Modeling the hadronization processes in HIC (based on the Nambu Jona-Lasinio Lagrangian)

in collaboration with
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• How one obtains the NJL Lagrangian
• How to construct mesons Mesons and Baryons
• Cross section for elastic scattering and hadronisation
• Expanding plasma: How quarks hadronize
• Realistic simulations
circumstantial evidence:

For beam energies $> \approx 100$ AGeV a plasma of quark and gluons (QGP) is formed.

The challenge:
How to come from quarks to hadrons?

As PHSD calculations see a heavy ion reaction is there local equilibrium?

Courtesy: P. Moreau 2015
QCD: The theory which contains the solution

\[ L_{QCD}(x) = \bar{\psi}(x) \left( i \gamma^\mu \left[ \partial_\mu - ig t^a A_\mu^a \right] - \hat{M}^0 \right) \psi(x) - \frac{1}{4} G^a_{\mu\nu}(x) G^{\mu\nu^a}(x) \]

Gluonic field strength tensor:

\[ G^a_{\mu\nu}(x) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu(x) A^c_\nu \]

\( \psi(x) \) - quark field

**flavor space**  **Dirac space**  **color space**

\( q = u, d, s \)  \( \mu = 0, 1, 2, 3 \)  \( c = r, b, g \)

In flavor space (3 flavors):

\[ \psi(x) = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \]

Mass term:

\[ \hat{M}^0 = \begin{pmatrix} m^0_u & 0 & 0 \\ 0 & m^0_d & 0 \\ 0 & 0 & m^0_s \end{pmatrix} \]

3x3 diagonal matrix in flavor space with the bare quark masses on the diagonal
1) local $SUc(3)$ color gauge transformation (by construction)

2) global $SU_f (3)$ flavor symmetry

3) for massless quarks ONLY:
   - chiral invariance of QCD Lagrangian: $SU_f (3)_V \times SU_f (3)_A$

However, chiral symmetry is a spontaneously broken since quarks have non-zero masses.

⇒ To explore more simple effective Lagrangians with the same symmetries for the quark degrees of freedom, however, discarding the gluon dynamics completely.
Euler-Lagrange equations:

\[
\frac{\partial L}{\partial \varphi} - \partial_\mu \left[ \frac{\partial L}{\partial (\partial^\mu \varphi)} \right] = 0
\]  

for any field \( \varphi \) (the same equation for \( \bar{\varphi} \)): e.g. \( \varphi = \Psi(x) \) or \( A_\mu^a(x) \).

1) Consider quark field \( \bar{\Psi}(x) \)

\[
\frac{\partial L}{\partial \bar{\Psi}} - \partial_\mu \left[ \frac{\partial L}{\partial (\partial^\mu \bar{\Psi})} \right] = 0
\]

\[\Rightarrow \frac{\partial L}{\partial \bar{\Psi}} = 0,\]

since the second term in Eq.(4) is equal to zero while no terms with \( \partial^\mu \bar{\Psi} \).

From eqs.\((1,5)\) follows that

\[
(i \gamma^\mu \partial_\mu - \hat{M}^0) \Psi_q(x) = -g \gamma^\mu t^a A_\mu^a(x) \Psi_q(x).
\]
2) Consider field $A^a_\nu(x)$:

Euler-Lagrange equation for gluon field:

$$\frac{\mathcal{L}}{\partial A^a_\nu(x)} - \partial_\mu \left[ \frac{\mathcal{L}}{\partial (\partial^\mu A^a_\nu(x))} \right] = 0.$$  \hspace{1cm} (7)

- Using (1) $\rightarrow$ first term in eq. (7):

$$\frac{\mathcal{L}}{\partial A^a_\nu(x)} = g \bar{\Psi} \gamma^\nu t^a \Psi + \Pi_g.$$  \hspace{1cm} (8)

where $\Pi_g$ is the 'self-energy' of gluons:

$$\Pi_g = \frac{\partial}{\partial A^a_\nu(x)} \left[ - \frac{1}{4} G^a_{\mu\nu}(x) G^{\mu\nu a}(x) \right]$$  \hspace{1cm} (9)

$$G^a_{\mu\nu}(x) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu(x) A^c_\nu$$

- Using (1) $\rightarrow$ second term in eq. (7):

$$\frac{\mathcal{L}}{\partial (\partial^\mu A^a_\nu(x))} = \partial^\mu A^a_\nu(x)$$  \hspace{1cm} (10)

- Substitute (8), (10) into (7):

$$\partial_\mu \partial^\mu A^a_\nu(x) = -g \bar{\Psi} \gamma_\nu t^a \Psi - \Pi_{g,\nu}$$  \hspace{1cm} (11)
• Approximation: scalar terms dominates and is positive: $\Pi_{g,\nu} = M_g^2$

constituent gluon mass $\neq 0$ due to self-interactions of gluons.

Then from eq. (11) $\Rightarrow$

$$\partial_\mu \partial^\mu A^a_\nu(x) = -g\bar{\Psi}\gamma_\nu t^a\Psi - \Pi_{g,\nu}$$

$$= -g\bar{\Psi}\gamma_\nu t^a\Psi$$

(12)

• Solution of eq. (12): $A^a_\nu(x) = -\int d^4x\ G(x-x')\ g\bar{\Psi}(x')\gamma_\nu t^a\Psi(x')$

(13)

Green function:

$$G(x-x') = -\int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x-x')}}{q^2 - M_g^2}$$

(14)

Approximation: consider low energy physics: i.e. small momentum or large distance

$q^2 \ll M_g^2$

In this limit:

$$G(x-x') = -\int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq(x-x')}}{q^2 - M_g^2} \bigg|_{q^2 \to 0} \approx \frac{1}{M_g^2} \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-x')} \delta(x-x')$$

(15)
From eq. (15) \( \Rightarrow \)
\[
G(x - x') \Rightarrow \delta^4(x - x') \cdot M_g^{-2}.
\]

\(\equiv\) (16)

- Substitute (16) into (13):
\[
A_\nu^a(x) = -\frac{g}{M_g^2} \bar{\Psi}(x) \gamma_\nu t^a \Psi(x).
\]

- Substitute (17) into (6):
\[
(i \gamma^\mu \partial_\mu - \hat{M}^0) \Psi(x) - G^2_c \gamma^\mu t^a (\bar{\Psi}(x) \gamma_\mu t^a \Psi(x)) \Psi(x) = 0
\]
where the low energy coupling constant:
\[
G^2_c = g^2 / M_g^2
\]

\(\equiv\) (18)

\(\equiv\) (19)

\[\mathcal{L}_{eff} = \bar{\Psi}(x) (i \gamma^\mu \partial_\mu - \hat{M}^0) \Psi(x) - G^2_c \sum_{a=1}^{8} \left( \bar{\Psi}(x) \gamma^\mu t^a \Psi(x) \right)^2.\]

\(\equiv\) (20)

**NJL Lagrangian**

interaction between gluons \(\rightarrow\) approximated by a gluon mass \(M_g\)

\(q^2 < M_g\)
The NJL Lagrangian is given by:

$$\mathcal{L}_{int} = -G_c^2 \left[ \bar{\Psi}_i \gamma^\mu T^a \delta_{ij} \Psi_j \right] \left[ \bar{\Psi}_k \gamma_\mu T^a \delta_{kl} \Psi_l \right]$$

where $i, j = 1 \ldots N_f = 3$ flavor index; $T^a$ : color generators $a = 1 \ldots N_c^2 - 1 = 8$ ($N_c = 3$).

Symmetries of the massless NJL Lagrangian:

$$SU_V(3) \otimes SU_A(3) \otimes U_V(1) \otimes U_A(1)$$

$U_A(1)$ symmetry not realized in nature ( $\eta$ and $\eta'$ would have the same mass).

$U_A(1)$ symmetry is broken (quantum fluctuations violate axial current conservation)

The breaking of the $U_A(1)$ symmetry can be obtained by adding the ‘t Hooft Lagrangian

$$\mathcal{L}'_{\text{t Hooft}} = H \det_{ij} [\bar{\Psi}_i (1 - \gamma_5) \Psi_j] - H \det_{ij} [\bar{\psi}_i (1 + \gamma_5) \psi_j]$$

For $N_f = 3$:
Six point interaction
taking into account on the mean field level

$H$ is determined by the experimental $\eta-\eta'$
mass gap
$\mathcal{L}_{NJL} = \bar{\Psi}_i (i\gamma_\mu \partial^\mu - \hat{M}_0) \Psi_i - G_c^2 \left[ \bar{\Psi}_i \gamma^\mu T^a \delta_{ij} \Psi_j \right] \left[ \bar{\Psi}_k \gamma_\mu T^a \delta_{kl} \Psi_l \right] + H \det_{ij} \left[ \bar{\Psi}_i (1 + \gamma_5) \Psi_j \right] - H \det_{ij} \left[ \bar{\psi}_i (1 - \gamma_5) \psi_j \right]$

$\mathcal{L}_{NJL}$ : Shares the symmetries with the QCD Lagrangian (color we discuss later) 
Allows for calculating effective quark masses:

$\mathbf{M} = \hat{M}_0 - 4G < \bar{\psi}\psi > + 2H < \bar{\psi}'\psi' > < \bar{\psi}''\psi'' >$

But contains only quarks
no gluons and
no hadrons
So not very obvious how of use for hadronisation.
Brief survey on thermal field theory

How to calculate physical quantities at final temperature and final chemical potential?

Imaginary time formalism (one introduces $0 \leq i\tau \leq \beta = \frac{1}{T}$ ($T =$ temperature))

In all momentum space integrals replace

$$k_0 \rightarrow i\omega_n, \quad \int \frac{d^4k}{(2\pi)^4} \rightarrow iT \sum_{n \in \mathbb{Z}} \int \frac{d^3k}{(2\pi)^3}$$

With the fermionic Matsubara frequencies

$$i\omega_n = i\pi T(2n + 1)$$

A chemical potential can be introduced by the Lagrangian

$$\mathcal{L}_\mu = \sum_{ij} \bar{\psi}_i \mu_{ij} \gamma_0 \psi_j \quad \mu_{ij} = \text{diag} (\mu_u, \mu_d, \mu_s)$$
First results: Quark masses

![Graph showing the variation of strange and up-down quark masses with temperature.]

![3D plot showing the variation of quark mass with temperature and chemical potential.]

- $m_{\text{strange}}$
- $m_{\text{up,down}}$
Polyakov NJL: gluons on a static level


It is not possible to introduce gluons as dynamical degrees of freedom without spoiling the simplicity of the NJL Lagrangian which allows for real calculations but one can introduce gluons through an effective potential for the Polyakov loop

\[
\frac{U(T, \Phi, \bar{\Phi})}{T^4} = -\frac{b_2(T)}{2} \bar{\Phi}\Phi - \frac{b_3}{6} (\Phi^3 + \bar{\Phi}^3) + \frac{b_4}{4} (\bar{\Phi}\Phi)^3
\]

\[
b_2(T) = a_0 + a_1 \frac{T_0}{T} + a_2 \left( \frac{T_0}{T} \right)^2 + a_3 \left( \frac{T_0}{T} \right)^3
\]

\[a_0 = 6.75, a_1 = -1.95, a_2 = 2.625, a_3 = -7.44, b_3 = 0.75, b_4 = 7.5\]

Parameters -> right pressure in the SB limit

\[\Phi = \frac{1}{N_c} \text{Tr}_c \langle P \exp \left( - \int_0^\beta d\tau A_0(x, \tau) \right) \rangle\]
In PNJL the transition is steeper than in NJL
How can we get mesons?

Trick: Fierz transformation of the original Lagrangian

Fierz Transformation allows for a reordering of the field operators in 4 point contact interactions. It is simultaneously applied in Dirac, colour and flavour space.

Example in Dirac space:

\[
(\bar{\chi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\chi) = (\bar{\chi}\chi)(\bar{\psi}\psi) - \frac{1}{2} (\bar{\chi}\gamma^\mu\chi)(\bar{\psi}\gamma_\mu\psi) - \frac{1}{2} (\bar{\chi}\gamma^\mu\gamma_5\chi)(\bar{\psi}\gamma_\mu\gamma_5\psi) - (\bar{\chi}\gamma_5\chi)(\bar{\psi}\gamma_5\psi)
\]

Scalar  vector  pseudovector  pseudoscalar
How to get mesons?

Fierz transformation transforms original Lagrangian to one for mesons

\[ \mathcal{L}_{\text{int}} = -G_c^2 \left[ \bar{\Psi}_i \gamma^\mu T^a \delta_{ij} \Psi_j \right] \left[ \bar{\Psi}_k \gamma_\mu T^a \delta_{kl} \Psi_l \right] \]

Fierz transformation transforms original Lagrangian to one for mesons

\[ \mathcal{L}_{\text{Pseudo scalar}} = G \left( \bar{\Psi}_i \tau^a_{il} 1_c i \gamma_5 \Psi_l \right) \left( \bar{\Psi}_k \tau^a_{kj} 1_c i \gamma_5 \Psi_j \right) ; \quad G = \frac{N_c^2 - 1}{N_c^2 g} \]

Singlet in color mixing of flavour

Similar terms can be obtained for

Vector mesons \( \gamma^\mu \)

Scalar Mesons \( 1 \)

Pseudovector mesons \( \gamma_\mu \gamma_5 \)
For calculations: Include the ‘t Hooft term to an effective coupling

\[
G_{\text{eff}} = G + \frac{1}{2} H \langle \bar{s}s \rangle
\]

Define an eff. coupling constant

Define interaction kernel \( \mathcal{H} = \Omega \ 2G_{\text{eff}} \bar{\Omega} \)

Which contains color, flavour and Dirac matrices

\[
\Omega = 1_c \otimes \tau^a \otimes \{1, i\gamma_5, \gamma_\mu, \gamma_5\gamma_\mu\}
\]
How to get mesons? III

and use $\mathcal{K}$ as a kernel for a Bethe-Salpeter equation (relativistic Lippmann-Schwinger eq.)

$$T(p) = \mathcal{K} + i \int \frac{d^4 k}{(2\pi)^4} \mathcal{K} S \left( k + \frac{p}{2} \right) S \left( k - \frac{p}{2} \right) T(p)$$

In (P)NJL one can sum up this series analytically:

$$T(p) = \frac{2G_{\text{eff}}}{1 - 2G_{\text{eff}} \Pi(p)} , \quad \Pi(p_0, p) = -\frac{1}{\beta} \sum_n \int \frac{d^3 k}{(2\pi)^3} \Omega S \left( k + \frac{p}{2} \right) \Omega S \left( k - \frac{p}{2} \right)$$

$\equiv \Pi$
The **meson pole mass** and the **width** one obtains by solving:

$$1 - 2G_{\text{eff}} \Pi(p_0 = M_{\text{meson}} - i\Gamma_{\text{meson}}/2, p = 0) = 0$$

When $\pi$'s become unstable they develop a width.

Masses of pseudoscalar mesons and of quarks at $\mu = 0$.

At $T=0$ physical and calculated mass agree quite well.
Looking back

We have seen that the NJL model describes quite well meson properties. For this one has to fix the 5 parameters of the model:

\( \Lambda \) = upper cut off of the internal momentum loops
\( g \) = coupling constant
\( M_0 \) = bare mass of u,d and s quarks
\( H \) = coupling constant ‘t Hooft term

These parameters have been adjusted to reproduce:

- Masses of \( \pi \) and \( K \) in the vacuum, as well as the \( \eta \)-\( \eta' \) mass splitting
- \( \pi \) decay constant, \( q\bar{q} \) condensate \((-241 \text{ MeV})^3\)

Therefore:
All properties of masses, cross sections etc. at finite \( \mu \) and \( T \) follow without any new parameters from ground state observables.
The Fierz transformation produces also a term for scalar diquarks

\[ \mathcal{L}_{qq} = G_{\text{DIQ}} \left( \bar{\Psi}_i \tau_A t_{A'} i \gamma_5 C \Psi_k^T \right) \left( \Psi_j^T \tau_A t_{A'} C i \gamma_5 \Psi_1 \right), \quad G_{\text{DIQ}} = (N_c + 1)g/(2N_c) \]

\[ C = i\gamma_0 \gamma_2; \quad t_a, \tau_a: \text{ Antisymmetric SU}(3) \text{ matrices in color and flavour} \]

as well as for axial diquarks.

Mass is determined like for mesons (Bethe Salpeter equation with elementary interaction kernel)

\[ T(p) = \frac{2G_{\text{DIQ}}}{1 - 2G_{\text{DIQ}} \Pi(p)} \]

\[ \Pi(p) = i \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \bar{\Omega} S \left( k + \frac{p}{2} \right) \Omega S^T \left( \frac{p}{2} - k \right) \right] \]

\[ \Omega = \text{color} \otimes \text{flavour} \otimes \{1, i\gamma_5, \gamma_\mu, \gamma_5\gamma_\mu\} \]
Diquarks – the road to baryons II

Scalar diquarks

Diquarks are bound

\[ T_c \ [qq] = 256 \text{ MeV} \]
\[ T_c \ [qs] = 273 \text{ MeV} \]

Strange diquarks melt at higher temperature

Diquarks form together with a quark the baryons

\[ \mathbf{3} \otimes (\bar{\mathbf{3}} \oplus \mathbf{6}) = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \]
(P)NJL reduces the 3-body Fadeev eq. to a two body quark-diquark eq.

\[ G(P) = G_0 + G_0 Z G(P) \]

\[ G_0 = \text{free quark propagator} \quad S_q \times \text{free diquark propagator} \quad t_D \]

\[ Z = \text{elementary interaction} \quad Z = \Omega S_q \Omega \]

\[ t_D(q) = -\frac{g_{\text{eff}}^2}{q_0^2 - \bar{q}_0^2 - m_{\text{DIQ}}^2} \]

\[ G(P) = \frac{G_0}{1 - G_0 Z} \]

\[ 1 - G_0 Z(P_0 = M_{\text{baryon}}, P = 0) = 0 \]
Omitting Dirac and flavour structure:

\[
\left[1 - \frac{2}{m_{\text{quark}}} \frac{1}{\beta} \sum_n \int \frac{d^3q}{(2\pi)^3} S_q(i\omega_n, q) t_D(i\nu_1 - i\omega_n, -q) \right] \bigg|_{i\nu_1 - P_0 + i\epsilon = M_{\text{Baryon}}} = 0
\]

where we approximated the quark propagator for the exchanged quark by:

\[
S_q(q) = \frac{1}{q - m_{\text{quark}}} \rightarrow -\frac{\Pi_{\text{Dirac}}}{m_{\text{quark}}}
\]

5% error (Buck et al. (92))

The more strange quarks the higher the melting temperature
Baryons II

With 5 parameters fixed to mesonic vacuum physics (+ 2 diquark coupling constants for baryons)

(P)NJL can describe

the vacuum masses of all pseudoscalar mesons + all octet and decouplet baryons

with a precision of less than 5%

The T and μ dependence of all these hadrons

It predicts: melting temperature depends on the hadrons species

<table>
<thead>
<tr>
<th>Hadron</th>
<th>PDG mass (MeV)</th>
<th>PNJL mass (T=0) (MeV)</th>
<th>NJL $T_c$ (MeV)</th>
<th>PNJL $T_c$ (MeV)</th>
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Perspectives

NJL Lagrangian:
transition between quarks and hadrons
Cross over at $\mu = 0$
1st order transition $\mu >> 0$
sudden change of $q$ and meson mass

Details have not been explored yet
Cross sections

Having the Lagrangian we can derive in the usual way the Feynman rules and can calculate cross sections.

Example: $uu \rightarrow uu$ matrix elements

But also elastic cross sections like $uu \rightarrow uu$

hadronization cross sections $q\bar{q} \rightarrow MM$ $M=\pi, K, \eta, \eta', \rho$ ...

hadronization cross sections $Diq \ Diq \rightarrow$ baryons +q etc.
Cross sections

\[ -i M_s = \delta_{c_1, c_2} \delta_{c_3, c_4} \bar{u}(p_2) T u(p_1) \left[ iD_s^S(p_1 + p_2) \right] \bar{u}(p_3) T v(p_4) \\
+ \delta_{c_1, c_2} \delta_{c_3, c_4} \bar{u}(p_2) (i \gamma_5 T) u(p_1) \left[ iD_s^P(p_1 + p_2) \right] \bar{u}(p_3) (i \gamma_5 T) v(p_4) \]

\[ -i M_t = \delta_{c_1, c_3} \delta_{c_2, c_4} \bar{u}(p_3) T u(p_1) \left[ iD_t^S(p_1 - p_3) \right] \bar{v}(p_2) T v(p_4) \\
+ \delta_{c_1, c_3} \delta_{c_2, c_4} \bar{u}(p_3) (i \gamma_5 T) u(p_1) \left[ iD_t^P(p_1 - p_3) \right] \bar{v}(p_2) (i \gamma_5 T) v(p_4) \]

\[ D(p_0, \vec{p}) \propto \frac{2G}{1 - 2G \Pi(p_0, \vec{p})} \]

Cross section up to 100 mb close to cross over due to resonant s-channel
otherwise small (5-10 mb)

Hadronization cross sections

\[ q \bar{q} \rightarrow M M \]

\[ -i M_s = g^2_{Mqq'} f_s \bar{u}_2 u_1 \Gamma_v (i \not{D}^S_M) \Gamma^v_{q_1 q_2 q_3} + \ldots \]

\[ -i M_t = g^2_{Mqq'} f_t \bar{u}_2 \Gamma_v \frac{i(p_1 - p_3 + m_t)}{(p_1 - p_3)^2 - m_t^2} \Gamma^v u_1 \]

\[ -i M_u = g^2_{Mqq'} f_u \bar{u}_2 \Gamma_v \frac{i(p_1 - p_4 + m_t)}{(p_1 - p_4)^2 - m_t^2} \Gamma^v u_1 \]

\[ \Gamma_{q_1 q_2 q_3} \quad \text{triangle vertex} \]

\[ \Gamma_v \quad \text{appropriate } \gamma \text{ matrix} \]
Hadronization cross sections

These s-channel resonances create as well very large hadronization cross section close to $T_c$

Consequence:
If an expanding plasma comes to $T_c$ quarks are converted into hadrons despite of the NJL Lagragian does not contain confinement
Using 7 parameters fitted to ground state properties of mesons and baryons the NJL model allows for calculating

Quark masses (T,μ)
Hadron masses (T,μ)
Elastic cross sections (T,μ)
Hadronization cross sections (T,μ)

So we have all ingredients for a transport theory

Problem:
With a mass of 2 MeV and temperatures > 200 MeV the quarks move practically with the speed of light.

So we have to construct a fully relativistic transport theory
Relativistic Transport theory

Hamiltonian formulation

Hamilton-Jacobi eqs.: Eqs. for the time evolution of a particle in phase space \((p,q)\)

\[
\begin{align*}
\frac{dq}{dt} &= \frac{\partial \mathcal{H}}{\partial p}, & \frac{dp}{dt} &= -\frac{\partial \mathcal{H}}{\partial q}
\end{align*}
\]

with Hamiltonian \(\mathcal{H}(q,p)\)

On the trajectory of the particle the energy is conserved

Time evolution of observables \(A(p,q)\):

\[
\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A,\mathcal{H}\} \quad \text{with} \quad \{A,B\} = \sum_k^N \frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k}
\]

\[
\begin{align*}
\frac{dq}{dt} &= \{q,\mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p}, & \frac{dp}{dt} &= \{p,\mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial q}
\end{align*}
\]

Problem: Hamilton eqs. cannot be extended to a relativistic approach:

\(q, p, \mathcal{H}\) are components of 4-vectors

eqs. cannot be Lorentz transformed
What we can extend to 4-vectors is

a) the Poison bracket:

\[ \{ A, B \} = \sum_{k=1}^{N} \frac{\partial A}{\partial q^\mu_k} \frac{\partial B}{\partial p_{k\mu}} - \frac{\partial A}{\partial p^\mu_k} \frac{\partial B}{\partial q_{k\mu}} \]

\[ \{ q^\mu_a, q^\nu_b \} = \{ p^\mu_a, p^\nu_b \} = 0 \]

\[ \{ q^\mu_a, p^\nu_b \} = \delta_{ab} g^{\mu\nu} \]

b) the geometrical interpretation that

\[ \frac{dq}{dt} = \{ q, H \} = \frac{\partial H}{\partial p} \]

\[ \frac{dp}{dt} = \{ p, H \} = -\frac{\partial H}{\partial q} \]

describes the trajectory in the \((p,q)\) phase space on which the Hamiltonian \( H(q,p) \) is conserved:

\[ \frac{dq^\mu(\tau)}{d\tau} = \lambda \{ q^\mu(\tau), K \} \]

\[ \frac{dp^\mu(\tau)}{d\tau} = \lambda \{ p^\mu(\tau), K \} \]

describes the trajectory in the 8-dim phase space on which the Lorentz inv. quantity \( K \) is conserved.

\( \tau \) is not a time but a parameter which characterizes the trajectory.
Example: One free particle:
We need a trajectory in the 6+1 dimensional phase space \((q, p, t)\)

Starting point: Choose 2 Lorentz invariant constraints \(K = p_\mu p^\mu = m^2\) and \(\chi(p_\mu, q_\mu, \tau) = 0\)

\[
\frac{d\chi}{d\tau} = \frac{\partial \chi}{\partial \tau} + \lambda \{\chi(\tau), K\} = 0
\]

\[
\lambda = -\frac{\partial \chi}{\partial \tau} \{\chi, K\}^{-1}
\]

Constraint

\[
\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \lambda \{f, K\}
\]

\[
\frac{dq_\mu}{d\tau} = -\frac{\partial \chi}{\partial \tau} \{q_\mu, K\}
\]

Free particle

\[
\frac{dp_\mu(\tau)}{d\tau} = 0
\]

All depends now on \(\chi\)

\(a\) \(\chi = q^0 - \tau = 0\) \(\rightarrow \frac{dp_\mu(\tau)}{d\tau} = \frac{p_\mu}{p^0}\)

\(b\) \(\chi = x_\mu p^\mu - m \tau = 0\) \(\rightarrow \frac{dp_\mu(\tau)}{d\tau} = \frac{p_\mu}{m}\)

Diff. \(\chi \rightarrow \text{diff. eqs. of motion}; \ \tau\ \text{is not time but parameter of trajectory}\)

Before fixing constraints: rel. dynamics is incomplete
This concept can be extended to N interacting particles (PRC87,034912)

\[ K_i = p_{i\mu}p_i^\mu - m^2 + \sum V_{ij} = 0 ; \quad \chi_i \in \mathcal{N}(q_1, \ldots, q_N; p_1, \ldots, p_N) ; \quad \chi_N(q_1, \ldots, q_N, p_1, \ldots, p_N, \tau) \]

with the eqs. of motion

\[
\frac{dq_i^\mu(\tau)}{d\tau} = \{q_i^\mu(\tau), K_j\} S_{ij} \frac{d\chi_l}{d\tau} \\
\frac{dp_i^\mu(\tau)}{d\tau} = \{p_i^\mu(\tau), K_j\} S_{ij} \frac{d\chi_l}{d\tau}
\]

with \( S_{ij} = \{\chi_j, K_i\}^{-1} \)

The 2N constraints \( K_i, \chi_i \) reduce the phasespace \n
\[ 8N \dim(q_\mu, p_\mu) \to (6N + 1) \dim(q, p, \tau) \]

reduction not unique \( \Rightarrow \) different eqs. of motions \( \Rightarrow \) different trajectories
Eqs. of motion with the constraints (which assures cluster separability):

\[ K_i = p_i^\nu p_{i\nu} - m_i^2 + V_i(qT^2) = 0 \]

\[ \chi_i = \frac{\sum_{j \neq i} q_{Tij}^\nu}{N} U_\nu = 0 \]

\[ \chi_N = \frac{\sum_j q_{Nj}^\nu}{N} U_\nu - \tau = 0 \]

\[ qT_{ij}^\mu = q_{ij}^\mu - [(q_{ij})_\sigma u_{ij}^\sigma] u_{ij}^\mu \]

\[ U^\mu = \frac{P^\mu}{\sqrt{P^2}} \]

\[ u_{ij}^\mu = \frac{p_{ij}^\mu}{\sqrt{p_{ij}^2}} \]

Time evolution of a thermal mini plasma

\[ t=0 \rightarrow q+\bar{q}=160 \]

\[ \pi \]

Surface emission

\[ q+\bar{q} \]

\[ D+\bar{D} \times 50 \]

\[ B+\bar{B} \times 100 \]
Hadronization:

Not at a fixed $T$ but broad $T$ distribution

Particles are produced over a wide mass range

Come to vacuum mass during expansion
Expansion of a plasma

For realistic calculations we use the initial configuration of the PHSD approach and compare NJL with PHSD calculations.

NJL

\[ m_q \approx 5 \text{ MeV} \]

no gluons

g fix

Hadronization by cross section

\[ q\bar{q} \rightarrow m_1 + m_2 \]

PHSD

\[ 400 \text{ MeV} \leq m_q \leq 800 \text{ MeV} \]

gluons

g running

\[ q\bar{q} \rightarrow m \text{ (or ”string”)}; qqq \rightarrow b \text{ (or”string”) } \]

Initial energy distr.

Au-Au @ 200 GeV - \( b=2 \text{ fm} \)

(a) transverse

(b) longitudinal
Expansion of a plasma with PHSD initial cond. I

(a) \( \pi^+ \) distribution

(b) \( \pi^+ \) distribution

(c) \( K^+ \) distribution

(d) \( K^+ \) distribution
NJL (RSP) has no hadronic rescattering without rescattering.

NJL (RSP) and PHSD have about the same $v_2$.

Time evolution completely different.
Summary of our long way

Starting point: NJL Lagrangian which shares the symmetries with QCD

Fierz transformation -> color less meson channel and qq channels

Bethe Salpeter equation in $q\bar{q}$ mesons as pole masses
Bethe Salpeter equation in qq diquarks as pole masses
diquark-quark Bethe Salpeter equation → baryons as pole masses
All masses described (10% precision) by 7 parameters fitted to ground state properties

Extension of all masses to finite $T$ and $\mu$ without new parameter
cross section (elastic and hadronisation) as well without any new parameter

Relativistic molecular dynamics approach based on constraints gives
time evolution equations of particles in a 6+1 dim. phase space

Studies of hadronization in realistic plasmas:
No sudden transition between quarks and hadrons
experimental results reasonably well reproduced (quite astonishing)