# RENORMALIZATION GROUP

## 1 The main formulae

Consider the multiplicative renormalization of the Green function

$$\Gamma(\{p^2\},\mu^2,g_\mu) = Z_\Gamma(\Lambda^2,\mu^2,g_\mu)\Gamma_{Bare}(\{p^2\},\Lambda,g_{Bare}) \tag{1}$$

or in dimensional regularization

$$\Gamma(\{p^2\},\mu^2,g_\mu) = Z_\Gamma(1/\varepsilon,g_\mu)\Gamma_{Bare}(\{p^2\},1/\varepsilon,g_{Bare}), \tag{2}$$

where the Green function is normalized by

$$\Gamma(\{p^2\},\mu^2,0) = 1.$$

The bare quantities do not depend on $\mu$. Differentiating eq.(2) with respect to $\mu$ one has:

$$\mu^2\frac{d}{d\mu^2}\Gamma = \left(\mu^2\frac{\partial}{\partial\mu^2} + \mu^2\frac{\partial g}{\partial\mu^2}\frac{\partial}{\partial g}\right)\Gamma = \mu^2\frac{d\ln Z_\Gamma}{d\mu^2}Z_\Gamma\Gamma_{Bare},$$

or

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \beta(g)\frac{\partial}{\partial g} + \gamma_\Gamma\right)\Gamma(\{p^2\},\mu^2,g_\mu) = 0, \tag{3}$$

where the so-called $\beta$ function $\beta(g)$ and the anomalous dimension $\gamma_\Gamma(g)$ are defined by

$$\beta = \mu^2\frac{dg}{d\mu^2}|_{g_{bare}}, \tag{4}$$

$$\gamma_\Gamma = -\mu^2\frac{d\ln Z_\Gamma}{d\mu^2}|_{g_{bare}}. \tag{5}$$

Eq.(3) is called the *renormalization group equation.*

Solution of the renormalization group equation can be written in terms of characteristics:

$$\Gamma\left(e^t\frac{\{p^2\}}{\mu^2},g\right) = \Gamma\left(\frac{\{p^2\}}{\mu^2},\bar{g}(t,g)\right)e^{\int_0^t \gamma_\Gamma(\bar{g}(t,g))dt}, \tag{6}$$

where the characteristic equation is

$$\frac{d}{dt}\bar{g}(t,g) = \beta(\bar{g}),\quad \bar{g}(0,g) = g. \tag{7}$$

The quantity $\bar{g}(t,g)$ is called the *effective charge.*

Since the vertex function usually comes with the coupling, one can consider the product

$$g\Gamma\left(\frac{\{p^2\}}{\mu^2},g\right). \tag{8}$$

If $\Gamma$ is n-point function, then the renormalization of the coupling $g$ is given by

$$g_{Bare} = Z_\Gamma Z_2^{-n/2} g,$$

and the product (8) is renormalized by

$$g\Gamma = Z_2^{n/2} g_{Bare}\Gamma_{Bare}.$$

Hence one has the same equation as (2) with the solution (6), but with $Z_\Gamma = Z_2^{n/2}$ and $\gamma_\Gamma = -n/2\gamma_2$. (Recall that the anomalous dimension $\gamma_2$ is defined with respect to the renormalization constant $Z_2^{-1}$.)

Furthermore, one can construct the so-called *invariant charge* by multiplying the product (8) by the corresponding propagators

$$\xi = g\Gamma\left(\frac{\{p^2\}}{\mu^2},g\right)\prod_i^n D^{1/2}\left(\frac{p_i^2}{\mu^2},g\right). \tag{9}$$

The invariant charge $\xi$, being RG-invariant, obeys the RG equation without the anomalous dimension and plays the important role in the formulation of the renormalization group together with the effective charge. In some cases the effective and invariant charges coincide.

The usefulness of the solution (6) is that it allows one to sum up an infinite series of logs coming from the diagrams in the infrared ($t \to -\infty$) or ultraviolet ($t \to \infty$) regime and to improve the usual perturbation theory expansions.

To demonstrate the power of the RG let us consider the invariant charge. Let the perturbative expansion to be

$$\xi(\frac{p^2}{\mu^2},g) = g(1 + bg\ln\frac{p^2}{\mu^2} + ...). \tag{10}$$

The $\beta$ function in one loop is given by

$$\beta(g) = bg^2. \tag{11}$$

Notice that the coefficient $b$ in front of the logarithm in eq.(10) coincides with that of the $\beta$ function. Alternatively the $\beta$ function can be defined as the derivative of the invariant charge with respect to logarithm

$$\beta(g) = p^2\frac{d}{dp^2}\xi(\frac{p^2}{\mu^2},g)|_{p^2=\mu^2}. \tag{12}$$

According to eq.(6) (with vanishing anomalous dimension) the RG-improved expression for the invariant charge corresponding to the perturbative expression (10) is:

$$\xi_{RG}(\frac{p^2}{\mu^2},g) = \xi_{PT}(1,\bar{g}(\frac{p^2}{\mu^2},g)) = \bar{g}(\frac{p^2}{\mu^2},g), \tag{13}$$

where we have put in eq.(6) $p^2 = \mu^2$ and $t = \ln p^2/\mu^2$. The effective coupling is a solution of the equation

$$\frac{d}{dt}\bar{g}(t,g) = b\bar{g}^2, \quad \bar{g}(0,g) = g, \quad t \equiv \ln\frac{p^2}{\mu^2}. \tag{14}$$

Solution of this equation is

$$\bar{g}(t,g) = \frac{g}{1-bgt}. \tag{15}$$

Being expanded over $t$, the geometrical progression (15) reproduces the expansion (10), however it sums the infinite series of terms of the form $g^n t^n$. This is called the leading log approximation (LLA) in QFT. To get the correction to the LLA one has to consider the next term in the expansion of the $\beta$ function. Then one can sum up the next series of terms of the form $g^n t^{n-1}$, which is called next to leading log approximation (NLLA), etc. This procedure allows one to describe the leading asymptotics of the Green functions for $t \to \pm\infty$.

Consider now the Green function with non-zero anomalous dimension. Let its perturbative expansion to be

$$\Gamma(\frac{p^2}{\mu^2}, g) = 1 + cg\ln\frac{p^2}{\mu^2} + ... \tag{16}$$

Then in the one-loop approximation the anomalous dimension is

$$\gamma(g) = cg. \tag{17}$$

Again the coefficient in front of the logarithm coincides with that of the anomalous dimension. In analogy with eq.(12) the anomalous dimension can be defined by

$$\gamma(g) = p^2\frac{d}{dp^2}\ln\Gamma(\frac{p^2}{\mu^2}, g)|_{p^2=\mu^2}. \tag{18}$$

Substituting (17) to eq.(6), one has in the exponent

$$\int_0^t \gamma(\bar{g}(t,g)dt = \int_g^{\bar{g}} \frac{\gamma(g)}{\beta(g)}dg = \int_g^{\bar{g}} \frac{cg}{bg^2}dg = \frac{c}{b}\ln\frac{\bar{g}}{g}.$$

This gives for the Green function

$$\Gamma_{RG} = \left(\frac{\bar{g}}{g}\right)^{-c/b} = \left(\frac{1}{1-bgt}\right)^{c/b} \approx 1 + ct + ... \tag{19}$$

Thus one again reproduces the perturbative expansion, but expression (19) again contains the whole infinite sum of the leading logs. To get the NLLA one has to take into account the next term in eq.(17) together with the expansion of the $\beta$ function (11).

All the formulas can be easily generalized to the case of multiple couplings and masses.

## 2 Effective coupling

Due to a central role playing by effective coupling in RG formulas consider it in more detail. The behaviour of the effective coupling is determined by the $\beta$ function. Qualitatively the $\beta$ function can exhibit the following behaviour shown in Fig.1.

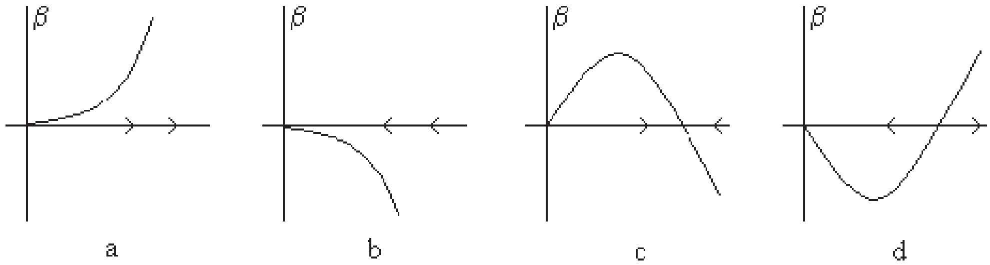


*Fig.1*: The shape of the $\beta$ function. The arrows show the behaviour of the effective charge in the UV regime ($t \to \infty$)

One can distinguish the Landau pole (a), the asymptotic freedom (b), the UV fixed point (c), and IR fixed point (d) behaviour.

In perturbation theory, the one loop $\beta$ function is given by eq.(11). Depending on the sign of the coefficient $b$ one has: the UV Landau pole and the infrared freedom ($b > 0$), or the asymptotic freedom and the IR Landau pole ($b < 0$) behaviour given by eq.(15). (The Landau pole is the pole in eq.(15) for a finite value of $t$, where the coupling goes to infinity for a finite value of momenta.)

## 3 Dimensional regularization and the $\overline{MS}$ scheme

Consider now how to calculate the $\beta$ function and the anomalous dimensions in some particular models within dimensional regularization and the minimal subtraction scheme.

In $\overline{MS}$ scheme the renormalization constants are given by the pole terms in $1/\varepsilon$ expansion and so does the bare coupling. One has:

$$Z_\Gamma = 1 + \sum_{n=1}^{\infty} \frac{c_n(g)}{\varepsilon^n} = 1 + \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{c_{nm} g^m}{\varepsilon^n}, \tag{20}$$

$$g_{Bare} = (\mu^2)^\varepsilon \left[ g + \sum_{n=1}^{\infty} \frac{a_n(g)}{\varepsilon^n} \right] = (\mu^2)^\varepsilon \left[ g + \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{a_{nm} g^m}{\varepsilon^n} \right]. \tag{21}$$

Differentiating eq.(20) with respect to $\ln \mu^2$ and having in mind the definitions (4) and (5), one has:[1]

$$-[1 + \sum_{n=1}^{\infty} \frac{c_n(g)}{\varepsilon^n}] \gamma_\Gamma(g) = [-\varepsilon g + \beta(g)] \frac{d}{dg} \sum_{n=1}^{\infty} \frac{c_n(g)}{\varepsilon^n}.$$

[1] note that in dimensional regularization since the dimension is changed from 4 to $4 - 2\varepsilon$ the $\beta$ function has the form $\beta_{4-2\varepsilon}(g) = -\varepsilon g + \beta_4(g)$

Equalizing the coefficients of equal powers of $\varepsilon$, one finds

$$\gamma_\Gamma(g) = g\frac{d}{dg}c_1(g), \tag{22}$$

$$g\frac{d}{dg}c_n(g) = [\gamma_\Gamma(g) + \beta(g)\frac{d}{dg}]c_{n-1}(g), \quad n \geq 2. \tag{23}$$

One sees that the coefficients of the higher poles $c_n,\ n \geq 2$ are completely defined by that of the lowest pole $c_1$ and the $\beta$ function. In its turn the $\beta$ function is also defined by the lowest pole. To see this, consider eq.(21). Differentiating it with respect to $\ln\mu^2$ one has:

$$\varepsilon\left[g + \sum_{n=1}^{\infty}\frac{a_n(g)}{\varepsilon^n}\right] + [-\varepsilon g + \beta(g)]\left[1 + \frac{d}{dg}\sum_{n=1}^{\infty}\frac{a_n(g)}{\varepsilon^n}\right] = 0. \tag{24}$$

Equalizing the coefficients of equal powers of $\varepsilon$, one finds

$$\beta(g) = (g\frac{d}{dg} - 1)a_1(g), \tag{25}$$

$$(g\frac{d}{dg} - 1)a_n(g) = \beta(g)\frac{d}{dg}a_{n-1}(g), \quad n \geq 2. \tag{26}$$

These equations are easily generalized for the multiple couplings case. In case of a single coupling they are simplified. In particular, substituting in the last equation the perturbative expansion (21) one can solve the recurrent equation and find for the highest pole term

$$a_{nn} = a_{11}^n, \tag{27}$$

which reflects the fact that the effective coupling in the LLA is given by a geometrical progression.

Consider now some particular models.

## 3.1 $\phi^4$ theory

The renormalization constants in $\overline{MS}$ scheme up to two loops are given by the following expressions: $(g \equiv \lambda/16\pi^2)$

$$Z_4 = 1 + \frac{3}{2\varepsilon}g + g^2(\frac{9}{4\varepsilon^2} - \frac{3}{2\varepsilon}), \tag{28}$$

$$Z_2^{-1} = 1 + \frac{g^2}{24\varepsilon}, \tag{29}$$

$$Z_g = 1 + \frac{3}{2\varepsilon}g + g^2(\frac{9}{4\varepsilon^2} - \frac{17}{12\varepsilon}). \tag{30}$$

$$\tag{31}$$

Notice that the higher pole coefficient $a_{22} = 9/4$ in the last expression is the square of the lowest pole one $a_{11} = 3/2$ in accordance with eq.(27).

Applying eqs.(22) and (25) this gives

$$\begin{aligned}\gamma_4(g) &= \frac{3}{2}g - 3g^2, &(32)\\ \gamma_2(g) &= \frac{1}{12}g^2, &(33)\\ \beta(g) &= g(\gamma_4 + 2\gamma_2) = \frac{3}{2}g^2 - \frac{17}{6}g^2. &(34)\end{aligned}$$

One can see from eq.(34) that the first coefficient of the $\beta$ function is 3/2, i.e. one has a Landau pole behaviour in the $\phi^4$ model in the LLA shown in Fig.1a. In the two-loop approximation (NLLA) the $\beta$ function gets a non-trivial zero and the effective coupling possesses an UV fixed point like the one shown in Fig.1c. However, this fixed point is unstable with respect to the higher orders and is not reliable.

## 3.2 QED

In QED due to the Ward identities the renormalization of the coupling is given by the photon wave function renormalization constant $Z_3$

$$\alpha_{Bare} = Z_3^{-1}\alpha, \quad \alpha \equiv \frac{e^2}{16\pi^2}.$$

In the one-loop approximation one has:

$$Z_3^{-1} = 1 + \frac{4}{3\varepsilon}\alpha, \tag{35}$$

hence

$$\gamma_3 = \frac{4}{3}\alpha, \quad \beta(\alpha) = \frac{4}{3}\alpha^2. \tag{36}$$

Thus, in QED, like in the $\phi^4$ theory one has a Landau pole behaviour in the LLA.

## 3.3 QCD

In QCD the calculation of the $\beta$ function can be based on various vertices. The result should be the same due to the gauge invariance. To simplify the calculations we choose the ghost-ghost-vector vertex. The renormalization of the coupling is then given by

$$\alpha_{Bare} = \tilde{Z}_1^2 \tilde{Z}_2^{-2} Z_3^{-1}\alpha, \quad \alpha \equiv \frac{g^2}{16\pi^2}.$$

The renormalization constants are as follows:

$$\begin{aligned}\tilde{Z}_1 &= 1 - \frac{C_2}{2\varepsilon}\alpha, &(37)\\ \tilde{Z}_2^{-1} &= 1 - \frac{C_2}{2\varepsilon}\alpha, &(38)\\ Z_3^{-1} &= 1 - \alpha(\frac{5}{3\varepsilon}C_2 - \frac{2}{3\varepsilon}n_f), &(39)\end{aligned}$$

which gives

$$\tilde{\gamma}_1 = -\frac{C_2}{2}\alpha, \tag{40}$$

$$\tilde{\gamma}_2 = -\frac{C_2}{2}\alpha, \tag{41}$$

$$\gamma_3 = -\alpha(\frac{5}{3}C_2 - \frac{2}{3}n_f), \tag{42}$$

$$\beta = \alpha(2\tilde{\gamma}_1 + 2\tilde{\gamma}_2 + \gamma_3) = -(\frac{11}{3}C_2 - \frac{2}{3}n_f)\alpha^2, \tag{43}$$

where $C_2$ is the quadratic Casimir operator of the gauge group (for SU(3) equal to 3), and $n_f$ is the number of flavours of quarks.

One can see from eq.(43) that if the number of flavours is less than $\frac{11}{2}C_2 = \frac{33}{2}$ the $\beta$ function is negative and the theory is asymptotically free.

# 4 $\Lambda_{QCD}$

Consider the one-loop expression for the effective coupling in a gauge theory. It can be written as

$$\alpha(\frac{Q^2}{\mu^2}, \alpha_\mu) = \frac{\alpha_\mu}{1 - \beta_0 \alpha_\mu \ln \frac{Q^2}{\mu^2}} = \frac{1}{\frac{1}{\alpha_\mu} - \beta_0 \ln \frac{Q^2}{\mu^2}} \equiv -\frac{1}{\beta_0 \ln \frac{Q^2}{\Lambda^2}} = \alpha(\frac{Q^2}{\Lambda^2}), \tag{44}$$

where we have introduced the notation

$$\Lambda^2 = \mu^2 e^{-\frac{1}{\beta_0 \alpha_\mu}}. \tag{45}$$

This quantity is called $\Lambda_{QCD}$ in quantum chromodynamics and can be introduced in any model.

Eq.(45) can be generalized to any number of loops. For this purpose let us rewrite the RG equation for the effective coupling in the Gell-Mann – Low form. One has

$$\ln \frac{Q^2}{\mu^2} = \int_{\alpha_\mu}^{\alpha_Q} \frac{d\alpha}{\beta_\alpha(\alpha)}. \tag{46}$$

Combining the lower limit with $\ln \mu^2$ one gets

$$\ln \frac{Q^2}{\Lambda^2} = \int^{\alpha_Q} \frac{d\alpha}{\beta_\alpha(\alpha)}, \tag{47}$$

where

$$\Lambda^2 = \mu^2 exp\left(\int^{\alpha_\mu} \frac{d\alpha}{\beta_\alpha(\alpha)}\right). \tag{48}$$

is a generalization of eq.(45) to any loop order.

The quantity $\Lambda$, introduced this way, is $\mu$-independent, but renormalization scheme dependent due to scheme dependence of the $\beta$ function. However, scheme dependence of $\Lambda$ is given exactly (!) in one-loop order. Indeed, consider the limit $\alpha \to 0$. Then the $\beta$ function is

$$\beta_\alpha(\alpha) = \beta_0 \alpha^2 + \beta_1 \alpha^3 + ...$$

or

$$\int \frac{d\alpha}{\beta(\alpha)} = -\frac{1}{\beta_0 \alpha} + \ln \alpha + O(\alpha).$$

In this limit the ratio of two parameters $\Lambda$ belonging to two different schemes is

$$\ln \frac{\Lambda_1^2}{\Lambda_2^2} = -\frac{1}{\beta_0}\left[\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right] = -\frac{1}{\beta_0}\left[c_1 - c_2\right], \tag{49}$$

where the coefficients $c_1$ and $c_2$ are calculated in one loop order. They can be found from perturbative expansion of any physical quantity in two different schemes

$$\begin{aligned} R &= \alpha_1(1 + c_1\alpha_1 + ...) \\ &= \alpha_2(1 + c_2\alpha_2 + ...). \end{aligned}$$

Since $\Lambda$ does not depend on $\alpha$, one can take any value of $\alpha$, and eq.(49) is always valid. The difference $c_1 - c_2$ does not depend on particular choice of $R$ and is universal.

# 5 Running Masses

One can also introduce the mass renormalization. We follow the approach when masses are treated like additional couplings. This corresponds to the so-called massless renormalization scheme. In this case the masses are renormalised like couplings, i.e. multiplicatively

$$m_{Bare} = Z_m m,$$

where the renormalization constant $Z_m$ is independent on the mass parameters and depends only on dimensionless couplings.

Then, in full analogy with eq.(7), one can introduce the *effective* or running mass

$$\frac{d}{dt}\bar{m}(t,\alpha) = \bar{m}\gamma_m(\bar{\alpha}), \quad \bar{m}(0,\alpha) = m_0. \tag{50}$$

Solving this equation together with eq.(7) one has

$$\bar{m}(t,\alpha) = m_0 e^{\int\limits_0^t \gamma_m(\bar{\alpha}(t,\alpha))dt} = m_0 e^{\int\limits_\alpha^{\bar{\alpha}} \frac{\gamma(\alpha)}{\beta(\alpha)}d\alpha} \tag{51}$$

In the one-loop order

$$\beta(\alpha) = b\alpha^2, \quad \gamma_m(\alpha) = c\alpha$$

and the solution is

$$m(t) = m_0 \left(\frac{\alpha(t)}{\alpha_0}\right)^{c/b}.$$

This is the running mass!

The natural question arise: what is the physical mass measured at experiment and how is it related to the running mass and at which scale?

To answer this question, consider why the mass is running. This is due to the radiative corrections. If one considers the value of momenta which is bigger than the mass, i.e. $Q^2 > m^2$, then the particles are created, they are running inside the loops and give contribution to the running. On the contrary, if $Q^2 < m^2$, particles are not created, they "decouple" and do not contribute to the running. Hence, it is quite natural in the minimal scheme to stop the running at the value of $Q^2 = m^2$ and to identify the physical mass with the running mass at the scale of the mass, i.e

$$m^2 = \bar{m}^2(m^2).$$

However, this is true only up to finite corrections. More appropriate is to take not the running mass at the mass scale, but the so-called pole mass: the position of the pole of a propagator of a particle, i.e.

$$\boxed{\text{physical mass} \equiv \text{pole mass}}$$

This definition of a mass is scheme independent and may have physical meaning. The pole mass can be expressed through the running mass at the scale of a mass with finite and calculable corrections.

<u>Example</u>

Consider as an example the quark mass in QCD. The quark propagator can be graphically presented as in Fig.1.

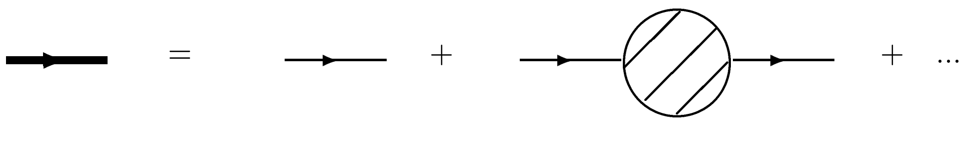

Figure 1: Quark propagator

The corresponding expression is

$$\begin{aligned} &= \frac{i}{\hat{p}-m} + \frac{i}{\hat{p}-m}(iA\hat{p} + iBm)\frac{1}{\hat{p}-m} + ... \\ &= \frac{i}{\hat{p}-m}\left[1 - \frac{A\hat{p}+Bm}{\hat{p}-m} + ...\right] = \frac{i}{\hat{p}-m}\,\frac{1}{1+\frac{A\hat{p}+Bm}{\hat{p}-m}} = \frac{i}{\hat{p}-m+A\hat{p}+Bm}. \end{aligned}$$

The pole mass now is defined by equation

$$\hat{p}(1 + A(p^2)) - m(1 - B(p^2)) = 0,$$

which gives

$$m_{pole} = m\frac{1-B(m^2)}{1+A(m^2)} = m[1-A(m^2)-B(m^2)].$$

Consider one-loop QCD diagram. It is shown in Fig.2.

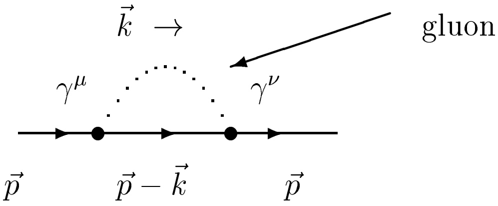


Figure 2: Quark self-energy graph

The corresponding expression is

$$\Sigma = -\frac{g_s^2}{(2\pi)^4}C_F\int\frac{dk\ \gamma^\mu(\hat{p}-\hat{k}+m)\gamma^\nu}{[(p-k)^2-m^2]}\frac{g^{\mu\nu}}{k^2} \tag{52}$$

Performing the $\gamma$-matrix algebra

$$\gamma^\mu(\hat{p}-\hat{k})\gamma^\mu = -2(1-\varepsilon)(\hat{p}-\hat{k}), \quad \gamma^\mu m\gamma^\mu = (4-2\varepsilon)m,$$

one gets

$$\begin{aligned}\Sigma &= \frac{g_s^2}{(2\pi)^4}C_F(\mu^2)^\varepsilon\int d^{4-2\varepsilon}k\frac{2(1-\varepsilon)(\hat{p}-\hat{k})-(4-2\varepsilon)m}{[(p-k)^2-m^2]k^2}\\ &= \frac{g_s^2}{(2\pi)^4}C_F(\mu^2)^\varepsilon\int_0^1 dx\int d^{4-2\varepsilon}k\frac{2(1-\varepsilon)(\hat{p}-\hat{k})-(4-2\varepsilon)m}{[k^2-2pkx+p^2x-m^2x]^2}\\ &= \frac{g_s^2}{(2\pi)^4}C_F\frac{i\pi^2}{\varepsilon}(\mu^2)^\varepsilon\int_0^1 dx\frac{2(1-\varepsilon)(\hat{p}-\hat{p}x)-(4-2\varepsilon)m}{[p^2x(1-x)-m^2x]^\varepsilon}\end{aligned}$$

Thus,

$$A(p^2) = \frac{2(1-\varepsilon)}{\varepsilon}\int_0^1\frac{dx(1-x)(\mu^2)^\varepsilon}{[p^2x(1-x)-m^2x]^\varepsilon}, \quad B(p^2) = -\frac{4-2\varepsilon}{\varepsilon}\int_0^1\frac{dx(\mu^2)^\varepsilon}{[p^2x(1-x)-m^2x]^\varepsilon}.$$

Substituting $p^2=m^2$ one finds

$$A(m^2) = \frac{2(1-\varepsilon)}{\varepsilon}\left(\frac{\mu^2}{m^2}\right)^\varepsilon\int_0^1 dx(1-x)x^{-2\varepsilon} = \frac{1}{\varepsilon}+2+\ln\frac{\mu^2}{m^2},$$

$$B(m^2) = -\frac{4-2\varepsilon}{\varepsilon}\left(\frac{\mu^2}{m^2}\right)^\varepsilon\int_0^1 dxx^{-2\varepsilon} = -\frac{4}{\varepsilon}-6-4\ln\frac{\mu^2}{m^2}.$$

Performing minimal subrtaction one finally obtains

$$A^{\overline{MS}}(m^2) = 2+\ln\frac{\mu^2}{m^2}, \quad B^{\overline{MS}}(m^2) = -6-4\ln\frac{\mu^2}{m^2}, \quad A^{\overline{MS}}(m^2)+B^{\overline{MS}}(m^2) = -4-3\ln\frac{\mu^2}{m^2}$$

Thus, for the pole mass one finds

$$m_{pole} = m(\mu)\left[1+\frac{\alpha_s C_F}{4\pi}(4+3\ln\frac{\mu^2}{m^2})\right]. \tag{53}$$

Substituting $C_F = 4/3$ and $\mu^2 = m^2$ one obtains the desired relation between the pole mass and the running mass at the mass scale

$$m_{pole} = m(m)\left[1+\frac{4}{3}\frac{\alpha_s}{\pi}\right]. \tag{54}$$