Hidden Symmetries and Separation of Variables in Higher Dimensional Black Holes

Valeri P. Frolov

University of Alberta

Dubna, July 23-29, 2007

Outline of Lectures

Lecture I

I. Motivations and main results

II. Why large extra dimensions?

Gravity in higher dimensions; Hierarchy problem; Brane worlds; Black holes as probes of extra dimensions; Black hole Zoo in HDs

III. Mathematical Introduction

Symmetries, hidden symmetries, and separation of variables; Killing and Killing-Yano tensors and their properties; Petrov types

Lecture II

VI. 4D rotating black holes

Miraculous properties of the Kerr metric

Debever-Carter form of 4D Kerr-NUT-(A)dS metrics

Hidden symmetries and algebraic properties

V. Higher-dimensional rotating black hole

Higher dimensional Kerr-NUT-(A)dS metrics

Principal Killing-Yano tensor

Complete integrability of particle equations of motion

Separation of variables in the Hamilton-Jacobi and Klein-Gordon equations

VI. Open problems

Based on:

V. F., D.Kubiznak, "Hidden symmetries of higher-dimensional rotating black holes". Phys.Rev.Lett.98:011101,2007; gr-qc/0605058

D. Kubiznak, V. F., "Hidden Symmetry of Higher Dimensional Kerr-NUT-AdS Spacetimes". Class.Quant.Grav.24:F1-F6,2007; gr-qc/0610144

D. N. Page, D. Kubiznak, M. Vasudevan, P. Krtous, "Complete integrability of geodesic motion in general Kerr-NUT-AdS spacetimes". Phys.Rev.Lett.2007; hep-th/0611083

P. Krtous, D. Kubiznak, D. N. Page, V. F., "Killing-Yano Tensors, Rank-2 Killing Tensors, and Conserved Quantities in Higher Dimensions". JHEP 0702:004,2007; hep-th/0612029

P. Krtous, D. Kubiznak, D. N. Page, M. Vasudevan “Constants of geodesic motion in higher-dimensional black-hole spacetimes”. e-Print: arXiv:0707.0001

V. F., P. Krtous , D. Kubiznak , "Separability of Hamilton-Jacobi and Klein-Gordon Equations in General Kerr-NUT-AdS Spacetimes", JHEP 0702:005,2007; hep-th/0611245

`Alberta Separatists’

Lecture I

Motivations

Separation of variables allows one to reduce a physical problem to a simpler one in which physical quantities depend on less number of variables. In case of complete separability original partial differential equations reduce to a set of ordinary differential equations

Separation of variables in the Kerr metric is used for study:

(1) Black hole stability

(2) Particle and field propagation

(3) Quasinormal modes

(4) Hawking radiation

Main Results

Rotating black holes in higher dimensions in many aspects are very similar to the 4D Kerr black holes

In the most general $(Kerr-NUT-(A)dS)$ *higher-dimensional black hole spacetime*:

(1) *Geodesic motion is completely integrable*.

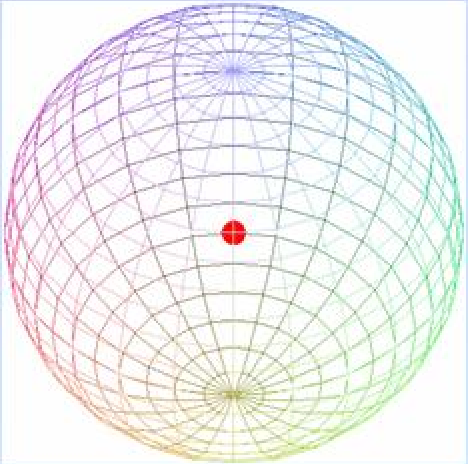
(2) *Hamilton-Jacobi and Klein-Gordon equations allow the complete separation of variables*

Why Larger Extra Dimensions?

From Fomenko collection

Gravity in Higher Dim. Spacetime

$$\Delta^{(4+k)}\Phi = -a_k G^{(k)} M \delta^{3+k}(\vec{x})$$



$$\vec{F} = -\nabla\Phi$$

$$F \sim \frac{G^{(k)}M}{r^{2+k}}$$

No bounded orbits (prove!)

Gravity at small scales is stronger than in 4D

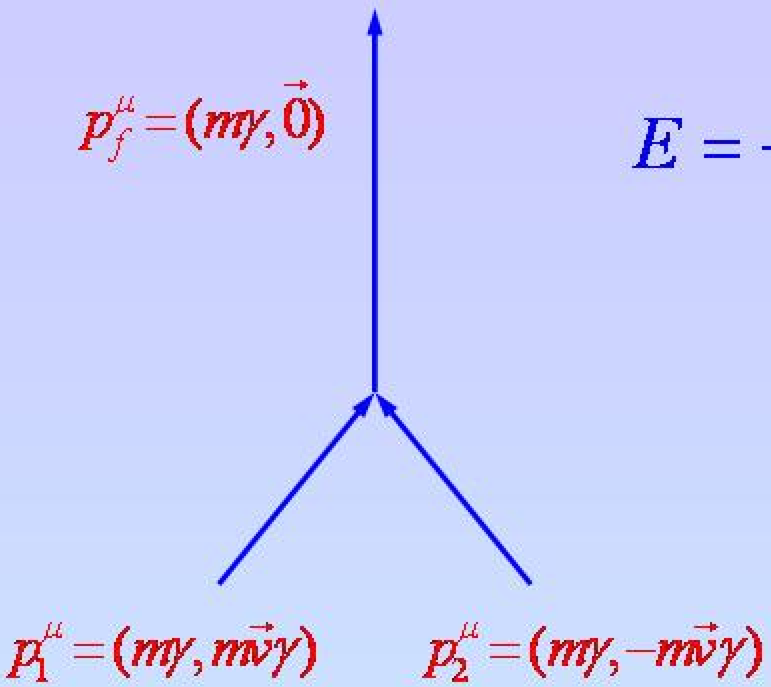
`Running coupling constant'

$$G(r) = \frac{G^{(k)}}{r^k}, \quad G = G(l), \quad G(r) = G\frac{l^k}{r^k}$$

$$\frac{G(r_*)m_p^2}{r_*^2} = \frac{e^2}{r_*^2} \quad \Rightarrow \quad r_* = \left(\frac{Gm_p^2}{e^2}\right)^{1/k} l$$

$$\left(\frac{Gm_p^2}{e^2}\right) \sim 10^{-36}$$ `Hierarchy problem'

$$For\ l \sim 0.01\,cm\ and\ k = 2 \quad r_* \sim 10^{-20}\,cm$$



$$E = -p_1^{\mu} p_{2\mu} / m, \quad E_* = 2m\gamma, \quad E_* \sim (Em)^{1/2}$$

$$For\ E \sim \hbar / r_* \quad E_* \sim 1\,TeV$$

New fundamental scale

$$M_* = M_{Pl} (l_{Pl} / l)^{\frac{k}{k+2}},$$

$$For\ k = 2\ and\ l \sim 0.01cm \quad M_* \sim 1\,TeV$$

4D Newton law is confirmed for r>l.

Q.: How to make gravity strong (HD) at small scales without modifying it at large scales?

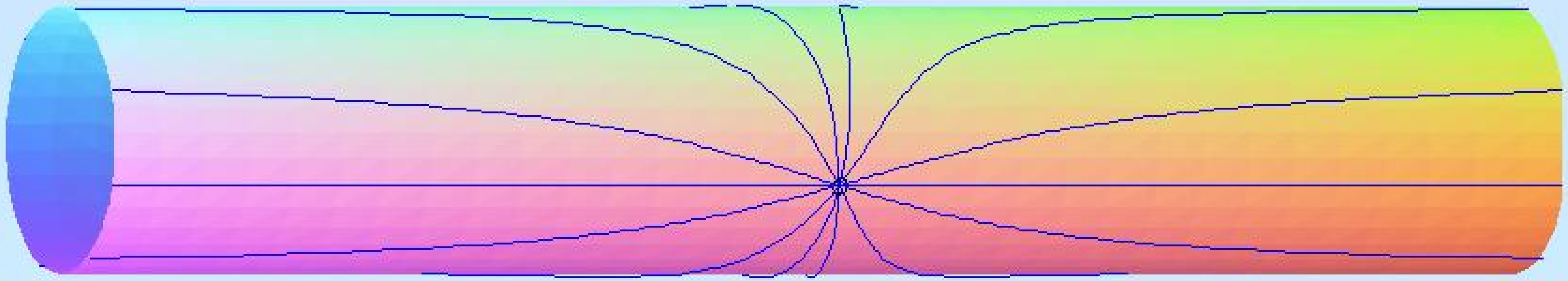
A.: Compactification of extra dimensions

Gravity in ST with Compact Dims

Example: $\Delta^{(4)}\Phi = 0, \quad \mathrm{M}^4 = R^3 \times S^1$

$$\Phi \sim G^* M \sum_{n=-\infty}^{\infty} \frac{1}{\vec{r}^{\,2} + (z + nl)^2} = \frac{G^* M \pi}{lr} \frac{\sinh(\pi r / l)\cosh(\pi r / l)}{\cosh^2(\pi r / l) - \cos^2(\pi z / l)}$$

$$\Phi(\vec{r}, z = 0) \sim \frac{G^* M \pi}{lr} \coth(\pi r / l), \quad G = G^* \pi / l$$



Kaluza-Klein tower

$$\Phi(x^\mu, y) = \Phi(x^\mu, y + 2\pi L) \quad \Rightarrow \quad \Phi(x^\mu, y) = \sum_n \Phi_n(x^\mu)\, e^{iny/L}$$

$$W = -\tfrac{1}{2} \int dx^4\, dy\, [\, \nabla\Phi\nabla\Phi^* + m_0^2\, \Phi\Phi^*]$$

$$W = -\tfrac{1}{2} \sum_n \int dx^4\, [\, \nabla_\mu \Phi_n \nabla^\mu \Phi^* + (m_0^2 + \frac{n^2}{L^2}) \Phi\Phi^*]$$

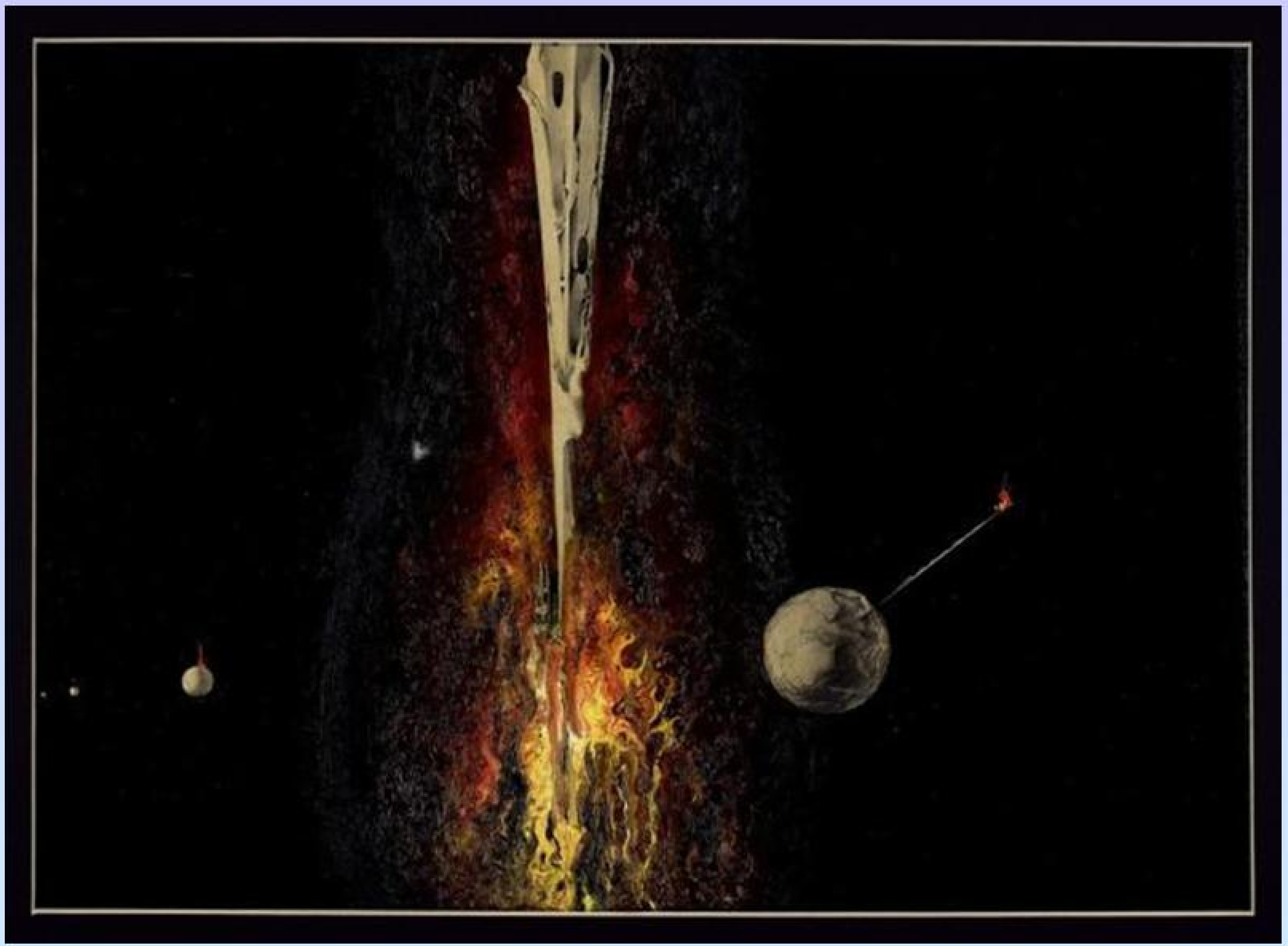
KK tower with the mass spectrum $m^2 = m_0^2 + n^2 / L^2$

Brane World Paradigm

Bosons, fermions and gauge fields are localized within the 4D brane

Gravity is not localized and `lives' in (4+k)-D bulk space

Fundamental scale of order of TeV. Large extra dimensions generate Planckian scales in 4D space



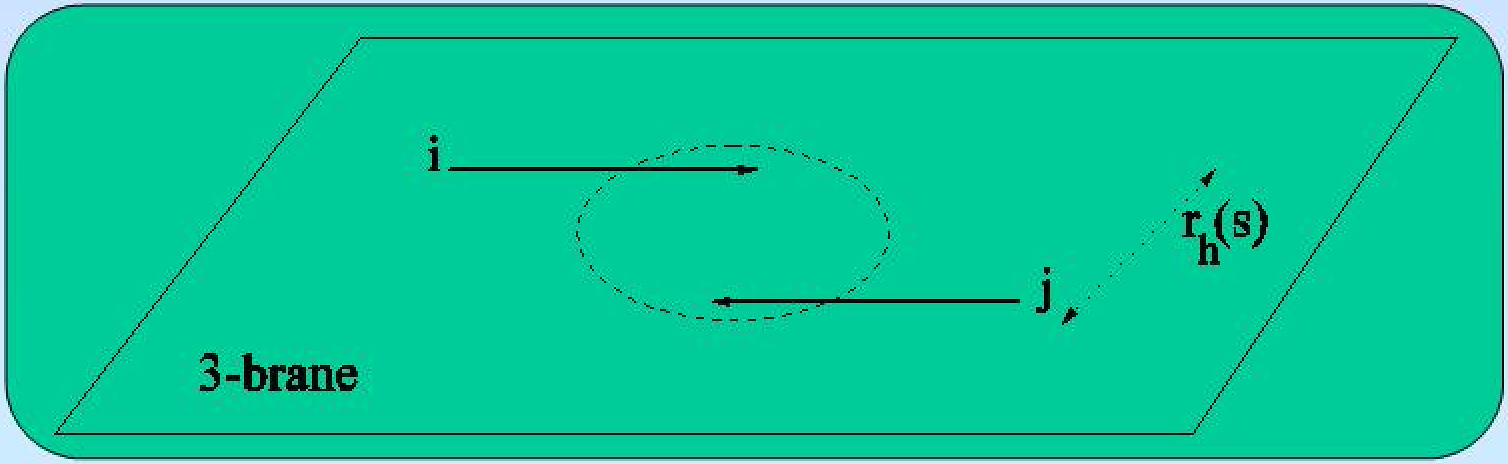
From Fomenko collection

Black Holes as Probes of Extra Dims

We consider first black holes in the mass range

$$10^{-21} g \ll M \ll 10^{27} g$$

Mini BHs creation in colliders



BH formation

Bolding Phase

Thermal (Hawking) decay

Large Hadronic Collider (LHC)

The collider is contained in a 27 km circumference tunnel located underground at a depth ranging from 50 to 175 m. The tunnel was formerly used to house the LEP, an electron-positron collider.

The three metre diameter, concrete-lined tunnel actually crosses the border between Switzerland and France at four points, although the majority of its length is inside France.

The two beams travel in opposite directions around the ring. The protons will each have an energy of 7 TeV, giving a total collision energy of 14 TeV. It will take around ninety microseconds for an individual proton to travel once around the collider.

Rather than continuous beams, the protons will be "bunched" together into approximately 2,800 bunches, so that interactions between the two beams will take place at discrete intervals never shorter than twenty-five nanoseconds apart.

CERN 2003–001
28 February 2003
Theoretical Physics
Division

ORGANISATION EUROPÉENNE POUR LA RECHERCHE NUCLÉAIRE
CERN EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

STUDY OF POTENTIALLY DANGEROUS EVENTS DURING HEAVY-ION COLLISIONS AT THE LHC: REPORT OF THE LHC SAFETY STUDY GROUP

J.-P. Blaizot
CEA/Saclay-Orme des Merisiers, Gif-sur-Yvette, France

J. Iliopoulos
École Normale Supérieure, Paris, France

J. Madsen,
University of Aarhus, Århus, Denmark

G.G. Ross,
University of Oxford, Oxford, UK

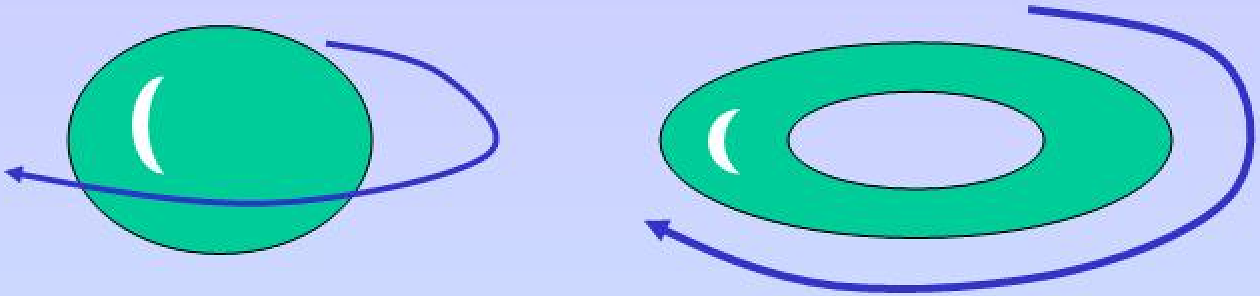
P. Sonderegger,
CERN, Geneva, Switzerland

H.-J. Specht,
University of Heidelberg, Heidelberg, Germany

GENEVA
2003

"We find no basis for any conceivable threat." If black holes are produced, they are expected to evaporate almost immediately via Hawking radiation and thus be harmless, although the existence of Hawking radiation is currently unconfirmed. Perhaps the strongest argument for the safety of colliders such as the LHC comes from the simple fact that cosmic rays with energies up to twenty million times the LHC's 1.4×10^{13} eV capacity have been bombarding the Earth, Moon and other objects in the solar system for billions of years with no such effects.

Black Objects in Higher Dimensions



→ Lack of uniqueness in higher dimensions

Black strings

Black rings

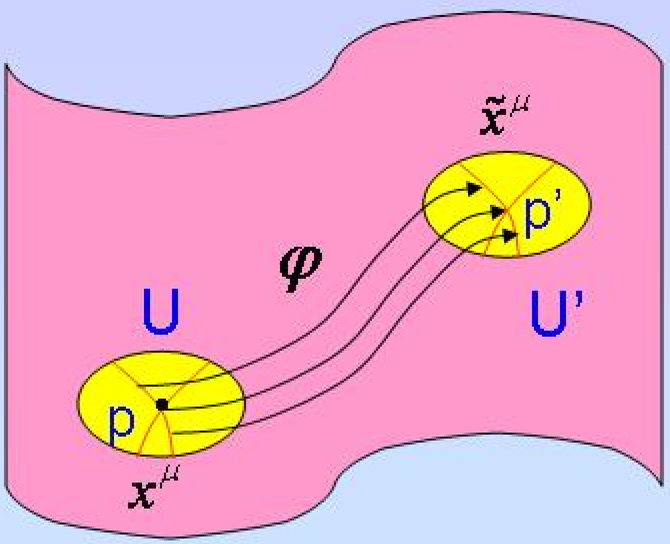
Black Saturns

Technical tools are required for study HD rotating black holes

Integrals of motion and separability in a ST of HD black holes

Mathematical Introduction:
Symmetries, Hidden Symmetries,
and Petrov types

Spacetime Symmetry



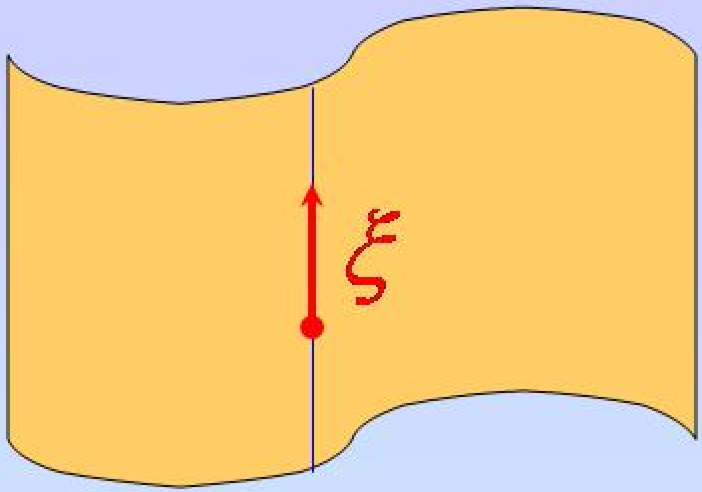
$$Diffeo: \quad \varphi(p) = p'$$

φ *is a symmetry tranformation if the metric* $g(p')$ *in* $\tilde{x}^{\mu}$ *coordinates has the same components as* $g(p)$ *in* x^{μ} *coordinates*

$$\varphi : x^{\mu} \to x^{\mu} + \xi^{\mu}\, \delta t$$

ξ^{μ} *is a generator of a symmetry transformation if* $\mathcal{L}_{\xi} g_{\mu\nu} = 0$

Spacetime Symmetries



$$\mathcal{L}_\xi g_{\mu\nu} = 0$$

$$\mathcal{L}_\xi g_{\mu\nu} = \alpha g_{\mu\nu}$$

$$\xi_{(\mu;\nu)} = 0 \quad \text{(Killing equation)}$$

$$\xi_{(\mu;\nu)} = \tilde{\xi}\, g_{\mu\nu} \quad \text{(conformal Killing eq)}$$

$$\tilde{\xi} \equiv D^{-1} \xi^{\nu}_{\ ;\nu}$$ D is # of ST dimensions

Symmetries and Integrals of Motion

$$u^\nu u_{\mu;\nu} = 0 \quad \text{(geodesic motion)}$$

$$\varepsilon = u^\mu u_\mu = -1 \text{ or } 0$$

$$u^\nu (\xi_\mu u^\mu)_{;\nu} = u^\nu u^\mu \xi_{(\mu;\nu)} + u^\nu u^\mu{}_{;\nu} \xi_\mu = \varepsilon\, \tilde{\xi}$$

Symmetries and Separation of Variables

If ξ is a Killing vector then there exist such coordinates (t, x^i) in which $\xi = \xi^\mu \partial_\mu = \partial_t$ and $\partial_t g_{\mu\nu} = 0$.

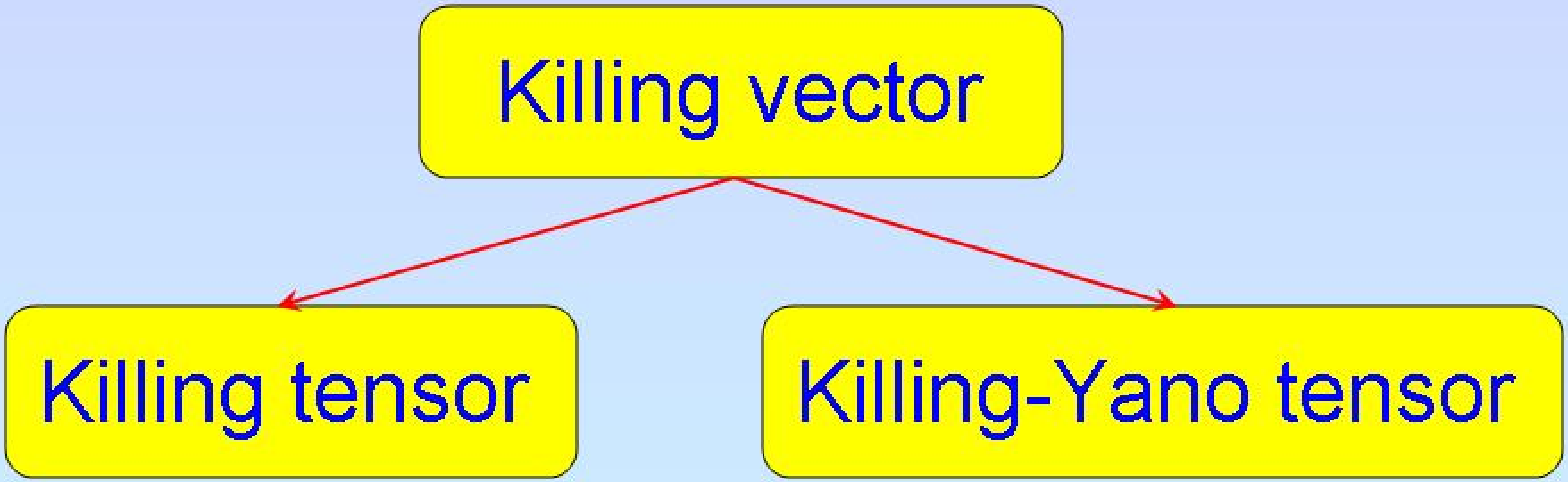
In this coordinates the equation $\Box\Phi = 0$ and other similar equations allow separation of variables $\Phi = T(t)\, R(x^i)$

For large enough symmetry group the field equations allow a complete separation when the partial DE are reduces to a set of ordinary DE

It happens, e.g., if there exist D commuting Killing vectors

Hidden Symmetries

$$\xi_{(\mu;\nu)} = g_{\mu\nu}\,\tilde{\xi}$$



Symmetric generalization

CK=Conformal Killing tensor

$$K_{\mu_1\mu_2\ldots\mu_n} = K_{(\mu_1\mu_2\ldots\mu_n)}, \quad \tilde{K}_{\mu_2\ldots\mu_n} \sim \nabla^{\mu_1} K_{\mu_1\mu_2\ldots\mu_n}$$

$$K_{(\mu_1\mu_2\ldots\mu_n;\nu)} = g_{\nu(\mu_1}\tilde{K}_{\mu_2\mu_3\ldots\mu_n)}$$

Integral of motion

$$u^\nu (K_{\mu_1\mu_2\ldots\mu_n} u^{\mu_1} u^{\mu_2} \ldots u^{\mu_n})_{;\nu} = \varepsilon\, \tilde{K}_{\mu_1\mu_2\ldots\mu_{n-1}} u^{\mu_1} u^{\mu_2} \ldots u^{\mu_{n-1}}$$

Antisymmetric generalization

CY=Conformal Killing-Yano tensor

$$k_{\mu_1\mu_2\ldots\mu_n} = k_{[\mu_1\mu_2\ldots\mu_n]}, \quad \tilde{k}_{\mu_2\ldots\mu_n} \sim \nabla^{\mu_1} k_{\mu_1\mu_2\ldots\mu_n}$$

$$\nabla_{(\mu_1} k_{\mu_2)\mu_3\ldots\mu_{n+1}} = g_{\mu_1\mu_2}\tilde{k}_{\mu_3\ldots\mu_{n+1}} - (n-1) g_{[\mu_3(\mu_1}\tilde{k}_{\mu_2)\ldots\mu_{n+1}]}$$

If the rhs vanishes f=k is a Yano tensor

$K_{\mu\nu} = f_{\mu\mu_2\ldots\mu_n} f_\nu{}^{\mu_2\ldots\mu_n}$ is the Killing tensor

Proof:

$$K_{(\mu\nu;\lambda)} = (-1)^{n-1}[f_{\mu_2\ldots\mu_n(\mu;\lambda} f_{\nu)}{}^{\mu_2\ldots\mu_n} + f_{\mu_2\ldots\mu_n(\nu;\lambda} f_{\mu)}{}^{\mu_2\ldots\mu_n}] = 0$$

If $f_{\mu_1\ldots\mu_n}$ is a Yano tensor then for a geodesic motion the tensor $p_{\mu_1\ldots\mu_{n-1}} = f_{\mu_1\ldots\mu_n} u^{\mu_n}$ is parallelly propagated along a geodesic.

Proof:

$$u^\lambda p_{\mu_1\ldots\mu_{n-1};\lambda} = f_{\mu_1\ldots(\mu_n;\lambda)} u^{\mu_n} u^\lambda + f_{\mu_1\ldots\mu_n} u^{\mu_n}{}_{;\lambda} u^\lambda = 0$$

Differential forms

$$\omega \quad \Leftrightarrow \quad \omega_{[\mu_1 \dots \mu_p]}$$

$$\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p \quad \Leftrightarrow \quad \alpha_{[\mu_1 \dots \mu_p} \beta_{\mu_{p+1} \dots \mu_{p+q}]}$$

$$d\omega \quad \Leftrightarrow \quad \omega_{[\mu_1 \dots \mu_p, \nu]} = \omega_{[\mu_1 \dots \mu_p ; \nu]}$$

$$X \lrcorner \omega \quad \Leftrightarrow \quad X^{\mu_1} \omega_{[\mu_1 \dots \mu_p]}$$

$$* \omega_p \quad \Leftrightarrow \quad e_{\mu_1 \dots \mu_{D-p}}{}^{\mu_{D-p+1} \dots \mu_D} \omega_{\mu_{D-p+1} \dots \mu_D}$$

$$\delta\omega = (-1)^p \varepsilon_p * d * \omega \quad \Leftrightarrow \quad \omega_{\mu_1 \dots \mu_p}{}^{;\mu_p}$$

$$e_{\mu_1 \dots \mu_D} = \sqrt{|\det(g)|}\, \varepsilon_{\mu_1 \dots \mu_D}$$

$$** \omega_p = \varepsilon_p \omega_p , \quad \varepsilon_p = (-1)^{p(D-p)} \varepsilon_g , \quad \varepsilon_g = \frac{\det(g)}{|\det(g)|}$$

CY Equation

$$\nabla_X \omega = \frac{1}{p+1} X \lrcorner d\omega - \frac{1}{D-p+1} X^\vee \wedge \delta\omega$$

$$X \lrcorner {}^*\omega = {}^*(\omega \wedge X^\vee)$$

$$\nabla_X ({}^*\omega) = \frac{1}{p_*+1} X \lrcorner d({}^*\omega) - \frac{1}{D-p_*+1} X^\vee \wedge \delta({}^*\omega)$$

Dual to a CY is again a CY

Dual to a CCY is a Y

4 Important Properties of CY

(1) $If\ \omega\ is\ a\ CY,\ then\ *\omega\ is\ again\ a\ CY$

(2) $If\ \omega\ is\ a\ CY\ and\ d\omega = 0\ then$

$*\omega\ is\ a\ Yano\ tensor$

(3) $If\ d\omega_1 = 0\ and\ d\omega_2 = 0\ then\ d(\omega_1 \wedge \omega_2) = 0$

(4) $If\ \omega_1\ and\ \omega_2\ are\ CYs\ then\ \omega = \omega_1 \wedge \omega_2$

$is\ a\ CY\ and\ d(\omega) = 0$

(4) *implies that for any 2 given Yano tensors* Y_1 *and* Y_2 *it is always possible to construct a new Yano tensor* Y_3

Proof of (4):

Step 1:

$$If\ \nabla_X \omega = X^\vee \wedge \Omega \quad then \quad \Omega = -\frac{1}{D-p+1}\delta\omega$$

Step 2:

$$If\ \alpha\ and\ \beta\ are\ CYs\ and\ d\alpha = d\beta = 0\ then\ d(\alpha \wedge \beta) = 0$$

$$and \quad \nabla_X(\alpha \wedge \beta) = X^\vee \wedge \Omega$$

Petrov-Pirani Classification

$$[\alpha\beta] \leftrightarrow A, \quad [\gamma\delta] \leftrightarrow B$$

$$C_{[\alpha\beta][\gamma\delta]} \leftrightarrow C_{AB} = C_{(AB)}$$

$$g_{\alpha\gamma} g_{\beta\delta} - g_{\beta\gamma} g_{\alpha\delta} \leftrightarrow g_{AB} = g_{(AB)}$$

$$\varepsilon_{\alpha\beta\gamma\delta} \leftrightarrow \varepsilon_{AB} = \varepsilon_{(AB)}$$

$$g^{AB} C_{AB} = 0, \quad \varepsilon^{AB} C_{AB} = 0 \; (in\ 4D)$$

$$C_A^B X_B = \lambda X_A$$

In 4D there exist not more than 4 diff. eigenvalues

Bel Criteria

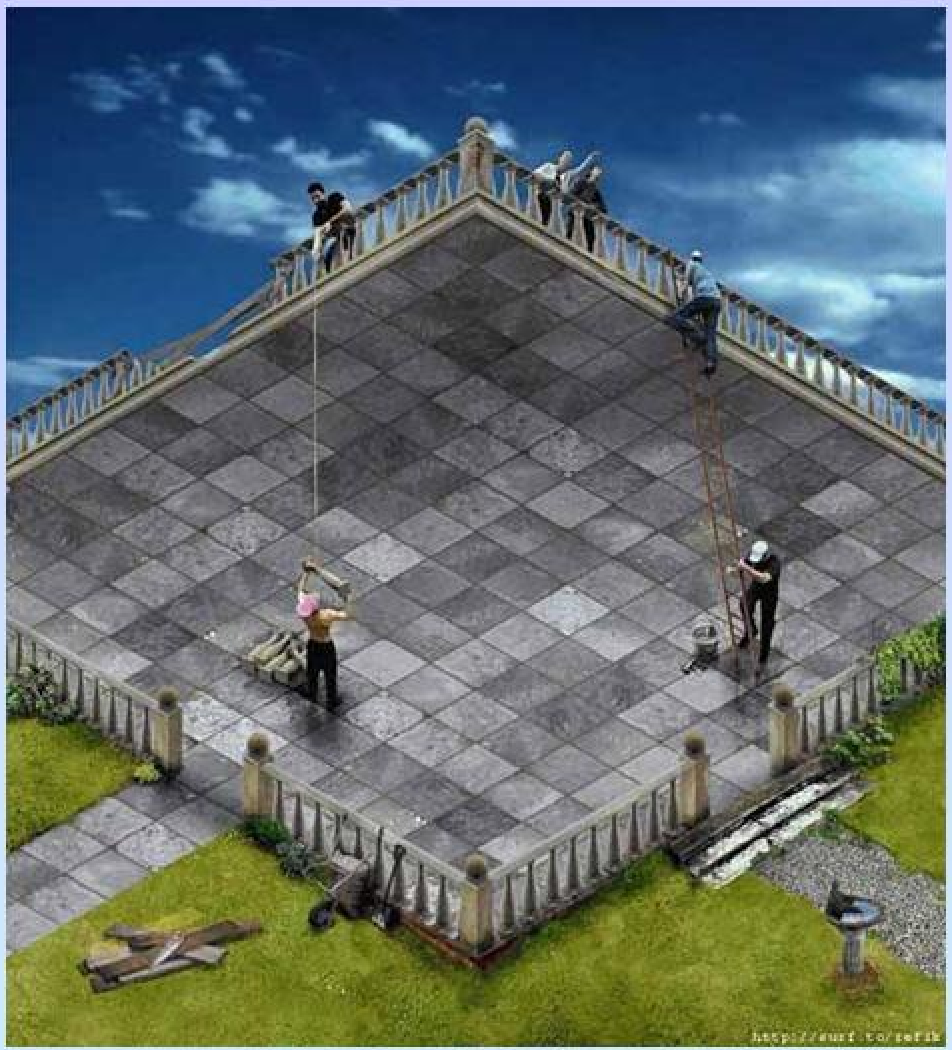
$Type\ N: \quad C_{\alpha\beta\gamma\delta}k^{\delta} = 0 \rightarrow k\ is\ null\ and\ unique\ (up\ to\ rescaling)$

$Type\ III: \quad C_{\alpha\beta\gamma\delta}k^{\beta}k^{\delta} = 0 \rightarrow k\ is\ null\ and\ unique\ (up\ to\ rescaling)$

$Type\ II: \quad C_{\alpha\beta\gamma\delta}k^{\beta}k^{\delta} = ak_{\alpha}k_{\gamma},\ {}^{*}C_{\alpha\beta\gamma\delta}k^{\beta}k^{\delta} = bk_{\alpha}k_{\gamma},\ ab \neq 0 \rightarrow k\ is\ null$

Type D: *There are 2 independent null vectors obeying the same equations as for Type II with different values of a and b*

Lecture II



Rotating Black Holes in 4D

`Miraculous' properties of Kerr metric

Hamilton-Jacobi and scalar field equations allow separation of variables [Carter, 1968]

Higher spin massless field equations can be decoupled and the decoupled equations allow a separation of variables [Teukolsky, 1973]

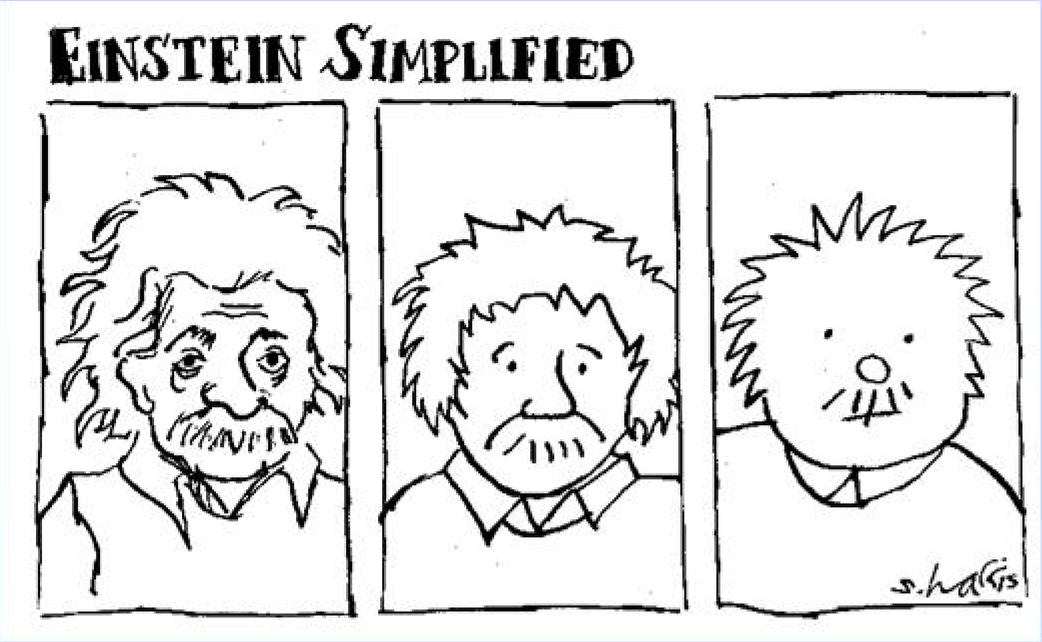
Stationary string equations allow a separation of variables [Carter, Frolov, 1989]

Spacetime symmetries generated by Killing vectors are not sufficient to explain separability

Kerr spacetime possesses additional `hidden' symmetries [Carter, 1968]

Hidden symmetries are responsible for the `miracles'

Separability is a property which takes place in a special system of coordinates



4D Kerr-NUT-(A)dS for dummies

4D Kerr-NUT-(A)dS Derivation in 3 Simple Steps

Step 1: Write flat ST metric in ellipsoidal coordinates

$$dS^2 = -dt^2 + dX^2 + dY^2 + dZ^2$$

$$X = \sqrt{r^2 + a^2}\,\sin\theta\cos\phi, \quad Y = \sqrt{r^2 + a^2}\,\sin\theta\sin\phi,$$

$$Z = r\cos\theta, \quad \frac{X^2 + Y^2}{r^2 + a^2} + \frac{Z^2}{r^2} = 1$$

$$dS^2 = -dt^2 + (r^2 + a^2\cos^2\theta)\,(\frac{dr^2}{r^2 + a^2} + d\theta^2) +$$

$$+ (r^2 + a^2)\sin^2\theta\, d\phi^2$$

Step 2: Rewrite this metric in `algebraic’ form

$$y = a\cos\theta, \qquad \tau = t - a\phi, \quad \psi = a^{-1}\phi$$

$$ds^2 = -\frac{X}{r^2+y^2}(d\tau + y^2 d\psi)^2 + \frac{Y}{r^2+y^2}(d\tau - r^2 d\psi)^2$$

$$+\frac{(r^2+y^2)\,dr^2}{X} + \frac{(r^2+y^2)\,dy^2}{Y}$$

$$X = r^2 + a^2, \quad Y = a^2 - y^2$$

(i) Coefficients are rational functions;
(ii) `Almost symmetric’ form

Step 3: Carter-Plebanski form of Kerr-NUT-(A)dS metric

$$X \to (r^2 + a^2)(1 + \lambda r^2) - 2Mr,$$

$$Y \to (a^2 - y^2)(1 - \lambda y^2) + 2Ny$$

$$R_{\mu\nu} = -3\lambda g_{\mu\nu}$$

$$\partial_r^2 X + \partial_y^2 Y = 12\lambda(r^2 + y^2)$$

a – rotation parameter, M – mass,

N – `NUT' parameter, $\lambda$ – `cosmological term'

[Carter, Comm. Math. Phys, 10, 280, (1968); Plebanski, Ann. Phys. 90, 196 (1975)]

Symmetric form of the metric

$$r = ip, \quad M = iN_p, \quad N = N_y$$

$$ds^2 = \frac{P}{p^2 - y^2}(d\tau + y^2 d\psi)^2 + \frac{Y}{y^2 - p^2}(d\tau + p^2 d\psi)^2$$

$$+ \frac{(p^2 - y^2)\, dp^2}{P} + \frac{(y^2 - p^2)\, dy^2}{Y}$$

$$P = (a^2 - p^2)(1 - \lambda p^2) + 2N_p p,$$

$$Y = (a^2 - y^2)(1 - \lambda y^2) + 2N_y y$$

Hidden Symmetries in Kerr-NUT-(A)dS

Step 1: Flat ST in Cartesian coordinates

Potential for CY

$$b = \tfrac{1}{2}[(X^2 + Y^2 + Z^2)\,dt + a(XdY - YdX)]$$

$$k = db = (XdX + YdY + ZdZ) \wedge dt \ (+a\, dX \wedge Y) \quad (rank\ 2\ CY)$$

$$f = {*k} \quad (rank\ 2\ Y), \quad f_{ij} = \varepsilon_{ijk} X^k$$

$$K_{\mu\nu} = f_{\mu\lambda} f^{\lambda}_{\ \nu} \quad (rank\ 2\ Y), \quad K_{ij} = X_i X_j - R^2 \delta_{ij}$$

In flat ST Killing tensor K is reducible

$$K_{ij} = -\sum_k \xi_{(k)i}\xi_{(k)j}, \quad \xi_{(k)i} = \varepsilon_{kij}X^j$$

Conserved quantity: Total angular momentum

$$K_{\mu\nu}p^\mu p^\nu = \vec{L}^2 = \sum_k L_{(k)}^2, \quad L_{(k)} = \xi_{(k)\mu}p^\mu$$

Step 2: Ellipsoidal and Carter-Plebanski coordinates

$$b = (r^2 + a^2 \sin^2\theta)\, dt - a \sin^2\theta (r^2 + a^2) d\phi$$

$$= (r^2 - y^2)\, d\tau + r^2 y^2 d\psi \; (+a^2 d\tau)$$

Step 3: Use b (without changes) to construct K and Y tensors in 4D Kerr-NUT-(A)dS

In the general case K is irreducible any more

Carter's separation constant

Separation of Variables in the 4D Kerr-NUT-(A)ds

$$Klein-Gordon\ equation: \quad (\Box + \mu^2)\Phi = 0$$

$$Separation\ of\ variables: \quad \Phi = e^{iE\tau + im\psi} R(r) Z(y)$$

GRTensor calculations give:

$$\sqrt{-g}\,(\Box\Phi + \mu^2\Phi)/\Phi = 0$$

$$(XR')' + [\frac{(Er^2+m)^2}{X} + \mu^2 r^2 + \lambda]R = 0$$

$$(Y\dot{Z})^{\cdot} + [-\frac{(Ey^2-m)^2}{Y} + \mu^2 y^2 - \lambda]Z = 0$$

Lesson

Kerr-NUT-(A)dS metric can be generated by a simple algebraic modification of 2 metric functions in the flat ST metric written in the Carter-Plebanski coordinates

If we preserve the form of CY tensor of hidden symmetries of a flat ST, after this transformation it is again CY but it becomes non-trivial (non-reducible)

Petrov Type and Hidden Symmetry

The Kerr-NUT-(A)dS spacetime is of type D

All type-D vacuum metrics (except C-metric) admit a Killing tensor [Walker & Penrose, 1970, Hughston et al, 1972]

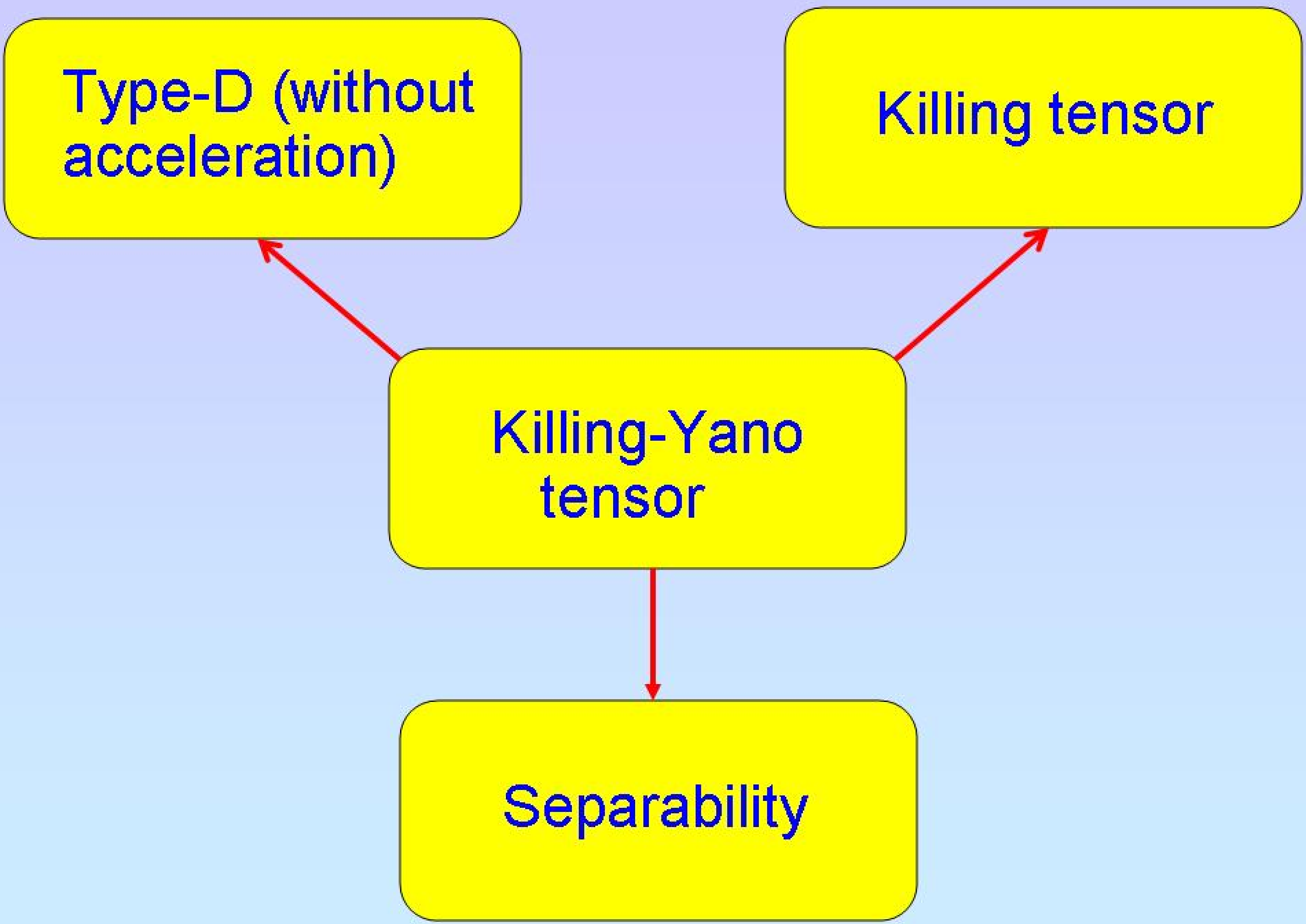
In all type-D vacuum metrics admitting a Killing tensor of rank=2 it can be written as a `square' of the Killing-Yano tensor [Collinson, 1976, Stephani, 1978]

The Weyl tensor of a ST admitting a Killing-Yano tensor belongs to the Petrov-Pirani type D (or type N) [Dietz & Rudiger,1981)]

A vacuum type-D ST (without acceleration) admits a separability structure (HJ equation allows a complete separation of variables) [Demianski & Francaviglia, 1980]

Hamilton-Jacobi and Klein-Gordon separable solutions of Einstein equations [Carter, 1968]

Separation of variables in higher-spin equations [Teukolsky, 1972, 1973]


Type-D (without acceleration)
Killing tensor
Killing-Yano tensor
Separability

Higher Dimensional BHs

Brief History of Higher-Dim BHs

Higher-Dim BHs solutions

1963	*Tangerlini*	$\Lambda = 0, \quad J = 0, \quad N = 0$
1976	*Gonzales − Diaz*	$\Lambda \neq 0, \quad J = 0, \quad N = 0$
1986	*Myers − Perry*	$\Lambda = 0, \quad J \neq 0, \quad N = 0$
2004	*Gibbons − Lu − Page − Pope*	$\Lambda \neq 0, \quad J \neq 0, \quad N = 0$
2006	*Chen − Lu − Pope*	$\Lambda \neq 0, \quad J \neq 0, \quad N \neq 0$

Coordinates on Higher Spheres

Hyper-elliptical coordinates

$$S^N : \quad O(N+1) \supset O(2) \times O(2) \times \ldots O(2)$$

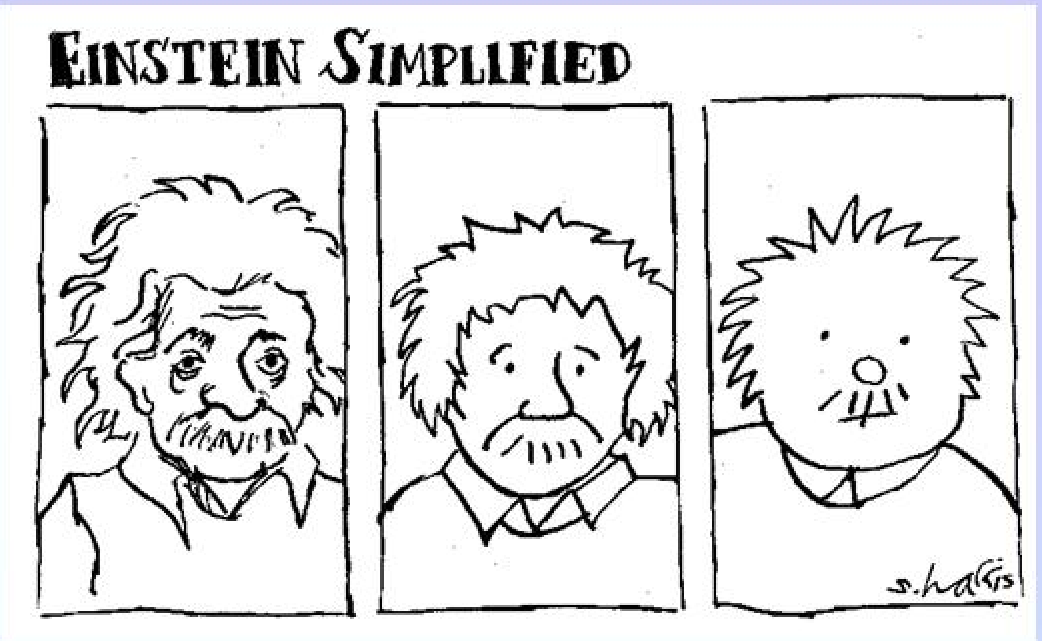
$$([(N+1)/2] \; times)$$

$$ds^2 = -dt^2 + \sum_{i=1}^{n} (dX_i^2 + dY_i^2) + \varepsilon dZ^2$$

$$X_i = \sqrt{r^2 + a_i^2}\ \mu_i \sin\phi_i ,$$

$$Y_i = \sqrt{r^2 + a_i^2}\ \mu_i \cos\phi_i , \quad Z = r\ \mu$$

[see e.g. Kei-Ichi Maeda & M. Tanabe,
NP, B738, 184 (2006);
if all *a* vanish – hyperspherical coords;
MP – Quasi-spheroidal coords]



Higher-Dimensional Kerr-NUT-(A)dS for Dummies

Derivation in 3 `Simple' Steps

Step 1: Write flat metric in hyper-elliptic coordinates

$$ds^2 = -dt^2 + F\,dr^2 + r^2 d\omega_{D-2}^2 + \sum_i a_i^2 (d\mu_i^2 + \mu_i^2 d\phi_i^2),$$

$$d\omega_{D-2}^2 = \sum_i (d\mu_i^2 + \mu_i^2 d\phi_i^2) + \varepsilon\, d\mu^2$$

$$F = 1 - \sum_i \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}, \quad \sum_i \mu_i^2 + \varepsilon \mu^2 = 1$$

Step 2: Use unconstrained Jacobi coordinates on a unit sphere

$$(\mu_i, \mu) \to (x_\alpha) \qquad \mu_i^2 = \frac{\prod_{\alpha=1}^{n-1}(a_i^2 - x_\alpha^2)}{\prod_{k=1}^{'n}(a_i^2 - a_k^2)}$$

In these coordinates the metric on a init sphere is diagonal. The metric coefficients are rational functions of x. Laplace equation allows separation of variables.

$$d\omega_{D-2}^2 = \sum_{\alpha=1}^{n-1} g_\alpha dx_\alpha^2 + \sum_{i=1}^{[(D-1)/2]} \mu_i^2 d\phi_i^2$$

$$g_\alpha = -\frac{x_\alpha^2 \prod_{\beta=1}^{'n-1}(x_a^2 - x_\beta^2)}{\prod_{k=1}^{n}(a_k^2 - x_\alpha^2)}$$

Step 3: Make a linear transformation to Carter-Debever coordinates

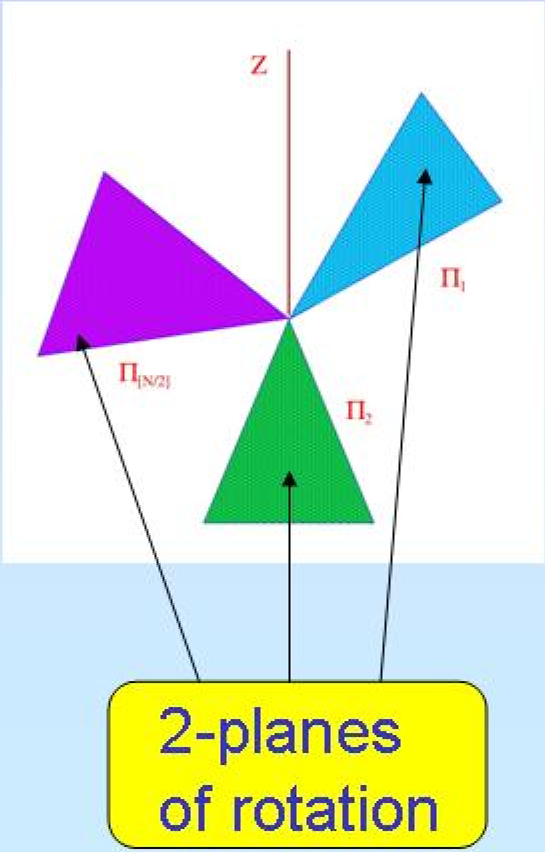
$$(t,\phi_i) \to (\tau = \psi_0, \psi_i)$$

An additional transformation $r \to ix_0,\ M \to iM_0$ *put the metric into symmetric form.*

Coordinates

$$Killing \quad coordinates: \quad \psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_n$$

$$Essential \quad coordinates: \quad x_1, \ldots, x_n$$



Orthonormal form:

$$\mathbf{g} = \sum_{a=1}^{D} \mathbf{e}^a \mathbf{e}^a = \sum_{\mu=1}^{n} (\mathbf{e}^\mu \mathbf{e}^\mu + \mathbf{e}^{\hat{\mu}} \mathbf{e}^{\hat{\mu}}) + \varepsilon \mathbf{e}^{2n+1} \mathbf{e}^{2n+1} ,$$

($\hat{\mu} = n + \mu$)

$$\mathbf{e}^\mu \; = \; Q_\mu^{-1/2} dx_\mu , \quad \mathbf{e}^{\hat{\mu}} = Q_\mu^{1/2} \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k ,$$

$$\mathbf{e}^{2n+1} \; = \; (-c/A^{(n)})^{1/2} \sum_{k=0}^{n} A^{(k)} d\psi_k ,$$

where:

$$n = [D/2], \quad \varepsilon = D - 2n, \quad m = n - 1 + \varepsilon.$$

$$Q_\mu = \frac{X_\mu}{U_\mu}, \quad U_\mu = {\prod_{\nu=1}^{n}}' (x_\nu^2 - x_\mu^2), \quad c = \prod_{k=1}^{m} a_k^2,$$

$$X_\mu = (-1)^{1-\varepsilon} \frac{1 + \lambda x_\mu^2}{x_\mu^{2\varepsilon}} \prod_{k=1}^{m} (a_k^2 - x_\mu^2) + 2M_\mu (-x_\mu)^{1-\varepsilon},$$

$$A_\mu^{(k)} = {\sum_{\nu_1 < \cdots < \nu_k}}' x_{\nu_1}^2 \dots x_{\nu_k}^2, \; A^{(k)} = \sum_{\nu_1 < \cdots < \nu_k} x_{\nu_1}^2 \dots x_{\nu_k}^2 \; .$$

\`Higher dimensional analogue of Carter-Plebanski solution'

"General Kerr-NUT-AdS metrics in all dimensions“, Chen, Lü and Pope, Class. Quant. Grav. 23 , 5323 (2006).

$$n = \left[D/2 \right],\ D = 2n + \varepsilon$$

$$R_{\mu\nu} = (D-1)\lambda g_{\mu\nu}$$

$\lambda, M - mass,\ a_k - (n-1+\varepsilon)$ *rotation parameters*,

$M_\alpha - (n-1-\varepsilon)$ *`NUT' parameters*

Total # of parameters is $D - \varepsilon$

Hidden Symmetry and Separation of Variables in HD Kerr-NUT-(A)dS BHs

Main result : *Besides* $n+\varepsilon$ *Killing vectors this metric has* $n-1$ *rank 2 non-trivial Killing tensors. This together with the trivial Killing tensor* $g_{\mu\nu}$ *provides one with D integrals of motion. The Poisson brackets of all pairs of these D constants are zero, so geodesic motion is completely integrable. The Hamilton-Jacobi and Klein-Gordon equations allow the separation of variables.*

Separability in HD BHs

Separability structures in D-dimensional ST [Benenti & Francaviglia, 1979]

Killing tensor and separation of variables in 5D rotating black holes [V.F. & Stojjkovic, 2003 a,b]

Separability in 5D Kerr-(A)dS [Kunduri and Lucietti, 2005]

Special cases (sets of equal rotation parameters)

Chong, Gibbons, Lu, & Page, 2004;
Vasudevan & Stevens, 2005;
Vasudevan, Stevens & Page, 2005 a,b;
Chen, Lu, & Pope, 2006;
Davis, 2006

1. Principal Yano tensor

V. F., D.Kubiznak, Phys.Rev.Lett. 98: 011101, 2007; gr-qc/0605058; D. Kubiznak, V. F., Class.Quant.Grav. 24: F1, 2007; gr-qc/0610144.

$$b = \frac{1}{2}\sum_{k=0}^{n-1} A^{(k+1)}\, d\psi_k$$

(The same as in a flat ST in the Carter-Plebanski-type coordinates)

$$k = db, \quad k = \sum_{\mu=1}^{n} x_\mu e^\mu \wedge e^{\hat{\mu}}, \quad k \text{ is a rank 2 closed CY}$$

Its dual $f = {}^*k$ *is rank* $(D-2)$ *Yano tensor*

2. Complete integrability of geodesic motion in general Kerr-NUT-AdS spacetimes

D. N. Page, D. Kubiznak, M. Vasudevan, P. Krtous, Phys.Rev.Lett. 98 :061102, 2007; hep-th/0611083

Full set of D independent constants of motion in involution

D-n+1 – Killing vectors plus n-1 rank 2m (m=1, …) Killing tensors obtained from PY

Vanishing Poisson brackets for integrals of motion

3. Complete set of rank 2 Killing tensors

P. Krtous, D. Kubiznak, D. N. Page, V. F., JHEP 0702: 004, 2007; hep-th/0612029

Take the principal conformal Yano tensor k.

It is closed. Construct a tower of closed CY

tensors of the rank $2j,\ j = 2, 3, \ldots:$

$$k^{(2)} = k \wedge k, \quad k^{(3)} = k \wedge k \wedge k, \quad \ldots$$

$$f^{(j)} = {}^{*}k^{\wedge j}, \quad \mathbf{K}^{(j)}_{\alpha\beta} = f^{(j)}{}_{\alpha\ldots} f^{(j)}{}_{\beta}{}^{\ldots}$$

$$\mathbf{K}^{(j)}_{\alpha\beta} = \sum_{\mu=1}^{n} A^{(j)}_{\mu} (e^{\mu}_{\alpha} e^{\mu}_{\beta} + e^{\hat{\mu}}_{\alpha} e^{\hat{\mu}}_{\beta}) + \varepsilon A^{(j)} e^{2n+1} \wedge e^{2n+1}$$

4. Separability of Hamilton-Jacobi and Klein-Gordon equations in Kerr-NUT-(A)dS ST

V. F., P. Krtous , D. Kubiznak , JHEP 0702:005,2007; hep-th/0611245

Klein-Gordon equation

$$\Box\Phi = \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}g^{ab}\partial_b\Phi) = m^2\Phi.$$

Multiplicative separation

$$\Phi = \prod_{\mu=1}^{n} R_\mu(x_\mu) \prod_{k=0}^{m} e^{i\Psi_k\psi_k}.$$

$$(X_\mu R'_\mu)' + \epsilon\,\frac{X_\mu}{x_\mu}R'_\mu - \frac{R_\mu}{X_\mu}\Big(\sum_{k=0}^{m}(-x_\mu^2)^{n-1-k}\Psi_k\Big)^2 - \sum_{k=0}^{m} b_k(-x_\mu^2)^{n-1-k}R_\mu = 0\ .$$

$$b_0 = m^2$$

Open Problems

Massless fields of higher spin in HD Kerr-NUT-(A)dS ST:

- Decoupling
- Separation of Variables

Hidden symmetries and separation of variables in 5D black rings, black saturns, etc

Petrov type I_i [*A. Coley et al*, 2004, 2005]

Relation between Petrov type, hidden symmetry, and separation of variables in higher dimensional curved spaceime