QCD Thermodynamics on the lattice

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Day I:
- **Introduction**: Dense Matter and Heavy Ion collisions
- **Finite-T lattice QCD**: Chiral symmetry and the hadron spectrum

Day II:
- **Chiral (phase) transition**: O(4) scaling and $T_c$
- **Deconfinement**: Polyakov loop and $Z(3)$ symmetry, baryon number and electric charge fluctuations, the QCD equation of state
  - thermodynamics at $\mu_B \neq 0$ (lectures by C. Schmidt)

Helmholtz Summer School
Lattice QCD, Hadron Structure and Hadronic Matter
Dubna, Russia, 5-18 September, 2011
some review articles


The Phases of Nuclear Matter

**Early Universe**
- Future LHC Experiments

**Quark-Gluon Plasma**
- Current RHIC Experiments

**Hadron Gas**
- Future FAIR Experiments

**Critical Point**
- RHIC Energy Scan

**Crossover**

**Color Superconductor**

**Nuclear Matter**

**Neutron Stars**

**Vacuum**

**Baryon Chemical Potential**

**The Phases of QCD**

- physics of the early universe
  - hot: \( T \sim 10^{12} K \)
- experimentally accessible in Heavy Ion Collisions at SPS, RHIC, LHC, FAIR
- properties of compact stars
  - dense: \( n_B \sim 10 n_{NM} \)
temperatures in the early universe after $10^{-6}$ sec: $\sim 10^{12}$ K

density of neutron stars: $\sim (3-10)$-times nuclear matter density
From Hadronic Matter to the Quark Gluon Plasma
with the help of QCD

hadron gas

dense hadronic matter

Quantum Chromo Dynamics
(Fritsch, Gell-Mann, 1972)

$n_f$ quarks;
$(N_c^2 - 1)$ gluons;
confinement;
asymptotic freedom;
chiral symmetry breaking;

J.C. Collins, M.J. Perry, Superdense Matter: Neutrons and asymptotically free quarks?
PRL 34 (1975) 1353

N. Cabibbo, G. Parisi, Exponential Hadronic Spectrum and Quark Liberation, PL B59 (1975) 67
Creating hot and dense matter in heavy ion collisions

Creating a QGP in A-A Collisions (RHIC)

beam energy: 200 GeV/A (for Au)
\( \sim \mathcal{O}(1000) \) particles/event at central rapidity

initial (thermalized) energy density
\( \epsilon(\tau_0) \sim 10 \text{ GeV/fm}^3 \)
\( \tau_0 \sim (0.5 - 1.0)\text{fm} \)

initial temperature; baryon density
\( \sim 1.5 T_c \); \( \mu_B \approx 50 \text{ MeV} \)
\( \sim 250 \text{ MeV} \)

phase transition at \( T_c \approx 170 \text{ MeV} \) back to the ordinary QCD vacuum

”measured” in experiment; using Bjorken formula
derydodynamc expansion at constant \( S, N_B \)
need EoS: \( p(\epsilon) \Rightarrow v_s \)
(transport coefficients)

hydro: \( \epsilon(\tau) \)
lattice QCD: \( \epsilon(T) \)
\( \Rightarrow \epsilon(\tau_0), T_f \equiv T_c, \tau_f \)

observable properties of QGP?
Bjorken formula:
Estimating the energy density of the dense (thermalized) matter created in an A-A collision

\[
\varepsilon_{Bj} = \frac{1}{\pi} \frac{1}{R^2} \frac{dE_T}{dy}
\]

\( R \simeq 1.2 \ A^{1/3}\text{fm} \): transverse radius
\( \tau_0 \): equilibration time \( \cong \) time after collision at which "some form of equilibrium" is reached
\( dE_T/dy \simeq \langle E_T \rangle dN/dy \): transverse energy per unit rapidity
Bjorken formula:
Estimating the energy density of the dense (thermalized) matter created in an A-A collision

\[ \mathcal{E}_{Bj} = \frac{1}{\pi} \frac{1}{R^2} \frac{dE_T}{\tau_0 dy} \]

- \( R \simeq 1.2 \ A^{1/3} \text{fm} \): transverse radius
- \( \tau_0 \): equilibration time \( \simeq \) time after collision at which ”some form of equilibrium” is reached
- \( dE_T/ dy \simeq \langle E_T \rangle dN/ dy \): transverse energy per unit rapidity

- \( R_{Au} \simeq 7 \text{ fm} \)
- \( \tau_0 \simeq 1 \text{ fm} \)
- \( \langle E_T \rangle \simeq 1 \text{ GeV} \)
- \( dN/ dy \simeq 1000 \)

\[ \downarrow \]

\[ \epsilon_{Bj} \simeq 7 \text{ GeV/fm}^3 \]
Transverse energy in A-A collisions

RHIC $\Rightarrow$ LHC: transverse energy ($\Leftrightarrow$ energy density) doubles;

initial temperature increase by at least 20% $\Rightarrow T_0 \sim (2 - 2.5)T_c$
Heavy Ion Collisions at the SPS/LHC@CERN:

A-A collisions since 1986

Pb-Pb beams: $\sqrt{s} = 17.4$ GeV/A (SPS)
2.7 TeV/A (LHC)
Heavy Ion Collisions at the SPS/LHC@CERN:

A-A collisions since 1986

Pb-Pb beams: $\sqrt{s} = 17.4$ GeV/\(A\) (SPS)

2.7 TeV/\(A\) (LHC)

estimated temperature: $T_0 \simeq (1-1.2) T_c$

estimated initial energy density:

$\epsilon_0 \simeq (1 - 2) \text{ GeV/fm}^3$
Heavy Ion collisions at the RHIC@BNL:

AU-AU beams: $\sqrt{s} = 130, \ 200 \text{ GeV/A}$
Heavy Ion collisions at the RHIC@BNL:

AU-AU beams: $\sqrt{s} = 130, 200$ GeV/A

estimated temperature: $T_0 \simeq (1.5-2)T_c$

estimated initial energy density:

$$\epsilon_0 \simeq (5 - 15) \text{ GeV/fm}^3$$
Heavy Ion Collisions at the SPS/LHC@CERN:

A-A collisions since 1986

Pb-Pb beams: $\sqrt{s} = 2.7 \text{ TeV/A}$

(LHC)

estimated temperature:

$T_0 \simeq (2 - 3) \, T_c$
Particle ratios and freeze out conditions

resonance gas: \[ Z(T, V, \mu_i) = \text{Tr} e^{-\beta(H - \sum_i \mu_i Q_i)} \]
describes observed particle ratios and freeze out conditions

Particle ratios and freeze out conditions

\[ \ln Z(T, V, \mu_B, \ldots) = \sum_{m_i} \ln Z_i(T, V, \mu_B, \ldots) \]

- Is the freeze out temperature the critical temperature of the QCD transition?
- Which role do resonances play for the occurrence of the transition to the QGP?

resonance gas describes observed particle ratios and freeze out conditions
QCD thermodynamics at non-zero temperature and density

THERMODYNAMICS:  \( Z(V, T, \mu) = \text{Tr}_V e^{-\frac{1}{T}(\hat{H} - \mu \hat{N})} \)

Euclidean path integral:  \( \tau \equiv it \implies \tau \in [0, 1/T) \)

Partition function:  \( Z(V, T, \mu) = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_E} \)

\[
S_E = \int_0^{1/T} dx_0 \int_V d^3x \; L_E(A, \psi, \bar{\psi}, \mu)
\]

QCD:

\[
L_E = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{\psi}_{j,a} \left( \sum_{\nu=0}^{3} \gamma_\nu \left( i\partial_\nu + \frac{g}{2} A_\nu - i\mu \delta_{0,\nu} \right) - m_j \right)^{a,b} \psi_{j,b}
\]

\( a, b = 1, \ldots, N_c^2 - 1, \) colour
\( j = 1, \ldots, n_f, \) flavour
Analyzing hot and dense matter on the lattice: $N^3_\sigma \times N_\tau$

Quantum Chromo Dynamics partition function: 

$$Z(V, T, \mu) = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \ e^{-S_E}$$

$$S_E = \int_0^{1/T} dx_0 \int_V d^3x \mathcal{L}_E(A, \psi, \bar{\psi}, \mu)$$

$V^{1/3} = N_\sigma a$

$1/T = N_\tau a$

Analyzing hot and dense matter on the lattice: $N^3_\sigma \times N_\tau$

Quantum Chromo Dynamics

partition function: $Z(V, T, \mu) = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \ e^{-S_E}$

$S_E = \int_0^{1/T} dx_0 \int_V d^3 x \ \mathcal{L}_E(A, \psi, \bar{\psi}, \mu)$

- $a$
- $V^{1/3} = N_\sigma a$
- $1/T = N_\tau a$
QCD Thermodynamics: Simulating hot and dense matter

the lattice: $N^3_{\sigma} \times N_{\tau}$

lattice spacing: $a$
gauge coupling: $\beta = 6/g^2$

the problem: fermion determinant requires large scale computing

particularly difficult problem: finite density QCD (complex determinant)

$Z(V, T, \mu) = \int \mathcal{D}A \ DetM(A, \mu) \ e^{-S_G}$

$S_E = \int_0^{1/T} dx_0 \int_V d^3x \ \mathcal{L}_E(A, \psi, \bar{\psi}, \mu)$

temperature  volume  chemical potential

$\mathcal{O}(10^6)$ grid points; $\mathcal{O}(10^8)$ d.o.f.; integrate eq. of motion

- p. 1
QCD Thermodynamics: Simulating hot and dense matter

the lattice: $N^3_\sigma \times N_\tau$

lattice spacing: $a_\sigma, a_\tau$

gauge coupling: $\beta = 6/g^2$

bulk thermodynamics:

$$\frac{p}{T^4} = -\frac{1}{VT^3} \ln Z$$
$$\frac{\epsilon}{T^4} = -\frac{1}{VT^4} \frac{\partial}{\partial T^{-1}} \ln Z$$
$$\frac{n_q}{T^3} = \frac{1}{VT^3} \frac{\partial}{\partial \mu_q/T} \ln Z$$
$$\frac{\chi_q}{T^2} = \frac{1}{VT^3} \frac{\partial^2}{\partial (\mu_q/T)^2} \ln Z$$

$$= \frac{T}{V} \left( \langle N^2_q \rangle - \langle N_q \rangle^2 \right)$$

partition function:

$$Z(V, T, \mu) = \int DA \ DetM(A, \mu) e^{-S_G}$$

$$S_E = \int_0^{1/T} dx_0 \int_V d^3x \ L_E(A, \psi, \bar{\psi}, \mu)$$

temperature, volume, chemical potential.
Detecting the QCD phase transition on the lattice

Deconfinement vs.

chiral symmetry restoration

phase transition \iff breaking/restoration of global symmetries

exist only for \( m_q = 0 \) and \( m_q \to \infty \)

B. Svetitsky, L.G. Yaffe, NPB210, 423 (1982)


global symmetries – suggest order of the phase transition

– control universal behaviour at second order transition

\( m_q = 0, n_f = 2: SU(2) \times SU(2) \simeq O(4) \Rightarrow 2^{\text{nd}} \text{order (possible)} \)

\( m_q = 0, n_f = 3: SU(3) \times SU(3), \text{no fixed point} \Rightarrow 1^{\text{st}} \text{order} \)
Critical behavior in hot and dense matter: QCD phase diagram: chiral limit \((m_l = 0)\)

\[ T \quad \text{m=0} \]

\[ 2^{\text{nd}} \text{order} \]

\[ 1^{\text{st}} \text{order} \]

\[ \mu \]

already \(\mu_B = 0\) is not fully explored

\[ N_f = 2 \]

\[ N_f = 2 + 1 \]

\[ N_f = 3 \]

\[ N_f = 1 \]

\[ \infty \]

\[ m_s \]

\[ m_{s_{\text{tric}}} \]

\[ 2^{\text{nd}} \text{order} \quad O(4) ? \]

\[ 2^{\text{nd}} \text{order} \quad Z(2) \]

\[ \text{phys. point ?} \]

\[ N_f = 2 + 1 \]

\[ m_u, m_d \]

\[ 1^{\text{st}} \text{order} \quad Z(2) \]

\[ \text{crossover} \]

\[ \text{crossover} \]

\[ \infty \]

- p. 20
Phase diagram for $\mu_B = 0$

- already the $\mu_B = 0$ phase diagram is not fully explored
- phase boundary is known to be very sensitive to cut-off effects

\[ N_f = 2 \]
\[ N_f = 3 \]
\[ N_f = 1 \]
\[ N_f = 2 + 1 \]
\[ N_f = 4 \]

influence of $U_A(1)$ breaking on QCD transition in the chiral limit; may change $O(4)$ to $O(4) \times O(2)$, can induce 1st order transition

$N_T = 4$, standard staggered fermions:
\[ \Rightarrow m_{ps}^{\text{crit}} \approx 300 \text{ MeV} \] for $n_f = 3$, i.e. larger than physical $m_\pi$

FK, E. Laermann, C Schmidt, PL B520 (2001) 41
already the $\mu_B = 0$ phase diagram is not fully explored

phase boundary is known to be very sensitive to cut-off effects:

$N_\tau = 4$, standard staggered fermions $\Rightarrow m_{ps}^{\text{crit}} \simeq 300$ MeV

for $n_f = 3$, i.e. larger than physical $m_\pi$

physical point may be above $m_s^{\text{tric}}$

$N_\tau = 4, 6$; improved actions:

$\Rightarrow m_{ps}^{\text{crit}} \lesssim 70$ MeV


(also $N_\tau = 6$, unimp.)
Phase diagram for $\mu_B = 0$

- $m_s / m_{ud}$
- $m_{phys}$
- $m_{crit}$
- $n_f = 3$
- Physical point may be above $m_s^{tric}$

$N_\tau = 4, 6$; improved actions:

$$\Rightarrow m_{ps}^{crit} \lesssim 70 \text{ MeV}$$

(also $N_\tau = 6$, unimp.)
Phase diagram for $\mu_B = 0$

Is physics at the physical quark mass point sensitive to (universal) properties of the chiral phase transition?

Physical point may be above $m_{s}^{crit}$

$N_\tau = 4, 6$; improved actions:

$\Rightarrow m_{ps}^{crit} \lesssim 70$ MeV

(also $N_\tau = 6$, unimp.)
Symmetries of the QCD Lagrangian

\[ U_V(1) \times U_A(1) \times SU_L(n_f) \times SU_R(n_f) \]

\[ \mathcal{L}_F \sim \bar{\psi}_L \not{D}_\mu \psi_L + \bar{\psi}_R \not{D}_\mu \psi_R - m_q(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \]

chiral projection:

\[ P_\epsilon = \frac{1}{2} (1 + \epsilon \gamma_5) , \epsilon = \pm 1 , \quad P_\epsilon^2 = P_\epsilon , \quad P_+ P_- = 0 \]

\[ \psi = \psi_L + \psi_R \]

\[ \psi_L = P_+ \psi \quad , \quad \psi_R = P_- \psi \]

\[ \bar{\psi}_L = \bar{\psi} P_- \quad , \quad \bar{\psi}_R = \bar{\psi} P_+ \]
Symmetries of the QCD Lagrangian

\[ U_V(1) \times U_A(1) \times SU_L(n_f) \times SU_R(n_f) \]

\[ \mathcal{L}_F \sim \bar{\psi}_L \not{D}_\mu \psi_L + \bar{\psi}_R \not{D}_\mu \psi_R - m_q(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L) \]

**U_V(1):** baryon number

\[ \psi^\Theta = e^{i\Theta} \psi, \quad \bar{\psi}^\Theta = \bar{\psi} e^{-i\Theta} \]

**U_A(1):** axial symmetry

\[ \psi^\Theta = e^{i\Theta \gamma_5} \psi, \quad \bar{\psi}^\Theta = \bar{\psi} e^{i\Theta \gamma_5} \]

**SU_L,R(n_f):** flavour symmetry

\[ G_\epsilon \equiv P_{-\epsilon} \cdot 1 + P_\epsilon U_\epsilon, \quad U_\epsilon \in U(n_f) \]

\[ G \equiv G_+(U_+)G_-(U_-) : \]

\[ \psi' = G \psi, \quad \bar{\psi}' = \bar{\psi} G^\dagger \]

\[ \psi \equiv (\psi_1, ... \psi_{n_f}) \]
The QCD mass term

\[ \bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \]

- \( U_L(n_f) \times U_R(n_f) \) transformation:
  \[ \bar{\psi}'\psi' = \bar{\psi}_R U_+^\dagger U_- \psi_L + \bar{\psi}_L U_+^\dagger U_- \psi_R = \bar{\psi}_R V^\dagger \psi_L + \bar{\psi}_L V \psi_R \]
  \[ V \equiv U_+^\dagger U_- \equiv e^{i\Theta_a T_a}, \ a = 1, ..., n_f^2 - 1 \]

- infinitesimal transformation:
  \[ \delta \bar{\psi}\psi = -i\Theta_a \bar{\psi}_R T_a \psi_L + i\Theta_a \bar{\psi}_L T_a \psi_R \]
  \[ = i\Theta_a \bar{\psi} \gamma_5 T_a \psi + \mathcal{O}(\Theta^2) \]
The QCD mass term

\[ \bar{\psi} \psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L \]

- \( U_L(n_f) \times U_R(n_f) \) transformation:
  \[ \bar{\psi}' \psi' = \bar{\psi}_R U_+^\dagger U_- \psi_L + \bar{\psi}_L U_+^\dagger U_+ \psi_R = \bar{\psi}_R V^\dagger \psi_L + \bar{\psi}_L V \psi_R \]
  \[ V \equiv U_-^\dagger U_+ \equiv e^{i \Theta_a T_a}, \quad a = 1, \ldots, n_f^2 - 1 \]

- infinitesimal transformation:
  \[ \delta \bar{\psi} \psi = -i \Theta_a \bar{\psi}_R T_a \psi_L + i \Theta_a \bar{\psi}_L T_a \psi_R \]
  \[ = i \Theta_a \bar{\psi} \gamma_5 T_a \psi + O(\Theta^2) \]
  mixes flavour components
  adds pseudo-scalar component to scalar
The QCD mass term

\[ \bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \]

- **\(U_L(n_f) \times U_R(n_f)\) transformation:**

\[ \bar{\psi}'\psi' = \bar{\psi}_R U_+^\dagger U_- \psi_L + \bar{\psi}_L U_+^\dagger U_- \psi_R = \bar{\psi}_R V^\dagger \psi_L + \bar{\psi}_L V \psi_R \]

\[ V \equiv U_+^\dagger U_- \equiv e^{i\Theta_a T_a}, \quad a = 1, ..., n_f^2 - 1 \]

- **infinitesimal transformation:**

\[ \delta \bar{\psi}\psi = -i\Theta_a \bar{\psi}_R T_a \psi_L + i\Theta_a \bar{\psi}_L T_a \psi_R \]

\[ = \quad i\Theta_a \bar{\psi} \gamma_5 T_a \psi + \mathcal{O}(\Theta^2) \]

\[ \Rightarrow \quad \langle \bar{\psi}\psi \rangle = 0, \text{ if } \chi\text{-symmetry not spontaneously broken} \]

\[ \Rightarrow \quad T = 0 : \quad \lim_{m_q \to 0} \langle \bar{\psi}\psi \rangle \neq 0 \quad \Leftrightarrow \quad \text{Goldstone particle} \]
Topology, $U_A(1)$: A primer

$U_A(1)$ Symmetry Restoration

- **Topological charge:**

$$Q = \frac{g^2}{32\pi^2} \int d^4 x F^a_{\mu\nu} \bar{F}^a_{\mu\nu}, \quad \bar{F}^a_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_a^{\rho\sigma}$$

$$Q = \int d^4 x q(x), \quad q(x) = \frac{g^2}{32\pi^2} F^a_{\mu\nu} \bar{F}^a_{\mu\nu}$$

- **Topological charge fluctuations**

$$\chi_{top} \equiv \frac{1}{V_4} \langle Q^2 \rangle = \int d^4 x \langle q(x) q(0) \rangle, \quad \chi_{top}^{T=0} \simeq (180 \text{MeV})^4$$

$$\frac{2n_f}{f_\pi^2} \chi_{top} = m_{\eta'}^2 + m_\eta^2 - 2m_K^2, \quad \text{Witten – Veneziano rel.}$$

- **Axial current:**

$$J_5^\mu(x) = \bar{\psi}(x) \gamma_\mu \gamma_5 \psi(x)$$

$$\partial_\mu J_5^\mu = -\frac{g^2}{16\pi^2} F^a_{\mu\nu} \bar{F}^a_{\mu\nu}, \quad U_A(1) \text{ breaking} \Rightarrow m_{\eta'} \gg m_\pi$$
Meson Spectrum and Chiral Symmetry Restoration

scalar, flavor singlet operator: \( O_\sigma = \bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \)

- \( U_L(n_f) \times U_R(n_f) \) transformation:
  \[
  \bar{\psi}'\psi' = \bar{\psi}_R U_+^\dagger U_- \psi_L + \bar{\psi}_L U_+^\dagger U_+ \psi_R = \bar{\psi}_R V^\dagger \psi_L + \bar{\psi}_L V \psi_R
  \]
  \[
  V \equiv U_+^\dagger U_+ \equiv e^{i\Theta_a T_a}, \quad a = 1, \ldots, n_f^2 - 1
  \]

- choose transformation: \( \Theta_a = \pi/2 \)
  \[
  \bar{\psi}'\psi' = -i\frac{\pi}{2} \Theta_a (\bar{\psi}_R T_a \psi_L + \bar{\psi}_L T_a \psi_R) \sim \bar{\psi}\gamma_5 T_a \psi \equiv O_\pi
  \]
  pseudo-scalar, flavor non-singlet

\[\Rightarrow\]
\[
G_\pi(x, T) = \left< O_\pi(0) O_\pi^\dagger(x) \right> \sim e^{-m_\pi(T)x}
\]
\[\Rightarrow\]
\[
G_\sigma(x, T) = \left< O_\sigma(0) O_\sigma^\dagger(x) \right> \sim e^{-m_\sigma(T)x}
\]

\( \chi \)-symmetry restoration: \( G_\pi(x, T) \equiv G_\sigma(x, T) \)
Meson Spectrum and Chiral Symmetry Restoration

\[ \chi_{5,\text{con}} : \bar{q} \frac{\gamma_5}{2} q \rightarrow \sigma : \bar{q} q \rightarrow \chi_{\text{con}} + \chi_{\text{disc}} \]

\[ \chi_{\text{con}} : \bar{q} \frac{\tau}{2} q \leftarrow \eta^\dagger : \bar{q} \gamma_5 q \rightarrow \chi_{5,\text{con}} - \chi_{5,\text{disc}} \]

SU(2) \times SU(2) \quad \text{flavor singlet}

SU(2) \times SU(2) \quad \text{flavor non-singlet}

U(1) \quad \text{SU(2) } \text{L} \times \text{SU(2) } \text{R}

\text{correlation functions:}

\[ G_\delta(x) = -\text{tr} \langle M_l^{-1}(x, 0) M_l^{-1}(0, x) \rangle \]

\[ G_\sigma(x) = G_\delta(x) + \langle \text{tr} M_l^{-1}(x, x) \text{tr} M_l^{-1}(0, 0) \rangle - \langle \text{tr} M_l^{-1}(x, x) \rangle \langle \text{tr} M_l^{-1}(0, 0) \rangle \]

\text{susceptibilities:}

\[ \frac{\chi_\sigma}{T^2} = \frac{\chi_{\text{con}}}{T^2} + \frac{\chi_{\text{disc}}}{T^2} = N_\tau^2 \sum_x G_\sigma(x, T) \]

\[ \frac{\chi_\delta}{T^2} = \frac{\chi_{\text{con}}}{T^2} \]
Vector Meson Spectrum and Chiral Symmetry Restoration

- testing $SU(2)_L \times SU(2)_R$ restoration with correlation functions is difficult as the calculation of "disconnected correlation functions" is difficult (noisy)

- test $U(1)_A$ is more straightforward as only connected correlation functions are involved

$\Rightarrow$ I) test $SU(2)_L \times SU(2)_R$ restoration in the vector/axial-vector channels

$$O_{\rho,\mu}(x) = \bar{u}\gamma_\mu d(x), \quad O_{a1,\mu}(x) = \bar{u}\gamma_5\gamma_\mu d(x)$$

$\Rightarrow$ II) test $SU(2)_L \times SU(2)_R$ restoration using susceptibilities; disconnected contributions much easier to handle

$\chi_\sigma$ is related to chiral susceptibility $\chi_m = d\langle \bar{\psi}\psi \rangle/dm$
Effective $U_A(1)$ symmetry restoration above $T_c$

\[ \pi : J_{PS} \sim \bar{q}\gamma_5\tau q \quad \Leftrightarrow \quad \delta : J_S \sim \bar{q}\tau q \]

Chiral symmetry breaking below $T_c$ ⇒ light pseudo-scalar pion, heavy scalar ($\delta$);

discrepancy decreases with increasing temperature
Effective $U_A(1)$ symmetry restoration above $T_c$

\[ \pi : J_{PS} \sim \bar{q} \gamma_5 \tau q \quad \Leftrightarrow \quad \delta : J_S \sim \bar{q} \tau q \]

chiral symmetry restoration $\Leftrightarrow$ degeneracy of correlation functions

effective $U_A(1)$ restoration

\[ m_\delta(T) \rightarrow m_\pi(T) \]
Meson Spectrum and Chiral Symmetry Restoration

calculations with $\mathcal{O}(\alpha^2)$ improved staggered fermions (p4-action):

screening masses: $G_H(z) \sim e^{-m_H z}$

$SU(2)_L \times SU(2)_R$ restoration

$U(1)_A$ ”effective restoration”

$m_{SC} \neq m_{PS} \iff U(1)_A$ not restored at $T_c$ for chiral symmetry restoration

The flavor non-singlet correlation functions pseudo-scalar ($\pi$) versus scalar ($\delta$)

What generates the differences between $G_\delta(x, T)$ and $G_\pi(x, T)$ in the $SU(2)_L \times SU(2)_R$ symmetric phase?

$$G_{\delta(\pi)}(x) = \langle \bar{u}_L d_R(x) \bar{d}_L u_R(0) + \bar{u}_R d_L(x) \bar{d}_R u_L(0) \rangle$$

$$\pm \langle \bar{u}_L d_R(x) \bar{d}_R u_L(0) + \bar{u}_R d_L(x) \bar{d}_L u_R(0) \rangle$$

under $U(1)_A$ transformation terms are variant / invariant

DWF calculation; HotQCD preliminary
The flavor non-singlet correlation functions pseudo-scalar ($\pi$) versus scalar ($\delta$)

$U(1)_A$ variant contribution: $\Delta(x) = (G_\pi(x) + G_\delta(x))/2T^3$

$\Delta(x) \neq 0$ ’only’ on configurations with non-trivial topology: $|Q_{top}| \neq 0$

DWF calculation; HotQCD preliminary
Eigenvalue spectrum of the fermion matrix

chiral condensate

\[ \langle \bar{\psi} \psi \rangle_l = \frac{n_f}{4} \frac{1}{N^3} \text{Tr} \langle M_l^{-1} \rangle = \frac{n_f}{4} \frac{1}{N^3} \sum_j \frac{1}{m_l + i\lambda_j} \]

\[ = \int d\lambda \ \rho_V(\lambda) \frac{2m_l}{m_l^2 + \lambda^2} \]

chiral limit:

\[ \langle \bar{\psi} \psi \rangle_l = \pi \lim_{m_l \to 0} \lim_{V \to \infty} \rho_V(0) \equiv \pi \rho(0) \]

(Banks-Casher relation)

\[ \Delta_{\pi-\delta} \equiv (\chi_{\pi} - \chi_{\delta}) / T^2 = \int d\lambda \ \rho_V(\lambda) \frac{4m_l^2}{(m_l^2 + \lambda^2)^2} \]

\[ U(1)_A \text{ remains broken, if } \rho(\lambda) \sim \lambda \text{ (problematic) or } \rho(\lambda) \sim m^2 \delta(\lambda) \ldots ?? \]

\[ U(1)_A \text{ restored, if (for instance) } \rho(\lambda) \text{ has a gap or } \rho(\lambda) \sim \lambda^a, \ a > 1 \]
Eigenvalue spectrum of the fermion matrix

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\[ T = 145.82 \text{ MeV, } m_l/m_s = 1/20 \]
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![Graph](image.png)
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Summary: T-dependence of correlation functions

- Analysis of correlation functions with meson quantum numbers gives insight into the T-dependence of $SU(2)_L \times SU(2)_R$ as well as $UA(1)$ symmetry breaking.

- Current studies suggest that a 'significant' $UA(1)$ symmetry breaking persists at the time of $SU(2)_L \times SU(2)_R$ restoration.

- HOWEVER: The $V \to \infty$, $m \to 0$ limits in the $UA(1)$ sector are subtle. No final conclusion on the effective $UA(1)$ restoration can be drawn at present; studies with chiral fermion formulations may be crucial....