Eternal inflation
(l >> 3 x 10^{26} GPc from us)

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Classical force and quantum fluctuations

$$V(\phi) = \frac{m^2}{2}\phi^2$$

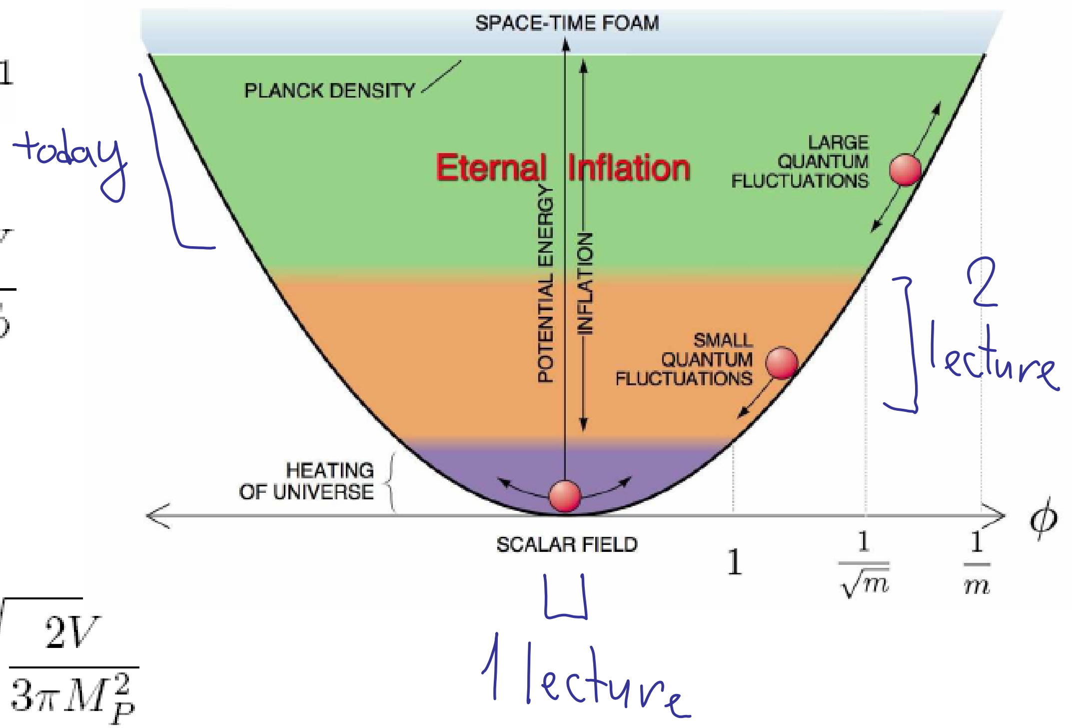
During one Hubble time $\Delta t \sim H^{-1}$
the inflaton decreases by

$$\Delta\phi \sim \dot{\phi}\Delta t \sim \frac{1}{3H^2}\frac{\partial V}{\partial \phi} \sim \frac{M_P^2}{8\pi V}\frac{\partial V}{\partial \phi}$$

However, during the same time the fluctuations of the inflaton are generated with charact. wavelength

$$l \sim k^{-1} \sim H^{-1}$$

and typical amplitude $|\delta\phi| \sim \frac{H}{2\pi} \sim \sqrt{\frac{2V}{3\pi M_P^2}}$



It looks like in different Hubble patches the EVs of the inflaton measured by local observers are different. In some Hubble patches, due to the stochastic kicks produced by generated fluctuations, the EV of the inflaton may grow. Let's prove it.

Note: (presently unknown) What if the energy density inside a particular Hubble patch reaches Planckian values?

Random walk of inflaton and the Langevin equations 1

EOM for the inflaton (*): $\left(\frac{\partial^2}{\partial t^2}+3H\frac{\partial}{\partial t}-\frac{1}{a(t)^2}\Delta\right)\phi(\boldsymbol{x},t)+\frac{\partial V}{\partial\phi}(\boldsymbol{x},t)=0$

Let's subtract the IR mode (super-Hubble scale which is indistinguishible from the background mode from the point of view of an observer):

$$\phi(\boldsymbol{x},t)=\varphi(\boldsymbol{x},t)+\sqrt{\hbar}\,\varphi_s(\boldsymbol{x},t)\,,\qquad \dot\phi(\boldsymbol{x},t)=v(\boldsymbol{x},t)+\sqrt{\hbar}\,v_s(\boldsymbol{x},t)$$

such that

$$\varphi_s(\boldsymbol{x},t)=\int\frac{d^3k}{(2\pi)^{3/2}}\theta(k-\epsilon a(t)H)\phi_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,,\quad v_s(\boldsymbol{x},t)=\int\frac{d^3k}{(2\pi)^{3/2}}\theta(k-\epsilon a(t)H)\dot\phi_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Fourier modes are defined as $\phi_{\boldsymbol{k}}(t)=a_{\boldsymbol{k}}\varphi_{\boldsymbol{k}}(t)+a_{\boldsymbol{k}}^{\dagger}\varphi^*_{-\boldsymbol{k}}(t)$

EOM for them is $\ddot\varphi_{\boldsymbol{k}}(t)+3H\dot\varphi_{\boldsymbol{k}}(t)+\left(\frac{k^2}{a(t)^2}+M^2\right)\varphi_{\boldsymbol{k}}(t)=0$

Substituting IR decomposition into (*), one has EOM for the IR mode

$$\dot\varphi=v+\sqrt{\hbar}\,\sigma$$

$$\dot v=-3Hv+\frac{1}{a^2}\Delta\varphi-V'(\varphi)+\sqrt{\hbar}\,\tau$$

where

$$\sigma(\boldsymbol{x},t)=\epsilon aH^2\int\frac{d^3k}{(2\pi)^{3/2}}\delta(k-\epsilon aH)\phi_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\tau(\boldsymbol{x},t)=\epsilon aH^2\int\frac{d^3k}{(2\pi)^{3/2}}\delta(k-\epsilon aH)\dot\phi_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Random walk of inflaton and the Langevin equations 2

Commutation relations for noises can be derived taking the initial condition as the Bunch-Davies vacuum:

$j_0(x) = \frac{\sin x}{x}$

$$\langle 0|\sigma(x_1)\sigma(x_2)|0\rangle \approx \epsilon^{(2M^2/3H^2)} \frac{H^3}{4\pi^2} j_0(\epsilon a H|\boldsymbol{x}_1-\boldsymbol{x}_2|)\delta(t_1-t_2)$$

$$\langle 0|\tau(x_1)\tau(x_2)|0\rangle \approx \epsilon^{(2M^2/3H^2)}\left(\frac{M^2}{3H^2}+\epsilon^2\right)^2 \frac{H^5}{4\pi^2} j_0(\epsilon a H|\boldsymbol{x}_1-\boldsymbol{x}_2|)\delta(t_1-t_2)$$

$$\langle 0|\sigma(x_1)\tau(x_2)+\tau(x_2)\sigma(x_1)|0\rangle$$

$$\approx -2\epsilon^{(2M^2/3H^2)}\left(\frac{M^2}{3H^2}+\epsilon^2\right)\frac{H^4}{4\pi^2} j_0(\epsilon a H|\boldsymbol{x}_1-\boldsymbol{x}_2|)\delta(t_1-t_2),$$ so one has

$$[\sigma(x_1),\, \sigma(x_2)] = [\tau(x_1),\, \tau(x_2)] = 0\,, \quad [\sigma(x_1),\, \tau(x_2)] = i\epsilon^3 \frac{H^4}{4\pi^2} j_0(\epsilon a H|\boldsymbol{x}_1-\boldsymbol{x}_2|)\delta(t_1-t_2)$$

For $\exp\left(-\frac{3H^2}{|M^2|}\right) \ll \epsilon^2 \ll \frac{|M^2|}{3H^2}$ the quantum nature of noises becomes negligible (effectively they commute!), dependence on ε also disappears in the leading order.

In addition, effectively we have $\tau \approx -\frac{M^2}{3H}\sigma$

and the final result is $$\dot{\varphi} = v + \sqrt{\hbar}\,\sigma\,, \quad \dot{v} = -3Hv + \frac{1}{a^2}\Delta\varphi - V'(\varphi) - \frac{M^2}{3H}\sqrt{\hbar}\,\sigma$$

$$\langle \sigma(x_1)\sigma(x_2)\rangle = \frac{H^3}{4\pi^2} j_0(\epsilon a H|\boldsymbol{x}_1-\boldsymbol{x}_2|)\delta(t_1-t_2)$$

Fokker-Planck equation for inflationary spacetime

In what follows, we will use the model with scalar field potential of the form

$$V(\phi) = V_0 + \delta V(\phi) \quad \text{where} \quad |\delta V(\phi)| \ll V_0$$

The EV of the inflaton in a given Hubble patch satisfies the Langevin equation

$$\dot{\phi} = -\frac{1}{3H_0}\frac{\partial \delta V}{\partial \phi} + f(t)$$

where the stochastic force f is distributed according to the gaussian law, i.e., has the correlation properties

$$\langle f(t) f(t') \rangle = \frac{H_0^3}{4\pi^2}\delta(t - t')$$

The corresp. **Fokker-Planck equation** $$\frac{\partial \rho(\phi, t)}{\partial t} = \frac{H_0^3}{8\pi^2}\frac{\partial^2 \rho}{\partial \phi^2} + \frac{1}{3H_0}\frac{\partial}{\partial \phi}\left(\frac{\partial V}{\partial \phi}\rho\right)$$

defines the probability distribution to have a given value of the inflaton within a given Hubble patch (to measure a given value of the cosmological constant). How to derive it?

Fokker-Planck equation: derivation

Suppose you have a stochastic process described by the Langevin equation

$\dot{\xi} = h(\xi, t) + g(\xi, t)\Gamma(t)$, where $\langle \Gamma(t) \rangle = 0$ and $\langle \Gamma(t)\Gamma(t') \rangle = 2\delta(t - t')$

The quantity $\rho(t) = \delta(\xi(t) - x)$ satisfies the equation

$$\partial\rho(t)/\partial t \equiv \dot{\rho}(t) = [A(t) + B(t)\Gamma(t)]\rho(t)$$

where

$$A(t) = -\frac{\partial}{\partial x} h(x, t), \qquad B(t) = -\frac{\partial}{\partial x} g(x, t)$$

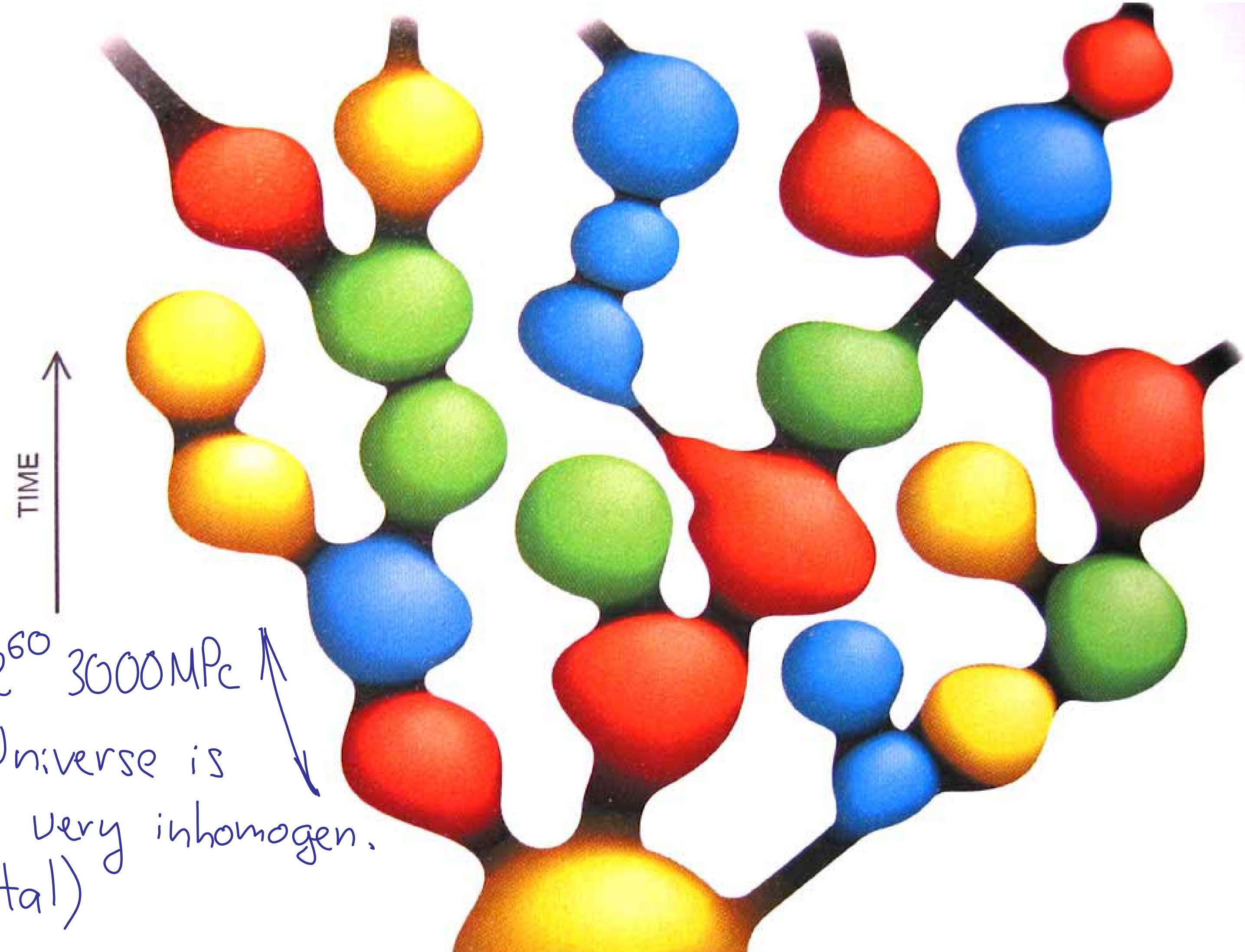
Representing this equation as integro-differential, averaging and differentiating back, we will have $\langle \dot{\rho}(t) \rangle = [A(t) + B^2(t)]\langle \rho(t) \rangle$

or

$$\dot{W} = L_{\mathrm{FP}} W,$$

$$L_{\mathrm{FP}} = A + B^2 = -\frac{\partial}{\partial x} h(x, t) + \frac{\partial}{\partial x} g(x, t) \frac{\partial}{\partial x} g(x, t)$$

Where $W(x, t) = \langle \rho(t) \rangle = \langle \delta(\xi(t) - x) \rangle$ - the distribution function



$\ell \gg e^{60}$ 3000 MPc
the Universe is
again very inhomogen.
(fractal)

SELF-REPRODUCING COSMOS appears as an extended branching of inflationary bubbles. Changes in color represent "mutations" in the laws of physics from parent universes. The properties of space in each bubble do not depend on the time when the bubble formed. In this sense, the universe as a whole may be stationary, even though the interior of each bubble is described by the big bang theory.

Solution of the Fokker-Planck equation

$$\rho = \exp\left(-\frac{4\pi^2\delta V(\phi)}{3H_0^4}\right)\sum_n c_n\psi_n(\phi)\exp\left(-\frac{E_nH_0^3(t-t_0)}{4\pi^2}\right)$$

where ψ's are eigenfunctions of the following Schrödinger equation:

$$\frac{1}{2}\frac{\partial^2\psi_n}{\partial\phi^2}+(E_n-W(\phi))\psi_n=0$$ and $$W(\phi)=\frac{8\pi^4}{9H_0^8}\left(\frac{\partial\delta V}{\partial\phi}\right)^2-\frac{2\pi^2}{3H_0^4}\frac{\partial^2\delta V}{\partial\phi^2}$$

is the "superpotential".

1. The eigenvalues E are all **positive definite** (supersymmetric form of the Hamiltonian). If the wavefunctions are normalizable, then the ground state corresponds to zero energy, i.e., is time-independent.
2. **Steady state:** contributions from higher eigenstates become negligible exponentially quickly. However, if the spectrum of the Hamiltonian is very dense, then higher eigenstates are also important at finite time scales. For example n first states are important for dynamics at $\Delta t \lesssim 1/E_n$.

The Fokker-Planck equation for chaotic inflation; the Hawking-Moss instanton

In the general case (potential of arbitrary form) it is convenient to derive the Fokker-Planck equation in the reference frame with time measured in the units of inverse H. One finds

$$\frac{\partial \rho}{\partial \log a} = \frac{G}{3\pi}\frac{\partial^2}{\partial \phi^2}(V\rho) + \frac{1}{8\pi G}\frac{\partial}{\partial \phi}\left(\frac{\partial \log V}{\partial \phi}\rho\right)$$

Its solution also exponentially rapidly approaches the steady state which is often denoted as **the Hawking-Moss instanton** (was first derived by Hawking and Moss in the context of euclidean quantum gravity):

note the sign!

$$\rho_o \sim V^{-1} \exp\left(\frac{3}{8G^2V}\right)$$

Note: the steady state which is achieved rapidly means very weak dependence on initial conditions! It look like we loose information about what happens near the singularity during eternal inflation (eternal inflation gives a kind of ”solution” to the singularity problem)

Some problems with our approach

Note that the steady state distribution function is not normalizable (normalization integral diverges at both large and small values of the inflaton):

$$\rho_o \sim V^{-1} \exp\left(\frac{3}{8G^2V}\right), \qquad N = \int_{-\infty}^{+\infty} \rho_0 d\phi = 2\int_0^{+\infty} \rho_0 d\phi$$

Take $V(\varphi) = \frac{m^2\varphi^2}{2}$ for example

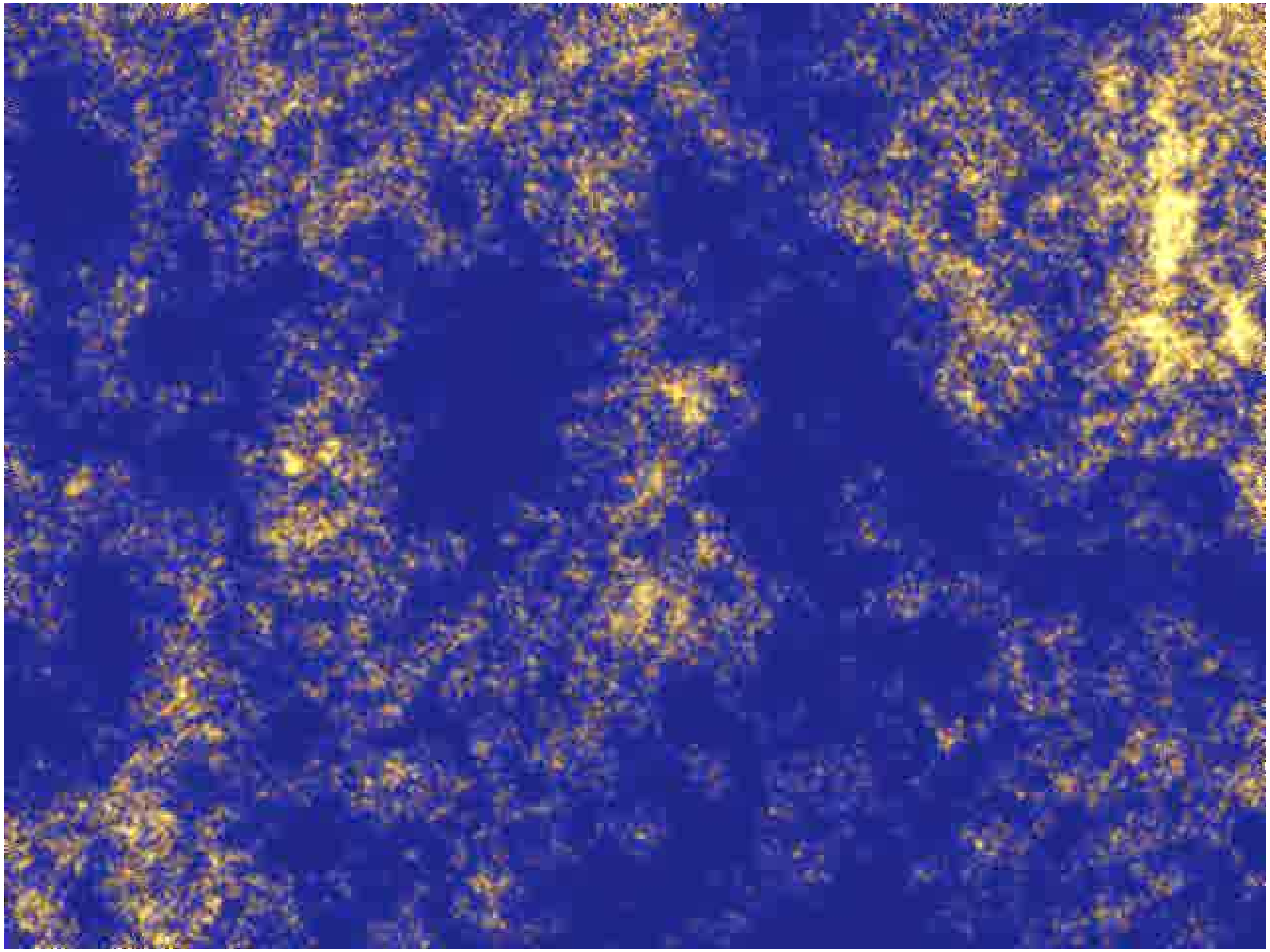
The issue of small values is solved by taking volumes of different Hubble patches into account, i.e., by changing the FP equation in the following way:

$$\frac{\partial \rho}{\partial t} = \hat{L}\rho + 3(H - \langle H \rangle)\rho$$

At large values of the inflaton energy density grows and can become Plankian (mentioned earlier). The boundary conditions are unknown.

The problem of measuring eternal inflation is currently under debates.

Fractal structure of eternally inflating spacetime



Scales are $l \gg e^{60} \times 3000$ MPc = 3×10^{26} GPc

A bit about string theory landscape

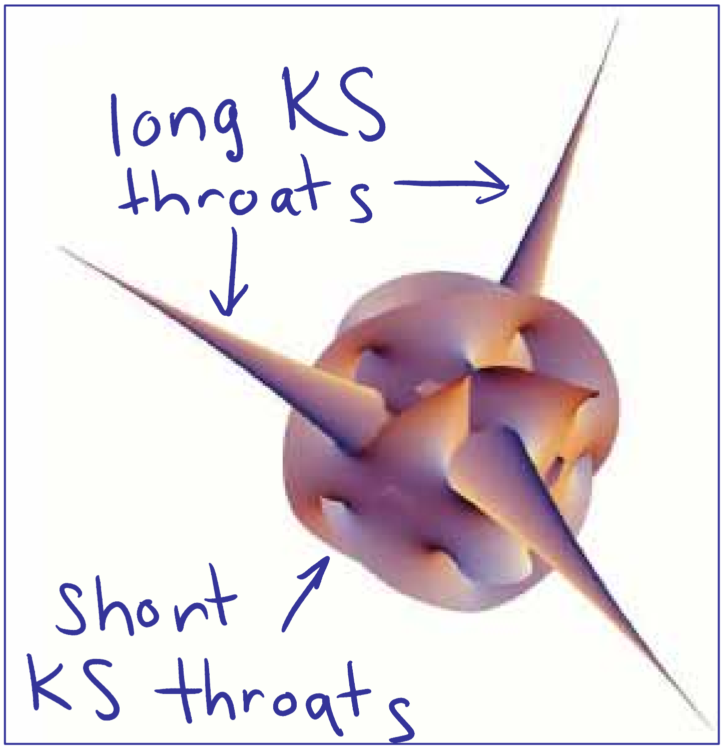
- Metastable vacua with positive effective cosmological constant (dS)
- True vacua with vanishing (Dine-Seiberg-Minkowski) or negative cosmological constant (AdS)
- The overall number is huge:

$$10^{100} \div 10^{1000}$$

- Possibility to tunnel from one vacuum to another
- AdS vacua are ”sinks” – the bubbles of collapsing spacetime

Already the statistical problem of counting vacua on the landscape (or calculating distribution functions of vacua) is very complicated (NP hard). However, we want more than that - to understand the dynamics of fields on the landscape, and in particular - how eternal inflation is realized in this setup.

Tunneling between vacua on the landscape 1



One relatively simple geometric realization of the landscape: mutithroat scenario (warped AdS geometry inside each KS throat). Main players are D3- and anti D3-branes, fluxes.

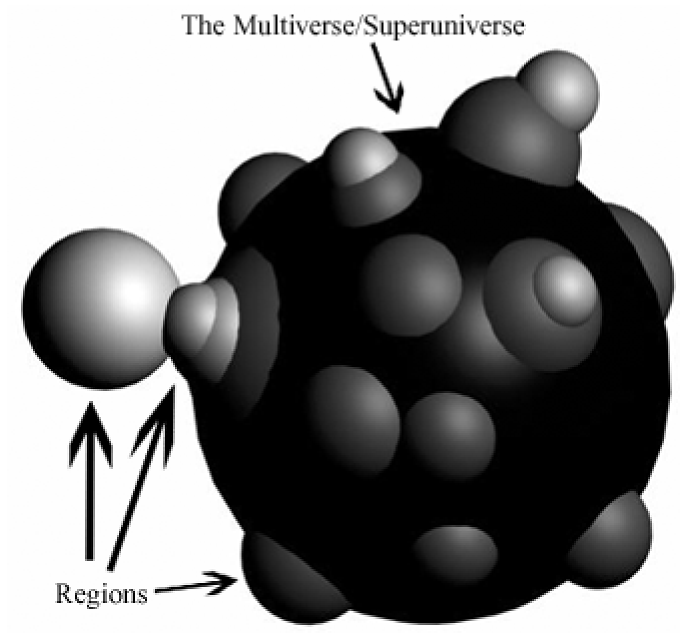
Classifying vacua:

1) Inside each throat: could be different number of fluxes; effective cosmological constant is related to the number of fluxes by

$$\Lambda = -\Lambda_0 + \frac{1}{2}\sum_{i=1}^{J} q_i^2 n_i^2$$

Polchinski-Bousso transitions between them: Coleman - di Luccia instantons, bubbles of new vacua (old inflation)

(Question: why this ”old inflation” can be eternal?)



1. False vacua are metastable and exponetially decay
2. Volumes of Hubble patches with false vacua expand exponentially rapidly
3. Rate of expansion >> rate of decay → volumes of false vacua increase with time

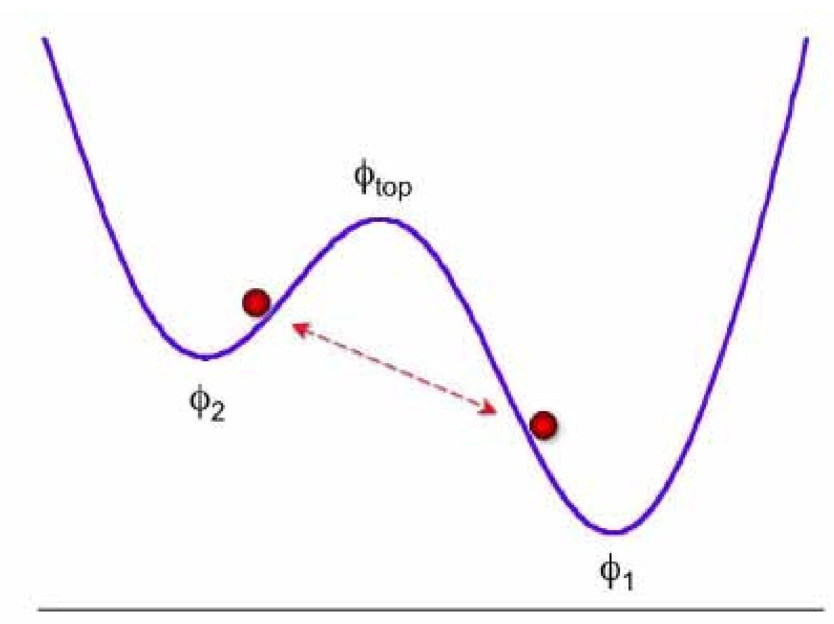
Tunneling between vacua on the landscape 2

2) Tunneling between different throats. Model it by a single inflaton field. The tunneling rate between two ajacent vacua is defined by the action on the corresponding Hawking-Moss instanton:

$$\Gamma_{12} = e^{-S_{\rm top}+S_1} = \exp\left(-\frac{24\pi^2}{V(\phi_1)} + \frac{24\pi^2}{V(\phi_{\rm top})}\right)$$

Both tunnelings 1→2 and 2→1 are possible; for the system of two minima one has

$$\frac{P_2}{P_1} = e^{-S_2+S_1} = \exp\left(-\frac{24\pi^2}{V_1} + \frac{24\pi^2}{V_2}\right)$$



For an arbitrary number of dS vacua (and AdS sinks) one has the "vacuum dynamics" equations:

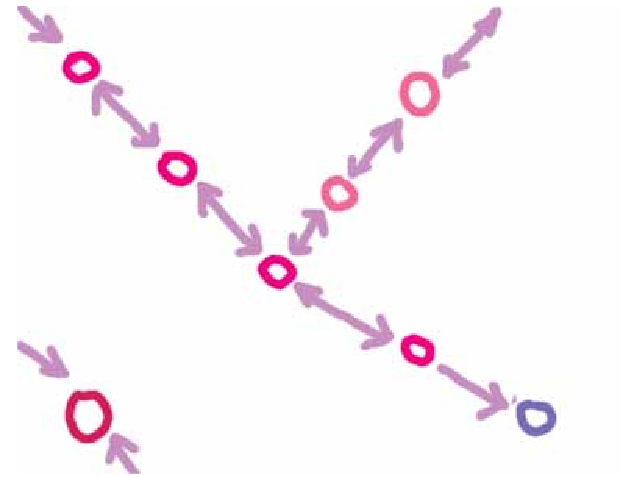
$$\frac{dP_i}{dt} = -\sum_{j\neq i}\Gamma_{ij}P_i + \sum_{j\neq i}\Gamma_{ji}P_j - \Gamma_{is}P_i$$

Tunneling between vacua on the landscape 3

But the number of equations is $10^{100} \div 10^{1000}$!
If we are interested in time scales t such that

$$t_{\text{AdS}} \gg t \gg t_{\tau} \gg t_{\text{tunnel}}$$

then physical answers are given by the average over disorder on the landscape.
Classifying parts of the landscape according to Hausdorff dimension of the corresponding tunneling graph: quasi 1 dim, quasi 2 dim, etc.



Quasi-one-dimensional: two nearest neigbors (neglecting AdS sinks)

$$\partial_t P_i = -\Gamma_{i,i+1} P_i + \Gamma_{i+1,i} P_{i+1} - \Gamma_{i,i-1} P_i + \Gamma_{i-1,i} P_{i-1}$$

Let us supply this system of equations with delta-function-like initial conditions:

$$P_i(0) = 1, \;\; P_{j \neq i}(0) = 0$$

Then, old result from the theory of diffusion on random lattices says that

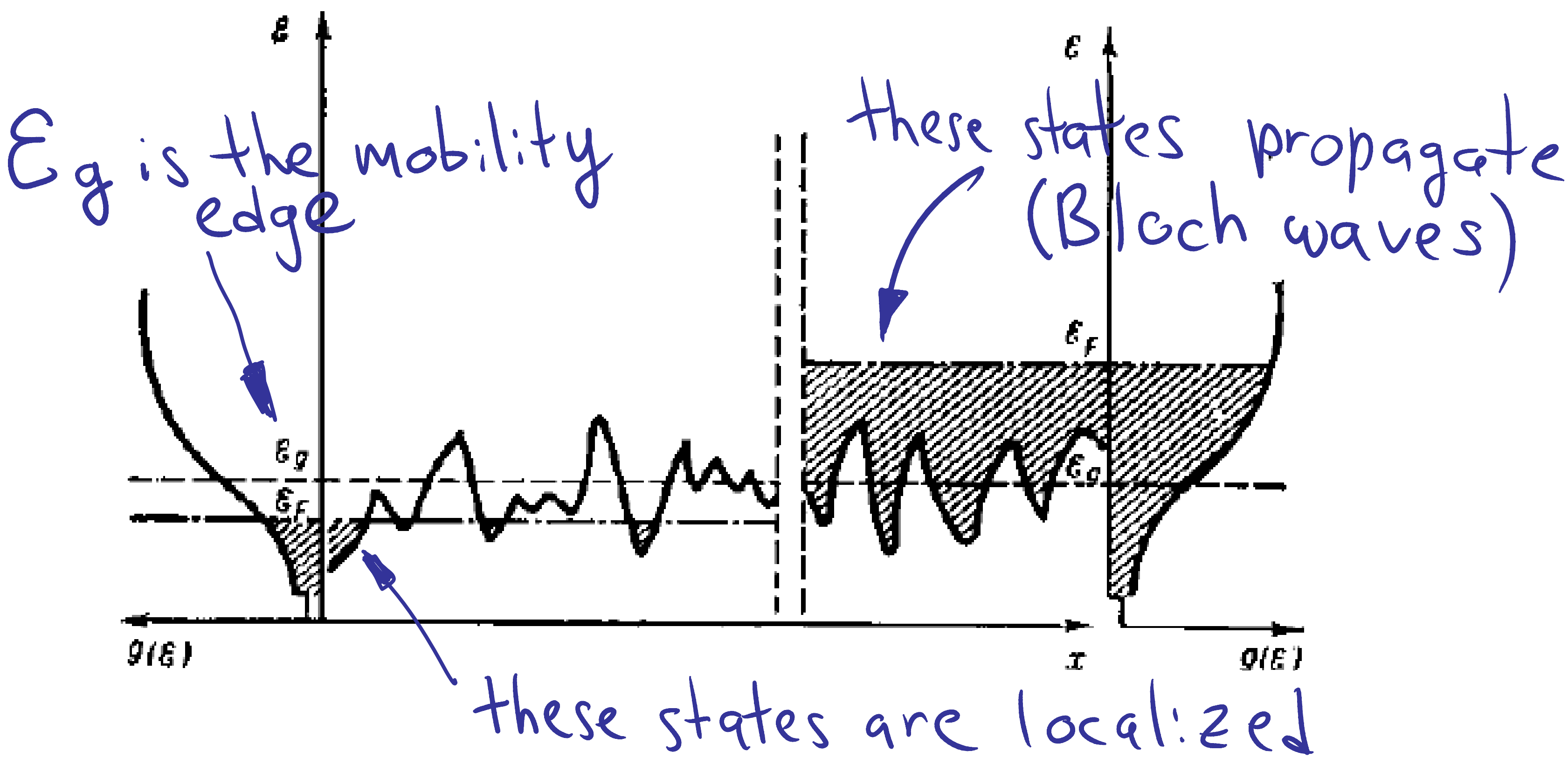
the probability distribution spreads out with time much slower then diffusively:

$$\langle n^2(t) \rangle \sim \log^4 t$$

Localization in disordered quantum systems 1

The motion of carriers (electrons) in random potential of impurities is governed by Schrödinger equation with the potential

$\langle u(r)u(r')\rangle = \frac{1}{2\nu\tau}\delta(r-r'), \langle u(r)\rangle = 0$. Mean free time: $\tau \sim (na_0^2)^{-1}$



Localization in disordered quantum systems 2

If wave package of electron in ultra-pure metallic wire is located near the origin in the initial moment of time, then the probability density will spread out as

$\langle R^2(t) \rangle \sim t$ (usual diffusive behavior of the width of the probability density)

In the case of Anderson localization (metal with impurities) one has

$\rho(R) \sim \exp(-R/L)$ where L is the localization length (the same order of magnitude as the mean free path)

This happens in 1 dimension for **arbitrarily weak disorder.**

The reason for anomalous diffusion of the distribution function for eternal inflation is Anderson localization in the dual quantum problem: wave functions ψ behave as

$$\psi_n(\phi) \sim \exp\left(-\frac{|\phi - \phi_0|}{L}\right)$$

Anderson localization on the landscape

1. Quasi-one-dimensional islands: Strong dependence of eternal inflation history on initial conditions due to the effect analogous to the Anderson localization. Weak logarithmic spreading of the distribution function:

$$\langle \phi^2(t) \rangle \sim \log^4 t$$

2. Quasi-two-dimensional islands: all states are localized but the localization length grows exponentially with energy leading to subdominant log corrections to the linear diffusion law:

$$\langle \phi^2(t) \rangle \sim t \left(1 + c_1 \frac{1}{\log^{\alpha} t} + \cdots \right)$$

Recalling initial conditions at later times.

3. Quasi-higher-dimensional islands: edge of mobility. Recalling initial conditions at later times:

$$\Delta t \gg E_g^{-1}$$