Unity of Physics and Cosmology...

B.B. $\rho_{Pl} \sim 10^{94}$ GUT; EW; Baryon asymm.; $\rho_{Nuc} \sim 10^{15}$; photons

10^{-43} sec $\rightarrow$ 10^{-34} s. $\rightarrow$ 10^{-10} s. - - - - 10^{2} s. - - - 10^{13} s

Q.G. Superstrings ?; Inflation ?; Confinement Nucleons (Antiquarks die); Helium (positrons die); atoms matter

10^{9} y. - - - → $3 \cdot 10^{9}$ year → $(15 \div 20)\ 10^{9}$ years

Galaxies, Stars; Supernovae; TODAY

This is VERY SCHEMATIC!

- Main SCALES: $t_{Pl} \sim 10^{-44}$ s, $E_{Pl} \sim 10^{19}$ GeV, $T_{Pl} \sim 10^{32}$ K

Univ. $R_{Pl} \sim 10^{-3}$

GUT: $t_G \sim 10^{-36}$ s., $E_G \sim 10^{15}$ GeV, $T_G \sim 10^{28}$ (R ~ 10 cm)

present day EW: $t_{EW} \sim 10^{-10}$, $E_{EW} \sim 100$, $T_{EW} \sim 10^{15}$ ($R \sim 10^{14}$)

- Available energies: $\lesssim 10^{4}$ GeV (accelerators)

$\sim 10^{7}$ GeV (cosmic rays)

(to reach this energy in accel. the size should be $\gtrsim$ 5000 km !)

Particle Phys. ⟷ Cosmology

mutual constraints

deep connection on th. and exp.

Becoming indivisible

0a Philosophy • Motivation: L-H, L.D., E.I. July 2007
+ short history and ref.

- Introduction: • standard dim. red.;
  - $(1+1) \to (0+1)/(1+0)$ dilaton grav. (DG) from h. dim. Gravity, SUGRA, Superstr.:
  - overview of integrable D.G.
- 'Sofisticated' vs. 'Naive' dim. red.
  - Separation of variables and waves in gravity.
- Static - Cosmological DUALITY -
  - examples of 'unusual' static and cosm. solutions
- Cosmologies, B.H. (static states)
  - and Waves in an explicitly integrable D.G. (N-Liouville) $(1+1) \begin{matrix} \nearrow (1+0) \\ \dashrightarrow \text{waves} \\ \searrow (0+1) \end{matrix}$
- Extension of this 'Triality' (?)
  WAVES
  COSM. = STATIC
  to more general models (Toda)
  - and attempts to find it in 'realistic' theories

+ various REMARKS

'Philosophy' and motivation

- Lagrangian-Hamiltonian dynamis as important as geometry in grav.
- Einstein eqs. → dynamical eqs. in $\mathcal{L}$-$\mathcal{H}$ framework (esp. in 1+1, 1+0/0+1 dim.)
- Explicitly integrable approxim. esp. important for Quantum Th. (and for perturb. theory)
- Reducing to $\mathcal{L}$-$\mathcal{H}$ dyn. to lower. dim. ↦ explicitly integrable approxim (for B.H., C. and Waves)
- Dim. red. of SUGRA and H.D. grav. ↦ interesting integrable and non-integrable 1+1, 1+0/0+1 dim. $\mathcal{L}$-$\mathcal{H}$ models, in which
  - Geometry 'degrades' to scalar matter fields and 'cosmological potentials'
- In lower dim. it is easier to uncover hidden relations between C. — B.H(static) — Waves

July 2007

Math. models of B.H., C., W.

	B.H.	Cosm.	Waves	Dim. red.
'16	Schwarzschild		—	'21 Kaluza
'17	R-N	Einstein, de Sitter	—	'26 Mandel. Fock. O. Klein
'22	—	Friedmann	—	KMKF-reduct
'27-'36	–	Lemaitre, R-W.... ('29 Hubble)	–	
'39	Oppenheimer and Snyder	('48 Gamow)	Einstein and Rosen	
'62-63	Kerr	'65 Penzias, Wilson	Ehlers, Kundt	
'65	Newman	EXP. observ.	Thorne	
'68	Ernst		—	
'72	—	(W-S – theory '74 GUT; '76 SUGRA)	Szekeres	

~ '78 Integrable models of grav.

Maison; Belinsky, Zakharov; ....

later Jackiw; Nicolai; Alekseev; ....

~ '88 Dimensional reduction ('86 Superstring)

Breitenlohner, Gibbons, Maison;

later dim. red. in SUGRA and Superstr.

KMKF (Kaluza); compactifications etc.

→ mainly, appl. to B.H. and C.

→ development of Dilaton Gravity (DG)

• Short (subjective) history of Dilaton Gravity: exact solutions and applications to Black Holes (B.H.) and Cosmology (C.)

'91 – '92 'String inspired' D.G. in 1+1
before and after } E. Witten; (Verlinde)^2; C.G.H.S.
→ many papers on related models:
e.g. Barbashov, Nesterenko; Jackiw, Teitelboim; Kuhař; Tseitlin; Kummer e.a.

'95 – '96 Cavaglià, de Alfaro, A.T.F. (B.H ↔ C.)

'96 – '97 A.T.F. (integrable 1+1, 0+1/1+0 models relat. C. ↔ B.H.) — on solutions

'98 A.T.F. + V. Ivanov (elem. waves in integrable models
(C ↔ Waves)

'97 – '99 Other relations between C. and B.H (p-branes)
Lukas, Ovrut, Waldram
Larsen, Wilczek
Lü, Mukherdji, Pope } — observation on solutions 0+1 . branes ↔ 1+0 cosm.

'98 – 2002 String cosmology Veneziano Schwarz...

• This rep. is based on work of 2002–2007
resp. hep-th: 0307269, 0504101, 0612258
ATF+VdA; and A.T.F 050506, 0605276 + some new results.

J.

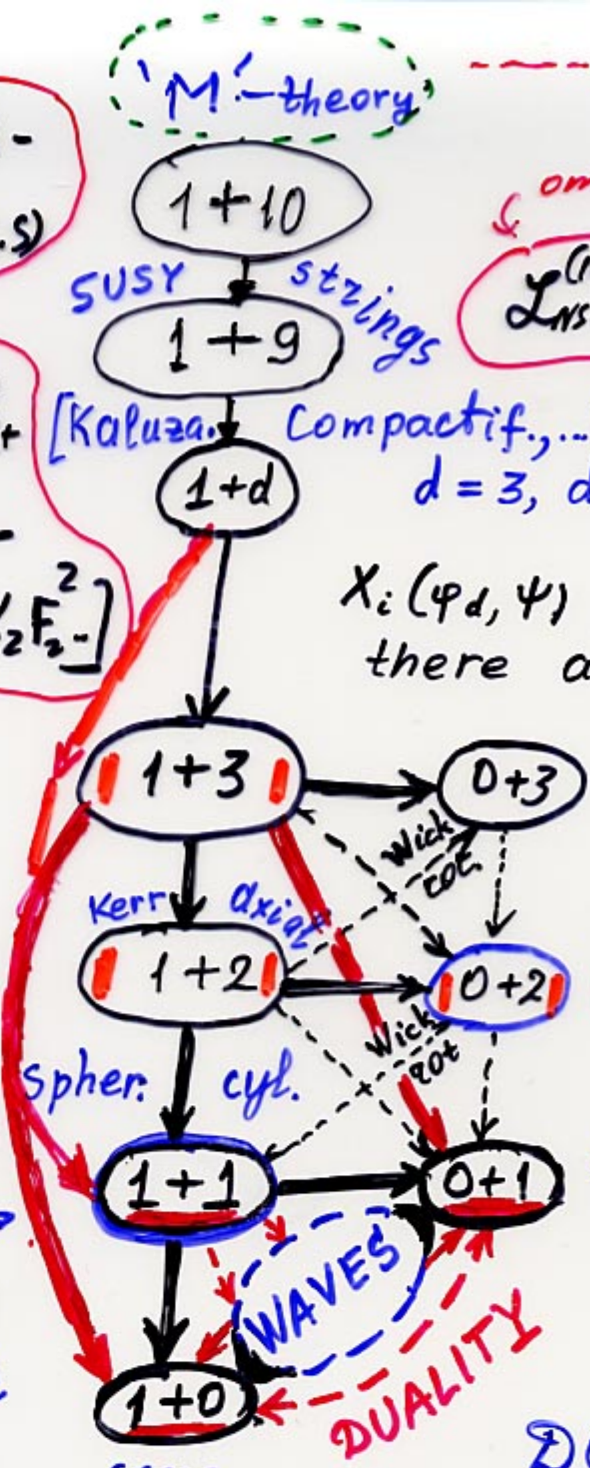
$$\mathcal{L}^{(11)} = \sqrt{-g}\Big[R - \frac{1}{48}F_4^2\Big] + (C.S)$$

- - - - ((B.A + A.F.) naive (11 dim grav + W.E. + strong

omitting C-S. dilaton

$$\mathcal{L}^{(10)}_{NS-NS} = \sqrt{-g}\Big[R + 4(\nabla\phi)^2 - \frac{1}{12}H_3^2\Big]e^{-2\phi}$$

$$\mathcal{L}^{(d)} = \sqrt{g}\cdot e^{-2\phi_d}\Big[R + 4(\nabla\phi_d)^2 - (\nabla\psi)^2 - \underbrace{X_0}_{\text{Cosm. potential}} - X_1(\nabla\sigma)^2 - X_2F_2^2 - \Big]$$



d = 3, d = 4, d = 5 most pop.

$X_i(\varphi_d, \psi)$, $F_2 = dA_2$, there are higher forms

general static

Ernst (static axial)

static. states B. H.

Direct reduction (spherical, cylindrical)

Direct reduction (homogeneous, isotropic)

Waves - special topic

D.G. = pure dilaton gravity

topol. integrable, autom. reducing to 1+0/0+1

$$\mathcal{L}^{(2)} = \sqrt{-g}\Big[U(\varphi)R(g) + V(\varphi) + W(\varphi)(\nabla\varphi)^2 + X(\varphi,\psi,F^2_{(1)},\dots,F^2_{(A)}) + Y(\varphi,\psi) + \sum Z_{mn}(\varphi,\psi)\,\bar\nabla\psi_m\cdot\bar\nabla\psi_n\Big]$$

generalized B.-I. $\longrightarrow$ $Y(\varphi,\psi;Q_1,\dots,Q_A)$

$\varphi(t,r)$, $\psi(t,r)$, ... Naive red.: $\varphi = \varphi(t)$ ... (1+0) or $\varphi = \varphi(r)$ ... (0+1)

But there exist more sophisticated reductions!

4 Dimensional reduction and relation between static - cosmological - wavy (SCW-triality) in integrable models of grav.

- Different sorts of dim. reduction:
  - 'Symmetry' reduction (spherical, axial, plane ....)
  - K-M-K-F reduction + compactif. → cylindrical, axial (unusual application to D=4)
  - 'Naive' reductions ($f(r,t) \to f(r)$, $\to f(t)$ etc.) (not wrong but usually incomplete)
  - Reducing by separation (a rather general approach but classification is not complete)
  - 'Dynamical' reduction (reduc. solutions) ↳ waves? (in integrable theories = 'moduli space' reduction)

0+1 / 1+0 reductions are often realistic
(1+1) realistic - noninteg.

{ Applications to non-integrable } ?

- Applications to real world are possible but require more labour
- Realistic computations require perturb. theory that is rather nontrivial. (not discussed here!)

5. In LC gauge, with Weyl transf. to $W \equiv 0$.

E.O.M. $(\varphi_{,u} \equiv \partial_u \varphi,\ \varphi_{,uv} \equiv \partial_u \partial_v \varphi$, etc.)

(1) $\varphi_{,uv} + f V(\varphi, \psi) = 0$ $\qquad \boxed{ds^2 = -4f \cdot du\, dv}$

(2) $f\left(\frac{\varphi_{,i}}{f}\right)_{,i} = \sum_n \overset{n}{Z}\, \underbrace{\overset{n}{\psi}{}^2_{,i}}_{\to 0}$, $i = u, v$ $\Big\}$ **CONSTRAINTS**

(3) $(\overset{n}{Z}\overset{n}{\psi}_{,u})_{,v} + (\overset{n}{Z}\overset{n}{\psi}_{,v})_{,u} + f \overset{\to 0}{V_{,\psi}} = \sum_m \overset{m}{Z}_{,\psi}\, \underbrace{\overset{m}{\psi}_u \overset{m}{\psi}_v}_{\to 0}$

(4) $(\ln|f|)_{,uv} + f V_{,\varphi}(\varphi, \psi) = \sum_n \overset{n}{Z}_{,\varphi}\, \underbrace{\overset{n}{\psi}_u \overset{n}{\psi}_v}_{\to 0}$

((4) follows from (1)–(3))

'Topological' integrability:

- The solution of these eqs. for $\psi_n = \psi_n^{(0)}$, $V_\psi(\varphi, \psi_n^{(0)}) = 0$ coincides with the solution of the pure dilaton gravity (without scalar matter).

- The key eqs. $\boxed{\partial_i\left(\frac{1}{f}\partial_i \varphi\right) = 0}$; $i = u, v$.

$\Rightarrow \varphi = F(\phi)$, $f = F'(\phi)\phi_u \phi_v$, $\phi_{uv} = 0$

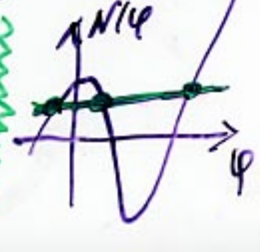
i.e. $\phi = a(u) + b(v)$

$F(\phi)$ is found from (1):

- $F'(\phi) + N(F) = M\ (= \text{const})$ $\qquad N(F) \equiv \int V(F, -)\, dF$

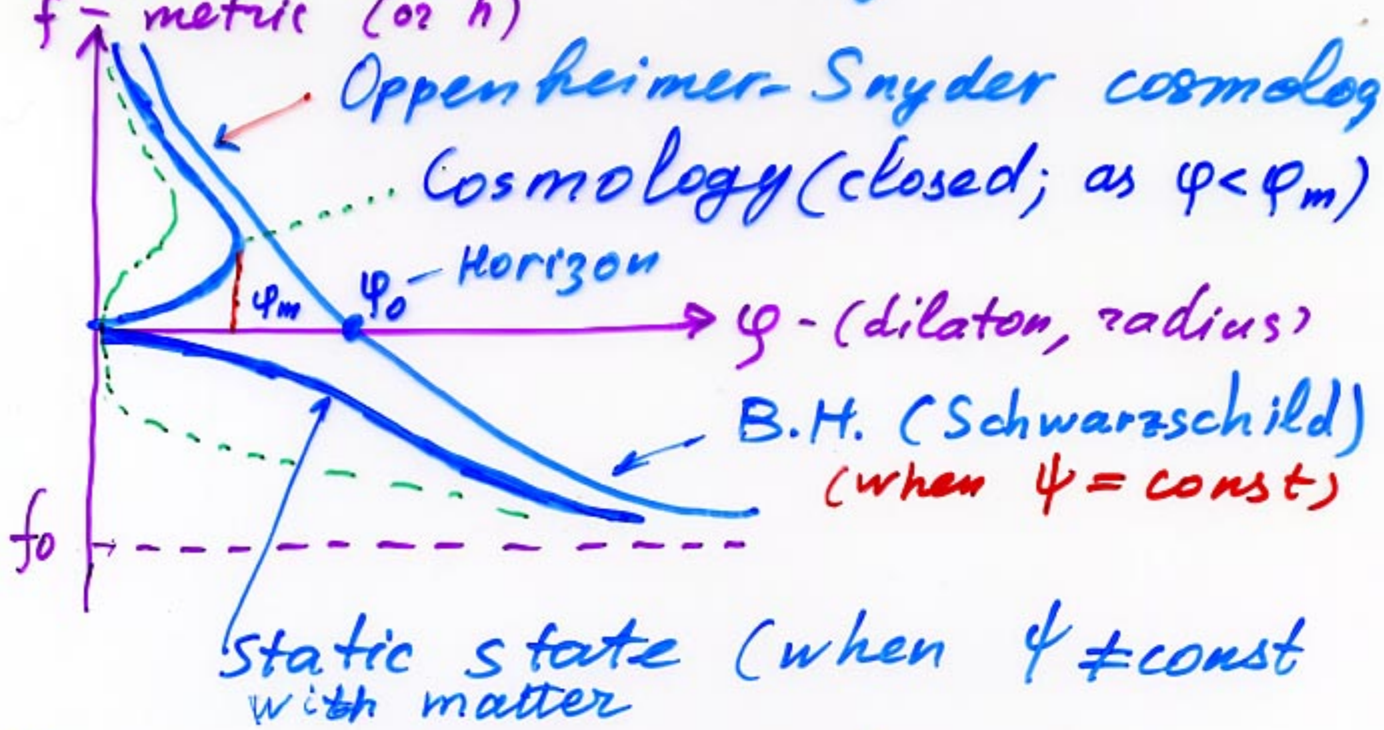
Thus: $f = (M - N(\varphi))\, a'(u)\, b'(v)$

$F(\phi)$ is found from: $\int \frac{dF}{M - N(F)} = \phi - \phi_0$



NOT ALL !

'Naive' reduction gives static states and Cosmologies



NB: Only closed cosmologies are derived by this naive reduction!

Question: How to derive other cosmologies (e.g. FRW-typ.

Answer: Use more general dim. reduction from (1+1) to (1+0) and (0+1) theory (see below)

8.

Reducing by 'separating' in spherical gravity (see. App. I)

Removing δ-term ($\delta \to -\infty$) after varying $\mathcal{L}_2$ in δ, we get:

Constraint: $\boxed{2\dot{\beta}' + \dot{\beta}\beta' - \dot{\beta}\gamma' - \dot{\alpha}\beta' = \frac{1}{2} Z_4 \dot{\psi}\psi'}$ (*)

($\dot{\beta} \equiv \partial_t \beta$, $\gamma' \equiv \partial_r \gamma$, ...; $Z_4 \mapsto$ const)

Then: • $\mathcal{L}_{eff} = -e^{\alpha + 2\beta - \gamma}(2\dot{\beta}^2 + 2\dot{\beta}\dot{\alpha} + Z_4 \dot{\psi}^2)$

($\alpha \leftrightarrow \gamma$, $\partial_t \leftrightarrow \partial_r$) $\quad + e^{-\alpha + 2\beta + \gamma}(2\beta'^2 + 2\beta'\gamma' + Z_4 \psi'^2) + V_{eff}$

S-C duality of the Lagrangian!

• $V_{eff} = V_4 e^{\alpha + 2\beta + \gamma} + 2\kappa e^{\alpha + \gamma}$

($\kappa = \pm 1, 0$)

• Vary in $\alpha, \beta, \gamma, \psi \mapsto$ E.O.M.
  • E.O.M. + constraint (*) ~ Einst. eqs.

• 'Natural' separation:

$\alpha(t,r) = \alpha_0(t) + \alpha_1(r)$, $\beta = \beta_0 + \beta_1$; $\gamma = \gamma_0 + \gamma_1$

($\alpha_1(r) \mapsto 0$, $\gamma_0(t) \mapsto 0$)

ψ: either $\psi = \psi_0 + \psi_1$ (additive)

or $\psi = \psi_0 \cdot \psi_1$ (multipl.)

for exp. potent. $\quad$ for polyn. pot.

- Equations now read:

$$(*)\quad \dot{\alpha}_0 \beta_1' + \dot{\beta}_0(\gamma_1' - \beta_1') - \dot{\psi}_0 \psi_1' = 0 \qquad \left\{ \begin{matrix} \text{addit.} \\ \text{and} \\ (z_4 = -2) \end{matrix} \right.$$

E.O.M. (Einstein eqs.)

$$(1)\quad B_1 - B_0 = g\, v_0 v_1 + k e_0 e_1$$

$$(2)\quad \tilde{B}_1 + \tilde{B}_0 = -E_1 - E_0$$

$$(3)\quad A_1 - A_0 = -k e_0 e_1 + E_0 - E_1$$

$v_0 = e^{a\psi_0} + \dots$ if $V_4 = g e^{a\psi} + \dots$ $(v_1 = e^{a\psi_1} + \dots)$

$e_0 \equiv e^{2(\alpha_0 - \beta_0)}$, $e_1 \equiv e^{2(\gamma_1 - \beta_1)}$

$E_0 \equiv e^{2\alpha_0}(\dot{\psi}_0^2 + \dots)$ $\quad E_1 \equiv e^{2\gamma_1}(\psi_1'^2 + \dots)$

$B_1 \equiv e^{2\gamma_1}(\beta_1'' + 2\beta_1'^2 + \beta_1'\gamma_1')$, $B_0 (\alpha_0 \leftrightarrow \gamma_1, \partial_t \leftrightarrow \partial_r)$

$\tilde{B}_0, \tilde{B}_1, A_1, A_0$ — similar expressions

- All the eqs. have the form:

$$(S)\quad \boxed{\sum_{n=1}^{N} T_n(t) \cdot R_n(r) = 0}$$

- NB: eqs. (*) and (1), (3) are more restrictive, esp. (*)
- NB: for one field ψ, eqs. for ψ follow from above eqs. for addit. scalars should be added. (varying)

10.

Solution of the (S)-eq.

* a/ THEOREM:

- Define for any m $(1 \le m \le N-1)$

$$\Psi \equiv \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \quad \Psi_+ \equiv \begin{pmatrix} R_1 \\ \vdots \\ R_m \end{pmatrix}, \quad \Psi_- \equiv \begin{pmatrix} T_{m+1} \\ \vdots \\ T_N \end{pmatrix}$$

$$\bar{\Psi} \equiv \begin{pmatrix} \bar{\Psi}_+ \\ \bar{\Psi}_- \end{pmatrix}, \quad \bar{\Psi}_+ \equiv \begin{pmatrix} T_1 \\ \vdots \\ T_m \end{pmatrix}, \quad \bar{\Psi}_- \equiv \begin{pmatrix} R_{m+1} \\ \vdots \\ R_N \end{pmatrix}$$

- Then (obviously) eq. (S) is equiv. to

$$\boxed{\bar{\Psi}^{Tr} \cdot \Psi = 0}$$

- Choose an arbitrary vector Ψ and derive $\boxed{\bar{\Psi} = \hat{C}\,\Psi}$, $\hat{C} = \begin{pmatrix} 0 & \hat{C}_0 \\ -\hat{C}_0^T & 0 \end{pmatrix}$

where $\hat{C}_0$ is an arb. const. matrix $\boxed{m \times (N-m)}$

→ • Then we have a solution $T_i(t), R_i(r)$

→ • To get all solutions take all possible divisions of $R_1, \dots, R_N$ into two groups $R_{m_1}, \dots, R_{m_n}; R_{m_{n+1}}, \dots, R_{m_N}$ (sim. T-div)

→ NB: proof-by induction $(N \to N+1)$ ■

↦ Using this theorem try to classify all possible solutions (with 1 scalar φ)

- **Note:** • for $k=0$ (flat case) we get Static, Cosmological, and Wave solutions;
  • for $k\neq 0$: • Static and Cosm. EASY

(technical problem?) • Waves – NOT yet clear how to derive all the waves (?)

$k\neq 0$ - Applications start with eq. (3), which tells I. $\dot e_0 = 0$, $\dot\alpha_0 = \dot\beta_0$ or II. $e_1' = 0$, $\gamma_1' = \beta_1'$

... other scalars

Then:

(*) $\longmapsto$ $\dot\beta_0\,\gamma_1' = \dot\psi_0\psi_1' + \ldots$ $\quad \dot\alpha_0\,\beta_1' = \dot\psi_0\psi_1' + \ldots$.

(if $\dot\psi_0\psi_1'$ – then other eqs greatly simplify)

- EXAMPLE: FRW cosmology with one scalar

I, $\dot\alpha_0 = \dot\beta_0$, $\gamma_1' = 0$ (as $\psi = \psi_0$) $e_0 = \text{const}$

Then:

$$\left.\begin{aligned} &\beta_1'^2 = -\varkappa + k e_0 e^{-2\beta_1} \\ &2\beta_1'' + 3\beta_1'^2 = C_1 + k e_0 e^{-2\beta_1} \end{aligned}\right\}\ C_1 = -3\varkappa \ ! \text{ for consistenc}$$

$$\boxed{\beta_1'^2 - k e_0 e^{-2\beta_1} = -\varkappa}$$ (integral of motion

define $\beta_1(r)$)

Of 3 eqs for α_0, ψ_0 only 2 are independen

FRW eqs.
$$\begin{cases} \dot\alpha_0^2 + \varkappa e^{-2\alpha_0} = -\frac{1}{6}(V + Z\dot\psi^2) \overset{\text{def}}{=} \frac{2}{3}\rho \\ \ddot\alpha_0 - \varkappa e^{-2\alpha_0} = \frac{1}{2} Z\dot\psi^2 \overset{\text{def}}{=} -(P+\rho) \end{cases}$$

- $\dot\alpha_0^2 \equiv H^2(t)$, $\dfrac{\ddot\alpha_0 + \dot\alpha_0^2}{\dot\alpha_0^2} \equiv q$ (deceleration param.)

(eq. for ψ – by differentiation) { Generaliz are possib with many ψ.

Other integrable (1+1) - D.G.
(beside topological and 1-dimensional)

- **σ - case:** $V \equiv 0$, Z-terms give an integrable σ - model
  - Example: cylindr. reduction (sol. by Inverse scatt. meth.; twistor approach, etc.; related to Y-M-th.)
  - Explicitly integrable: $\underline{V \equiv 0,\ Z_{mn} = \delta_{mn} Z(\varphi)}$ where $Z(\varphi)$ is such that eqs. for ψ_n are explicitly solved in terms of free massless fields $\phi_n \equiv a_n(u) + b_n(v)$.

(A.T.F. + Ivanov)

- **Minimal Z-coupling:**

$\underline{Z_{mn} = -\,\delta_{mn} \cdot (\overset{\to 1}{\text{const.}})}$, V - special

  - Examples: → CGHS ($V = g$)
  - → Barbashov-Nesterenko-Jackiw-Teitelboim ($V = g\varphi$)
  - → Russo - Tseitlin ($V = g e^{\varphi}$)
  - → A.T.F.: → $V = g_+ e^{g\varphi} + g_- e^{-g\varphi}$
  - → Generalization: $V = \sum g_n e^{q_n^{(0)}}$ (N-Liouville)

$$q_n^{(0)} = a_n \varphi + \sum_{m=3}^{N} \psi_m a_{mn} \quad \left(\sum_{3}^{N} a_{lm} a_{ln} - 2(a_m + a_n) = \gamma_n^{-1} \delta_{mn} \right).$$

  - → Toda - models (lectures at this School)

12. • Explicitly integrable 1+1 D.G. + scal.

(N-Liouville)

• $Z_{mn} = -\delta_{mn}$, • $V = \sum_{n=1}^{N} g_n \exp(q_n^{(0)})$

$$q_n^{(0)} = a_n \varphi + \sum_{m=3}^{N} \psi_m a_{mn}; \quad \left[\sum_{\ell=3}^{N} a_{\ell m} a_{\ell n} - 2(a_m + a_n) = \gamma_n^{-1} \delta_{mn} \right.$$

$$q_n(u,v) \equiv F(u,v) + q_n^{(0)}(u,v). \quad \begin{cases} F = \ln|f| \\ ds^2 = -4f\,du\,dv \end{cases}$$

L. • $\partial_u \partial_v q_n = \tilde{g}_n e^{q_n}$, $\tilde{g}_n \equiv \varepsilon g_n \gamma_n^{-1}$

$X_n \equiv e^{-q_n/2}$ (•) $\boxed{X_n \partial_u \partial_v X_n - \partial_u X_n \partial_v X_n = -\frac{1}{2}\tilde{g}_n}$

• Constraints: (••) $\boxed{\sum_{n=1}^{N} \gamma_n X_n^{-1} \partial_i^2 X_n = 0, \quad (i = u, v)}$

• To solve (•) differentiate in u (and in v)

$$\Rightarrow \left(\frac{X_{,uu}}{X}\right)_{,v} = \left(\frac{X_{,vv}}{X}\right)_{,u} = 0. \quad \begin{cases} \frac{X_{uu}}{X} = U(u) \\ \frac{X_{vv}}{X} = V(v) \end{cases}$$

Generalization $\Rightarrow$ $\boxed{X_n = a_{n(u)}^{(i)} \underbrace{C_{ij}^{(n)}} b_n^{(j)}(v)}$ $\underbrace{\det \hat{C}_n = -\frac{\tilde{g}_n}{2}}$

$[a_n''(u) - U a_n = 0, \quad b_n''(v) - V b_n = 0]$

$$\begin{cases} a_n(u) = \left|\sum \gamma_m \mu_m(u)\right|^{-1/2} \exp \int du\, \mu_n(u), & \boxed{\sum \gamma_n \mu_n^2 = 0} \\ b_n(v) = \left|\sum \gamma_m \nu_m(v)\right|^{-1/2} \exp \int dv\, \nu_n(v), & \boxed{\sum \gamma_n \nu_n^2 = 0} \end{cases}$$

• $\mu_n(u)$, $\nu_n(v)$ – arb. funct. satisfying

13.

The structure of the moduli space

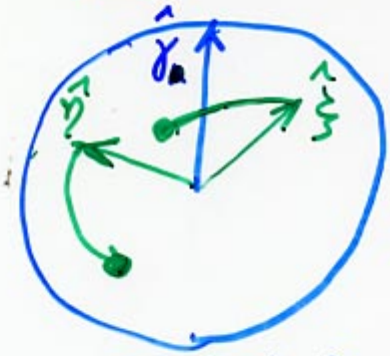
By the coord. transform. $u \to \alpha(u)$, $v \to \beta(v)$ one may fix $\boxed{\mu_1(u) = \nu_1(v) = 1}$

Then, using the 'sum rule' $\left\{ \begin{array}{l} \sum_{n=1}^{N} \gamma_n = 0 \\ (\gamma_1 < 1, \gamma_{n \geq 2} > 0) \end{array} \right.$ one defines the 'reduced' moduli (completely independent)

- $\hat{\xi}_k(u) \equiv \hat{\gamma}_k \mu_k(u)$; $\hat{\eta}_k(v) \equiv \hat{\gamma}_k \nu_k(v)$, $k = 2, \dots, N$

where $\hat{\gamma}_k \equiv \sqrt{\gamma_k / |\gamma_1|}$, $\sum_2^N \hat{\gamma}_k^2 = 1$; $\sum_2^N \hat{\xi}_k^2 = \sum_2^N \hat{\eta}_k^2 = 1$

$\mapsto \boxed{\hat{\xi} \in S^{(N-2)}}, \boxed{\hat{\eta} \in S^{(N-2)}}$



$\left\{ \begin{array}{l} \text{Any solution is defined} \\ \text{by a pair of unit} \\ (N-2)\text{-dim. vectors } (\hat{\xi}(u), \hat{\eta}(v)) \end{array} \right.$

The True moduli space

$\left\{ \begin{array}{l} \hat{\xi}_k = \hat{\xi}_k^{(0)} = \text{const} \\ \hat{\eta}_k = \hat{\eta}_k^{(0)} = \text{const} \end{array} \right.$ Wave solutio[ns]

- If $\hat{\xi}_k^{(0)} = \hat{\eta}_k^{(0)} \longrightarrow$ we have Static or Cosmolog. sol.

- Horizons: $\hat{\xi}_k^{(0)} = \hat{\eta}_k^{(0)} = \hat{\gamma}_k$, i.e. $\left\{ \mu_n = \nu_n = 1 \right.$, $n = 1, \dots, N$

(2 horizons)

14. $\bar{\lambda}_n = \frac{1}{2}(\mu_n - \nu_n)$ $\lambda_n = \frac{1}{2}(\mu_n + \nu_n)$

- There exist <u>NONSINGULAR</u> waves!

(the general solution gives singular waves)

$$X_n = \frac{1}{\sqrt{\mu_n \nu_n}} \Big\{ C_n^+ \cosh(\lambda_n \tau + \bar{\lambda}_n t + \delta_n^+) + + C_n^- \cosh(\lambda_n t + \bar{\lambda}_n \tau + \delta_n^-) \Big\} \quad (*)$$

$\tau = u + v$, $t = u - v$

$$(C_n^+)^2 - (C_n^-)^2 = -\frac{1}{2}\tilde{g}_n$$ (definitions above)

For a special choice of $\lambda_n, \bar{\lambda}_n$ it is possible to construct the waves that are finite also for $t \to \pm\infty$, $\tau \to \pm\infty$

(see hep-th/0612258) (simple but tedious and lengthy!)

alternatively:

(*) Try SEPARATION of the $\mathcal{L}$ eqs.

- $\tau_n \overset{\text{def}}{=} \mu_n u + \nu_n t$, $t_n \overset{\text{def}}{=} \mu_n u - \nu_n v$

$q_n'' - \ddot{q}_n = \tilde{G}_n e^{q_n}$, $q_n = \xi_n(t_n) + \eta_n(\tau_n)$ (?)

$\to$ 1/2 of the solutions!

- Try $\boxed{X_n = \xi_n(t_n) + \eta_n(\tau_n)}$, we get

$$\begin{cases} \ddot{\xi}_n(t_n) - C_n \xi_n(t_n) = A_n \\ \eta_n''(\tau_n) - C_n \eta_n(t_n) = B_n \end{cases} \Bigg| \; A_n + B_n = 0$$

This gives all the solutions (*) with constant moduli NOT using integrability!

a strange separation

Explicitly integrable 'Toda-models'

simplest A_N Toda system: $x_i(u,v)$

$$e^{-x_1} \overset{\text{def}}{=} \varepsilon_1 X; \quad e^{-x_2} \overset{\text{def}}{=} \varepsilon_2 \Delta_2; \quad e^{-x_3} \overset{\text{def}}{=} \varepsilon_3 \Delta_3; \quad \text{etc.}$$

A_2-case: $\boxed{x_3 = 0 \leftrightarrow \Delta_3 = \varepsilon_3^{-1}}$ (ε_i - useful)

Def: $\Delta_1 \equiv X$; $\Delta_2 \equiv X X_{,uv} - X_{,u} X_{v}$;

$$\Delta_3 \equiv \begin{vmatrix} X & X_u & X_{uu} \\ X_v & X_{uv} & X_{uuv} \\ X_{vv} & X_{uvv} & X_{uuvv} \end{vmatrix}, \quad \text{etc.}$$

Then:
$$\begin{cases} x_{1,uv} = -\varepsilon_1^2 \varepsilon_2^{-1} e^{2x_1 - x_2} \\ x_{2,uv} = -\varepsilon_1^{-1} \varepsilon_2^2 \varepsilon_3^{-1} e^{2x_2 - x_1} \end{cases}$$
Toda equations (A_2)

• Ansatz: (try to prove!) (that it gives the general sol.)
$$\boxed{X = \sum_{i=1}^{3} a_i(u)\, b_i(v)}$$
(a sort of separation!)

$$\longmapsto \Delta_3 = W[a_1, a_2, a_3] \cdot W[b_1, b_2, b_3] \equiv w_a w_b = \varepsilon_3^{-1}$$

$$W^{(a)} \equiv W[a_1, a_2, a_3] = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1' & a_2' & a_3' \\ a_1'' & a_2'' & a_3'' \end{vmatrix}, \quad \underset{W^{(b)}}{\underset{\parallel}{W[b_1, b_2, b_3]}} = \begin{vmatrix} b_1 & b_2 & b_3 \\ b_1' & b_2' & b_3' \\ b_1'' & b_2'' & b_3'' \end{vmatrix}$$

$\boxed{W^{(a)} = w_a}$ is to be solved w.r.t. $a_3(u)$ for given a_1, a_2

$$a_3 = \sum_{k=1}^{2} a_k(u)\, I_k(u), \quad I_k(u) = w_a \int du \, \frac{a_k(u)}{W^2(u)}$$
$$W(u) \equiv W[a_1, a_2] \equiv a_1 a_2' - a_1' a_2$$
(similar solution for $b_3(v)$)

simple generali of th Liouville

- The most general form (general. of $\mathcal{L}$)

$$\boxed{X = \sum_{i,j=1}^{3} a_i(u)\, C_{ij}\, b_j(v)\,, \quad \det(C_{ij}) = 1}$$

compare to Liouville!

'Dynamical' reduction to waves:

- $a_i(u) = \alpha_i e^{\mu_i u}$; $b_i(v) = \beta_i e^{\nu_i v}$.
- $X = \sum_1^3 f_i e^{\mu_i u + \nu_i v}$; $\sum \mu_i = \sum \nu_i = 0$

$$\begin{cases} f_1 \cdot f_2 \cdot f_3\, \tilde{\Delta}_\mu \tilde{\Delta}_\nu = w_a w_b = \varepsilon_3^{-1}, \text{ where} \\ \tilde{\Delta}_\mu \equiv \prod_{i>j} (\mu_i - \mu_j)\,, \quad \tilde{\Delta}_\nu \equiv \prod_{i>j} (\nu_i - \nu_j) \end{cases}$$

- This is a generalization of the N-Liouville

There exist nonsingular waves (when $X \neq 0$ and $\Delta_2(X) \neq 0$ for any finite (u, v))

D.G. reducible to A_2-Toda:

$V \equiv V(\psi_1, \psi_2)$, $Z = \text{const}$

(ψ_1, ψ_2) – linear comb. of (x_1, x_2)

$V = g_1 e^{2x_1 - x_2} + g_2 e^{-x_1 + 2x_2}$ | see App.

- Easily generalizable to A_N-Toda.

(6) Wave-like solutions in nonintegrable 'realistic' models: (1 scalar, 1-term in V)

$$V = g\underbrace{\varphi^{1+a}}_{\text{not exponential}} e^{\lambda\psi}, \quad Z = -\varphi \leftarrow \text{Bub}$$

def.: $\varphi \equiv e^{\phi}$, $\Psi \equiv \psi + \lambda\phi/2$, $\mu^2 \equiv \frac{\lambda^2}{4} - 1 - a$

Special Waves (not general)

$$\begin{cases} \phi = \frac{\alpha+\beta}{2}(r + V_1 t) & V_1 = \frac{\alpha-\beta}{2} \\ \Psi = \frac{1}{2}\left[(\mu - \frac{\lambda}{2})\alpha + (\mu + \frac{\lambda}{2})\beta\right](r + V_2 t) \\ F = -\frac{1}{2}\left[\alpha(a - \frac{\lambda^2}{2} + \lambda\mu) + \beta(a - \frac{\lambda^2}{2} - \lambda\mu)\right](r + V_3 t) \end{cases}$$

- With $\alpha = \beta$: $V_1 = 0$, $V_2 = -\frac{2\mu}{\lambda}$, $V_3 = -\frac{2\mu}{\lambda}\left(1 - \frac{2a}{\lambda^2}\right)^{-1}$

(Ψ may be finite for $r \to \pm\infty$)

{We have to look for (S C W) triality in realistic theories using either high-dim. equations or (prefer.) (1+1)-dim reduction

NB: [If we do not fix the gauge before writing equations, both approaches are equivalent

- For example, in spherically symm. th. S-C duality has been proved in both approaches

[A.T.F. hep-th/0605276]

See: V. de Alfaro + ATF → hep-th/0612258

(also: /050506 (Theor. M. Phys.)

Discussion / Summary

- S=C duality O.K. (for integrable and non-integrable models)
- $S \overset{W}{=} C$ needs further consideration (O.K. in N-Liouville and Toda)
- 'Naive' reduction often (usually, always?) incomplete.
- More general – reduction by separating (difficult in realistic non-integrable models), Most difficult: waves in non-integrable models (N-Liouv. and Toda show why)
- At the moment, we have only patterns of separation, not a 'theory'
- Possibly, a perturbation theory around an integrable model (or 'around a pattern') will suffice for getting interesting physics ?

HOPE!

APPENDIX I: MAIN EQUATIONS

$\boxed{+W(\varphi)(\nabla\varphi)^2}$

- $\mathcal{L}^{(2)} = \sqrt{-g_2}\left[\varphi R + V(\varphi;\psi) + \sum_{mn} Z_{mn} g^{ij} \partial_i \psi^m \partial_j \psi^n\right]$

LC coord. $ds^2 = -4f(u,v)\,du\,dv$ $(\varepsilon = \pm 1)$ $\boxed{f \equiv \varepsilon e^F}$

- $\partial_u \partial_v \varphi + fV = 0$; Suppose: $Z_{mn} = \delta_{mn} Z_n$
- $f\partial_i(\partial_i\varphi / f) = \sum Z_n (\partial_i \psi_n)^2$, $(i = u, v)$
- $\partial_u(Z_n \partial_v \psi_n) + (u \leftrightarrow v) + f\dfrac{\partial V}{\partial \psi_n} = \sum_m \underbrace{Z_{m,\psi_n}}_{\partial Z_m/\partial \psi_n} \partial_u \psi_m \partial_v \psi_m$

The Liouville equation:

- $\begin{cases} \partial_u \partial_v q = g e^q, \quad X \equiv e^{-q/2} \\ \boxed{X \cdot \partial_u \partial_v X - \partial_u X \partial_v X = -\frac{1}{2} g} \end{cases}$ Bilinear form

- $\mathcal{L}^{(4)} = \sqrt{-g_4}\left[R(g_4) + V_4(\psi) + Z_4(\psi)(\nabla\psi)^2\right]$

  Spherical $(D=4)$ reduction: $\alpha(t,r), \beta(t,r) \ldots\ldots$

  $ds_4^2 = e^{2\alpha} dr^2 + e^{2\beta} d\Omega_k^2(\theta,\phi) - e^{2\gamma} dt^2 + 2e^{2\delta} dt\,dr$

- In: $d = 1+1$ $(\delta \to \infty)$

eff.: $V(\varphi,\psi) = V_4 \cdot e^{2\beta} + 2k$ $(k = \pm 1, 0)$ spher. pseudo spher. flat

eff: $Z_{mn} \to \delta_{mn} Z_4\, e^{2\beta}$

APPENDIX II: Dim. red.

Block-diagonal $\bar{g} = \left[\begin{array}{c|c} g_{ij} & 0 \\ \hline 0 & h_{mn} \end{array}\right]$, $\left\{ \bar{g}_{MN},\ M = (i, m) \right.$

$$R(\bar{g}) = R(g) + R(h) - \frac{2}{\sqrt{h}} \nabla_g^2 \sqrt{h} +$$

∇_g – covar. der.

$$+ \frac{1}{4} g^{ij} \partial_i h^{mn} \partial_j h_{mn} + \frac{1}{4} g^{ij} (h^{mn} \partial_i h_{mn})(h^{\tilde{m}\tilde{n}} \partial_j h_{\tilde{m}\tilde{n}})$$

+ similar terms with $(ij) \leftrightarrow (mn)$, $g \leftrightarrow h$, ...

Kaluza: ("toroidal" reduction)
$$\bar{g} = \begin{bmatrix} g_{ij} + A_i^{\tilde{m}} A_j^{\tilde{n}} h_{\tilde{m}\tilde{n}} & A_{im} \\ A_{mj} & h_{mn} \end{bmatrix}, \quad \begin{cases} h_{mn}(x^k) \\ g_{ij}(x^k) \\ A_{im}(x^k) \end{cases}$$

with additional terms ($R(h) = 0$, but $\sim F^2$ appear)

$$-\frac{1}{4} h_{mn} F^m_{ij} F^{nij}, \quad F^m_{ij} \equiv \partial_i A^m_j - \partial_j A^m_i .$$

- **Kaluza** in $D = 1 + 3$ (generalized cylindr. symm)

- $\mathcal{L}_4 = \sqrt{-g_4} \left[R_4 + V_4 + Z_4(\psi)(\nabla\psi)^2 \right]$, $(h_{mn} \equiv \varphi \sigma_{mn})$

$\left[x^i = (r, t);\ h_{mn}(r, t),\ \psi(r, t) \right]$ dilaton

- $$\mathcal{L}_2 = \sqrt{-g} \Big\{ \varphi \left[R(g) + V_4 + Z_4 (\nabla\psi)^2 \right] + \frac{1}{2\varphi} (\nabla\varphi)^2 - \underbrace{\frac{\varphi}{4} \mathrm{tr}(\nabla\sigma \cdot \sigma^{-1} \cdot \nabla\sigma \cdot \sigma^{-1})}_{\sigma\text{-model}} - \underbrace{\frac{\varphi^2}{4} \sigma_{mn} F^m_{ij} F^{nij}}_{\text{'gauge' field term}} \Big\}$$

$(ij = 0, 1)$

| Gauge terms gives 'cosmological potential (effective)' |

APP. III. The (fairly) general integrable theory (1+0)

(0+1) $$\mathcal{L} = -\frac{1}{\ell}\left(\dot F \dot\varphi + \sum_{n=3}^{N} Z_n \dot\psi_n^2\right) + \ell\left(\sum_{n=1}^{N} \frac{1}{2} g_n e^{q_n}\right)$$

$$q_n = \sum_{m=1}^{N} \psi_m a_{mn} = F + a_n\varphi + \sum_{m=3}^{N} \psi_m a_{mn}$$

- $F = \psi_1 + \psi_2$, $\varphi = \psi_1 - \psi_2$, $a_{1n} = 1 + a_n$, $a_{2n} = 1 - a_n$

(Orth.) $$\boxed{\sum_{m=1}^{N} \varepsilon_m a_{mn} a_{mn'} = \gamma_n^{-1}\delta_{nn'}}; \quad \varepsilon_1 = -1,\ \varepsilon_2 = \varepsilon_3 = \ldots = +1$$

So, $\mathcal{L} = \frac{1}{\ell}\sum_{m=1}^{N} \frac{\varepsilon_m}{2}\dot\psi_m^2 + \bar\ell \sum_{m=1}^{N} g_m e^{q_m}$, $(Z_n = -1)$

$$\Rightarrow \boxed{\ddot q_n = \frac{g_n}{\gamma_n} e^{q_n}}, \text{ where } \text{'}\cdot\text{'} \equiv \frac{d}{d\tau}, \quad \tau = \int^t \bar\ell(t)\,dt$$

(N – Liouville theory)

(Orth.) can be solved explicitly for arb. N

$\frac{N(N-1)}{2}$ coef. a_{mn} are arbitrary

Integrals: $$\dot q_n^2 - 2\frac{g_n}{\gamma_n} e^{q_n} = C_n \Rightarrow \boxed{\frac{C_n\gamma_n}{g_n}\left(1 + \varepsilon \operatorname{ch}\sqrt{C_n}\,\tau_{(n)}\right)^{-1} = e^{q_n}, \quad \tau_{(n)} = \tau - \tau_n}$$

$\boxed{C_n, \tau_n - \text{integrals}}$

Constraint: $$\boxed{\sum_{n=1}^{N} \gamma_n C_n = 0} \qquad \Sigma\gamma_n = 0$$

- **Remark**: if $Z_n = -\frac{\gamma_n}{y'(\varphi)}$, $g_n \sim y'(\varphi) e^{a_n y}$ we also have 0+1 d. integrable system.

(1+1) $$\mathcal{L} = \sqrt{-g}\left(\varphi R + \frac{1}{2}\sum_{1}^{N} g_n e^{q_n - F} + Z_n \psi^{(n)}_{;i}\psi^{(n)}_{;j} g^{ij}\right)$$

N-Liouv. ($e^F = f$ in (u,v) variables, Z_n = constants)
(constraints solved!)