The deformation philosophy, quantization and noncommutative space-time structures

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[This talk summarizes many joint works (some, in progress) that would not have been possible without Flato's deep insight on the role of deformations in physics]

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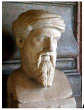
Abstract

The role of deformations in physics and mathematics lead to the deformation philosophy promoted in mathematical physics by Flato since the 70's, exemplified by deformation quantization and its manifold avatars, including quantum groups and the "dual" aspect of quantum spaces. Deforming Minkowski space-time and its symmetry to anti de Sitter has significant physical consequences that we sketch (e.g. singleton physics). We end by presenting an ongoing program in which anti de Sitter would be quantized in some regions, speculating that this might explain baryogenesis in a universe in accelerated expansion.

The Earth is not flat

Act 0. Antiquity (Mesopotamia, ancient Greece).

Flat disk floating in ocean, or Atlas. Similar physical assumption in (ancient) China (Φ).



Act I. Fifth century BC: Pythogoras, theoretical astrophysicist. Pythagoras is often considered as the first mathematician; he and his students believed that everything is related to mathematics. On aesthetic (and democratic?) grounds he conjectured that **all** celestial bodies are spherical.

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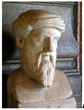

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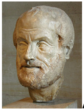

Act II. 3^{rd} century BC: Aristotle, phenomenologist astronomer. Travelers going south see southern constellations rise higher above the horizon, and shadow of earth on moon during the partial phase of a lunar eclipse is always circular: fits physical model of sphere for Earth.

Eratosthenes “Experiment”

Act III. ca. 240 BC: Eratosthenes, “experimentalist”.

Chief librarian of the Great Library in Alexandria. At summer solstice (21 June), knew that sun (practically) at vertical in Aswan and angle of $\frac{2\pi}{50}$ in Alexandria, “about” (based on estimated average daily speed of caravans of camels?) 5000 stadions “North;” assuming sun is point at ∞ (all not quite), by simple geometry got circumference of 252000 “stadions”, 1% or 16% off correct value (Egyptian or Greek stadion). Computed distance to sun as 804,000 kstadions and distance to moon as 780 kstadions, using data obtained during lunar eclipses, and measured tilt of Earth’s axis 11/83 of 2π.

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In China, ca. same time, different context: measure similarly distance of earth to sun assuming earth is flat (the prevailing belief there until 17^{th} century).

Riemann's Inaugural Lecture


Quotation from Section III, §3. 1854 [Nature **8**, 14–17 (1873)]

See http://www.emis.de/classics/Riemann/

The questions about the infinitely great are for the interpretation of nature useless questions.

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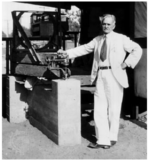
The questions about the infinitely great are for the interpretation of nature useless questions. But this is not the case with the questions about the infinitely small. . . .

It seems that the empirical notions on which the metrical determinations of space are founded, . . . , cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena.

Relativity




Paradox coming from Michelson & Morley experiment (1887) resolved in 1905 by Einstein with special theory of relativity. Experimental need triggered theory. In modern language: Galilean geometrical symmetry group of Newtonian mechanics ($SO(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4$) is deformed, in Gerstenhaber's sense, to Poincaré group ($SO(3,1) \cdot \mathbb{R}^4$) of special relativity.

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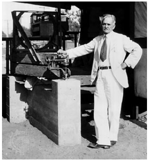


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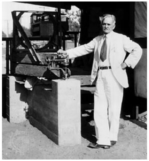

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Flato's deformation philosophy


Physical theories have domain of applicability defined by the relevant distances, velocities, energies, etc. involved. The passage from one domain (of distances, etc.) to another doesn't happen in an uncontrolled way: experimental phenomena appear that cause a paradox and contradict [Fermi quote] accepted theories.

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Philosophy?

Mathematics and physics are two communities separated by a common language. In mathematics one starts with axioms and uses logical deduction therefrom to obtain results that are absolute truth in that framework. In physics one has to make approximations, depending on the domain of applicability.
As in other areas, a *quantitative* change produces a *qualitative change*. *(So we should deform, not extrapolate!)* Engels (i.a.) developed that point and gave a series of examples in Science to illustrate the transformation of quantitative change into qualitative change *at critical points* (see `http://www.marxists.de/science/mcgareng/engels1.htm`).
That is also a problem in psychoanalysis that was tackled using Thom's catastrophe theory. Robert M. Galatzer-Levy, *Qualitative Change from Quantitative Change: Mathematical Catastrophe Theory in Relation to Psychoanalysis*, J. Amer. Psychoanal. Assn., **26** (1978), 921–935.

Deformation theory is an algebraic mathematical way to deal with that "catastrophic" situation, most relevant to physics.

Why, what, how

Why Quantization? In physics, experimental need.
In mathematics, because physicists need it (and gives nice maths).
In mathematical physics, deformation philosophy.

What is quantization? In (theoretical) physics, expression of "quantum" phenomena appearing (usually) in the microworld.
In mathematics, passage from commutative to noncommutative.
In (our) mathematical physics, deformation quantization.

How do we quantize? In physics, correspondence principle.
For many mathematicians (Weyl, Berezin, Kostant, . . .), functor (between categories of algebras of "functions" on phase spaces and of operators in Hilbert spaces; take physicists' formulation for God's axiom; but physicists are neither God nor Jesus; stones. . .). Even Witten. . .
In mathematical physics, deformation (of composition laws)

Classical Mechanics and around

What do we quantize?

Non trivial phase spaces $\to$ Symplectic and Poisson manifolds.

Symplectic manifold: Differentiable manifold M with nondegenerate closed 2-form ω on M. Necessarily $\dim M = 2n$. Locally: $\omega = \omega_{ij} dx^i \wedge dx^j$; $\omega_{ij} = -\omega_{ji}$; $\det \omega_{ij} \neq 0$; $Alt(\partial_i \omega_{jk}) = 0$. And one can find coordinates (q_i, p_i) so that ω is constant: $\omega = \sum_{i=1}^{i=n} dq^i \wedge dp^i$.
Define $\pi^{ij} = \omega_{ij}^{-1}$, then $\{F, G\} = \pi^{ij} \partial_i F \partial_j G$ is a Poisson bracket, i.e. the bracket $\{\cdot, \cdot\} \colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a skewsymmetric ($\{F, G\} = -\{G, F\}$) bilinear map satisfying:

- Jacobi identity: $\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$
- Leibniz rule: $\{FG, H\} = \{F, H\} G + F\{G, H\}$

Examples: 1) $\mathbb{R}^{2n}$ with $\omega = \sum_{i=1}^{i=n} dq^i \wedge dp^i$;
2) Cotangent bundle T^*N, $\omega = d\alpha$, where α is the canonical one-form on T^*N (Locally, $\alpha = -p_i dq^i$)

Poisson manifolds

Poisson manifold: Differentiable manifold M, and skewsymmetric contravariant 2-tensor (not necessarily nondegenerate) $\pi = \sum_{i,j} \pi^{ij} \partial_i \wedge \partial_j$ (locally) such that $\{F, G\} = i(\pi)(dF \wedge dG) = \sum_{i,j} \pi^{ij} \partial_i F \wedge \partial_j G$ is a Poisson bracket.

Examples:

1) Symplectic manifolds ($d\omega = 0 = [\pi, \pi] \equiv$ Jacobi identity)
2) Lie algebra with structure constants C_{ij}^k and $\pi^{ij} = \sum_k x^k C_{ij}^k$.
3) $\pi = X \wedge Y$, where (X, Y) are two commuting vector fields on M.

Facts : Every Poisson manifold is "foliated" by symplectic manifolds. If π is nondegenerate, then $\omega_{ij} = (\pi^{ij})^{-1}$ is a symplectic form.

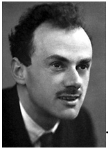
A Classical System is a Poisson manifold (M, π) with a distinguished smooth function, the Hamiltonian $H : M \longrightarrow \mathbb{R}$.

Quantization in physics

   Planck and black body radiation [ca. 1900]. Bohr atom [1913]. Louis de Broglie [1924]: “wave mechanics” (waves and particles are two manifestations of the same physical reality).

  Traditional quantization (Schrödinger, Heisenberg) of classical system $(\mathbb{R}^{2n}, \{\cdot,\cdot\}, H)$: Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n) \ni \psi$ where acts “quantized” Hamiltonian $\mathbf{H}$, energy levels $\mathbf{H}\psi = \lambda\psi$, and von Neumann representation of CCR. Define $\hat{q}_\alpha(f)(q) = q_\alpha f(q)$ and $\hat{p}_\beta(f)(q) = -i\hbar \frac{\partial f(q)}{\partial q_\beta}$ for f differentiable in $\mathcal{H}$. Then (CCR) $[\hat{p}_\alpha, \hat{q}_\beta] = i\hbar\delta_{\alpha\beta} I \quad (\alpha, \beta = 1, \ldots, n)$.

Orderings, Weyl, Wigner

The couple $(\hat{q}, \hat{p})$ quantizes the coordinates (q, p). A polynomial classical Hamiltonian H is quantized once chosen an operator ordering, e.g. (Weyl) complete symmetrization of $\hat{p}$ and $\hat{q}$. In general the quantization on $\mathbb{R}^{2n}$ of a function $H(q, p)$ with inverse Fourier transform $\tilde{H}(\xi, \eta)$ can be given by (Hermann Weyl [1927] with weight $\varpi = 1$):

$H \mapsto \mathbf{H} = \Omega_\varpi(H) = \int_{\mathbb{R}^{2n}} \tilde{H}(\xi, \eta) \exp(i(\hat{p}.\xi + \hat{q}.\eta)/\hbar)\varpi(\xi, \eta) d^n\xi d^n\eta.$

E. Wigner [1932] inverse $H = (2\pi\hbar)^{-n}\mathrm{Tr}[\Omega_1(H) \exp((\xi.\hat{p} + \eta.\hat{q})/i\hbar)]$.

Ω_1 defines an isomorphism of Hilbert spaces between $L^2(\mathbb{R}^{2n})$ and Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. Can extend e.g. to distributions.

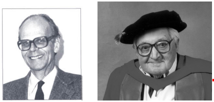
Constrained systems e.g. constraints $f_j(p, q) = 0$ ($\Rightarrow$ also algebraic varieties and manifolds with corners): Dirac formalism [1950].

Dirac quote

"... One should examine closely even the elementary and the satisfactory features of our Quantum Mechanics and criticize them and try to modify them, because there may still be faults in them. The only way in which one can hope to proceed on those lines is by looking at the basic features of our present Quantum Theory from all possible points of view. Two points of view may be mathematically equivalent and you may think for that reason if you understand one of them you need not bother about the other and can neglect it. But it may be that one point of view may suggest a future development which another point does not suggest, and although in their present state the two points of view are equivalent they may lead to different possibilities for the future. Therefore, I think that we cannot afford to neglect any possible point of view for looking at Quantum Mechanics and in particular its relation to Classical Mechanics. Any point of view which gives us any interesting feature and any novel idea should be closely examined to see whether they suggest any modification or any way of developing the theory along new lines.

A point of view which naturally suggests itself is to examine just how close we can make the connection between Classical and Quantum Mechanics. That is essentially a purely mathematical problem – how close can we make the connection between an algebra of non-commutative variables and the ordinary algebra of commutative variables? In both cases we can do addition, multiplication, division..." **Dirac**, *The relation of Classical to Quantum Mechanics* (2$^{\text{nd}}$ Can. Math. Congress, Vancouver 1949). U.Toronto Press (1951) pp 10-31.

Classical ↔ Quantum correspondence



The correspondence $H \mapsto \Omega(H)$ is not an algebra homomorphism, neither for ordinary product of functions nor for the Poisson bracket P ("Van Hove theorem"). Take two functions u_1 and u_2, then (Groenewold [1946], Moyal [1949]):

$\Omega_1^{-1}(\Omega_1(u_1)\Omega_1(u_2)) = u_1 u_2 + \frac{i\hbar}{2}\{u_1, u_2\} + O(\hbar^2)$, and similarly for bracket.

More precisely Ω_1 maps into product and bracket of operators (resp.):

$u_1 *_M u_2 = \exp(tP)(u_1, u_2) = u_1 u_2 + \sum_{r=1}^{\infty} \frac{t^r}{r!} P^r(u_1, u_2)$ (with $2t = i\hbar$),

$M(u_1, u_2) = t^{-1}\sinh(tP)(u_1, u_2) = P(u_1, u_2) + \sum_{r=1}^{\infty} \frac{t^{2r}}{(2r+1)!} P^{2r+1}(u_1, u_2)$

We recognize formulas for deformations of algebras.

Deformation quantization: forget the correspondence principle Ω and work in an *autonomous* manner with "functions" on phase spaces.

Some other mathematicians' approaches

Geometric quantization (Kostant, Souriau). [1970's. Mimic correspondence principle for general phase spaces M. Look for generalized Weyl map from functions on M:] Start with "prequantization" on $L^2(M)$ and tries to halve the number of degrees of freedom using (complex, in general) polarizations to get Lagrangian submanifold $\mathcal{L}$ of dimension half of that of M and quantized observables as operators in $L^2(\mathcal{L})$. Fine for representation theory (M coadjoint orbit, e.g. solvable group) but few observables can be quantized (linear or maybe quadratic, preferred observables in def.q.).

Berezin quantization. (ca.1975). Quantization is an algorithm by which a quantum system corresponds to a classical dynamical one, i.e. (roughly) is a functor between a category of algebras of classical observables (on phase space) and a category of algebras of operators (in Hilbert space).

Examples: Euclidean and Lobatchevsky planes, cylinder, torus and sphere, Kähler manifolds and duals of Lie algebras. [Only (M, π), no H here.]

The framework

Poisson manifold (M, π), deformations of product of functions.

Inspired by deformation philosophy, based on Gerstenhaber's deformation theory [Flato, Lichnerowicz, Sternheimer; and Vey; mid 70's] [Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer, LMP '77 & Ann. Phys. '78]

• $\mathcal{A}_t = C^\infty(M)[[t]]$, formal series in t with coefficients in $C^\infty(M) = A$. Elements: $f_0 + tf_1 + t^2 f_2 + \cdots$ (t formal parameter, not fixed scalar.)
• Star product $\star_t : \mathcal{A}_t \times \mathcal{A}_t \to \mathcal{A}_t$; $f \star_t g = fg + \sum_{r \geq 1} t^r C_r(f, g)$
- C_r are bidifferential operators null on constants: $(1 \star_t f = f \star_t 1 = f)$.
- $\star_t$ is associative and $C_1(f, g) - C_1(g, f) = 2\{f, g\}$, so that $[f, g]_t \equiv \frac{1}{2t}(f \star_t g - g \star_t f) = \{f, g\} + O(t)$ is Lie algebra deformation.

Basic paradigm. Moyal product on $\mathbb{R}^{2n}$ with the canonical Poisson bracket P:
$F \star_M G = \exp\left(\frac{i\hbar}{2} P\right)(f, g) \equiv FG + \sum_{k \geq 1} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k P^k(F, G)$.

Applications and Equivalence

Equation of motion (time τ): $\frac{dF}{d\tau} = [H, F]_M \equiv \frac{1}{i\hbar}(H \star_M F - F \star_M H)$
Link with Weyl's rule of quantization: $\Omega_1(F \star_M G) = \Omega_1(F)\Omega_1(G)$

Equivalence of two star-products $\star_1$ and $\star_2$.

- Formal series of differential operators $T(f) = f + \sum_{r\geq 1} t^r T_r(f)$.
- $T(f \star_1 g) = T(f) \star_2 T(g)$.

For symplectic manifolds, equivalence classes of star-products are parametrized by the 2nd de Rham cohomology space $H^2_{dR}(M)$: $\{\star_t\}/\sim = H^2_{dR}(M)[[t]]$ (Nest-Tsygan [1995] and others). In particular, $H^2_{dR}(\mathbb{R}^{2n})$ is trivial, all deformations are equivalent.
Kontsevich: $\{$Equivalence classes of star-products$\} \equiv \{$equivalence classes of formal Poisson tensors $\pi_t = \pi + t\pi_1 + \cdots\}$.

Remarks: - The choice of a star-product fixes a quantization rule.
- Operator orderings can be implemented by good choices of T (or ϖ).
- On $\mathbb{R}^{2n}$, all star-products are equivalent to Moyal product (cf. von Neumann uniqueness theorem on projective UIR of CCR).

Existence and Classification

Let (M, π) be a Poisson manifold. $f \tilde{\star} g = fg + t\{f, g\}$ does not define an associative product. But $(f \tilde{\star} g) \tilde{\star} h - f \tilde{\star} (g \tilde{\star} h) = O(t^2)$.

Is it always possible to modify $\tilde{\star}$ in order to get an associative product?

Existence, symplectic case:
– DeWilde-Lecomte [1982]: Glue local Moyal products.
– Omori-Maeda-Yoshioka [1991]: Weyl bundle and glueing.
– Fedosov [1985,1994]: Construct a flat abelian connection on the Weyl bundle over the symplectic manifold.

General Poisson manifold M with Poisson bracket P:
Solved by Kontsevich [1997, LMP 2003]. "Explicit" local formula: $(f, g) \mapsto f \star g = \sum_{n \geq 0} t^n \sum_{\Gamma \in G_{n,2}} w(\Gamma) B_\Gamma(f, g)$, defines a differential star-product on $(\mathbb{R}^d, P)$; globalizable to M. Here $G_{n,2}$ is a set of graphs Γ, $w(\Gamma)$ some weight defined by Γ and $B_\Gamma(f, g)$ some bidifferential operators.

Particular case of Formality Theorem. Operadic approach

This is Quantization

A star-product provides an *autonomous* quantization of a manifold M. BFFLS '78: Quantization is a deformation of the composition law of observables of a classical system: $(A, \cdot) \to (A[[t]], \star_t)$, $A = C^\infty(M)$.

Star-product $\star$ ($t = \frac{i}{2}\hbar$) on Poisson manifold M and Hamiltonian H; introduce the star-exponential: $\mathrm{Exp}_\star(\frac{\tau H}{i\hbar}) = \sum_{r\geq 0} \frac{1}{r!}(\frac{\tau}{i\hbar})^r H^{\star r}$.

Corresponds to the unitary evolution operator, is a singular object i.e. belongs not to the quantized algebra $(A[[t]], \star)$ but to $(A[[t, t^{-1}]], \star)$. Singularity at origin of its trace, Harish Chandra character for UIR of semi-simple Lie groups.

Spectrum and states are given by a spectral (Fourier-Stieltjes in the time τ) decomposition of the star-exponential.

Paradigm: Harmonic oscillator $H = \frac{1}{2}(p^2 + q^2)$, Moyal product on $\mathbb{R}^{2\ell}$.

$\mathrm{Exp}_\star(\frac{\tau H}{i\hbar}) = \left(\cos(\frac{\tau}{2})\right)^{-1} \exp\left(\frac{2H}{i\hbar}\tan(\frac{\tau}{2})\right) = \sum_{n=0}^\infty \exp\left(-i(n + \frac{\ell}{2})\tau\right)\pi_n^\ell$.

Here ($\ell = 1$ but similar formulas for $\ell \geq 1$, L_n is Laguerre polynomial of degree n)

$\pi_n^1(q, p) = 2\exp\left(\frac{-2H(q,p)}{\hbar}\right)(-1)^n L_n\left(\frac{4H(q,p)}{\hbar}\right)$.

Complements

The Gaussian function $\pi_0(q,p) = 2\exp\left(\frac{-2H(q,p)}{\hbar}\right)$ describes the vacuum state. As expected the energy levels of H are $E_n = \hbar(n+\frac{\ell}{2})$: $H \star \pi_n = E_n \pi_n$; $\pi_m \star \pi_n = \delta_{mn}\pi_n$; $\sum_n \pi_n = 1$. With normal ordering, $E_n = n\hbar$: $E_0 \to \infty$ for $\ell \to \infty$ in Moyal ordering but $E_0 \equiv 0$ in normal ordering, preferred in Field Theory.

• Other standard examples of QM can be quantized in an autonomous manner by choosing adapted star-products: angular momentum with spectrum $n(n+(\ell-2))\hbar^2$ for the Casimir element of $\mathfrak{so}(\ell)$; hydrogen atom with $H = \frac{1}{2}p^2 - |q|^{-1}$ on $M = T^*S^3$, $E = \frac{1}{2}(n+1)^{-2}\hbar^{-2}$ for the discrete spectrum, and $E \in \mathbb{R}^+$ for the continuous spectrum; etc.

• Feynman Path Integral (PI) is, for Moyal, Fourier transform in p of star-exponential; equal to it (up to multiplicative factor) for normal ordering) [Dito'90]. Cattaneo-Felder [2k]: Kontsevich star product as PI.

Cohomological renormalization (see below; "Subtract infinite cocycle.")

Conventional vs. deformation quantization

- It is a matter of practical feasibility of calculations, when there are Weyl and Wigner maps to intertwine between both formalisms, to choose to work with operators in Hilbert spaces or with functional analysis methods (distributions etc.) Dealing e.g. with spectroscopy (where it all started; cf. also Connes) and finite dimensional Hilbert spaces where operators are matrices, the operatorial formulation is easier.
- When there are no precise Weyl and Wigner maps (e.g. very general phase spaces, maybe infinite dimensional) one does not have much choice but to work (maybe "at the physical level of rigor") with functional analysis. Contrarily to what Gukov and Witten assert (`arXiv:0809.0305v1` p.10) deformation quantization is quantization: it permits (in concrete cases) to take for $\hbar$ its value, when there are Weyl and Wigner maps one can translate its results in Hilbert space, and e.g. for the 2-sphere there is a special behavior when the radius of the sphere has quantized values related to the Casimir values of $SO(3)$.

Some avatars

(Topological) Quantum Groups. Deform Hopf algebras of functions (differentiable vectors) on Poisson-Lie group, and/or their topological duals (as nuclear t.v.s., Fréchet or dual thereof).
Preferred deformations (deform either product or coproduct) e.g. G semi-simple compact: $A = C^\infty(G)$ (gets differential star product) or its dual (compactly supported distributions on G, completion of $\mathcal{U}\mathfrak{g}$, deform coproduct with Drinfeld twist); or $A = \mathcal{H}(G)$, coefficient functions of finite dimensional representations of G, or its dual.

"Noncommutative Gelfand duality theorem." Commutative topological algebra $A \simeq$ "functions on its spectrum." What about $(A[[t]], \star_t)$?
Woronowicz's matrix C^* pseudogroups. Gelfand's NC polynomials.

Noncommutative geometry vs. deformation quantization.
Strategy: formulate usual differential geometry in an unusual manner, using in particular algebras and related concepts, so as to be able to "plug in" noncommutativity in a natural way (cf. Dirac quote).

Overview

The deformation quantization of a given classical field theory consists in giving a proper definition for a star-product on the infinite-dimensional manifold of initial data for the classical field equation and constructing with it, as rigorously as possible, whatever physical expressions are needed.

As in other approaches to field theory, here also one faces serious divergence difficulties as soon as one is considering interacting fields theory, and even at the free field level if one wants a mathematically rigorous theory.

But the philosophy in dealing with the divergences is significantly different and one is in position to take advantage of the cohomological features of deformation theory to perform what can be called *cohomological renormalization*.

In the same way as we quantize by deforming the (commutative) product of observables to an $\hbar$-dependent star product, keeping the classical observables unchanged, the idea is to renormalize by deforming the normal star-product to another, coupling constant dependent, quantization.

Poisson structure and field equations

Poisson structures are known on infinite-dimensional manifolds since a long time; there is an extensive literature on this subject. A typical structure, for our purpose, is a weak symplectic structure such as that defined in 1974 by Segal and by Kostant on the space of solutions of a classical field equation like $\Box\Phi = F(\Phi)$, $\Box$ = d'Alembertian. Now for scalar-valued functionals Ψ over such a space, i.e., over the phase space of initial conditions $\varphi(x) = \Phi(x, 0)$ and $\pi(x) = \frac{\partial}{\partial t}\Phi(x, 0)$, a Poisson bracket can be defined by

$$P(\Psi_1, \Psi_2) = \int \left(\frac{\delta\Psi_1}{\delta\varphi} \frac{\delta\Psi_2}{\delta\pi} - \frac{\delta\Psi_1}{\delta\pi} \frac{\delta\Psi_2}{\delta\varphi} \right) dx \tag{1}$$

δ being the functional derivative. But while one can give a precise mathematical meaning to (1) by specifying an appropriate algebra of functionals, the formal extension to powers of P, needed to define the Moyal bracket, is highly divergent, already for P^2.

This is no surprise to physicists who know that the correct approach to field theory starts with normal ordering, and that there are infinitely many inequivalent representations of the canonical commutation relations, even if in recent physical literature some are working formally with Moyal product.

The idea of cohomological renormalization in deformation quantization

Starting with some star-product $\star$ (e.g. similar to the normal star-product on a manifold of initial data), we would like to interpret various divergences appearing in the theory in terms of coboundaries (or cocycles) for the relevant Hochschild cohomology. If we suspect that a term in a cochain of the product $\star$ is responsible for the appearance of divergences, applying an iterative procedure of equivalence, we can try to eliminate it, or at least get a lesser divergence, by subtracting at the relevant order a divergent coboundary; we would then get a better theory with a new star-product, "equivalent" to the original one. Furthermore, since in this case we expect to have at each order an infinity of non equivalent star-products, we can try to subtract a cocycle and then pass to a nonequivalent star-product whose lower order cochains are identical to those of the original one. We would then make an analysis of the divergences up to order $\hbar^r$, identify a divergent cocycle, remove it, and continue the procedure (at the same or hopefully a higher order). Along the way one should preserve the usual properties of a quantum field theory (Poincaré covariance, locality, etc.) and the construction of adapted star-products should be done accordingly. The complete implementation of this program should lead to a cohomological approach to renormalization theory.

It seems (e.g. looking at the formulas in Connes 2005 lectures at Collège de France)

that the Connes–Kreimer rigorous renormalization procedure could fit in this pattern.

Normal star-product and quantized fields

Let Φ be a (classical) free massive scalar field with initial data (φ, π) in the Schwartz space $\mathcal{S}$. Replace them by their Fourier modes $(\bar{a}, a)$, also in $\mathcal{S}$ seen as a real vector space. After quantization $(\bar{a}, a)$ become the usual creation and annihilation operators.

The normal star-product $\star_N$ can be written

$(F \star_N G)(\bar{a}, a) = \int_{\mathcal{S}' \oplus \mathcal{S}'} d\mu(\bar{\xi}, \xi) F(\bar{a}, a + \xi) G(\bar{a} + \bar{\xi}, a)$ where μ is the Gaussian measure on $\mathcal{S}' \oplus \mathcal{S}'$ and F, G are holomorphic functions with semi-regular kernels.

Creation and annihilation operators being operator-valued distributions, we take $(\bar{a}, a) \in \mathcal{S}' \oplus \mathcal{S}'$ (the distribution aspect is present in the definition of the cochains of the star-product). Fermionic fields can also be cast in that framework.

For the above normal product one can formally consider interacting fields. The star-exponential of the Hamiltonian turns out to be, up to a multiplicative well-defined function, equal to Feynman's path integral. For free fields, we have a mathematically meaningful equality between the star-exponential and the path integrals as both of them are defined by a Gaussian measure, hence well-defined. In the interacting fields case, giving a rigorous meaning to either of them would give a meaning to the other.

A toy model of cohomological renormalization

Work in that direction (free scalar fields, Klein–Gordon equation etc.) is done by Dito since the 90's including an example of cancellation of some infinities in $\lambda\varphi_2^4$-theory via a λ-dependent star-product formally equivalent to normal:
Take a $\lambda\varphi_2^4$ interacting Hamiltonian $H[\varphi,\pi] = H_0[\varphi,\pi] + \lambda V[\varphi]$ with $H_0 = \frac{1}{2}\int_{\mathbb{R}}(\pi^2(x) + |\nabla\varphi(x)|^2 + m^2\varphi^2(x))dx$, $V[\varphi] = \int_{\mathbb{R}}\varphi^4(x)dx$ or its equivalent form with $(\bar{a}, a)$. Singular terms appear in the $\star_N$-powers of H, not surprising since (Glimm–Jaffe) one needs an infinite renormalization of H in order to give a meaning to the operator expression of H.
We would like to leave H unchanged and define a new $\star$-product such that no singular terms occur in the $\star$-powers of H and, ultimately, that the $\star$-exponential of H is well defined. Dito (LMP 1993) constructed a $\star$-product equivalent to normal which gives a meaning to $H \star F(H)$, F an arbitrary polynomial function of H. The equivalence operator T, $T(F\star G) = TF \star_N TG$, is given by an expression $T(F) = \exp \hbar\lambda \int dkf(k)[\frac{\delta^2 F}{\delta a(k)\delta a(k)} - \frac{\delta^2 F}{\delta\bar{a}(k)\delta\bar{a}(k)}$ where f is a function adjusted in such a way to generate a counterterm for $C_4(H,H)$, the only singular term in $H \star_N H$ leading to an infinite constant. It however does not give divergenceless expressions for the $\star$-powers of H with $n \geq 3$ because these are not polynomials in H.

Epistemological remarks


Two quotes by Sir James Hopwood Jeans:

"The Great Architect of the Universe now begins to appear as a pure mathematician."

"We may as well cut out the group theory. That is a subject that will never be of any use in physics." [Discussing a syllabus in 1910.] [Physicists liberty with rigor vs. mathematicians lack of physical touch.]

Spectroscopy. In atomic and molecular physics we know the forces and their symmetries. Energy levels (spectral lines) classified by UIR (unitary irreducible representations) of $SO(3)$ or $SU(2)$, and e.g. with crystals that is refined (broken) by a finite subgroup. [Racah school, Flato's M.Sc.] The more indirect physical measurements become, the more one has to be careful. "Curse" of experimental sciences. Mathematical logic: if A and $A \to B$, then B. In real life, imagine model or theory A. If $A \to B$ and "B is nice" (e.g. verified & more), then A! [Inspired by Kolmogorov quote.] (It ain't necessarily so.)

Poincaré and anti De Sitter "external" symmetries

1930's: Dirac asks Wigner to study UIRs of Poincaré group. 1939: Wigner paper in Ann.Math. UIR: particle with positive and zero mass (and "tachyons"). Seminal for UIRs (Bargmann, Mackey, Harish Chandra etc.)

Deform Minkowski to AdS, and Poincaré to AdS group SO(2,3). UIRs of AdS studied incompletely around 1950's. 2 (most degenerate) missing found (1963) by Dirac, the singletons that we call Rac$= D(\frac{1}{2}, 0)$ and Di$= D(1, \frac{1}{2})$ (massless of Poincaré in 2+1 dimensions). In normal units a singleton with angular momentum j has energy $E = (j + \frac{1}{2})\rho$, where ρ is the curvature of the AdS_4 universe (they are naturally confined, fields are determined by their value on cone at infinity in AdS_4 space).

The massless representations of $\mathrm{SO}(2,3)$ are defined (for $s \geq \frac{1}{2}$) as $D(s+1, s)$ and (for helicity zero) $D(1,0) \oplus D(2,0)$. There are many justifications to this definition. They are kinematically composite: $(\mathrm{Di} \oplus \mathrm{Rac}) \otimes (\mathrm{Di} \oplus \mathrm{Rac}) = (D(1,0) \oplus D(2,0)) \oplus 2 \bigoplus_{s=\frac{1}{2}}^{\infty} D(s+1, s)$.
Also dynamically (QED with photons composed of 2 Racs, FF88).

Generations, “internal” symmetries

At first, because of the isospin *I*, a quantum number separating proton and neutron introduced (in 1932, after the discovery of the neutron) by Heisenberg, $SU(2)$ was tried. Then in 1947 a second generation of “strange” particles started to appear and in 1952 Pais suggested a new quantum number, the strangeness *S*. In 1975 a third generation (flavor) was discovered, associated e.g. with the τ lepton, and its neutrino ν_τ first observed in 2000. In the context of what was known in the 1960’s, a rank 2 group was the obvious thing to try and introduce in order to describe these “internal” properties. That is how in particle physics theory appeared $U(2)$ (or $SU(2) \times U(1)$, now associated with the electroweak interactions) and the simplest simple group of rank 2, $SU(3)$, which subsists until now in various forms, mostly as “color” symmetry in QCD theory.

Connection with space-time symmetries? (O’Raifeartaigh no-go “theorem” and FS counterexamples.) Reality is (much) more complex.

Composite leptons and flavor symmetry

The electroweak model is based on “the weak group”, $S_W = SU(2) \times U(1)$, on the Glashow representation of this group, carried by the triplet $(\nu_e, e_L; e_R)$ and by each of the other generations of leptons. Suppose that

(a) There are three bosonic singletons $(R^N R^L; R^R) = (R^A)_{A=N,L,R}$ (three “Rac”s) that carry the Glashow representation of S_W;

(b) There are three spinorial singletons $(D_\varepsilon, D_\mu; D_\tau) = (D_\alpha)_{\alpha=\varepsilon,\mu,\tau}$ (three “Di”s). They are insensitive to S_W but transform as a Glashow triplet with respect to another group S_F (the “flavor group”), isomorphic to S_W;

(c) The vector mesons of the standard model are Rac-Rac composites, the leptons are Di-Rac composites, and there is a set of vector mesons that are Di-Di composites and that play exactly the same role for S_F as the weak vector bosons do for S_W: $W_A^B = \bar{R}^B R_A$, $L_\beta^A = R^A D_\beta$, $F_\beta^\alpha = \bar{D}_\beta D^\alpha$.

These are initially massless, massified by interaction with Higgs.

Composite leptons massified

Let us concentrate on the leptons ($A = N, L, R$; $\beta = \varepsilon, \mu, \tau$)

$$(L^A_\beta) = \begin{pmatrix} \nu_e & e_L & e_R \\ \nu_\mu & \mu_L & \mu_R \\ \nu_\tau & \tau_L & \tau_R \end{pmatrix} . \tag{2}$$

A factorization $L^A_\beta = R^A D_\beta$ is strongly urged upon us by the nature of the phenomenological summary in (1). Fields in the first two columns couple horizontally to make the standard electroweak current, those in the last two pair off to make Dirac mass-terms. Particles in the first two rows combine to make the (neutral) flavor current and couple to the flavor vector mesons. The Higgs fields have a Yukawa coupling to lepton currents, $\mathcal{L}_{\rm Yu} = -g_{\rm Yu}\bar{L}^\beta_A L^B_\alpha H^{\alpha A}_{\beta B}$. The electroweak model was constructed with a single generation in mind, hence it assumes a single Higgs doublet. We postulate additional Higgs fields, coupled to leptons in the following way, $\mathcal{L}'_{\rm Yu} = h_{\rm Yu} L^A_\alpha L^B_\beta K^{\alpha\beta}_{AB} + {\rm h.c.}$. This model predicts 2 new mesons, parallel to the W and Z of the electroweak model (Frønsdal, LMP 2000). But too many free parameters.

Do the same for quarks (and gluons), adding color?

Questions and facts

Even if know "intimate structure" of particles (as composites of quarks etc. or singletons): How, when and where happened "baryogenesis"? [Creation of 'our matter', now 4% of universe mass, vs. 74% 'dark energy' and 22 % 'dark matter'; and matter–antimatter asymetry, Sakharov 1967.] Everything at "big bang"?! [Shrapnel of 'stem cells' of initial singularity?]

Facts: $SO_q(3,2)$ at even root of unity has finite-dimensional UIRs ("compact"?). Black holes à la 't Hooft: can communicate with them, by interaction at surface.

Noncommutative (quantized) manifolds. E.g. quantum 3- and 4-spheres (Connes with Landi and Dubois-Violette); spectral triples $(\mathcal{A}, \mathcal{H}, D)$).

Connes' Standard Model with neutrino mixing, minimally coupled to gravity. Space-time is Riemannian compact spin 4-manifold (Barrett has Lorentzian version) $\times$ finite (32) NCG. More economical than SUSYSM and predicts Higgs mass at upper limit (SUSYSM gives lower). [Ongoing with Marcolli and Chamseddine.]

Conjectures and a speculative answer

[Odessa Rabbi anecdote] Space-time could be, at very small distances, not only deformed (to AdS_4 with tiny negative curvature ρ, which does not exclude at cosmological distances to have a positive curvature or cosmological constant, e.g. due to matter) but also "quantized" to some $qAdS_4$. Such $qAdS_4$ could be considered, in a sense to make more precise (e.g. with some measure or trace) as having "finite" (possibly "small") volume (for q even root of unity). At the "border" of these one would have, for most practical purposes at "our" scale, the Minkowski space-time, obtained by $q\rho \to 0$. They could be considered as some "black holes" from which "q-singletons" would emerge, create massless particles that would be massified by interaction with dark matter or dark energy. That could (and should, otherwise there would be manifestations closer to us, that were not observed) occur mostly at or near the "edge" of our universe in accelerated expansion. These "qAdS black holes" ("inside" which one might find compactified extra dimensions) could be a kind of "shrapnel" resulting from the Big Bang (in addition to background radiation) and provide a clue to baryogenesis.

A NCG model for $qAdS_4$

To AdS_n, $n \geq 3$, we associate *naturally* a symplectic symmetric space (M, ω, s). The data of any invariant (formal or not) deformation quantization on (M, ω, s) yields canonically universal deformation formulae (procedures associating to a topological algebra $\mathbb{A}$ having a symmetry $\mathcal{G}$ a deformation $\mathbb{A}_\theta$ in same category) for the actions of a non-Abelian solvable Lie group $\mathcal{R}_0$ (one-dimensional extension of the Heisenberg group $\mathcal{H}_n$), given by an oscillatory integral kernel.

Using it we (P.Bieliavsky, LC, DS & YV) define a noncommutative Lorentzian spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ where $\mathcal{A}^\infty := (L^2_{\text{right}}(\mathcal{R}_0))^\infty$ is a NC Fréchet algebra modelled on the space $\mathcal{H}^\infty$ of smooth vectors of the regular representation on the space $\mathcal{H}$ of square integrable functions on $\mathcal{R}_0$, and D a Dirac operator acting as a derivation of the noncommutative bi-module structure, and for all $a \in \mathcal{A}^\infty$, the commutator $[D, a]$ extends to $\mathcal{H}$ as a bounded operator. The underlying commutative limit is endowed with a causal black hole structure (for $n \geq 3$) encoded in the $\mathcal{R}_0$-group action.

Some open problems and speculations

1. Define within the present Lorentzian context the notion of causality at the operator algebraic level.
2. Representation theory for $SO_q(2, n)$ (e.g. new reps. at root of unity, analogs of singletons, 'square root' of massless reps. of AdS or Poincaré, etc.)
3. Define a kind of trace giving finite "q-volume" for qAdS at even root of unity (possibly in TVS context).
4. Find analogs of all the 'good' properties (e.g. compactness of the resolvent of D) of Connes' spectral triples in compact Riemannian case, possibly with quadruples $(\mathcal{A}, \mathcal{E}, D, \mathcal{G})$ where $\mathcal{A}$ is some topological algebra, $\mathcal{E}$ an appropriate TVS, D some (bounded on $\mathcal{E}$) "Dirac" operator and $\mathcal{G}$ some symmetry.
5. Limit $\rho q \to 0$ ($\rho < 0$ being AdS curvature)?
6. Unify (groupoid?) Poincaré in Minkowski space (possibly modified locally by the presence of matter) with these $SO_q(2, n)$ in the qAdS "black holes".
7. Field theory on such q-deformed spaces, etc.