\partial_\mu, \quad \hat{\psi}_\mu = \tfrac{1}{\sqrt{2}}\,\gamma_\mu$$

$$\Psi = \Psi_a(x), \qquad \gamma^\mu \partial_\mu \Psi = 0, \qquad \partial^\mu \partial_\mu \Psi = 0 \qquad \text{Dirac field}$$

Quantization 1D matter fields in 1D supergravity background $\Rightarrow$ spin $\frac{1}{2}$ target space field

Note: Fermionic string action: $S_{f\text{–}string} = \int d^2\sigma \mathcal{L}_{f\text{–}string}$,

$$\mathcal{L}_{f\text{–}string} = T\sqrt{g}\left\{g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - \bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu - 2\bar{\chi}_\alpha \gamma^\beta \gamma^\alpha \psi^\mu \left(\partial_\beta X_\mu + \tfrac{1}{2}\,\bar{\chi}_\beta \psi_\mu\right)\right\}$$

$$\gamma^\alpha = e^\alpha_a \gamma^a, \qquad \alpha = 1,2, \qquad a = 1,2, \qquad \mu = 0,1,...,D-1$$

Superspace: Supersymmetry is realized by coordinate transformations

Q describes fermionic transformations $\rightarrow$ translations in odd direction of extended space

Usual 1D space: (t) $\Rightarrow$

$N{=}1$, 1D superspace: (t,θ), where $\theta=\bar\theta$ is Grassmann coordinate, $\theta\theta\equiv 0$

$$Q=Q^+=\partial_\theta+i\,\theta\,\partial_t\,,\qquad H=H^+=i\,\partial_t\,;\qquad \{Q,Q\}=2H\,,\qquad [H,Q]=0$$

$$\delta t=\varepsilon Q\cdot t,\quad \delta\theta=\varepsilon Q\cdot\theta\ :\qquad \delta t=i\,\varepsilon\theta,\qquad \delta\theta=\varepsilon$$

$N{=}1, 1D$ superfield : $\Phi(t,\theta)=\phi(t)+i\,\theta\psi(t)$

$$\Phi'(t',\theta')=\Phi(t,\theta),\quad \delta\Phi=\Phi(t',\theta')-\Phi(t,\theta)=\varepsilon Q\cdot\Phi=\delta\phi+i\,\theta\delta\psi\quad\Rightarrow$$

$$\delta\phi=i\,\varepsilon\psi,\quad \delta\psi=-\varepsilon\dot\phi$$

Integration over odd variable : $\int d\theta\, f(\theta)=\int d\theta\, f(\theta{+}\alpha)\quad\Rightarrow\quad \int d\theta\,\theta=1,\quad \int d\theta\,\alpha=0$

Covariant derivatives : $D_\theta=\partial_\theta-i\,\theta\,\partial_t\equiv D\,,\qquad D_t=\partial_t\,,\qquad \{Q,D\}=0\,,\quad [Q,\partial_t]=0$

$$S=\int dt\,d\theta\,\mathcal{L}(\Phi,\partial_t\Phi,D\Phi),\qquad \delta S=\int dt\,d\theta\,Q\,[...]=\int \underbrace{dt\,d\theta\,\partial_\theta\,[...]}_{=0,\,\partial_\theta[\ldots]\text{ contains no }\theta}+\int dt\,d\theta\ \underbrace{i\theta\partial_t\,[...]}_{\text{the total derivative}}$$

Any action, built from superfields and covariant derivatives ∂_t and D, is always supersymmetric

Examples of the $\mathcal{N}=1$ supermultiplets

$$\Phi(t,\theta) = \phi(t) + i\,\theta\psi(t) \qquad - \qquad \text{even superfield}$$

$$S = \tfrac{i}{2}\int dt\, d\theta\, \partial_t\Phi\, D\Phi = \tfrac{1}{2}\int dt \left(\dot\phi^2 + i\,\psi\dot\psi\right)$$

(1, 1, 0) supermultiplet

$$\Psi(t,\theta) = \psi(t) + \theta F(t) \qquad - \qquad \text{odd superfield}$$

$$S = \tfrac{1}{2}\int dt\, d\theta\, \Psi\, D\Psi = \tfrac{1}{2}\int dt \left(i\,\psi\dot\psi + F^2\right) \quad \stackrel{F=0}{\to} \quad \tfrac{i}{2}\int dt\, \psi\dot\psi$$

(0, 1, 1) supermultiplet

The supermultiplet $(m, n, n-m)$ contains $\begin{cases} m & \text{physical bosons} \\ n & \text{fermions} \\ n-m & \text{auxiliary bosons} \end{cases}$

Supergravity: local translations and local supertranslations $\Rightarrow$
general coordinate transformations in superspace

4D scalar matter in the curved space : $\sim \int d^4x \, \det(e^m_\mu)\, \eta^{mn} \underbrace{e^\mu_m \partial_\mu \phi}_{\mathcal{D}_m \phi} \cdot \underbrace{e^\nu_n \partial_\nu \phi}_{\mathcal{D}_n \phi}$

4D space $\to$ 1D superspace $\Downarrow$

$$S = \tfrac{i}{2} \int dt\, d\theta \,\mathrm{sdet}(E_M{}^A)\, \underbrace{E_{\underline{t}}{}^M \partial_M \Phi}_{\mathcal{D}_{\underline{t}} \Phi}\, \underbrace{E_{\underline{\theta}}{}^N \partial_N \Phi}_{\mathcal{D}_{\underline{\theta}} \Phi}$$

$E_M{}^A(t,\theta)$ – the super vierbein (supermatrix)

$M, N = (1,2) = (t,\theta)$ are curved indices; $A, B = (1,2) = (\underline{t}, \underline{\theta})$ are flat indices

$E_A{}^M(t,\theta)$ – the inverse super vierbein, $E_A{}^M E_M{}^B = \delta_A^B$

$\partial_M = (\partial_t, \partial_\theta)$ – curved derivatives

$$\left.\begin{array}{l} \mathcal{D}_{\underline{t}} = E_{\underline{t}}^M \partial_M = E_{\underline{t}}^t \partial_t + E_{\underline{t}}^\theta \partial_\theta \\ \mathcal{D}_{\underline{\theta}} = E_{\underline{\theta}}{}^N \partial_N = E_{\underline{\theta}}{}^t \partial_t + E_{\underline{\theta}}{}^\theta \partial_\theta \end{array}\right\} \quad \text{– covariant derivatives}$$

General coordinate transformations in superspace:

$$\delta E_M{}^A = \xi^N \partial_N E_M{}^A + \partial_M \xi^N E_N{}^A, \quad \delta E_A{}^M = \xi^N \partial_N E_A{}^M - E_A{}^N \partial_N \xi^M, \qquad \delta\Phi = \xi^M \partial_M \Phi, \qquad \xi^M(t,\theta) \ - \ \text{2 local parameters}$$

The extra symmetry which acts on tangent vectors:

$$\delta E_M{}^A = E_M{}^B \delta_B^t \alpha^A, \quad \delta E_A{}^M = -\delta_A^t \alpha^B E_B{}^M, \qquad \delta\,\mathrm{sdet}(E_M{}^A) = \alpha^t\,\mathrm{sdet}(E_M{}^A), \quad \delta\Phi = 0, \qquad \alpha^A(t,\theta) \ - \ \text{2 local parameters}$$

Gauge fixing for 3 local transformations eliminates 3 from 4 superfields in $E_M{}^A$. Possible choice:

$$E_M{}^t = E \ \overset{o}{E}_M{}^t, \qquad E_M{}^\theta = E^{1/2} \ \overset{o}{E}_M{}^\theta, \qquad E(t,\theta) \ - \ \text{residual gauge superfield}$$

$$\overset{o}{E}_M{}^A = \begin{pmatrix} 1 & 0 \\ i\theta & 1 \end{pmatrix}, \qquad \overset{o}{E}_A{}^M = \begin{pmatrix} 1 & 0 \\ -i\theta & 1 \end{pmatrix}, \qquad - \ \text{flat vielbein}$$

$$\overset{o}{E}_A{}^M \partial_M = (\partial_t, D), \qquad E_A{}^M = \begin{pmatrix} E^{-1} & 0 \\ -i\theta E^{-1/2} & E^{-1/2} \end{pmatrix}, \qquad \mathrm{sdet}(E_M{}^A) = E^{1/2}$$

$$S = \tfrac{i}{2} \int dt\, d\theta\, E^{-1}\, \dot\Phi\, D\Phi$$

$$E(t,\theta) = e(t) - i\theta\chi(t), \qquad \Phi(t,\theta) = \phi(t) + i\theta\psi(t), \qquad S = \tfrac{1}{2}\int dt\, e^{-1}(\dot\phi^2 + i\,\psi\dot\psi - i\,e^{-1}\chi\dot\phi\psi)$$

Replacement $\psi \to e^{1/2}\psi$, $\chi \to e^{3/2}\chi$ yields the component action considered.
Residual gauge transformations coincide with local SUSY of the component action considered.

$\mathcal{N}$-extended 1D superspace:

$$(t,\theta_i), \qquad \theta_k = (\overline{\theta_k}), \qquad \{\theta_i,\theta_k\} = 0, \qquad i,j,k = 1,...,N$$

Realization of super-Poincare algebra in superspace:

$$Q_k = Q_k^+ = \frac{\partial}{\partial\theta_k} + i\,\theta_k\,\frac{\partial}{\partial t}\,, \qquad H = H^+ = i\,\partial_t\,; \qquad \{Q_k,Q_j\} = 2\,\delta_{kj}\,H\,, \qquad [H,Q_k] = 0$$

$$\delta t = \varepsilon_k Q_k\cdot t, \quad \delta\theta_k = \varepsilon_j Q_j\cdot\theta_k \;\; : \qquad \delta t = i\,\varepsilon_k\theta_k, \qquad \delta\theta_k = \varepsilon_k$$

General supersfield:

$$\Phi(t,\theta_k) = \phi(t) + \theta_k\psi_k(t) + \theta_{k_1}\theta_{k_2}\phi_{k_1k_2}(t) + \theta_{k_1}\theta_{k_2}\theta_{k_3}\psi_{k_1k_2k_3}(t) + \cdots + \theta_{k_1}\ldots\theta_{k_N}\phi_{k_1\ldots k_N}(t)$$

Off-shell contents:

$$\left.\begin{array}{l} 2^{\mathcal{N}-1}\ \text{bosonic (fermionic) component fields}\ \phi,\phi_{k_1k_2},\ldots \\ 2^{\mathcal{N}-1}\ \text{fermionic (bosonic) component fields}\ \psi_{k_1},\psi_{k_1k_2k_3},\ldots \end{array}\right\}\quad \text{if}\ \Phi(t,\theta_k)\ \text{is bosonic (fermionic)}$$

Covariant derivatives:

$$D_k = \frac{\partial}{\partial\theta_k} - i\,\theta_k\,\frac{\partial}{\partial t}\,, \qquad \{Q_j,D_k\} = 0$$

$$F(D_k)\,\Phi = 0 \qquad - \qquad \text{covariant constraint}$$

On-shell (physical) contents of a model is defined by the action.

$\mathcal{N}$=2 1D supersymmetric models are similar to the models with $\mathcal{N}$=1 4D SUSY

Real $\mathcal{N}$=2, 1D superspace: (t,θ_1,θ_2), $\theta_1=\theta_1^+$, $\theta_2=\theta_2^+$

$$Q_1=\frac{\partial}{\partial\theta_1}+i\,\theta_1\,\partial_t\,,\quad Q_2=\frac{\partial}{\partial\theta_2}+i\,\theta_2\,\partial_t\,,\qquad H=i\,\partial_t\,;$$

$$\{Q_1,Q_1\}=2\,H,\qquad \{Q_2,Q_2\}=2\,H,\qquad \{Q_1,Q_2\}=0,\qquad [H,Q_1]=[H,Q_2]=0$$

$$\delta t=i\,(\varepsilon_1\theta_1+\varepsilon_2\theta_2),\qquad \delta\theta_1=\varepsilon_1,\qquad \delta\theta_2=\varepsilon_2$$

Complex $\mathcal{N}$=2, 1D superspace:

$$(t,\theta,\bar\theta),\qquad \theta=\tfrac{1}{\sqrt2}\,(\theta_1+i\,\theta_2),\ \bar\theta=\theta^+=\tfrac{1}{\sqrt2}\,(\theta_1-i\,\theta_2)$$

$$Q=\frac{\partial}{\partial\theta}+i\,\bar\theta\,\partial_t\,,\quad \bar Q=\frac{\partial}{\partial\bar\theta}+i\,\theta\,\partial_t\,,\qquad H=i\,\partial_t$$

$$\{Q,\bar Q\}=2\,H,\qquad \{Q,Q\}=\{\bar Q,\bar Q\}=0,\qquad [H,Q]=[H,\bar Q]=0$$

$$\delta t=i\,(\varepsilon\bar\theta+\bar\varepsilon\theta),\qquad \delta\theta=\varepsilon,\qquad \delta\bar\theta=\bar\varepsilon,\qquad \bar\varepsilon=\varepsilon^+$$

General $\mathcal{N}$=2, 1D superfield:

$$\Phi(t,\theta)=\phi(t)+\theta\psi(t)+\bar\theta\chi(t)+\theta\bar\theta F(t)$$

$$\delta\phi=\varepsilon\psi+\bar\varepsilon\chi,\qquad \delta\psi=-i\,\bar\varepsilon\dot\phi+\bar\varepsilon F,\qquad \delta\chi=-i\,\varepsilon\dot\phi-\varepsilon F,\qquad \delta F=-i\,\varepsilon\dot\psi+i\,\bar\varepsilon\dot\chi$$

Covariant derivatives : $D=\dfrac{\partial}{\partial\theta}-i\,\bar\theta\,\partial_t\,,\quad \bar D=\dfrac{\partial}{\partial\bar\theta}-i\,\theta\,\partial_t\,,\qquad \{D,Q\}=\{D,\bar Q\}=0$

$\Phi^+=\Phi$ – the real superfield; $\bar D\,\Phi=0$ – the chiral superfield

Real superfield:

$$\Phi(t,\theta) = \Phi^+ = \phi(t) + \theta\psi(t) - \bar\theta\bar\psi(t) + \theta\bar\theta F(t), \qquad \phi^+ = \phi, \quad F^+ = F, \quad \psi^+ = \bar\psi$$

Off-shell SUSY transformations:

$$\delta\phi = \varepsilon\psi - \bar\varepsilon\bar\psi, \qquad \delta\psi = -i\,\bar\varepsilon\dot\phi + \bar\varepsilon F, \qquad \delta\bar\psi = i\,\varepsilon\dot\phi + \varepsilon F, \qquad \delta F = -i\,(\varepsilon\dot\psi + \bar\varepsilon\dot\psi)$$

$$S = \tfrac{i}{2}\int dt\,d\theta\,d\bar\theta\,\bar D\Phi\,D\Phi = \tfrac{1}{2}\int dt\left\{\dot\phi^2 + i\,(\psi\dot{\bar\psi} - \dot\psi\bar\psi) + F^2\right\}$$

$$\mathrm{On-shell:} \qquad \ddot\phi = 0, \qquad \dot\psi = 0, \qquad \dot{\bar\psi} = 0, \qquad F = 0 \qquad (1,2,1)\ \mathrm{multiplet}$$

On-shell action:

$$S = \tfrac{1}{2}\int dt\left\{\dot\phi^2 + i\,(\psi\dot{\bar\psi} - \dot\psi\bar\psi)\right\}$$

On-shell SUSY transformations:

$$\delta\phi = \varepsilon\psi - \bar\varepsilon\bar\psi, \qquad \delta\psi = -i\,\bar\varepsilon\dot\phi, \qquad \delta\bar\psi = i\,\varepsilon\dot\phi$$

$$[\delta_1,\delta_2]\psi = i\,(\varepsilon_1\bar\varepsilon_2 - \varepsilon_2\bar\varepsilon_1)\,\dot\psi - \underbrace{2i\,\bar\varepsilon_1\bar\varepsilon_2\dot{\bar\psi}}_{=0\ \mathrm{on-shell}}$$

On-shell SUSY transformations are closed only on equations of motion.

Chiral superfield:

$$\bar{D}\,\Phi = 0 \quad \rightarrow \quad \Phi(t,\theta) = \phi(t) + \theta\psi(t) - i\,\theta\bar{\theta}\dot{\phi}(t), \qquad \phi,\,\psi \quad - \quad \text{complex fields}$$

(2, 2, 0) multiplet

$$\Phi(t,\theta) = \phi(t) + \theta\psi(t) - i\,\theta\bar{\theta}\dot{\phi}(t) = \phi(t_L) + \theta\psi(t_L) = \Phi(t_L,\theta)$$

Chiral N=2, 1D subspace:

$$(t_L,\theta), \qquad t_L \equiv t - i\,\theta\bar{\theta}$$

$$\delta t = i\,(\varepsilon\bar{\theta} + \bar{\varepsilon}\theta), \quad \delta\theta = \varepsilon, \quad \delta\bar{\theta} = \bar{\varepsilon} \quad \Rightarrow \quad \delta t_L = 2i\,\bar{\varepsilon}\theta, \qquad \delta\theta = \varepsilon$$

Supercharges in superspace $(t_L,\theta,\bar{\theta})$:

$$Q = \frac{\partial}{\partial\theta}\,, \qquad \bar{Q} = \frac{\partial}{\partial\bar{\theta}} + 2i\,\theta\,\partial_{t_L}$$

SUSY transformations of component fields:

$$\delta\phi = \varepsilon\psi\,, \qquad \delta\psi = -2i\,\bar{\varepsilon}\dot{\phi}$$

SUSY invariant action:

$$S = -\tfrac{1}{2}\int dt\,d\theta\,d\bar{\theta}\,\bar{D}\Phi\,\bar{D}\bar{\Phi} = \tfrac{1}{2}\int dt\,\Big\{4\dot{\phi}\dot{\bar{\phi}} - i\,(\psi\dot{\bar{\psi}} - \dot{\psi}\bar{\psi})\Big\}$$

In the near horizon limit the extreme ($M = Q$) Reissner-Nordström black hole solution of Einstein-Maxwell equations are (in the units with $G = 1$)

$$ds^2 = -\left(\frac{r}{M}\right)^2 dt^2 + \left(\frac{M}{r}\right)^2 dr^2 + M^2 d\Omega$$

But $$-\left(\frac{r}{M}\right)^2 dt^2 + \left(\frac{M}{r}\right)^2 dr^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

where $\eta_{\mu\nu} = diag(-,+,-), \quad \eta_{\mu\nu} x^\mu x^\nu = -M^2,$

$$x^0 = (2r)^{-1}[1 + r^2(M^2 - t^2)], \quad x^1 = (2r)^{-1}[1 - r^2(M^2 + t^2)], \quad x^2 = Mrt.$$

The near horizon limit the extreme Reissner-Nordström black hole possesses $AdS_2 \times S^2$ geometry.

AdS_2 part, having $SO(2,1)$ symmetry, is described by conformal mechanics.

Superconformal mechanics models describe motion of the particle with angular momentum (spin) near horizon of the extreme Reissner-Nordström black hole.

Conformal mechanics action:

$$S = \tfrac{1}{2}\int dt\,\left(\dot{x}^2 - \frac{g}{x^2}\right)$$

Conformal invariance:

$$\delta t = a + b\,t + c\,t^2 \equiv f(t), \qquad \delta x = \tfrac{1}{2}\,\dot{f}\,x\,, \qquad \delta S = \int dt\,\dot{\Lambda}\,, \quad \Lambda = \tfrac{1}{4}\,\ddot{f}\,x^2$$

Conserved charges $\Big(p = \dot{x}; \qquad \frac{d}{dt}\,(H\,\delta t - p\,\delta x + \Lambda) = 0\Big)$:

$$\begin{array}{rcl} H &=& \tfrac{1}{2}\,\left(p^2 + \frac{g}{x^2}\right) \\ D &=& tH - \tfrac{1}{2}\,xp \\ K &=& t^2H - txp + \tfrac{1}{2}\,x^2 \end{array}$$

$$\frac{d}{dt}K = \frac{\partial}{\partial t}K + \{K,H\}_P = 0, \quad \frac{d}{dt}D = \frac{\partial}{\partial t}D + \{D,H\}_P = 0, \quad H \ - \ \text{the Hamiltonian}$$

$$\{H,D\}_P = H, \quad \{K,D\}_P = -K, \quad \{H,K\}_P = 2D \qquad - \qquad \text{dynamical symmetry}$$

$$[\mathbf{A},\mathbf{B}] = i\{A,B\}_P : \quad [\mathbf{H},\mathbf{D}] = i\,\mathbf{H}, \quad [\mathbf{K},\mathbf{D}] = -i\,\mathbf{K}, \quad [\mathbf{H},\mathbf{K}] = 2i\,\mathbf{D} \quad - \quad sl(2,\mathbb{R}) \text{ algebra}$$

Properties of the conformal mechanics:

- If $\mathbf{H}\,|E>=E\,|E>$, then $\mathbf{H}\,e^{i\alpha D}\,|E>=e^{2\alpha}\,E\,|E>$ $\quad\Rightarrow$ the spectrum of **H** is continuous;
- The eigenspectrum of **H** includes all $E>0$ values, for each of which there exists a plane wave narmalizable state;
- The spectrum of **H** does not have an endpoint (ground state), the state with $E=0$ is not even plane wave normalizible.

It is obstacle to describe the conformal theory in terms of **H** eigenstates.

The $sl(2,\mathbb{R})$ algebra in the Virasoro form:

$$\mathbf{R}=\tfrac{1}{2}\,(a\,\mathbf{H}+\frac{1}{a}\,\mathbf{K}),\qquad \mathbf{L}_{\pm}=-\tfrac{1}{2}\,(a\,\mathbf{H}-\frac{1}{a}\,\mathbf{K}\mp i\,\mathbf{D});\qquad a\ \text{ is a parameter}$$

$$[\mathbf{R},\mathbf{L}_{\pm}]=\pm\,\mathbf{L}_{\pm},\qquad [\mathbf{L}_{+},\mathbf{L}_{-}]=-2\,\mathbf{R}$$

R is the $u(1)$ generator in $sl(2,\mathbb{R})\sim o(1,2)$ algebra.

The eigenvalues of

$$\mathbf{R}|_{t=0,\,a=1}=\tfrac{1}{2}\,\left(p^2+\frac{g}{x^2}+x^2\right)$$

are given by a discrete series

$$r_n=r_0+n,\quad n=0,1,2,\ldots;\qquad r_0=\tfrac{1}{2}\left(1+\sqrt{g+\tfrac{1}{4}}\right)$$

The absence of a normalizable ground states in the conformal mechanics and necessity to redefine the Hamiltonian are given in black hole interpretation.

In the black hole interpretation, eigenstates E of H describe the states with time-like energy-momentum vector, $p^\mu p_\mu > 0$.

The absence of a ground state at $E = 0$ can be interpreted as impossibility to cover null geodesics of the event horizon with $p^\mu p_\mu = 0$ by the static time coordinates adapted by H.

Thus, the passing from H to "the Hamiltonian" R is transition to "good" time coordinate near horizon.

Calogero model (multiparticle generalization of conformal mechanics)

$$S = \tfrac{1}{2} \int dt \left[\sum_{a=1}^{n} \dot{x}_a \dot{x}_a - \sum_{a \neq b} \frac{c^2}{(x_a - x_b)^2} \right], \qquad H = \tfrac{1}{2} \left[\sum_{a} p_a p_a + \sum_{a \neq b} \frac{c^2}{(x_a - x_b)^2} \right]$$

in the large n limit may provide the description of the extreme RN black holes in the near horizon limit.

The $\mathcal{N}=2$ superconformal group OSp(2|2)∼SU(1,1|1)

$$\{Q,\bar{Q}\}=2H,\quad \{S,\bar{S}\}=2K,\quad \{Q,\bar{S}\}=2(D-U),\quad \{S,\bar{Q}\}=2(D+U),$$

$$i\left[P,\begin{pmatrix}S\\ \bar{S}\end{pmatrix}\right]=-\begin{pmatrix}Q\\ \bar{Q}\end{pmatrix},\qquad i\left[K,\begin{pmatrix}Q\\ \bar{Q}\end{pmatrix}\right]=\begin{pmatrix}S\\ \bar{S}\end{pmatrix},$$

$$i\left[D,\begin{pmatrix}Q\\ \bar{Q}\end{pmatrix}\right]=\frac{1}{2}\begin{pmatrix}Q\\ \bar{Q}\end{pmatrix},\qquad i\left[D,\begin{pmatrix}S\\ \bar{S}\end{pmatrix}\right]=-\frac{1}{2}\begin{pmatrix}S\\ \bar{S}\end{pmatrix},$$

$$i\left[U,\begin{pmatrix}Q\\ \bar{Q}\end{pmatrix}\right]=\frac{1}{2}\begin{pmatrix}Q\\ -\bar{Q}\end{pmatrix},\qquad i\left[U,\begin{pmatrix}S\\ \bar{S}\end{pmatrix}\right]=-\frac{1}{2}\begin{pmatrix}S\\ -\bar{S}\end{pmatrix}$$

The closure of S, $\bar{S}$ with Q, $\bar{Q}$ $\Rightarrow$ the full OSp(2|2).

We obtain the superconformal transformations by nonlinear realization method.

Coset realization of $\mathcal{N}=2$ superspace:

$$\mathcal{G}=\{H,Q,\bar{Q},U\},\qquad \mathcal{H}=\{U\},\qquad \mathcal{K}=\{H,Q,\bar{Q}\}$$

$$\mathcal{K}(t,\theta,\bar{\theta})=e^{itH+\theta Q+\bar{\theta}\bar{Q}},\qquad t,\ \theta,\ \bar{\theta}\ \text{ are the coordinates on the coset}$$

$$e^{\varepsilon Q+\bar{\varepsilon}\bar{Q}}\,e^{itH+\theta Q+\bar{\theta}\bar{Q}}=e^{it'H+\theta' Q+\bar{\theta}'\bar{Q}}:\qquad \delta t=i(\varepsilon\bar{\theta}+\bar{\varepsilon}\theta),\quad \delta\theta=\varepsilon,\quad \delta\bar{\theta}=\bar{\varepsilon}$$

Note : $$e^A\,e^B=\exp\left\{A+B+\tfrac{1}{2}[A,B]+\tfrac{1}{12}([A,[A,B]]+[[A,B],B])+\cdots\right\}$$

Coset realization of SU(1,1|1):

$$\mathcal{G}=\{H,D,K,Q,\bar{Q},S,\bar{S},U\},\qquad \mathcal{H}=\{U\},\qquad \mathcal{K}=\{H,D,K,Q,\bar{Q},S,\bar{S}\}$$

$$\mathcal{K}=e^{itH}\,e^{\theta Q+\bar\theta\bar Q}\,e^{iuD}\,e^{izK}\,e^{\zeta S+\bar\zeta\bar S}$$

$$e^{\varepsilon Q+\bar\varepsilon\bar Q}\,\mathcal{K}=\mathcal{K}'\mathcal{H},\qquad e^{\eta S+\bar\eta\bar S}\,\mathcal{K}=\mathcal{K}'\mathcal{H}$$

Note : $\quad e^{A}\,B\,e^{-A}=e^{A}\wedge B,\qquad 1\wedge B\equiv B,\quad A\wedge B\equiv[A,B],\quad A^2\wedge B\equiv[A,[A,B]],\quad\cdots$

$$\delta t=i(\varepsilon\bar\theta+\bar\varepsilon\theta),\qquad \delta\theta=\varepsilon,\qquad \delta\bar\theta=\bar\varepsilon;$$

$$\delta' t=i(\eta\bar\theta+\bar\eta\theta)t,\qquad \delta'\theta=\eta(t-i\theta\bar\theta),\qquad \delta'\bar\theta=\bar\eta(t+i\theta\bar\theta)$$

$$\delta'(dtd^2\theta)=0,\qquad \delta' D=-2i\,\eta\bar\theta\,D,\quad \delta'\bar D=-2i\,\bar\eta\theta\,\bar D$$

$$\mathcal{X}=x(t)+\theta\psi-\bar\theta\bar\psi(t)+\theta\bar\theta F(t),\qquad \delta'\mathcal{X}=i(\eta\bar\theta+\bar\eta\theta)\,\mathcal{X}$$

$$S=\int dtd^2\theta\left(\tfrac12 D\mathcal{X}\bar D\mathcal{X}+\gamma\ \ln\mathcal{X}\right)=\tfrac12\int dt\left\{\dot x^2+i\,(\psi\dot{\bar\psi}-\dot\psi\bar\psi)-\frac{\gamma^2+\gamma\psi\bar\psi}{x^2}\right\}$$

Multi-particle generalization (N=2 superconformal Calogero):

$$S=\int dt\,d^2\theta\left(\tfrac12\sum_a D\mathcal{X}_a\bar D\mathcal{X}_a+\gamma\sum_{a\neq b}\ln|\mathcal{X}_a-\mathcal{X}_b|\right)$$

The standard $\mathcal{N}=4$, 1D superspace:

$$\left\{ t, \theta_k, \bar\theta^k = (\theta_k)^+ \right\}, \qquad k = 1,2$$

Supersymmetry transformations from the $\mathcal{N}=4$, 1D superconformal group $D(2,1;\alpha)$:

$$\delta t = i(\theta_k \bar\varepsilon^k - \varepsilon_k \bar\theta^k), \qquad \delta\theta_k = \varepsilon_k, \qquad \delta\bar\theta^k = \bar\varepsilon^k;$$

$$\delta' t = -i(\eta_k \bar\theta^k - \bar\eta^k \theta_k)t + (1+2\alpha)\theta_j\bar\theta^j(\eta_k\bar\theta^k + \bar\eta^k\theta_k),$$

$$\delta'\theta_k = \eta_k t - 2i\alpha\theta_k\theta_j\bar\eta^j + 2i(1+\alpha)\theta_k\bar\theta^j\eta_j - i(1+2\alpha)\eta_k\theta_j\bar\theta^j$$

$$\text{Covariant derivatives}: \qquad D^k = \frac{\partial}{\partial\theta_k} + i\,\bar\theta^k\partial_t \qquad \bar D_k = \frac{\partial}{\partial\bar\theta^k} + i\,\theta^k\partial_t$$

Some types of the $\mathcal{N}=4$, 1D superfields:

- $D^k D_k \mathcal{X} = m$, $\bar D^k \bar D_k \mathcal{X} = m$, $[D^k, \bar D_k]\mathcal{X} = 0$ - scalar superfield, (1,4,3) multiplet
- $D^{(i}V^{jk)} = 0$, $\bar D^{(i}V^{jk)} = 0$ - vector superfield, (3,4,1)

Superconformal models ($\mathcal{X} = (V^{ik}V_{ik})^{1/2}$ for vector superfield):

$$S \sim \int dt\, d^4\theta\; \mathcal{X}^{-1/2} \qquad \text{for } \alpha \neq -1; \qquad S \sim \int dt\, d^4\theta\; \mathcal{X}\ln\mathcal{X} \qquad \text{for } \alpha = -1$$

$$\text{In components}: \qquad S \sim \int dt \left[\dot x^2 + i\,(\psi_k\dot{\bar\psi}^k - \dot\psi_k\bar\psi^k) - \frac{g + F(\psi,\bar\psi)}{x^2}\right]$$

More general formulations of $\mathcal{N}=4$, 1D models is achieved in harmonic superspace

Lecture 3: $4D$ SUSY in superspace

- $\mathcal{N}=1$ $4D$ superspace.
- $\mathcal{N}=1$ $4D$ superfields action.
- $\mathcal{N}$-extended SUSY in superspace.
- $\mathcal{N}=2$ $4D$ in harmonic superspace formulation.
- Superstring action.

The Poincaré group has a natural realization in the Minkowski space

$$\mathcal{M}^{(4)} = (x^\mu)\,, \qquad \mu = 0,1,2,3$$

We can consider the Minkowski space as the coset

$$\mathcal{M}^{(4)} = \frac{\text{Poincare group}}{\text{Lorentz group}} = (P_\mu, L_{\mu\nu}) \,/\, (L_{\mu\nu})\,.$$

$$k(x^\mu) = \exp\{ix^\mu P_\mu\}\,, \qquad g(\varepsilon^\mu, \varepsilon^{\mu\nu}) \cdot k(x^\mu) = k(x'^\mu) \cdot \ell(x^\mu, \varepsilon^\mu, \varepsilon^{\mu\nu})$$

The Poincaré supergroup can be realized in the coset

$$\mathcal{M}^{(4|4)} = \frac{\text{Poincare supergroup}}{\text{Lorentz group}} = (P_\mu, L_{\mu\nu}, Q_\alpha, \bar{Q}_{\dot\alpha}) \,/\, (L_{\mu\nu})\,.$$

Exponential parametrization of it:

$$k(x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha}) = \exp\left\{ix^\mu P_\mu + i\theta^\alpha Q_\alpha - i\bar\theta^{\dot\alpha}\bar{Q}_{\dot\alpha}\right\},$$

where new spinor coordinates $\theta^\alpha, \bar\theta^{\dot\alpha}$ of coset are anticommuting parameters.

The extended manifold

$$\mathcal{M}^{(4|4)} = \left(x^\mu\,,\ \theta^\alpha\,,\ \bar\theta^{\dot\alpha}\right),$$

is called $\mathcal{N}=1$ **Minkowski superspace**.

The spinor coordinates are called odd or Grassmann coordinates and have the Grassmann parity -1, while x^m are even coordinates having the Grassmann parity $+1$

$$[\theta^\alpha, x^\mu] = [\bar\theta^{\dot\alpha}, x^\mu] = 0\,, \quad \{\theta^\alpha, \theta^\beta\} = \{\theta^\alpha, \bar\theta^{\dot\beta}\} = 0\,.$$

Important consequence: fermionic coordinates are nilpotent, $\theta_1\theta_1 = \theta_2\theta_2 = 0$.

From ($\epsilon^\alpha, \bar\epsilon^{\dot\alpha}$ are anticommuting parameters and spinor coordinates also anticommute with them)

$$\exp\left\{i\varepsilon^\alpha Q_\alpha - i\bar\varepsilon^{\dot\alpha}\bar Q_{\dot\alpha}\right\}\exp\left\{ix^\mu P_\mu + i\theta^\alpha Q_\alpha - i\bar\theta^{\dot\alpha}\bar Q_{\dot\alpha}\right\} = \exp\left\{ix'^\mu P_\mu + i\theta'^\alpha Q_\alpha - i\bar\theta'^{\dot\alpha}\bar Q_{\dot\alpha}\right\}$$

we obtain supersymmetry transformations on superspace

$$\theta^{\alpha\,\prime} = \theta^\alpha + \varepsilon^\alpha\,, \quad \bar\theta^{\dot\alpha\,\prime} = \bar\theta^{\dot\alpha} + \bar\varepsilon^{\dot\alpha}\,.$$

$$x^{\mu\,\prime} = x^\mu - i(\epsilon\sigma^m\bar\theta - \theta\sigma^m\bar\epsilon)\,.$$

Supersymmetry transformations are realized as translations in nilpotent directions of the superspace.

The general scalar $\mathcal{N}=1$ superfield

$$\Phi'(x',\theta',\bar\theta')=\Phi(x,\theta,\bar\theta)\,.$$

has the following finite series expansion in Grassmann coordinates

$$\begin{aligned}\Phi(x,\theta,\bar\theta) \;\; = \;\; & \phi(x)+\theta^\alpha\,\psi_\alpha(x)+\bar\theta_{\dot\alpha}\,\bar\chi^{\dot\alpha}(x)+\theta^2\,M(x)+\bar\theta^2\,N(x)\\ & +\theta\sigma^\mu\bar\theta\,A_\mu(x)+\bar\theta^2\,\theta^\alpha\,\rho_\alpha(x)+\theta^2\,\bar\theta_{\dot\alpha}\,\bar\lambda^{\dot\alpha}(x)+\theta^2\,\bar\theta^2\,D(x)\,,\end{aligned}$$

where $\theta^2:=\theta^\alpha\theta_\alpha=\epsilon_{\alpha\beta}\theta^\alpha\theta^\beta$, $\bar\theta^2=\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}=\epsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta}\bar\theta^{\dot\alpha}$, $\epsilon_{12}=\epsilon_{\dot1\dot2}=1$.
Here there are 8 bosonic and 8 fermionic independent complex component fields.

The reality condition

$$\overline{(\Phi)}=\Phi$$

implies the following reality conditions for the component fields

$$\begin{aligned}&\phi(x)=\overline{\phi(x)}\,,\;\;\bar\chi_{\dot\alpha}(x)=\overline{\psi_\alpha(x)}\,,\;\;M(x)=\overline{N(x)}\,,\;\;A_\mu(x)=\overline{A_\mu(x)}\,,\\ &\bar\lambda^{\dot\alpha}(x)=\overline{\rho^\alpha(x)}\,,\;\;D(x)=\overline{D(x)}\,.\end{aligned}$$

They leave in Φ just $(8+8)$ independent real components.

The transformation law $\Phi'(x,\theta,\bar\theta)=\Phi(x-\delta x,\theta-\epsilon,\bar\theta-\bar\epsilon)$ implies

$$\delta\Phi=-\epsilon^\alpha\frac{\partial\Phi}{\partial\theta^\alpha}-\bar\epsilon_{\dot\alpha}\frac{\partial\Phi}{\partial\bar\theta_{\dot\alpha}}-\delta x^m\frac{\partial\Phi}{\partial x^m}\equiv i\left(\epsilon^\alpha Q_\alpha+\bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha}\right)\Phi$$

and leads to the expressions for supercharges

$$Q_\alpha=i\frac{\partial}{\partial\theta^\alpha}+\bar\theta^{\dot\alpha}(\sigma^\mu)_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu}\,,\quad \bar Q_{\dot\alpha}=-i\frac{\partial}{\partial\bar\theta^{\dot\alpha}}-\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu}\,,$$

$$\{Q_\alpha,\bar Q_{\dot\alpha}\}=2P_\mu\,,\ \{Q_\alpha,Q_\beta\}=\{\bar Q_{\dot\alpha},\bar Q_{\dot\beta}\}=0\,,\quad P_\mu=-i\frac{\partial}{\partial x^\mu}\,.$$

The relevant transformations of component fields are obtained from the formula

$$\delta\Phi=\delta\phi+\theta^\alpha\delta\psi_\alpha+\ldots+\theta^2\bar\theta^2\delta D$$

and are

$$\delta\phi=-\epsilon\psi-\bar\epsilon\bar\chi\,,\quad \delta\psi_\alpha=-i(\sigma^m\bar\epsilon)_\alpha\partial_m\phi-2\epsilon_\alpha M-(\sigma^m\bar\epsilon)_\alpha A_m\,,\ldots\,,$$

$$\delta D=\frac{i}{2}\partial_m\rho\sigma^m\bar\epsilon-\frac{i}{2}\epsilon\sigma^m\partial_m\bar\lambda\,.$$

The supermultiplet of fields encompassed by $\Phi(x,\theta,\bar\theta)$ is reducible.
How to describe irreducible supermultiplets in the superfield language?

An important element of the superspace formalism are spinor covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\bar\theta^{\dot\alpha}(\sigma^\mu)_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu}, \quad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu}\,,$$
$$\{D_\alpha, \bar D_{\dot\alpha}\} = -2i(\sigma^m)_{\alpha\dot\alpha}\partial_m\,, \quad \{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0\,.$$

The covariant derivatives anticommute with supercharges,

$$\{D, Q\} = \{D, \bar Q\} = 0,$$

so $D_\alpha\Phi$ and $\bar D_{\dot\alpha}\Phi$ are again superfields:

$$\delta D_\alpha\Phi = D_\alpha\delta\Phi = D_\alpha i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha}\right)\Phi = i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha}\right)D_\alpha\Phi\,.$$

With the help of the covariant spinor derivatives it becomes possible to define superfield description of the irreducible supermultiplets.

Chiral superfields

Chirality condition

$$\bar{D}_{\dot{\alpha}}\Phi_L(x,\theta,\bar{\theta})=0$$

implies

$$\Phi_L(x,\theta,\bar{\theta})=\varphi_L(x_L,\theta)=\phi(x_L)+\theta^\alpha\psi_\alpha(x_L)+\theta\theta F(x_L)\,,$$

where

$$x_L^m=x^m+i\theta\sigma^m\bar{\theta}\,.$$

We obtain less independent fields: the set ϕ,ψ_α,F is closed under $\mathcal{N}=1$ supersymmetry:

$$\delta\phi=-\epsilon\psi\,,\quad \delta\psi_\alpha=-2i(\sigma^m\bar{\epsilon})_\alpha\partial_m\phi-2\epsilon_\alpha F\,,\quad \delta F=-i\bar{\epsilon}\tilde{\sigma}^m\partial_m\psi\,.$$

This is just the transformation law of the scalar $\mathcal{N}=1$ supermultiplet.

The geometric interpretation: the subspace (chiral $\mathcal{N}=1$ superspace)

$$(x_L^m,\theta^\alpha)$$

is closed under $\mathcal{N}=1$ supersymmetry:

$$\delta x_L^m=2i\theta\sigma^m\bar{\epsilon}\,,\quad \delta\theta^\alpha=\epsilon^\alpha\,.$$

In the basis $(x_L^m,\theta^\alpha,\bar{\theta}^{\dot{\alpha}})$: $\quad \bar{D}_{\dot{\alpha}}=\dfrac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}.$

- Scalar superfield Lagrangian:

$$\mathcal{L} = \mathcal{L}(\Phi, D_\alpha \Phi, \bar{D}_{\dot\alpha} \Phi, \partial_m \Phi, \ldots)\,, \quad \delta\mathcal{L} = i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha} \bar{Q}^{\dot\alpha}\right) \mathcal{L}\,.$$

- The variation of the higher component of any expansion of any superfield is a total derivative. This component is extracted by the Berezin integral. It is equivalent to differentiation in Grassmann coordinates. In the considered case of $\mathcal{N} = 1$ superspace it is defined by the rules

$$\int d^2\theta\, (\theta)^2 = 1\,, \quad \int d^2\bar\theta\, (\bar\theta)^2 = 1\,, \quad \int d^2\theta d^2\bar\theta\, (\theta)^4 = 1\,, \quad (\theta)^4 \equiv (\theta)^2 (\bar\theta)^2$$

- Invariant superfield action:

$$S = \int d^4x d^4\theta\, \mathcal{L}(\Phi, D_\alpha \Phi, \bar{D}_{\dot\alpha} \Phi, \partial_m \Phi, \ldots)\,, \quad \delta\mathcal{L} = i\left(\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha} \bar{Q}^{\dot\alpha}\right) \mathcal{L}\,.$$

Chiral superfield

The kinetic terms is as follows

$$S_{kin} = \int d^4x d^4\theta \, \Phi(x_L, \theta)\bar{\Phi}(x_R, \bar{\theta}) \,, \quad x_R^\mu = \overline{(x_L^\mu)} = x^\mu - i\theta\sigma^\mu\bar{\theta} \,.$$

After performing integration over Grassmann coordinates, one obtains

$$S \sim \int d^4x \left(\partial^\mu\bar{\phi}\partial_\mu\phi - \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} + F\bar{F} \right) .$$

The total Wess-Zumino model action is reproduced by adding, to this kinetic term, also potential superfield term

$$S_{pot} = \int d^4x_L d^2\theta \, \left(\frac{g}{3}\Phi^3 + \frac{m}{2}\Phi^2 \right) + \text{c.c.} \,.$$

This action is the only renormalizble action of the scalar $\mathcal{N}=1$ multiplet. In principle, one can construct more general actions, e.g., the action of Kähler sigma model and generalized potential term

$$\tilde{S}_{kin} = \int d^4x d^4\theta \, K\left[\Phi(x_L, \theta), \bar{\Phi}(x_R, \bar{\theta})\right] \,, \quad \tilde{S}_{pot} = \int d^4x_L d^2\theta \, P(\Phi) + \text{c.c.} \,.$$

Vector superfield ($\mathcal{N}=1$ SYM)

It is described by the real superfield $V(x,\theta,\bar\theta)$ possessing the gauge freedom

$$\delta V(x,\theta,\bar\theta)=i[\bar\lambda(x^\mu-i\theta\sigma^\mu\bar\theta,\bar\theta)-\lambda(x^\mu+i\theta\sigma^\mu\bar\theta,\theta)],$$

where $\lambda(x_L,\theta)$ is an arbitatry chiral superfild parameter. Using this freedom, one can choose the so called Wess-Zumino gauge

$$V_{WZ}(x,\theta,\bar\theta)=2\,\theta\sigma^\mu\bar\theta\,A_\mu(x)+2i\bar\theta^2\theta^\alpha\,\psi_\alpha(x)-2i\theta^2\bar\theta_{\dot\alpha}\,\bar\psi^{\dot\alpha}(x)+\theta^2\bar\theta^2\,D(x)\,.$$

Residual gauge invariance in WZ gauge: $A'_\mu(x)=A_\mu+\partial_\mu\lambda(x)$.

The invariant action is written as an integral over the chiral superspace

$$S^{N=1}_{gauge}=\frac{1}{16}\int d^4x_Ld^2\theta\,(W^\alpha W_\alpha)+\text{c.c.}\,,$$

$$\text{where}\quad W_\alpha=-\frac{1}{2}\bar D^2D_\alpha V\quad\text{is chiral}\quad \bar D_{\dot\alpha}W_\alpha=0\,.$$

The corresponding component off-shell action reads ($F^{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$)

$$S=\int d^4x\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-i\psi\sigma^\mu\partial_\mu\bar\psi+\frac{1}{2}D^2\right].$$

$\mathcal{N} = 1$ SYM theory + matter

Superfields:

$$V(x, \theta, \bar{\theta}) \quad + \quad \Phi(x_L, \theta)$$

Total action

$$S = S_{gauge} + S_{matter} \,,$$

$$S_{gauge} = \frac{1}{16} \int d^4x_L d^2\theta \, (W^\alpha W_\alpha) + \text{c.c.} \,,$$

$$S_{matter} = \int d^4x d^4\theta \, \bar{\Phi}_a e^V \Phi_a + \int d^4x_L d^2\theta \left(\frac{m_{ab}}{2} \Phi_a \Phi_b + \frac{g_{abc}}{3} \Phi_a \Phi_b \Phi_c \right) + \text{c.c.}$$

invariant with respect local superfield transformations

$$V \to e^{-i\bar{\lambda}} V e^{i\lambda} \,, \qquad \Phi_a \to e^{-it_a\lambda} \Phi_a$$

($m_{ab} = 0$ when $t_a + t_b \neq 0$ and $g_{abc} = 0$ when $t_a + t_b + t_c \neq 0$)

In nonabelian case:

$$W_\alpha = -\frac{1}{2} \bar{D}^2 e^{-V} D_\alpha e^V \,.$$

The difficulties in case of higher $\mathcal{N}$ supersymmetries arise because the relevant superspaces contain too many θ coordinates and many component fields,

$$2^n \quad \text{for} \quad n \quad \text{Grassmann coordinates}\,,$$

and it is a very complicated constraints to define the superfields which would correctly describe the relevant irreps.

Let us consider $\mathcal{N}=2$ case.

$$\mathcal{N}{=}2,\ 4D\ \text{SUSY algebra :} \qquad \Big\{ P_\mu,\ Q^k_\alpha,\ \bar{Q}_{\dot\alpha k}=(Q^k_\alpha)^+,\ L_{\mu\nu},\ \overbrace{\underbrace{J^{(ik)},}_{su(2)}}^{R-\text{symmetry}} \Big\}, \qquad i,k=1,2$$

$$\text{Standard } \mathcal{N}{=}2,\ 4D \text{ superspace :} \quad \Big\{ P_\mu,\ Q^k_\alpha,\ \bar{Q}_{\dot\alpha k},\ L_{\mu\nu},\ J^{ik} \Big\} / \Big\{ L_{\mu\nu},\ J^{ik} \Big\}$$

$$\text{Standard superspace coordinates :} \qquad \Big\{ x^\mu, \theta^\alpha_k, \bar\theta^{\dot\alpha k} = (\theta^\alpha_k)^+ \Big\}$$

Basic $\mathcal{N}=2$ superfield (hypermultiplet superfield) $q^j(x,\theta,\bar\theta)$ subjected by the constraints

$$D^{(i}_\alpha q^{j)}(x,\theta,\bar\theta)=0\,, \qquad \bar{D}^{(i}_{\dot\alpha} q^{j)}(x,\theta,\bar\theta)=0$$

does not have off-shell description (Lagrangian description) in standard superspace. It possesses an off-shell formulation only in the $\mathcal{N}=2$ harmonic superspace.

$$su_L(2)\ \text{algebra}: \qquad J^{(ik)} = \left\{ J^{\pm}, J^{0} \right\}, \qquad J^{0} \ - \ u(1)\ \text{generator}$$

$\mathcal{N}=4,\ 1D$ harmonic superspace : $\left\{ P_\mu,\ Q^k_\alpha,\ \bar{Q}_{\dot\alpha k},\ L_{\mu\nu},\ J^{ik} \right\} / \left\{ L_{\mu\nu},\ J^{0} \right\}$

Harmonic superspace coordinates : $\left\{ x^\mu, \theta^\alpha_k, \bar\theta^{\dot\alpha k}, u^\pm_i \right\}$

Harmonic coordinates parametrize $S^2 \sim SU(2)/U(1)$ by two $SU(2)$ spinors

$$u^\pm_i, \qquad u^-_i = \left(\overline{u^{+i}}\right)$$

which subject to the constraint $\quad u^{+i}u^-_i = 1 \quad \rightarrow \quad u^+_i u^-_k - u^+_k u^-_i = \epsilon_{ik}$
and are defined up to a $U(1)$ phase transformations

$$u^+_i \rightarrow e^{i\alpha} u^+_i, \qquad u^-_i \rightarrow e^{-i\alpha} u^-_i$$

$$||u|| = \begin{pmatrix} u^+_1 & u^-_1 \\ u^+_2 & u^-_2 \end{pmatrix} \in SU(2), \qquad ||u|| \rightarrow g\, ||u||\, h\,, \quad g \in SU(2), \quad h \in U(1)$$

Any function on $S^2 \sim SU(2)/U(1)$ must have a definite $U(1)$ charge q

$$\Phi^{(q)}(u) = \sum_{n=0}^{\infty} \phi^{i_1 \ldots i_{n+q} j_1 \ldots j_n} u^+_{i_1} \ldots u^+_{i_{n+q}} u^-_{j_1} \ldots u^-_{j_n} \qquad \text{for} \quad n \geq 0$$

Harmonic functions are defined up to the transformations $\Phi^{(q)} \rightarrow e^{i\alpha q}\Phi^{(q)}$.

The use of such parametrization of S^2 has the advantage of manifest $SU(2)$ covariance

Covariant derivatives on the harmonic sphere S^2:

$$D^{\pm\pm} = u_i^{\pm}\frac{\partial}{\partial u_i^{\mp}} \equiv \partial^{\pm\pm}\,, \qquad D^0 = u_i^{+}\frac{\partial}{\partial u_i^{+}} - u_i^{-}\frac{\partial}{\partial u_i^{-}} \equiv \partial^0$$

$$[D^{++}, D^{--}] = D^0\,, \qquad [D^0, D^{\pm\pm}] = \pm 2\, D^{\pm\pm}$$

Harmonic fields satisfy

$$D^0\, \Phi^{(q)} = q\, \Phi^{(q)}$$

Harmonic integrals:

$$\int du\, u^+_{(i_1} \ldots u^+_{i_m} u^-_{j_1} \ldots u^-_{j_n)} = 0\,,$$

$$\int du\, 1 = 1\,,$$

$$\int du\, F^{(q)} = 0 \qquad \text{if} \quad q \neq 0$$

Central basis in harmonic superspace:

$$\left\{ x^\mu, \theta^\alpha_k, \bar\theta^{\dot\alpha k}, u^\pm_i \right\} \equiv \{ z, u \}$$

The $\mathcal{N}=4$, 1D Poincare supersymmetry:

$$\delta x^\mu = i(\varepsilon^k \sigma^\mu \bar\theta_k - \theta^k \sigma^\mu \bar\varepsilon_k), \qquad \delta\theta^\alpha_k = \varepsilon^\alpha_k, \qquad \delta\bar\theta^{\dot\alpha k} = \bar\varepsilon^{\dot\alpha k}, \qquad \delta u^\pm_i = 0$$

Analytic basis in harmonic superspace:

$$\{ x^\mu_A, \theta^\pm_\alpha, \bar\theta^\pm_{\dot\alpha}, u^\pm_i \} \equiv \{ z_A, u \},$$

where $\theta^\pm_\alpha = \theta^i_\alpha u^\pm_i, \quad \bar\theta^\pm_{\dot\alpha} = \bar\theta^i_{\dot\alpha} u^\pm_i, \qquad x^\mu_A = x^\mu - i\left(\theta^+ \sigma^\mu \bar\theta^- + \theta^- \sigma^\mu \bar\theta^+\right)$

Analytic superspace (with half odd coordinates)

$$\{ x^\mu_A, \theta^+_\alpha, \bar\theta^+_{\dot\alpha}, u^\pm_i \} \equiv \{ \zeta, u \}$$

is closed under $\mathcal{N}=4$ Poincare SUSY (and under $\mathcal{N}=4$ superconformal symmetry)

$$\delta x^\mu_A = -2i(\varepsilon^k \sigma^\mu \bar\theta^+ + \theta^+ \sigma^\mu \bar\varepsilon^k) u^-_k, \qquad \delta\theta^+_\alpha = \varepsilon^i{}_\alpha u^+_i, \qquad \delta\bar\theta^+_{\dot\alpha} = \bar\varepsilon^i{}_{\dot\alpha} u^+_i, \qquad \delta u^\pm_i = 0$$

Covariant derivatives $D^{\pm}_{\alpha} = D^{i}_{\alpha} u^{\pm}_{i}$, $\bar{D}^{\pm}_{\dot\alpha} = \bar{D}^{i}_{\dot\alpha} u^{\pm}_{i}$ in analytic basis:

$$D^{+}_{\alpha} = \frac{\partial}{\partial\theta^{-\alpha}}\,, \quad \bar{D}^{+}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{-\dot\alpha}}\,, \quad D^{-}_{\alpha} = \frac{\partial}{\partial\theta^{+\alpha}} - 2i(\sigma_\mu\bar\theta^{-})_\alpha\partial^\mu_A\,, \quad \bar{D}^{-}_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{+\dot\alpha}} + 2i(\theta^{-}\sigma_\mu)_{\dot\alpha}\partial^\mu_A$$

Therefore,

$$D^{+}_{\alpha}\,\Psi(z,u) = \bar{D}^{+}_{\dot\alpha}\,\Psi(z,u) = 0 \qquad \Rightarrow \qquad \Psi = \Psi(\zeta, u)$$

– superfield that lives on the analytic superspace depends on half of the Grassmann variables and has a smaller number of component fields.

Hypermultiplet

It is described by the analytic superfield q^{+}, $D^{+}q^{+} = \bar{D}^{+}q^{+} = 0$

Invariant (free) action:

$$S = \int d\zeta_A du\; \widetilde{q^{+}} D^{++} q^{+}$$

Analytic basis:

$$D^{++}q^{+} = 0 \quad \Rightarrow \quad \text{equations of motion}$$

Central basis:

$$D^{++}q^{+} = 0 \quad \Rightarrow \quad q^{+} = q^{i}(z)u^{+}_{i}$$
$$D^{+}q^{+} = \bar{D}^{+}q^{+} = 0 \quad \Rightarrow \quad D^{(i}q^{k)} = \bar{D}^{(i}q^{k)} = 0 \quad \text{(resolution of the constraints)}$$

Self-interactions:

$$S_{int} = \int d\zeta_A du\, \left[a(q^{+})^4 + b(\widetilde{q^{+}}q^{+})^2 + (\widetilde{q^{+}})^4\right]$$

Fermionic string (Neveu-Schwarz-Ramond string) – $2d$ world-sheet supersymmetry
Superstring (Green-Schwarz superstring) – $D = 3, 4, 6, 10$ target space supersymmetry

$$\sigma^\alpha = (\sigma^1, \sigma^2) \; - \; \text{world-sheet coordinates}, \quad \gamma^\alpha = e^\alpha_a \gamma^a, \quad \alpha = 1, 2$$

$$X^\mu(\sigma),\; \Theta^A(\sigma) \; - \; \text{target-space 2 fields}, \quad \mu = 0, 1, ..., D-1, \quad A = 1, ..., 2^{\frac{D}{2}}$$

Target-space supersymmetry:

$$\delta\Theta^A = \varepsilon^A, \qquad \delta X^\mu = i\bar{\varepsilon}\Gamma^\mu\Theta,$$

where Γ^μ are Dirac matrices in D-dimensional space-time.
SUSY invariant one-forms

$$\omega^\mu = dX^\mu - i\bar{\Theta}\Gamma^\mu d\Theta, \qquad \omega^\mu \equiv d\sigma^\alpha \omega^\mu_{,\alpha}$$

$\mathcal{N} = 1$ GS superstring action $\quad S_{GS} = \int d^2\sigma \mathcal{L}_{GS}$,

$$\mathcal{L}_{GS} = T\left(\sqrt{g}g^{\alpha\beta}\,\omega^\mu_{,\alpha}\omega_{\mu,\beta} + 2\epsilon^{\alpha\beta}\partial_\alpha X^\mu\,\bar{\Theta}\Gamma_\mu\partial_\beta\Theta\right)$$

Quantum spectrum of the superstring contains infinite towers of states having higher spins
(SYM on ground state or SUGRA on ground state of $\mathcal{N} = 2$ GS superstring).

Higher spin fields will be considered in the lessons by M.Vasiliev and V.Didenko.

Lectures 4,5: The elements of twistor theory

- Symmetries of massless particle action.
- Twistor space and its geometry.
- Twistor transform.
- Supertwistors.
- (Super)twistors in HS theory and superstring theory.

Proposed in 1967 by R. Penrose the twistor theory provides a basis for a new mathematical tool in theoretical physics, in which the complex structure of quantum field theory would follow directly from the complex structure of the new base space (replacing the usual space-time).

Although there is still no consensus on the interpretation of the twistor theory,
or as a more suitable base (instead of the space-time approach) to build a future complete theory of fundamental interactions,
or as a powerful mathematical tool for analyzing the conventional theories,
the twistor methods are very popular in modern theoretical physics.

Twistor methods allowed to develop new methods of constructing solutions of the Einstein equations, have led to some progress in the construction of quantum gravity, yielded non-trivial results in the Yang-Mills theory.

Twistor theory is used extensively in the theory of monopoles and instantons and in the analysis of higher spin theory and superstring theory.

The ground positions of the twistor theory is best explored when analyzing massless particle and its quantization.

The action of the relativistic spinless particles in the first-order formalism

$$S_1 = \int d\tau \left(p_\mu \dot{x}^\mu - e(p^2 - m^2) \right) .$$

$x^\mu(\tau)$, $p_\mu(\tau)$ are the position and momentum variables; their Poisson brackets are $[x^\mu, p_\nu]_P = \delta^\mu_\nu$. τ is evolution parameter. $e(\tau)$ is Lagrange multiplier for the constraint

$$p^2 - m^2 \approx 0 \, ,$$

determining the mass of the particle.

Inserting equations of motion $p_\mu = \dot{x}_\mu/(2e)$ back in S_1 we obtain the action in the second-order formalism

$$S_2 = \frac{1}{2} \int d\tau \left(\frac{\dot{x}^\mu \dot{x}_\mu}{e} + e m^2 \right) .$$

It is 1D gravity-like action where $e(\tau)$ plays the role of 1D gravity field and second term is 1D analog of 4D term with cosmological constant.

Insertion of equation of motion $e = \sqrt{\dot{x}^\mu \dot{x}_\mu}/m$ back in S_2 yields the square-root action

$$S = m \int d\tau \sqrt{\dot{x}^\mu \dot{x}_\mu} \, .$$

The last action is only valid for a massive particle with $m \neq 0$.

Let us consider massless case $m = 0$.
The action S_1 is invariant with respect the following global transformations

$$\delta x^\mu = a^\mu + l^{\mu\nu} x_\nu + c x^\mu + 2(k \cdot x) x^\mu - x^2 k^\mu \,, \qquad \delta e = 2ce + 4(k \cdot x) e \,,$$

$$\delta p_\mu = l_{\mu\nu} p^\nu - c p_\mu + 2(k \cdot p) x_\mu - 2(k \cdot x) p_\mu - 2(x \cdot p) k_\mu \,.$$

Generators of these transformations (conserved charges)

$$P_\mu = p_\mu \,, \qquad L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \,, \qquad D = x^\mu p_\mu \,, \qquad K_\mu = 2(x \cdot p) x_\mu - x^2 p_\mu$$

form conformal algebra (here we will take $\eta_{\mu\lambda} = \mathrm{diag}(- + ++)$) with respect the Poisson brackets $[x^\mu, p_\nu]_P = \delta^\mu_\nu$:

$$[L_{\mu\nu}, L_{\lambda\sigma}]_P = \eta_{\mu\lambda} L_{\nu\sigma} - \eta_{\mu\sigma} L_{\nu\lambda} - (\mu \leftrightarrow \nu) \,,$$

$$[L_{\mu\nu}, P_\lambda]_P = \eta_{\mu\lambda} P_\nu - (\mu \leftrightarrow \nu) \,, \qquad [L_{\mu\nu}, K_\lambda]_P = \eta_{\mu\lambda} K_\nu - (\mu \leftrightarrow \nu) \,,$$

$$[P_\mu, D]_P = -P_\mu \,, \qquad [K_\mu, D]_P = K_\mu \,, \qquad [P_\mu, K_\nu]_P = 2M_{\mu\nu} - 2\eta_{\mu\nu} D \,.$$

Conformal boosts are non-linear transformations in the space-time. In space-time formulation it is a hidden symmetry of conformally invariant systems. For example, in the case of conformal transformations we have $\delta\Box = -4(kx)\Box + 4k^\mu \partial_\mu$, $\Box \equiv \partial^\mu \partial_\mu$. Therefore, the conformal invariance of the simplest Klein-Gordon equation $\Box\,\Phi(x) = 0$ assumes the following transformations of massless scalar field $\delta\Phi = -2(kx)\,\Phi$.

Conformal algebra has transparent representation.

Collecting 15 conformal generators in antisymmetric tensor $J_{MN} = -J_{NM}$, $M = (1', 0'; \mu) = (1', 0'; 0, 1, 2, 3)$ by

$$J_{\mu\nu} = L_{\mu\nu}\,, \quad J_{\mu 0'} = \tfrac{1}{2}(P_\mu + K_\mu)\,, \quad J_{\mu 1'} = \tfrac{1}{2}(P_\mu - K_\mu)\,, \quad J_{0'1'} = D\,,$$

we get the following realization of the conformal algebra

$$[J_{MN}, J_{KL}]_P = \eta_{MK} J_{NL} - \eta_{ML} J_{NK} - (M \leftrightarrow N)\,,$$

where η_{MN} has components $\eta_{\mu\nu}$ and $\eta_{\mu 0'} = \eta_{\mu 1'} = \eta_{0'1'} = 0$, $\eta_{0'0'} = -\eta_{1'1'} = +1$ and is metric tensor of 6-dimensional space with signature $(--++++)$.

That is, the conformal algebra is nothing but the algebra of the group $SO(2,4)$, which is Lorentz group of the 6-dimensional space with two times. Corresponding spinor group is $SU(2,2)$ group (similar to $SL(2,C) \cong SO(1,3)$).

Consequently, there is possibility to reformulation of conformal invariant systems in terms of the quantities transformed by the spin-tensorial representations of $SO(2,4)$ or $SU(2,2)$. Then all conformal transformations (shifts, linear and nonlinear transformations) are realized in the form of linear transformations: the 6-dimensional rotations or spinor transformations of 6-dimensional space-time. In fact, just for the solution of this problem in 1967 Penrose introduced the concept of the twistors.

Possibility of the consideration of half-integer spin fields necessarily requires the use of $SU(2,2)$-spinors, which, in fact, determine the twistor space.

In twistor theory, the conformal-invariant systems are formulated in space, parameterized commuting $SU(2,2)$-spinor Z_a, $a=1,...,4$. In fact, this space replaces usual phase space formed by 4-vectors x^μ and p_μ.

To obtain the results in terms of the usual 4*D* spin-tensor fields it is convenient to consider the representation, when $SU(2,2)$ spinor

$$Z_a = (\lambda_\alpha, \mu^{\dot\alpha})$$

is represented in the form of two commuting 4*D* Weyl spinors ($\alpha=1,2$, $\dot\alpha=1,2$) with opposite chiralities λ_α , $\mu^{\dot\alpha}$.

Conjugated 4*D* spinors $\bar\lambda_{\dot\alpha} = (\overline{\lambda_\alpha})$, $\bar\mu^\alpha = (\overline{\mu^{\dot\alpha}})$ form $SU(2,2)$ spinor

$$\bar Z_{\dot a} = (\bar\lambda_{\dot\alpha}, \bar\mu^\alpha)\,,$$

that transforms according to a complex-conjugate representation.

$SU(2,2)$-invariant tensor

$$g^{a\dot b} = \begin{pmatrix} 0 & \delta^\alpha_\beta \\ -\delta^{\dot\beta}_{\dot\alpha} & 0 \end{pmatrix},$$

allows us to determine $SU(2,2)$ spinor

$$\bar Z^a = g^{a\dot b}\bar Z_{\dot b} = (\bar\mu^\alpha, -\bar\lambda_{\dot\alpha})\,,$$

which is transformed by the inverse $SU(2,2)$ matrix.

The contraction of the spinor Z_a and its adjoint $\bar{Z}^a$ defines a Hermitian form

$$\Lambda \equiv \tfrac{i}{2}\,\bar{Z}^a Z_a = \tfrac{i}{2}\,g^{a\dot{b}}\bar{Z}_{\dot{b}} Z_a = \tfrac{i}{2}\,(\bar{\mu}^{\alpha}\lambda_{\alpha} - \bar{\lambda}_{\dot{\alpha}}\mu^{\dot{\alpha}})$$

which is $SU(2,2)$-invariant and defines the norm of $SU(2,2)$ spinor Z_a.

The twistor space **T** is a spinor space (the space $\mathbf{C}^4$) conformal group $SU(2,2)$ with Hermitian form Λ.

The twistors are $SU(2,2)$ spinors Z_a, defined on the twistor space.

Depending on the value of the Hermitian form Λ, there are the following subsets of the twistor space:

- the positive twistor space $\mathbf{T}_+$, where $\Lambda > 0$,
- the negative twistor space $\mathbf{T}_-$, where $\Lambda < 0$
- the isotropic twistor space $\mathbf{T}_0$, where $\Lambda = 0$.

We will see below that the twistor norm Λ defines the helicity of massless particle.

From the definition, conformal transformations are realized by linear transformations on the twistor space. The infinitesimal transformations of spinor components are

$$
\begin{aligned}
\delta\lambda_\alpha &= l_{\alpha\beta}\lambda^\beta - \tfrac{1}{2}c\,\lambda_\alpha - k_{\alpha\dot\beta}\mu^{\dot\beta}\,,\\
\delta\mu^{\dot\alpha} &= \bar l^{\dot\alpha\dot\beta}\mu_{\dot\beta} + \tfrac{1}{2}c\,\mu^{\dot\alpha} + a^{\dot\alpha\beta}\lambda_\beta\,.
\end{aligned}
$$

Defining the Poisson brackets in the twistor space

$$
[\bar Z^a, Z_b]_P = \delta^a_b\ : \qquad [\bar\mu^\alpha, \lambda_\beta]_P = \delta^\alpha_\beta\,, \quad [\mu^{\dot\alpha}, \bar\lambda_{\dot\beta}]_P = \delta^{\dot\alpha}_{\dot\beta}\,,
$$

we find that these transformation are generated by the following quantities

$$
P_{\alpha\dot\alpha} = \lambda_\alpha\bar\lambda_{\dot\alpha}\,, \qquad K^{\dot\alpha\alpha} = \mu^{\dot\alpha}\bar\mu^\alpha\,,
$$

$$
L_{\alpha\beta} = \lambda_{(\alpha}\bar\mu_{\beta)}\,, \qquad \bar L_{\dot\alpha\dot\beta} = \bar\lambda_{(\dot\alpha}\mu_{\dot\beta)}\,,
$$

$$
D = \tfrac{1}{2}\,(\bar\mu^\alpha\lambda_\alpha + \bar\lambda_{\dot\alpha}\mu^{\dot\alpha})\,.
$$

They form the conformal algebra and leave invariant the twistor norm. In terms of the 4-component twistor formalizm, these generators are presented in the form of a traceless product of twistor Z_a and its adjoint $\bar Z^a$:

$$
\bar Z^a Z_b - \tfrac{1}{4}\,\delta^a_b\,\bar Z^c Z_c\,.
$$

In $4D$ twistor theory there are mainly used two-spinor notation for all quantities. For 4-vectors of the position x^μ and momentum p_μ we use two-index quantities $x^{\dot\alpha\alpha}$ and $p_{\alpha\dot\alpha}$. We define their relationship in the following form

$$p_{\alpha\dot\alpha} = \tfrac{1}{\sqrt{2}}\, p_\mu \sigma^\mu_{\alpha\dot\alpha}\,, \qquad p_\mu = \tfrac{1}{\sqrt{2}}\, p_{\alpha\dot\alpha} \tilde\sigma_\mu^{\dot\alpha\alpha}\,,$$

$$x^{\dot\alpha\alpha} = \tfrac{1}{\sqrt{2}}\, x^\mu \tilde\sigma_\mu^{\dot\alpha\alpha}\,, \qquad x^\mu = \tfrac{1}{\sqrt{2}}\, x^{\dot\alpha\alpha} \sigma^\mu_{\alpha\dot\alpha}\,.$$

Present in these relations multiplier $\frac{1}{\sqrt{2}}$ enables the rapid transformation of expressions written in terms of vectors in expressions using the spinor indices. That is, we can formally make substitutions

$$p_\mu = p_{\alpha\dot\alpha}\,, \qquad x^\mu = x^{\dot\alpha\alpha}\,.$$

For example,

$$[x^\mu, p_\nu]_P = \delta^\mu_\nu \delta^{\dot\alpha}_{\dot\beta} \qquad \Leftrightarrow \qquad [x^{\dot\alpha\alpha}, p_{\beta\dot\beta}]_P = \delta^\alpha_\beta \delta^{\dot\alpha}_{\dot\beta}\,.$$

We use also the following conventions

$$P_{\alpha\dot\alpha} = \tfrac{1}{\sqrt{2}}\, P_\mu \sigma^\mu_{\alpha\dot\alpha}\,, \qquad K^{\dot\alpha\alpha} = \sqrt{2}\, K^\mu \tilde\sigma_\mu^{\dot\alpha\alpha}\,,$$

$$L_{\alpha\beta} = -\tfrac{1}{2}\, L_{\mu\nu} \sigma^{\mu\nu}_{\alpha\beta}\,, \qquad \bar L_{\dot\alpha\dot\beta} = \tfrac{1}{2}\, L_{\mu\nu} \tilde\sigma^{\mu\nu}_{\dot\alpha\dot\beta}\,,$$

$$\sigma^{\mu\nu} = \tfrac{1}{4}\left(\sigma^\mu\tilde\sigma^\nu - \sigma^\nu\tilde\sigma^\mu\right), \quad \tilde\sigma^{\mu\nu} = \tfrac{1}{4}\left(\tilde\sigma^\mu\sigma^\nu - \tilde\sigma^\nu\sigma^\mu\right).$$

That is, we have two formalism to describe the conformal-invariant systems: space-time description and twistor formulation.

The relationship of space-time and twistor variables is defined by the relations

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} \,,$$

$$\mu^{\dot{\alpha}} = x^{\dot{\alpha}\beta}\lambda_\beta \,, \qquad \bar{\mu}^{\alpha} = \bar{\lambda}_{\dot{\beta}} x^{\dot{\beta}\alpha} \,.$$

These relations are the Penrose twistor transform for the coordinates.

When twistor transform are valid, then

- twistor representation of conformal generators goes into space-time realization of them;
- conformal transformations of twistor variables yield conformal transformations of space-time quantities.
- space-time and twistor symplectic structures are compatible with one another: the Poisson brackets of any two quantities equal to each other when using the space-time Poisson brackets or twistor Poisson brackets.

The relations of twistor transform have clear physical and geometrical meaning.

Equations $p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$ automatically mean light-like 4-momentum of the particle $p^2 = 0$. This follows directly from the identity $\lambda^\alpha \lambda_\alpha = \epsilon^{\alpha\beta}\lambda_\beta\lambda_\alpha \equiv 0$, which is valid for commuting $4D$ spinors.

The conditions $\mu^{\dot{\alpha}} = x^{\dot{\alpha}\beta}\lambda_\beta\,, \quad \bar{\mu}^\alpha = \bar{\lambda}_{\dot{\beta}} x^{\dot{\beta}\alpha}$ establish the links between Minkowski space-time and twistor variables. Its are the incidence conditions. In particular, for a fixed twistors the solution for x

$$x^{\dot{\alpha}\alpha} = x_0^{\dot{\alpha}\alpha} + a\lambda^\alpha \bar{\lambda}^{\dot{\alpha}}$$

contains an arbitrary real constant a, parametrizing light-like line in Minkowski space with direction vector $\lambda^\alpha \bar{\lambda}^{\dot{\alpha}}$.

That is, (several) point of the twistor space correspond to light-like line in Minkowski space.

Incidence conditions $\mu^{\dot\alpha} = x^{\dot\alpha\beta}\lambda_\beta\,, \quad \bar\mu^{\alpha} = \bar\lambda_{\dot\beta} x^{\dot\beta\alpha}$ have important consequence: the twistor, appearing in them, is isotropic

$$\Lambda = \tfrac{i}{2}\,\bar Z^a Z_a = \tfrac{i}{2}\,(\bar\mu^{\alpha}\lambda_\alpha - \bar\lambda_{\dot\alpha}\mu^{\dot\alpha}) = 0\,.$$

This is achieved, in fact, because the matrix $x^{\dot\alpha\alpha}$ is Hermitian.

What is the isotropic twistor condition or, indeed, what is the physical meaning of the twistor norm?

The answer to this question is found after the calculation of the Pauli-Lubanski vector

$$W_\mu = \tfrac{1}{2}\,\varepsilon_{\mu\nu\lambda\rho} P^\nu M^{\lambda\rho}\,.$$

In twistor realization of the generators we get

$$W_{\alpha\dot\alpha} = \Lambda P_{\alpha\dot\alpha}\,,$$

where Λ is the twistor norm and is defined above.

Thus, the twistor norm coincides with the helicity of massless particle, which is described by this twistor.

The twistor action of massless spinless particle has the following form

$$S_0^{twistor} = \frac{1}{2}\int d\tau \left[\bar{Z}^a\dot{Z}_a - \dot{\bar{Z}}^a Z_a - il\bar{Z}^a Z_a\right] ,$$

where $l(\tau)$ is Lagrange multiplier for the constraint $\Lambda = \frac{i}{2}\bar{Z}^a Z_a \approx 0$ (vanishing helicity). Up to a total derivative, this action takes the following form in the 4*D* spinor notation

$$S_0^{twistor} = \int d\tau \left[\dot{\bar{\mu}}^\alpha\lambda_\alpha + \bar{\lambda}_{\dot{\alpha}}\dot{\mu}^{\dot{\alpha}} - \tfrac{i}{2} l(\bar{\mu}^\alpha\lambda_\alpha - \bar{\lambda}_{\dot{\alpha}}\mu^{\dot{\alpha}})\right] .$$

It should be noted that, up to a total derivative the kinetic term $\dot{\bar{\mu}}^\alpha\lambda_\alpha + \bar{\lambda}_{\dot{\alpha}}\dot{\mu}^{\dot{\alpha}}$ of this action takes the form of the space-time kinetic term $p_\mu\dot{x}^\mu$ after using the twistor transform.

Let us do the quantization of twistor particle: find twistor wave function and compare it with the scalar field obtained in the space-time formulation.

At the transition to the quantum theory, the Poisson brackets go to the commutators

$$[\hat{\bar{Z}}^a, \hat{Z}_b] = i\delta_b^a \,.$$

The quantization of the twistor particles is conveniently carried out in the holomorphic representation (the Penrose representation), where the operators $\hat{Z}_a$ diagonal, and $\hat{\bar{Z}}^a$ are realized as differential operators

$$\hat{\bar{Z}}^a = i\frac{\partial}{\partial Z_a}\,; \qquad \hat{\bar{\lambda}}_{\dot\alpha} = -i\frac{\partial}{\partial \mu^{\dot\alpha}}\,, \qquad \hat{\bar{\mu}}^{\alpha} = i\frac{\partial}{\partial \lambda_{\alpha}}\,.$$

Twistor wave function

$$\Psi(Z) = \Psi(\lambda, \mu)$$

satisfies the equation

$$\hat{\Lambda}\Psi(Z) = 0\,.$$

Taking Weyl ordering in the helicity operator

$$\Lambda = \tfrac{i}{2}\,\bar{Z}^a Z_a \qquad \to \qquad \hat{\Lambda} = \tfrac{i}{4}\,(\hat{\bar{Z}}^a \hat{Z}_a + \hat{Z}_a \hat{\bar{Z}}^a) = \tfrac{i}{2}\,\hat{Z}_a \hat{\bar{Z}}^a - 1 = -\tfrac{1}{2}\,Z_a \frac{\partial}{\partial Z_a} - 1\,,$$

we find that the twistor wave equation has the form

$$\tfrac{1}{2} Z_a \frac{\partial}{\partial Z_a}\Psi = -\Psi\,; \qquad\qquad \tfrac{1}{2}(\lambda_\alpha \frac{\partial}{\partial \lambda_\alpha} + \mu^{\dot\alpha}\frac{\partial}{\partial \mu^{\dot\alpha}})\Psi = -\Psi\,.$$

Thus, the twistor wave function of the considered system is a holomorphic homogeneous function of homogeneity degree (-2):

$$\Psi^{(-2)}(cZ) = c^{-2}\Psi^{(-2)}(Z)\,,$$

where c is arbitrary complex number.

Usual space-time field is obtained from the twistor field by the Penrose twistor transform for the fields. It is constructed as follows. In the twistor field the spinor μ is resolved by the incidence conditions

$$\Psi^{(-2)}(Z)\Big|_{\mu^{\dot\alpha}=x^{\dot\alpha\alpha}\lambda_\alpha} = \Psi^{(-2)}(\lambda_\alpha, x^{\dot\alpha\alpha}\lambda_\alpha)\,.$$

Due to the homogeneity, this function is defined on the complex projective space $\mathbf{CP}^1$ and efficiently depends on a single complex variable. For example, it depends on $z \equiv \lambda_1/\lambda_2$ at $\lambda_2 \neq 0$. Integrating the twistor field with respect to this variable, we get the usual space-time field. In the covariant record that does not depend on the choice of independent coordinates on $\mathbf{CP}^1$, the twistor field is integrated with the measure $d\lambda \equiv \lambda^\alpha d\lambda_\alpha$

$$\Phi(x) = \oint \lambda d\lambda\, \Psi^{(-2)}(\lambda_\alpha, x^{\dot\alpha\alpha}\lambda_\alpha)\,,$$

where the integrand is invariant under $\lambda \to c\lambda$. In this integral transformation the integration is performed along a closed loop in the space of independent complex variable, covering the pole of the twistor field.

Defined integral transformation is the Penrose twistor transform for scalar field.

It is important that the field $\Phi(x)$ automatically satisfies massless Klein-Gordon equation $\partial^\mu\partial_\mu\Phi(x)$. This is the result of depending the twistor field on $x^{\dot\alpha\alpha}$ only in combination $x^{\dot\alpha\alpha}\lambda_\alpha$ with commuting spinor λ_α, for which the identity $\lambda^\alpha\lambda_\alpha \equiv 0$ is valid.

We have considered both purely space-time formulation of massless particle with zero helicity or its purely twistor formulation.

There is also mixed formulation that uses both space-time and twistor variables.
This is so-called Shirafuji twistor formulation.

The action of the massless spin-zero particles in the Shirafuji formulation is, in fact, space-time action, in which the momentum is resolved through the twistor variables

$$S^0_{(mix)} = \int d\tau \lambda_\alpha \bar\lambda_{\dot\alpha} \dot x^{\dot\alpha\alpha} \,.$$

That is, the importance of twistor relation

$$p_{\alpha\dot\alpha} - \lambda_\alpha \bar\lambda_{\dot\alpha} \approx 0 \,.$$

is coded directly in mixed action.

space-time formulation

$$L = p_{\alpha\dot\alpha}\dot x^{\dot\alpha\alpha} - ep^2$$

↑

Penrose transform

$$p_{\alpha\dot\alpha} = \lambda_\alpha\bar\lambda_{\dot\alpha}$$

$$\mu^{\dot\alpha} = x^{\dot\alpha\beta}\lambda_\beta\,, \qquad \bar\mu^{\alpha} = \bar\lambda_{\dot\beta}x^{\dot\beta\alpha}$$

→

mixed formulation

$$L = \lambda_\alpha\bar\lambda_{\dot\alpha}\dot x^{\dot\alpha\alpha}$$

↓

twistor formulation

$$L = \dot{\bar\mu}^{\alpha}\lambda_\alpha + \bar\lambda_{\dot\alpha}\dot\mu^{\dot\alpha} - \tfrac{i}{2}l(\bar\mu^{\alpha}\lambda_\alpha - \bar\lambda_{\dot\alpha}\mu^{\dot\alpha})$$

twistor field		space-time field
$\Psi^{(-2)}(Z) = \Psi^{(-2)}(\lambda_\alpha, \mu^{\dot\alpha})$ $\left(Z_a \frac{\partial}{\partial Z_a} + 2\right) \Psi^{(-2)}(Z) = 0$	$\longrightarrow$	$\Phi(x) = \oint \lambda d\lambda\, \Psi^{(-2)}(Z)\vert_{\mu^{\dot\alpha} = x^{\dot\alpha\alpha}\lambda_\alpha}$ $\Box\Phi(x) = 0$

In the twistor formulation of the helicity of the particles is determined by the twistor norm. Consequently, the phase space of massless particle of helicity s has to be limited to the constraint

$$\Lambda - s = \tfrac{i}{2}\bar{Z}^a Z_a - s = \tfrac{i}{2}(\bar{\mu}^\alpha \lambda_\alpha - \bar{\lambda}_{\dot{\alpha}} \mu^{\dot{\alpha}}) - s \approx 0\,.$$

The action

$$S_s^{twistor} = \int d\tau \left[\tfrac{1}{2}\,(\bar{Z}^a \dot{Z}_a - \dot{\bar{Z}}^a Z_a) - l\,(\tfrac{i}{2}\,\bar{Z}^a Z_a - s) \right]\,,$$

determines the twistor formulation of a massless particle nonzero helicity s.

Quantization of the system is carried out by analogy with the spinless case. Twistor constraint yields the equation for the twistor wave function

$$\tfrac{1}{2} Z_a \frac{\partial}{\partial Z_a} \Psi = -(1+s)\Psi\,.$$

Thus, the twistor field of massless particle with helicity s is holomorphic homogeneous function of degree $(-2-2s)$

$$\Psi^{(-2-2s)}(Z)\,.$$

Corresponding usual space-time field can be obtained by using the incidence conditions and the Penrose field transform:

$$\Phi_{\alpha_1\ldots\alpha_{2s}}(x) = \oint (\lambda d\lambda)\,\lambda_{\alpha_1}\ldots\lambda_{\alpha_{2s}}\Psi^{(-2-2s)}(\lambda_\alpha, x^{\dot\alpha\alpha}\lambda_\alpha)\,.$$

In contrast to the zero-helicity case, here integrand contains $2s$ components of the spinor λ to compensate the negative weight of the twistor field.

The resulting space-time field is symmetric with respect to spinor indices because of commuting twistor component, $\Phi_{\alpha_1\ldots\alpha_{2s}} = \Phi_{(\alpha_1\ldots\alpha_{2s})}$, and automatically satisfies the Dirac equation

$$\partial^{\dot\alpha\alpha_1}\Phi_{\alpha_1\ldots\alpha_{2s}}(x) = 0\,.$$

That is, it is complex self-dual field strength of massless particle of helicity s.

If the massless case the light-like momentum vector is resolved in term of single spinor, spinor representation of the time-like momentum of a massive particle

$$p^2 = m^2$$

must use at least two spinors (here it is summation with respect of repeating index $i = 1, 2$)

$$p_{\alpha\dot{\alpha}} = \lambda^i_\alpha \, \bar{\lambda}_{\dot{\alpha}\, i},$$

where $\bar{\lambda}_{\dot{\alpha}\, i} = (\overline{\lambda^i_\alpha})$.

Interpreting, by analogy with the massless case, the spinor λ as half the twistor, we find that a massive particle should have bitwistor description. Furthermore, two spinor used should be limited additional constraint

$$|\lambda^{\alpha i}\lambda_{\alpha i}|^2 = m^2$$

or perhaps stronger conditions, such as

$$\lambda^{\alpha i}\lambda_{\alpha i} = m\,, \qquad \bar{\lambda}_{\dot{\alpha} i}\bar{\lambda}^{\dot{\alpha} i} = m\,,$$

which would violate the conformal symmetry to the Poincare group.

Superparticle action in the formalism of the first order is given by

$$S_0^{super} = \int d\tau \left(p_{\alpha\dot\alpha}\omega^{\dot\alpha\alpha} - e p_{\alpha\dot\alpha}p^{\dot\alpha\alpha} \right) , \qquad \omega^{\dot\alpha\alpha} \equiv \dot x^{\dot\alpha\alpha} - i\bar\theta^{\dot\alpha}\dot\theta^{\alpha} + i\dot{\bar\theta}^{\dot\alpha}\theta^{\alpha} .$$

The action is invariant under the following global transformations:

- Poincare transformations

$$\delta x^{\dot\alpha\alpha} = a^{\dot\alpha\alpha} + x^{\dot\alpha\beta} l_\beta{}^\alpha + \bar l^{\dot\alpha}{}_{\dot\beta} x^{\dot\beta\alpha} , \qquad \delta\theta^\alpha = \theta^\beta l_\beta{}^\alpha ;$$

- dilatations

$$\delta x^{\dot\alpha\alpha} = c x^{\dot\alpha\alpha} , \qquad \delta\theta^\alpha = \tfrac12 c\theta^\alpha ;$$

- conformal boosts

$$\delta x^{\dot\alpha\alpha} = x^{\dot\alpha\beta} k_{\beta\dot\beta} x^{\dot\beta\alpha} - 4\theta^\alpha \bar\theta^{\dot\alpha}\, \theta^\beta k_{\beta\dot\beta}\bar\theta^{\dot\beta} , \qquad \delta\theta^\alpha = \theta^\beta k_{\beta\dot\beta}(x^{\dot\beta\alpha} + i\bar\theta^{\dot\beta}\theta^\alpha) ;$$

- supertranslations

$$\delta x^{\dot\alpha\alpha} = -(\bar\theta^{\dot\alpha}\epsilon^\alpha - \bar\epsilon^{\dot\alpha}\theta^\alpha) , \qquad \delta\theta^\alpha = \epsilon^\alpha ;$$

- superconformal boosts

$$\delta x^{\dot\alpha\alpha} = 2i(\bar\theta^{\dot\alpha}\bar\eta_{\dot\beta} x^{\dot\beta\alpha} - x^{\dot\alpha\beta}\eta_\beta\theta^\alpha) - 4\bar\theta^{\dot\alpha}\theta^\alpha(\theta^\beta\eta_\beta + \bar\eta_{\dot\beta}\bar\theta^{\dot\beta}) ,$$

$$\delta\theta^\alpha = -4i\theta^\alpha\,\theta^\beta\eta_\beta + \bar\eta_{\dot\beta}(x^{\dot\beta\alpha} + i\bar\theta^{\dot\beta}\theta^\alpha) ,$$

and chiral spinor transformations $\delta\theta^\alpha = -\frac52 i\phi\theta^\alpha$.

These nonlinear transformations form superconformal group $SU(2,2|1)$.

Supertwistor formulation solves the basic task: superconformal symmetry in it is realized by linear transformations.

By analogy with the purely bosonic case, supertwistors defined as spinors of the superconformal group $SU(2,2|1)$. Among the five components supertwistors

$$\mathcal{Z}_A = (Z_a;\, \chi) = (\lambda_\alpha, \mu^{\dot\alpha};\, \chi)\,, \qquad A = 1,\ldots,5$$

four ones are c-numerical components, formed by usual twistor - $SU(2,2)$-spinor Z_a. Fifth component is Grassmann, complex Lorentz scalar $\chi\,, \quad \bar\chi = \overline{(\chi)}\,.$

Conjugated supertwistor

$$\bar{\mathcal{Z}}^A = (\bar Z^a; 2i\bar\chi) = (\bar\mu^\alpha, -\bar\lambda_{\dot\alpha};\, 2i\bar\chi)$$

can be represented by complex-conjugated supertwistor

$$\bar{\mathcal{Z}}^A = G^{A\dot B}\,\bar{\mathcal{Z}}_{\dot B}\,, \qquad \bar{\mathcal{Z}}_{\dot B} = (\bar\lambda_{\dot\alpha}, \bar\mu^\alpha;\, \bar\chi)\,,$$

and $SU(2,2|1)$-invariant tensor

$$G^{A\dot B} = \begin{pmatrix} g^{a\dot b} & 0 \\ 0 & 2i \end{pmatrix}\,,$$

where $g^{a\dot b}$ is $SU(2,2)$-invariant tensor.

$SU(2,2|1)$-invariant supertwistor norm is defined by

$$\mathcal{N} \equiv \tfrac{i}{2}\,\bar{\mathcal{Z}}^A\mathcal{Z}_A = \tfrac{i}{2}\,G^{A\dot B}\,\bar{\mathcal{Z}}_{\dot B}\mathcal{Z}_A = \tfrac{i}{2}(\mu^\alpha\lambda_\alpha - \bar\lambda_{\dot\alpha}\bar\mu^{\dot\alpha}) - \bar\chi\chi\,.$$

Conformal transformations act only on bosonic components of the supertwistor and are defined previously. Supertranslations and superconformal boosts, realized linearly on the supertwistor space, mix bosonic and fermionic components of the supertwistors

$$\delta\lambda_\alpha = 2i\eta_\alpha\chi\,, \qquad \delta\mu^{\dot\alpha} = 2i\bar\epsilon^{\dot\alpha}\chi\,, \qquad \delta\chi = \epsilon^\alpha\lambda_\alpha - \bar\eta_{\dot\alpha}\mu^{\dot\alpha}\,.$$

Chiral transformations of supertwistor components are

$$\delta\lambda_\alpha = \tfrac{i}{2}\,\phi\,\lambda_\alpha\,, \qquad \delta\mu^{\dot\alpha} = \tfrac{i}{2}\,\phi\,\mu^{\dot\alpha}\,, \qquad \delta\chi = i\phi\,\chi\,.$$

Introducing the (graded) symplectic structure with using the canonical Poisson brackets for bosonic component and

$$\{\chi,\bar\chi\}_{\rm P} = \tfrac{i}{2}$$

for Grassmann component, we obtain the following expressions for the generators of supertranslations

$$Q_\alpha = 2i\,\bar\chi\lambda_\alpha\,, \qquad \bar Q_{\dot\alpha} = -2i\,\chi\bar\lambda_{\dot\alpha}\,,$$

superconformal boosts

$$S^\alpha = 2i\,\chi\,\bar\mu^\alpha\,, \qquad \bar S^{\dot\alpha} = -2i\,\bar\chi\,\mu^{\dot\alpha}$$

and chiral transformations

$$A = \tfrac{i}{2}(\bar\mu^\alpha\lambda_\alpha - \bar\lambda_{\dot\alpha}\mu^{\dot\alpha}) - 2\bar\chi\chi\,.$$

These generators together with conformal generators form $SU(2,2|1)$.

In addition to the above considered algebra $SU(2,2)$, the superconformal algebra $SU(2,2|1)$ has non-zero Poisson brackets between the supertranslation generators and superconformal boosts

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\}_{\rm P} = 2iP_{\alpha\dot\alpha}\,, \qquad \{S^\alpha, \bar{S}^{\dot\alpha}\}_{\rm P} = 2iK^{\dot\alpha\alpha}\,,$$

$$\{Q_\alpha, S^\beta\}_{\rm P} = -2iL_\alpha{}^\beta - i(D-iA)\delta_\alpha{}^\beta\,, \qquad \{\bar{Q}_{\dot\alpha}, \bar{S}^{\dot\beta}\}_{\rm P} = -2i\bar{L}_{\dot\alpha}{}^{\dot\beta} - i(D+iA)\delta_{\dot\alpha}{}^{\dot\beta}\,.$$

The closure of the fermion symmetries generate all superalgebra $SU(2,2|\,1)$.

Other non-zero brackets of the fermionic generators are

$$\{Q_\alpha, K^{\dot\beta\beta}\}_{\rm P} = 2i\,\delta_\alpha^\beta\,\bar{S}^{\dot\beta}\,, \qquad \{S^\alpha, P_{\beta\dot\beta}\}_{\rm P} = 2i\,\delta_\beta^\alpha\,\bar{Q}_{\dot\beta}\,,$$

$$\{Q_\alpha, A\}_{\rm P} = 2i\,Q_\alpha\,, \qquad \{S^\alpha, A\}_{\rm P} = 2i\,S^\alpha$$

and their complex conjugates.

Links of supertwistor and superspace variables are defined by the supersymmetric generalization of the Penrose transform

$$p_{\alpha\dot\alpha} = \lambda_\alpha \bar\lambda_{\dot\alpha}\,;$$

$$\mu^{\dot\alpha} = x^{\dot\alpha\alpha}\lambda_\alpha + i\,\bar\theta^{\dot\alpha}\chi\,, \qquad \bar\mu^{\alpha} = \bar\lambda_{\dot\alpha}x^{\dot\alpha\alpha} - i\bar\chi\,\theta^{\alpha}\,;$$

$$\chi = \theta^\alpha\lambda_\alpha\,, \qquad \bar\chi = \bar\lambda_{\dot\alpha}\bar\theta^{\dot\alpha}\,.$$

In case of such relationship of the supercoordinates of two formulations, superconformal symmetry in supertwistor formulation become the symmetry of the superspace approach.

As in the case of usual (nonsupersymmetric) particles the supertwistor transformations include the resolution light-like momentum vector $p_{\alpha\dot\alpha} = \lambda_\alpha\bar\lambda_{\dot\alpha}$.

Supersymmetric generalization of incidence condition $\mu^{\dot\alpha} = x^{\dot\alpha\alpha}\lambda_\alpha + i\,\bar\theta^{\dot\alpha}\chi$ is a shift of the spinor $\mu^{\dot\alpha}$, defined in purely bosonic case, on the quantity which depends on the Grassmann variables. Note that this condition contains complex vector coordinate of chiral superspace

$$\mu^{\dot\alpha} = x_{L}^{\dot\alpha\alpha}\lambda_\alpha\,.$$

The condition $\chi = \theta^\alpha\lambda_\alpha$ determines Grassmann component of supertwistors as λ-projection of the spinor θ.

Supertwistor action of massless superparticle has the form

$$S_{s-tw} = \frac{1}{2}\int d\tau \left[\bar{\mathcal{Z}}^{A}\dot{\mathcal{Z}}_{A} - \dot{\bar{\mathcal{Z}}}^{\mathcal{A}}\mathcal{Z}_{\mathcal{A}} - il\bar{\mathcal{Z}}^{\mathcal{A}}\mathcal{Z}_{\mathcal{A}}\right] ,$$

or in term of the twistor components

$$S_{s-tw} = \int d\tau \left(\dot{\bar{\mu}}^{\alpha}\lambda_{\alpha} + \bar{\lambda}_{\dot{\alpha}}\dot{\mu}^{\dot{\alpha}} + i(\dot{\bar{\chi}}\chi - \bar{\chi}\dot{\chi}) - l\mathcal{N}\right) ,$$

where $l(\tau)$ is Lagrange multiplier for the constraint

$$\mathcal{N} \equiv \frac{i}{2}(\mu^{\alpha}\lambda_{\alpha} - \bar{\lambda}_{\dot{\alpha}}\bar{\mu}^{\dot{\alpha}}) - \bar{\chi}\chi \approx 0 .$$

Calculating generalized Pauli-Lubanski vector shows that the supertwistor norm coincides with superhelicity of massless superparticle described by this supertwistors.

Twistor superfield, obtained as the wave function of first-quantized system, is a homogeneous function of the supertwistor

$$\Psi^{(-2)}(\mathcal{Z}_A) = \Psi^{(-2)}(Z_a;\,\chi)\,.$$

Twistor field yields standard superfield, defined on superspace, by the integral transformation, which is generalization of Penrose field transform

$$\Phi(x_{_L},\theta) = \oint \lambda d\lambda\, \Psi^{(-2)}(\lambda_\alpha,\, x_{_L}^{\dot\alpha\alpha}\lambda_\alpha;\, \theta^\alpha\lambda_\alpha)\,,$$

$$\Psi^{(-2)}(\mathcal{Z})\Big|_{\left({\mu^{\dot\alpha}=x_{_L}^{\dot\alpha\alpha}\lambda_\alpha \atop \chi=\theta^\alpha\lambda_\alpha}\right)} = \Psi^{(-2)}(\lambda_\alpha,\, x_{_L}^{\dot\alpha\alpha}\lambda_\alpha;\, \theta^\alpha\lambda_\alpha)\,.$$

The resulting chiral superfield is defined on superspace and describes Wess-Zumino supermultiplet: in the χ expansion there are only two terms with scalar and spinor component fields.

Other massless multiplets are described by the twistor superfields $\Psi^{(n)}(\mathcal{Z}_A)$ having a different degree of homogeneity (like massless field of nonzero helicity).

We have seen that the generators of the conformal group $SU(2,2)$ are represented in terms of bilinears

$$\bar{Z}^a Z_b - \tfrac{1}{4}\,\delta^a_b\,\bar{Z}^c Z_c \qquad\qquad (P_\mu, L_{\mu\nu}, K_\mu, P)$$

in the twistor formalism.

But the higher-spin (HS) symmetry is a generalization of the conformal symmetry. For this reason, HS symmetry has natural and simple realization in terms of twistors.

Generators of infinite-dimensional HS symmetry are the monomials

$$Z_{a_1}\ldots Z_{a_n}\bar{Z}^{b_1}\ldots\bar{Z}^{b_k}\,.$$

All bilinear combinations

$$Z_a Z_b\,, \qquad Z_a\bar{Z}^b\,, \qquad \bar{Z}^a\bar{Z}^b$$

form $Sp(8)$ algebra, $Sp(8)\supset SU(2,2)$. It is basic symmetry in HS theory and HS generalization of conformal algebra.

Action for HS particle

$$S_{HS} = \int d\tau \left(\lambda_\alpha \bar{\lambda}_{\dot\alpha} \dot{x}^{\dot\alpha\alpha} + y^\alpha \dot{\lambda}_\alpha + \bar{y}^{\dot\alpha} \dot{\bar{\lambda}}_{\dot\alpha} \right) .$$

It is important that in this case we obtain twistor relation

$$p_{\alpha\dot\alpha} - \lambda_\alpha \bar{\lambda}_{\dot\alpha} \approx 0$$

as constraints without fixing helicity (here not present the constraint $\Lambda \approx const$).

Taking the representation

$$\hat{\lambda}_\alpha = -i \frac{\partial}{\partial y^\alpha} , \qquad \hat{\bar{\lambda}}_{\dot\alpha} = -i \frac{\partial}{\partial \bar{y}^{\dot\alpha}} .$$

we see that basic twistor relation yields the Valiliev unfolded equation

$$\left(i\partial_{\alpha\dot\alpha} - \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \bar{y}^{\dot\alpha}} \right) \Psi(x, y, \bar{y}) = 0$$

for HS field $\Psi(x, y, \bar{y})$. Its expansion in $y, \bar{y}$ produces tower of usual massless space-time fields of all helicities.

Recently, a great interest in the twistor theory is connected with the unexpected application of superstring theory.

In 2003 E.Witten showed that the simple holomorphic form of maximal helicity-violating (MHV) tree amplitudes in $\mathcal{N}=4$ super-Yang-Mills theory can be described by curves in supertwistor space, i.e. by twistor superstring.

Later, the amplitudes of the other processes have been described in terms of the twistor superstring.

Witten twistor superstring is described by $\mathcal{N}=4$ supertwistors

$$\mathcal{Z}_A = (Z_a;\, \chi^i) = (\lambda_\alpha, \mu^{\dot\alpha};\, \chi^i)\,, \qquad i = 1,\ldots,4\,.$$

Corresponding twistor transformations are

$$p_{\alpha\dot\alpha} = \lambda_\alpha \bar\lambda_{\dot\alpha}\,;$$

$$\mu^{\dot\alpha} = x^{\dot\alpha\alpha}\lambda_\alpha + i\,\bar\theta_i^{\dot\alpha}\chi^i\,, \qquad \bar\mu^{\alpha} = \bar\lambda_{\dot\alpha}x^{\dot\alpha\alpha} - i\bar\chi_i\,\theta^{i\alpha}\,;$$
$$\chi^i = \theta^{i\alpha}\lambda_\alpha\,, \qquad \bar\chi_i = \bar\lambda_{\dot\alpha}\bar\theta_i^{\dot\alpha}\,.$$

Bibliography

At the introductory level:

- J.Wess, J.Bagger, “Supersymmetry and Supergravity”
- P.West, “Introduction to Supersymmetry and Supergravity”
- M.Kaku, “Introduction to Superstrings”

At the comprehensive level:

- S.J.Gates, Jr., M.T.Grisaru, M.Rocek, W.Siegel, “Superspace or One Thousand and One Lessons in Supersymmetry”
- M.B.Green, J.H.Schwarz, E.Witten, “Superstring theory”
- A.S.Galperin, E.A.Ivanov, V.I Ogievetsky, E.S.Sokatchev “Harmonic superspace”
- R.Penrose, W.Rindler “Spinors and space-time”

THANK YOU FOR YOUR ATTENTION!