Consistent noncommutative gauge theories.

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$x_\mu \to [\hat{x}_\mu, \hat{x}^\nu] = i\hbar\theta_{\mu\nu}$
$\downarrow$
$l \gtrsim l_{min}$

(M. A. Markov, H. Snyder (1947), V. Kadyshewsky (1961).

Smearing of UV behaviour → nonlocal theory

problems: unitarity, causality.

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Noncommutative field theory =
= effective theory.

1) $N \to \infty$ QCD: reduced quenched model
(T. Eguchi, H. Kawai (1982); A Gonsales, C.P. Korthales (1983)
$\Downarrow$
Quantum Noncommutative Y-M model
(J. Ambjorn et. al 2000)

2) Compactified Matrix models
$\parallel$
supersymmetric YM on a noncommutative torus (A. Cones, M. R. Duglas, A. Schwartz (1998)

3) (Brane) string dynamics.

(N. Seiberg, E. Witten 1999)

4) Quantum Hall effect (L. Susskind 2001, A. Polychronakos 2001)

Noncommutative quantum field theory may violate some physical requirements, but must allow a consistent calculation scheme.

A simple example:

$$S = \int d^4x \left(\frac{1}{2} (\partial_\mu \varphi)^2 + \frac{m^2}{2} \varphi^2 + g\, \varphi^4 \right)$$

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \varphi(k)$$

$$\downarrow$$

$$x_\mu \to \hat{x}_\mu \; ; \quad [\hat{x}^\mu, \hat{x}^\nu] = i\,\xi\, \theta_{\mu\nu} \; ; \quad \theta_{\mu\nu} = -\theta_{\nu\mu} \; ;$$

$$\varphi(x) \to \hat{\varphi}(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\hat{x}} \varphi(k)\, d^4k$$

Weyl symbol

$\varphi(x)$ may be thought as a coordinate space representation of $\hat{\varphi}(x)$.

Multiplication law for operator symbols:

$$\hat{\varphi}_3(x) = \hat{\varphi}_1(x)\hat{\varphi}_2(x)$$

$$\downarrow$$

$$\varphi_3(x) = \varphi_1(x)\exp\left\{\frac{i}{2}\overleftarrow{\partial}_\mu \xi \theta_{\mu\nu} \overrightarrow{\partial}_\nu\right\}\varphi_2(x) \equiv \varphi_1(x) * \varphi_2(x)$$

Moyal product

$$S_{nc} = \int d^4x \left[\frac{1}{2}\partial_\mu\varphi * \partial_\mu\varphi + \frac{m^2}{2}\varphi * \varphi + g\varphi * \varphi * \varphi * \varphi\right] =$$

$$= \int d^4p \left[\frac{1}{2}\varphi(p)(p^2 - m^2)\varphi(p) + \int dp_1 \ldots dp_4 e^{-i\xi p_1 \tilde{p}_2 - i\xi p_3 \tilde{p}_4} \times\right.$$

$$\times \varphi(p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4)\delta(p_1 + p_2 + p_3 + p_4). \quad \tilde{p}_\mu \equiv \theta_{\mu\nu} p_\nu$$

— o —

Noncommutative gauge invariant models.

$U(1)$:

$$U = {}_* e^{i\lambda} = \sum_{n=0}^{\infty} \frac{1}{n!} {}_*(i\lambda)^n$$

Covariant derivative in adjoint representation

$$D_\mu\varphi = \partial_\mu\varphi - ig[A_\mu * \varphi - \varphi * A_\mu]$$

$$A_\mu \to U * A_\mu * U^+ + \frac{i}{g} U * \partial_\mu U^+$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu * A_\nu - A_\nu * A_\mu]$$

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu} F_{\mu\nu}.$$

Nonabelian model even for $U(1)$.

If $\theta_{0i} = 0$, the standard Hamiltonian formalism may be applied.

We choose $\theta_{12} = -\theta_{21} = 1$; $\theta_{3i} = 0$; $\theta_{0i} = 0$.

Quantization:

$$S_{gf} = -\int \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 d^4x$$

$$S_{FP} = \int \partial_\mu \bar{c} D_\mu c \, d^4x .$$

Feynman rules:

Propagators:

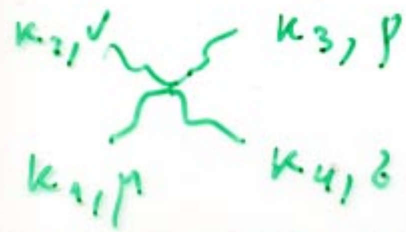
$$D_{\mu\nu} = \frac{1}{k^2} \left(g^{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \alpha \frac{k_\mu k_\nu}{k^4}$$

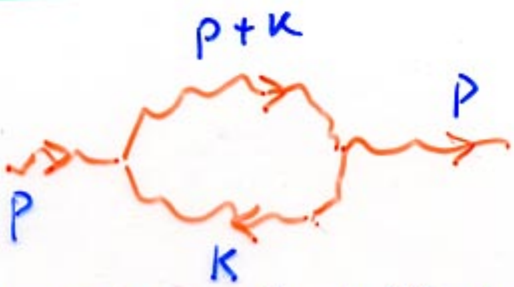
$$D_c = \frac{1}{k^2}$$

Vertices: YM vertices with the structure constants replaced by trigonometric factors:

$$-2ig \sin(\xi k_1 \tilde{k}_2) \left[(k_1 - k_2)_\rho g_{\mu\nu} + (k_2 - k_3)_\mu g_{\nu\rho} + (k_3 - k_1)_\nu g_{\mu\rho} \right]$$

$$4g^2 \left[(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) \sin(\xi k_1 \tilde{k}_2) \sin(\xi k_3 \tilde{k}_4) + \text{sym.} \right]$$





$$\Pi_{\mu\nu} = \int d^4k \, \sin^2(\xi k \tilde{p}) \frac{(k^2 + (k+p)^2 \ldots) \delta^{\mu\nu} + 10 k^\mu k^\nu + \ldots}{k^2 (k+p)^2}$$

$$\sin^2(\xi k \tilde{p}) = \frac{1}{2}(1 - \cos(2\xi \tilde{k} p))$$

divergent planar diagram | convergent nonplanar diagram

Planar diagrams are divergent and are renormalized in a usual way (charge, mass, wave function renormalization).

Nonplanar diagrams are convergent, but have poles at $p=0$

$$\Pi_{\mu\nu}(p)=\frac{g^2}{(2\pi^2)}\,\frac{\tilde{p}_\mu \tilde{p}_\nu}{\xi^2(\tilde{p}^2)^2}+\dots$$

$$\Gamma_{\mu\nu\rho}(p,q)\sim\cos(\xi p\tilde{q})\left\{\frac{\tilde{p}_\mu \tilde{p}_\nu \tilde{p}_\rho}{\xi(\tilde{p}^2)^2}+\text{sym}+\dots\right\}$$

Diagrams which in commutative theory diverge quadratically or linearly acquire infrared pole singularities

nonintegrable infrared singularity

(S. Minwalla, M. Van Raamsdonk, N. Seiberg (1999), M. Hayakawa (2000); I. Matusis, L. Susskind, N. Toumbas (2000); I. Arefieva et al (2000).

The theory does not exist beyond one loop!

Noncommutative U_N-models.

$A_\mu = \sum_a A_\mu^a T^a \quad ; \quad T^a \in SU(N)$

$[T^a; T^b] = t^{abc} T^c$

$[A_\mu, A_\nu] = B_{\mu\nu} \rightarrow$ commutative case

$[A_\mu * A_\nu - A_\nu * A_\mu] = B_{\mu\nu} + I . D_{\mu\nu}$

$SU(N)$ algebra is not closed under Moyal multiplication.

Only $U(N)$ gauge invariant noncommutative models may be constructed.

$$S = -\frac{1}{8} \operatorname{Tr} \int \mathcal{F}_{\mu\nu} \mathcal{F}_{\mu\nu}$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,(A_\mu * A_\nu - A_\nu * A_\mu).$$

Diagrams with $U(1)$ external lines exibit infrared singularities.

Diagrams with pure $SU(N)$ external lines have only logarithmic infrared singularities. (at least at one loop)

(A. Armoni 2001, L. Bonora, M. Salizzoni 2001)

Do quantum noncommutative gauge theories make sence?

1) Supersymmetric gauge models.

No pole infrared singularities at one loop.

(M. Sheikh-Jabbari 1999, D. Zanon 2001, I. Jack, D.R.T. Jones 2001); W-Z model (H. Girotti et.al, 2000)

2) Nonlocal counterterms?

Consistency? Ambiguity? Gauge invariance?

3) Modified noncommutative gauge models.

(A.A.S. 2003)

Requirements:

a) Gauge invariance.

b) Nonlocality may appear only via Moyal product.

c) In a classical commutative limit Lorentz invariance must be restored.

d) Power counting renormalizability.

e) No nonintegrable infrared poles.

New types of counterterms → classical Lagrangian is not complete.

E.g. $\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi + i\bar{\psi}\hat{\partial}\psi + g\bar{\psi}\gamma^5\psi\varphi$

 $\sim \lambda\varphi^4 \to \tilde{\mathcal{L}} = \mathcal{L} + \lambda\varphi^4$.

A more general power-counting renormalizable, gauge invariant action

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}F_{\mu\nu} + \beta\lambda(x)\theta_{\mu\nu}F_{\mu\nu}(x) + \gamma\left(\theta_{\mu\nu}F_{\mu\nu}(x)\right)^2 \right.$$

1) $\beta = 1$; $\gamma = 0$

Classical commutative limit $\xi \to 0$

$\theta_{12} = -\theta_{21} = 1$; $\theta_{i3} = \theta_{i0} = 0$. $\sum_i \tilde{\partial}_i^2 = \sum_{i=1,2} \partial_i^2 \equiv \tilde{\partial}^2$.

Generalized Hamiltonian system:

$$S_0 = \int d^4x \left[p_i\dot{A}_i - \frac{p_i^2}{2} - \frac{1}{4}(\partial_i A_j - \partial_j A_i)^2 + A_0\partial_i p_i + \lambda\tilde{\partial}_i A_i \right]$$

Coulomb gauge: $\partial_i A_i = 0$.

Constraints: $\partial_i p_i = 0$, $\tilde{\partial}_i A_i = 0$

$[H, \tilde{\partial}_i A_i] = \tilde{\partial}_i p_i$ → secondary constraint

$$\boxed{\tilde{S} = S_0 + \int \mu(x)\tilde{\partial}_i p_i \, d^4x}$$

$A_i = \partial_i\chi + \tilde{\partial}_i\psi + \partial^{-2}\partial_j\tilde{\partial}_k\varepsilon^{ijk}\varphi$.

$p_i = \partial_i p_\chi + \tilde{\partial}_i p_\psi + \varepsilon^{ijk}\partial^{-2}\partial_j\tilde{\partial}_k p_\varphi$

$$S_0 \equiv \int d^4x \left[p_\varphi\dot{\varphi} - \frac{p_\varphi^2}{2} - \frac{\partial_i\varphi\partial_i\varphi}{2} \right]$$

The usual free scalar field.

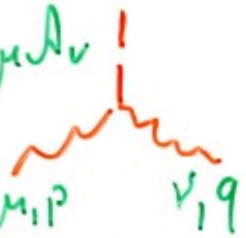
Diagram technique:

$\lambda\lambda$: $k^2\tilde{k}^{-2}$

λA_μ : $\tilde{k}_\mu(\tilde{k})^{-2}$

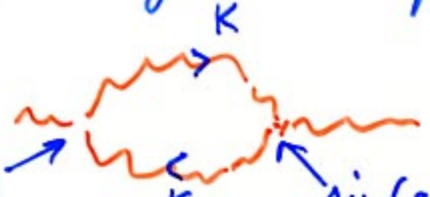
$A_\mu A_\nu$: $\frac{1}{k^2}\left(g^{\mu\nu}-\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right)$

$\lambda A_\mu A_\nu$



$ig\sin(\xi p\tilde{q})\theta_{\mu\nu}$

Polarization operator



$\sin(\xi p\tilde{k})$ $\sin(\xi p\tilde{k})$

$\Pi^{sing}_{\mu\nu}(p)$ (leading singularities)

$$\Pi^{sing}_{\mu\nu}(p)=\int d^4k\,\sin^2(\xi p\tilde{k})\,P_{\mu\nu}(k)$$

$$\sin^2(\xi p\tilde{k}) = \frac{1}{2}\left(1-\cos(2\xi p\tilde{k})\right)$$

planar infrared safe contribution — nonplanar infrared pole contribution

$$\Pi^{sing}_{\mu\nu}(p)\sim\frac{A g^{\mu\nu}\tilde{p}^2+B\tilde{p}_\mu\tilde{p}_\nu}{\xi^2(\tilde{p}^2)^2}+\ldots$$

Generalized Ward Identity

$p_\mu p_\nu \Pi^{sing}_{\mu\nu}(p) = 0 \rightarrow A = 0$

$$\Pi^{sing}_{\mu\nu} \sim \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2}$$

$= 0$, due to transversality of Yang-Mills propagator $\tilde{p}_\mu D_{\mu\nu}(p) = 0$.

$$\sim \frac{\sin^2(\xi p \tilde{q})}{(\tilde{p}^2)^2 \xi^2}$$

No infrared divergency

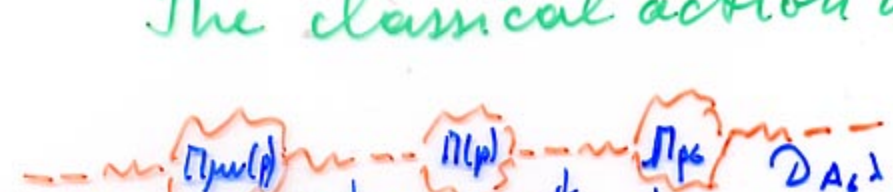
$\lim \xi \to 0$ nonsingular

logarithmically divergent → → no infrared poles.

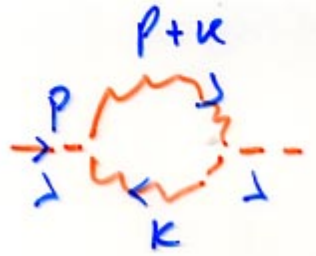
However: possible new counterterm $\sim \lambda\lambda$

↓

The classical action is not complete!



Accumulation of infrared poles may produce an infrared divergency.



$= \Pi(p)$

$$\Pi(p)\big|_{p\sim 0} \sim \xi^2 p^4 \ln(p^2 \xi)$$

$\Pi(p)$ compensates infrared poles of $\Pi_{\mu\nu}(p)$

No infrared divergency..

Three point function

$$\langle A_\mu(x) A_\nu(y) \partial_\rho A_\rho(z)\rangle = \langle \partial_\mu M^{-1}_{xz} A^a_\nu(y)\rangle +$$

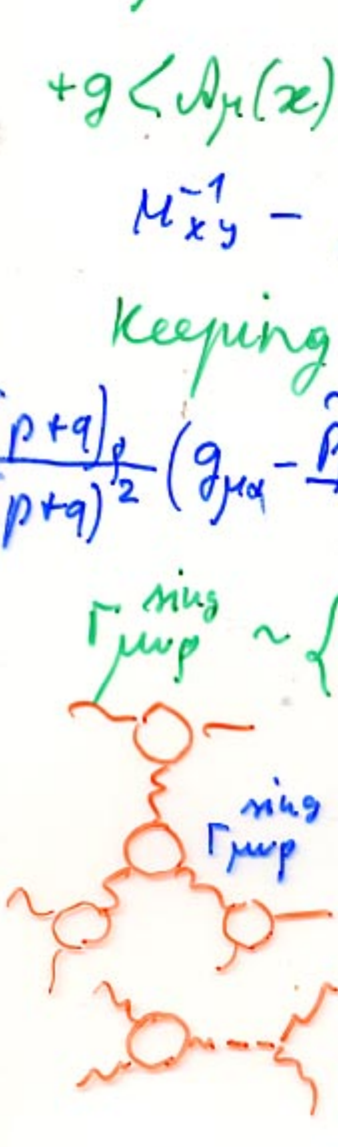
$$+ g\langle A_\mu(x) M^{-1}_{xz} A_\nu(y)\rangle + (\mu\to\nu,\ x\to y).$$

M^{-1}_{xy} – ghost green function

Keeping only pole singular terms

$$\frac{(p+q)_\rho}{(p+q)^2}\left(g_{\mu\alpha} - \frac{\tilde p_\mu \tilde p_\alpha}{\tilde p^2}\right)\left(g_{\nu\beta} - \frac{\tilde q_\nu \hat q_\beta}{\tilde q^2}\right)\Gamma^{sing}_{\alpha\beta\rho}(p,q) = 0$$

$$\Gamma^{sing}_{\mu\nu\rho} \sim \left\{ \frac{\tilde p_\mu \tilde p_\nu \tilde p_\rho}{\xi^2(\tilde p^2)^2} + (p\to q) + (p\to(-p-q)) \right\}$$



$\Gamma^{sing}_{\mu\nu\rho} = 0$

no infrared divergency.

U(N) model.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\,[A_\mu * A_\nu - A_\nu * A_\mu]$$

$$A_\nu = A_\nu^a T^a + A_\nu^0 I$$

$$S = \int d^4x \, \mathrm{tr}\left\{-\frac{1}{4} F_{\mu\nu} F_{\mu\nu}\right\} + \int \lambda(x) \underbrace{\mathrm{tr}\left(\theta_{\mu\nu} F_{\mu\nu}(x)\right)}_{\text{only } U(1) \text{ part contributes}} d^4x$$

The free action:

$$S_0 = \int \Big[\underbrace{-\frac{1}{4}\sum_a \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2}_{\text{The usual } SU(N) \text{ part}} \underbrace{- \frac{1}{4}\left(\partial_\mu A_\nu^0 - \partial_\nu A_\mu^0\right)^2 + \lambda(x)\tilde{\partial}_\mu A_\mu^0(x)}_{U(1) \text{ part}} \Big] d^4x$$

The usual SU(N) part → SU(N) vector bosons

U(1) part → scalar U(1) particle

Diagrams with at least one external U(1) line are analyzed as before → no infrared divergencies.

Diagrams with only SU(N) external lines do not produce infrared poles (at least at one loop) (A. Armoni, 2000).

Modified U(N) models are consistent!

Conclusion.

1. Nonsupersymmetric consistent quantum noncommutative $U(N)$ gauge theories do exist.

2. Their spectrum differes from naive one.
$U(1)$ gauge fields → scalar particles.

Questions

1. Rigorous renormalization to all orders.
2. Logarithmic infrared singularities → → commutative limit?
No breaking of Lorentz invariance in the commutative limit?
3. Explicit study of higher loops in $U(N)$ models.

4. Modified theory with $\gamma = 0$

$$\Delta S = -\frac{1}{8}\int (\theta_{\alpha\beta} F_{\alpha\beta})^2 d^4x.$$