Fluctuations and Correlations in Statistical Models of Hadron Production

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An extension of the standard concept of the statistical ensembles is suggested. Namely, the statistical ensembles with extensive quantities fluctuating according to an externally given distribution is introduced. Applications in the statistical models of multiple hadron production in high energy physics are discussed.

1. INTRODUCTION

A successful application of the statistical model to description of mean hadron multiplicities in high energy collisions (see, e.g., Ref. [1]) has stimulated investigations of properties of statistical ensembles. Whenever possible, one prefers to use the grand canonical ensemble (GCE) due to its mathematical convenience. The canonical ensemble (CE) [2] was applied when the number of carriers of conserved charges is small (of the order of 1), such as strange hadrons [3], anti-baryons [4], or charmed hadrons [5]. The micro-canonical ensemble (MCE) [6] has been used to describe small systems with fixed energy, e.g. mean hadron multiplicities in proton-antiproton annihilation at rest. In all these cases, calculations performed in different statistical ensembles yield different results. This happens because the systems are 'small' and they are 'far away' from the thermodynamic limit (TL). The mean multiplicity of hadrons in relativistic heavy ion collisions ranges from 10^2 to 10^4, and mean multiplicities (of light hadrons) obtained within GCE, CE, and MCE approach each other. One refers here to the thermodynamical equivalence of statistical ensembles and uses the GCE for the hadron yields.

Measurements of i-th hadron multiplicity distributions $P(N_i)$ open a new field of applications of the statistical models. Note that hadron species i may correspond to a special type of hadrons like π^-, K^+ etc., or to a group of hadrons like negatively (positively) charged hadrons etc. The particle multiplicity fluctuations are usually quantified by the ratio of variance to mean value of multiplicity distributions $P(N_i)$, the scaled variance, $\omega_i \equiv (\langle N_i^2\rangle - \langle N_i\rangle^2)/\langle N_i\rangle$, and are a subject of current experimental activities. In statistical models there is a qualitative difference in the properties of $\langle N_i \rangle$ and ω_i. It was recently found [7–12] that even in the TL the results for ω_i are different in different ensembles. Hence the equivalence of ensembles holds for mean values in the TL, but does not extend to fluctuations. It was shown in Ref. [12] that the form of the multiplicity distributions derived within GCE, CE, and MCE approaches the Gauss distribution:

$$P(N_i) \;\cong\; \frac{1}{\sqrt{2\pi\;\omega_i\;\langle N_i \rangle}}\;\exp\left[-\frac{(N_i\;-\;\langle N_i \rangle)^2}{2\;\omega_i\;\langle N_i \rangle}\right]\;, \tag{1}$$

at $\langle N_i \rangle \gg 1$. The widths of the Gaussians $\sqrt{\omega_i\;\langle N_i \rangle}$ depend on the choice of the statistical ensemble, while the expectation values $\langle N_i \rangle$ remain the same.

2. MULTIPLICITY FLUCTUATIONS IN NUCLEUS-NUCLEUS COLLISIONS

In this section we present the results of the hadron-resonance gas for the scaled variances in the GCE, CE, and MCE along the chemical freeze-out line in central PbPb (AuAu) collisions for the whole energy range from SIS to LHC. The model parameters are the volume V, temperature T, baryonic chemical potential μ_B, and the strangeness saturation parameter γ_S (see details in Ref. [11]). Once a suitable set of thermodynamical parameters is determined for each collision energy, the scaled variance of negatively and positively charged particles can be calculated in the GCE, CE, and MCE. The ω^- and ω^+ in different ensembles are presented in Fig. 1 as the functions of the center of mass energy $\sqrt{s_{\rm NN}}$ of the nucleon pair.

A comparison between the data and predictions of statistical models should be performed for results which correspond to nucleus-nucleus collisions with a fixed number of nucleon participants. In Fig. 2 our predictions are compared with the NA49 data for the 1% most central PbPb collisions at 20–158 AGeV [13] selected by the number of the projectile participants. In the experimental study of nuclear collisions at high energies only a fraction of all produced particles is registered. If detected particles are uncorrelated, the scaled variance for the accepted particles can be obtained as: $\omega_{\rm acc} = 1 - q + q \cdot \omega$, where q is the probability of a single particle to be accepted. From Fig. 2 it follows that the NA49 data for $\omega^{\pm}$ extracted from 1% of the most central PbPb collisions at all SPS energies are best described by the results of the hadron-resonance gas model calculated within the MCE.

3. EXTENSION OF THE CONCEPT OF STATISTICAL ENSEMBLES

A statistical system is characterized by the extensive quantities: volume V, energy E, and conserved charge(s) Q. In statistical description of hadron or quark-gluon systems, these conserved charges are usually the net baryon number, strangeness, and electric charge. The MCE is defined by the postulate that all micro-states with given V, E, and Q have equal probabilities of being realized. This is the basic postulate of the statistical mechanics. The MCE partition function just calculates the number of microscopic states with given fixed (V, E, Q) values. In the CE the energy exchange between the considered system and 'infinite thermal bath' is assumed. Consequently, a new parameter, temperature T is introduced. To define the GCE, one makes a similar construction for the conserved charge Q. An 'infinite chemical bath' and the chemical potential μ are introduced. The CE introduces the energy fluctuations. In the GCE, there are additionally the charge fluctuations. The MCE, CE, and GCE are most familiar statistical ensembles. In the textbooks (see, e.g., Ref. [14]), the pressure (or isobaric) canonical ensemble has been also discussed. The 'infinite bath of the fixed external pressure' p_0 is then introduced. This leads to the volume fluctuations around the average value.

In general, there are 3 pairs of variables (V, p_0), (E, T), (Q, μ) and, thus, the 8 statistical ensembles can be constructed. For several conserved charges the number of standard statistical ensembles is even larger, as each charge can be treated either canonically or grand canonically. The ensembles with fluctuating volume have been discussed in Ref. [15].

A more general concept of the statistical ensembles was suggested in Ref. [16]. The statistical ensemble is defined by an externally given distribution of extensive quantities, $P_\alpha(E, V, Q)$. The construction of distribution of any variable X in such an ensemble proceeds in two steps. Firstly, the MCE X-distribution, $P_{\rm mce}(X; \mathbf{A})$, is calculated at fixed values of the extensive quantities $\mathbf{A} = (V, E, Q)$. Secondly, this result is averaged over the external distribution $P_\alpha(\mathbf{A})$ [16],

$$P_\alpha(X) \;=\; \int d\mathbf{A}\; P_\alpha(\mathbf{A})\; P_{\rm mce}(X; \mathbf{A}) \;. \tag{2}$$

Fluctuations of extensive quantities $\mathbf{A}$ around their average values depend not on the system's physical properties, but rather on external conditions. One can imagine a huge variety of these conditions, thus, the standard statistical ensembles discussed above are only some special examples. Thermodynamics relates the average quantities of the statistical ensemble. Thus, it may work in these new ensembles. The ensembles defined by Eq. (2), the α-ensembles, include the standard statistical ensembles as particular cases.

4. MCE WITH SCALING VOLUME FLUCTUATIONS

In collisions at relativistic energies many new particles are produced. Their number, masses and charges as well as their momenta vary from event to event. Most of the experimental results concern single particle production properties averaged over many interactions. It is well established that some of these properties, namely, mean particle multiplicities and transverse momentum spectra, follow simple rules of statistical mechanics. In proton-proton pp collisions the single particle momentum distribution has an approximately Boltzmann form in the local rest frame of produced matter: $dN/(p^2dp) \sim \exp\left(-\sqrt{p^2+m^2}/T\right)$, where T, p and m are the temperature parameter, the particle momentum and its mass, respectively. At large momentum, $p \gg m$, this gives: $dN/(p^2dp) \sim \exp(-p/T)$. Integration over momentum yields the mean particle multiplicity, $\langle N\rangle$, which is also governed by the Boltzmann factor for $m \gg T$: $\langle N\rangle \sim (mT)^{3/2} \exp(-m/T)$. The approximate validity of the exponential distributions is confirmed at low transverse momentum ($p_T \leq 2$ GeV) and the low mass ($m \leq 2$ GeV) domains. However, the temperature parameter T extracted from the data on pp interactions is in the range 160–190 MeV. Thus, almost all particles are produced at low p_T and with low masses.

Along with evident successes there are obvious problems of the statistical approach. The probability $P(N)$ to create N particles in pp collisions obeys the so called KNO scaling [17], namely: $P(N) = \psi(z)/\langle N\rangle$, where the KNO scaling function $\psi(z)$ only depends on $z \equiv N/\langle N\rangle$. The mean multiplicity increases with increasing collision energy, whereas the KNO scaling function remains unchanged. The latter implies that the scaled variance ω grows linearly with the mean multiplicity: $\omega \propto \langle N\rangle$. A qualitatively different behavior is predicted within the existing statistical models where the scaled variance is expected to be independent of the mean multiplicity: $\omega \approx \text{const} \approx 1$. This contradiction between the data and the statistical models constitutes the first problem. The second and the third problems which will be addressed here concern particle production at high (transverse) momenta and with high masses, respectively. In these regions the single particle energy distribution seems to obey a power law behavior [18]: $dN/(p^2dp) \sim \left(\sqrt{p^2+m^2}\right)^{-K}$. At $p \gg m$ this gives: $dN/(p^2dp) \sim p^{-K_p}$, with $K_p = K$. Integration over particle momentum yields the mean multiplicity which follows a power law dependence on the particle mass: $\langle N\rangle \sim m^{-K_m}$, with $K_m = K_p - 3$. The above power laws describe the data on spectra of light particles at large (transverse) momenta ($p \geq 3$ GeV) and on the mean multiplicity of heavy ($m \geq 3$ GeV) particles, respectively. The parameters fitted to the data are $K_p \cong 8$ and $K_m \cong 5$ [18]. One observes a growing disagreement between the exponential behavior and power law dependence with increasing (transverse) momentum and/or mass. At $p = 10$ GeV or $m = 10$ GeV the statistical models underestimate the data by more than 10 orders of magnitude.

In Ref. [19] we made an attempt to extend the statistical model to the hard domain of high transverse momenta and/or high hadron masses. The proposal is inspired by statistical type regularities [18] in the high transverse mass region, as well as by the recent work on the statistical ensembles with fluctuating extensive quantities [16]. We postulate that the volume of the system created in pp collision changes from event to event (see also Refs. [15, 20]). The volume probability distribution is given by the scaling function, $P_\alpha(V) = \phi_\alpha(V/\overline{V})/\overline{V}$, where $\overline{V}$ is the scaling parameter. The energy of the system is assumed to be fixed. The model based on these assumptions will be referred as the Micro-Canonical Ensemble with scaling Volume Fluctuations, the MCE/sVF.

The MCE partition function for the system with N Boltzmann massless neutral particles reads [8]:

$$W_N(E,V) = \frac{1}{N!}\left(\frac{gV}{2\pi^2}\right)^N \int_0^\infty p_1^2 dp_1 \ldots \int_0^\infty p_N^2 dp_N\ \delta(E - \sum_{i=1}^N p_i) = \frac{1}{E}\ \frac{A^N}{(3N-1)!N!}\ , \quad (3)$$

where E and V are the system energy and volume, respectively, g is the degeneracy factor, and $A \equiv gVE^3/\pi^2$. The MCE multiplicity distribution is given by $P_{\rm mce}(N;E,V) = W_N(E,V)/W(E,V)$, where $W(E,V)$ is the total MCE partition function $W(E,V) \equiv \sum_{N=1}^\infty W_N(E,V)$. The mean multiplicity equals to $\langle N\rangle_{\rm mce} \equiv \sum_{N=1}^\infty N\ P_{\rm mce}(N;E,V) \cong (A/27)^{1/4}$ at $A \gg 1$, and $P_{\rm mce}(N;E,V)$ can be approximated by the normal distribution (1) with $\omega_{\rm mce} \cong 1/4$.

The numerical calculations presented in Fig. 3 are performed for $g = 1$ and the energy density which corresponds to the temperature parameter $T = 160$ MeV. The latter relates the values of E and V via equation: $E = 3V\ T^4/\pi^2$. The mean multiplicity $\langle N\rangle_{\rm mce}$ in the MCE is then approximately equal to the GCE value: $\overline{N} = V\,T^3/\pi^2$. This reflects the thermodynamic equivalence of the MCE and GCE. However, the value of $\omega_{\rm mce}$ is four times smaller than $\omega_{\rm gce} = 1$.

The single particle momentum spectrum in the GCE reads,

$$F_{gce}(p) \;\equiv\; \frac{1}{\overline{N}}\,\frac{dN}{p^2dp} \;=\; \frac{V}{2\pi^2\,\overline{N}}\;\exp\Big(-\,\frac{p}{T}\Big) \;=\; \frac{1}{2T^3}\;\exp\Big(-\,\frac{p}{T}\Big)\;, \tag{4}$$

whereas the corresponding spectrum in the MCE is given by [19]:

$$F_{\rm mce}(p) \equiv \frac{\langle N\rangle_{\rm mce}^{-1}dN}{p^2dp} = \frac{\langle N\rangle_{\rm mce}^{-1}}{2E^3}\sum_{N=2}^{\infty}\frac{N\,(3N-1)!}{(3N-4)!}\Big(1-\frac{p}{E}\Big)^{3N-4}P_{\rm mce}(N;E,V)\;. \tag{5}$$

Both (4) and (5) are normalized such that $\int_0^\infty p^2dp\;F_{gce}(p) = 1$ and $\int_0^E p^2dp\;F_{\rm mce}(p) = 1$.

The distribution of any quantity X can be calculated as:

$$P_\alpha(X;E) \;=\; \int_0^\infty dV\;P_\alpha(V)\;P_{\rm mce}(X;E,V)\;, \tag{6}$$

where $P_{\rm mce}(X;E,V)$ is the distribution of the quantity X in the MCE with fixed E and V. The MCE/sVF defined by Eq. (6) is a special case of α-ensembles (2). The multiplicity distribution in the MCE/sVF calculated with (6) can be presented as [19], $P_\alpha(N;E) \;\cong\; \psi_\alpha(z)/\langle N\rangle_\alpha$, where $z \equiv N/\langle N\rangle_\alpha$, and the scaling function $\psi_\alpha(y)$ will be required to satisfy two normalization conditions: $\int_0^\infty dy\;\psi_\alpha(y) \;=\; 1$, $\int_0^\infty dy\;y\,\psi_\alpha(y) \;=\; 1$. The first condition guarantees the proper normalization of the volume probability density function. The second condition keeps the mean multiplicity in the MCE/sVF equal to the MCE mean multiplicity, $\langle N\rangle_\alpha \;\cong\; \overline{N}$. For convenience, a simple analytical form of the scaling function will be used: $\psi_\alpha(y) \;=\; [k\;\Gamma^{-1}(k)]^k\;y^{k-1}\;\exp(-k\,y)$, where $\Gamma(k)$ is the Euler gamma function. This function with $k = 4$ approximately describes the experimental data on KNO scaling in pp interactions. Note, that both normalization conditions for $\psi(z)$ are satisfied for any $k > 0$. For $\overline{N} \gg 1$ one gets: $\omega_\alpha \;\cong\; \kappa\;\langle N\rangle_\alpha$, where $\kappa = \int_0^\infty dy\;(y-1)^2\;\psi_\alpha(y) \;=\; k^{-1} \;>\; 0$.

In Fig. 3a the multiplicity distributions obtained within the MCE and the MCE/sVF for $\overline{N} = 50$ are compared. The scaled variance of the MCE/sVF distribution for $\overline{N} = 50$ is about 12.5, whereas the scaled variance of the MCE distribution is $1/4$. This large difference in the width of the MCE/sVF and the MCE distributions is clearly seen in the figure. The volume fluctuations in the MCE/sVF significantly increase the width of the multiplicity distribution. They are also expected to modify the single particle momentum spectrum. This is because for a fixed system energy, the volume of the system determines the energy density, and consequently, the effective temperature of particles. The single particle momentum spectrum within the MCE/sVF can be directly calculated from Eq. (6) and it reads [19]:

$$F_\alpha(p) \;=\; \frac{\langle N\rangle_\alpha^{-1}}{2E^3}\sum_{N=2}^{\infty}\frac{N\,(3N-1)!}{(3N-4)!}\,\left(1\;-\;\frac{p}{E}\right)^{3N-4}\;P_\alpha(N;E)\;. \tag{7}$$

The formal structure of the expression (7) is similar to the structure of the corresponding expression derived within the MCE (5). The only, but the crucial, difference is that the narrow MCE multiplicity distribution used for averaging the particle spectrum in Eq. (5) is replaced by the broad MCE/sVF multiplicity distribution in Eq. (7). The single particle momentum spectrum (7) is shown in Fig. 3*b*. At $p \ll E$ it has the following analytical form:

$$F_\alpha(p)\;\cong\;\frac{k^k\;\Gamma(k+4)}{2\;\Gamma(k)}\;\overline{T}^{k+1}\;(p+\overline{T}k)^{-k-4}\;\cong\;11.27\;\mathrm{GeV}^5\;(\mathrm{p}+4\overline{\mathrm{T}})^{-8}\;, \tag{8}$$

where $\overline{T} = 160$ MeV and $k = 4$ are set in the last expression. A rapid decrease of the spectrum starts at $p \geq 20$ GeV, when the threshold value $p = 24$ GeV is approached.

5. CONCLUSIONS

We have suggested to extend the concepts of statistical ensembles. The class of ensembles defined by external distributions of extensive quantities was introduced. This construction was motivated by the statistical approach to the particle number fluctuations in high energy collisions. As an example the new statistical ensemble MCE/sVF was constructed. The particle number distributions and momentum spectra in the MCE/sVF resemble features of hadrons produced in high energy *pp* interactions. In Ref. [21] the selected properties of semi-inclusive events have been studied within statistical models: the GCE, CE, MCE, and MCE/sVF. In particular, the mean multiplicity of neutral particles and momentum spectra of charged particles are considered at a fixed charged particle multiplicity. Different statistical ensembles lead to qualitatively different results for these semi-inclusive quantities. We also hope that the concept of statistical ensembles with fluctuating extensive quantities may be appropriate in many other situations too.

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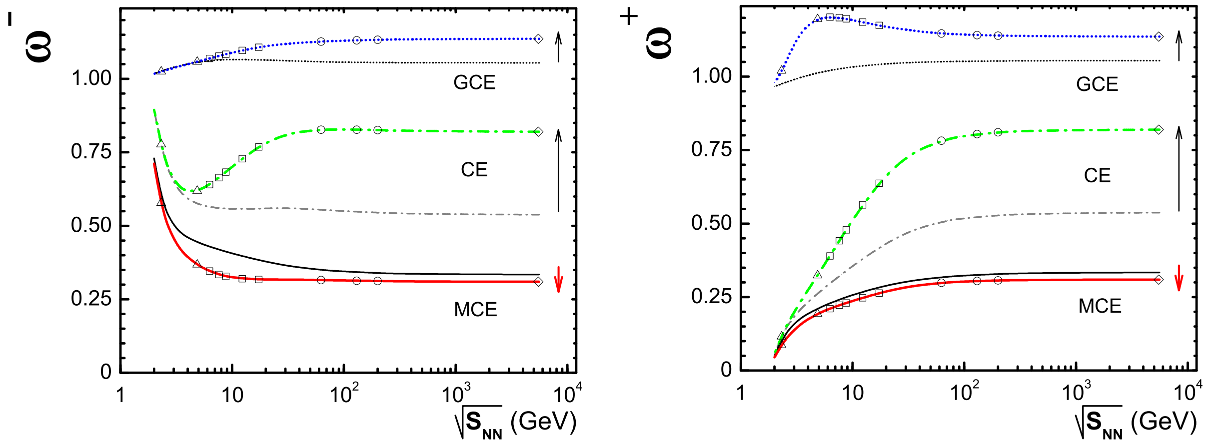

Figure 1. The scaled variances ω^- (a) and ω^+ (b), both primordial and final, along the chemical freeze-out line for central PbPb (AuAu) collisions. Different lines present the GCE, CE, and MCE results. Symbols at the lines for final particles correspond to the specific collision energies. The arrows show the effect of resonance decays.

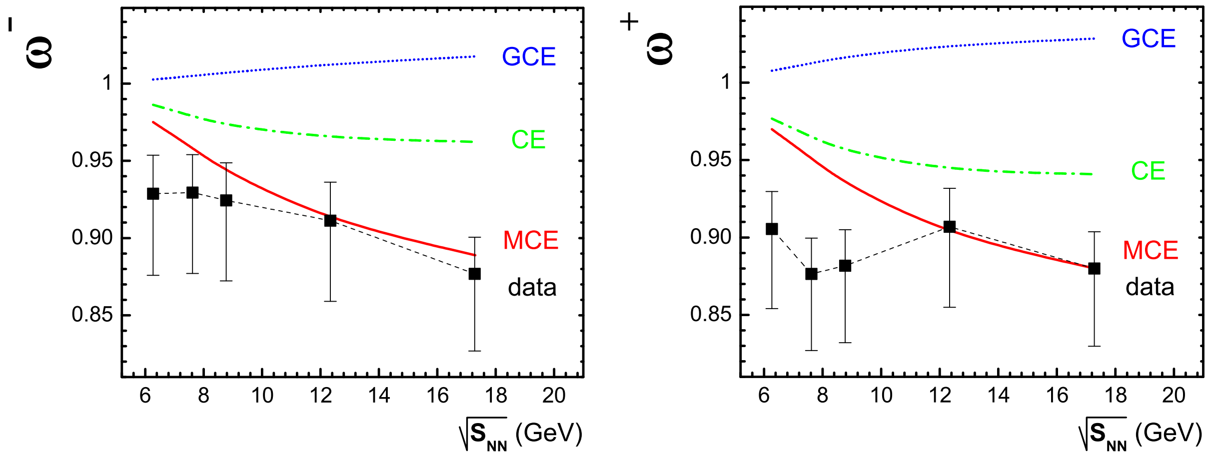

Figure 2. The scaled variances for negative (a) and positive (b) hadrons along the chemical freeze-out line for central PbPb collisions at the SPS energies. The points show the data of NA49 [13]. Lines show the GCE, CE, and MCE results calculated with the NA49 experimental acceptance.

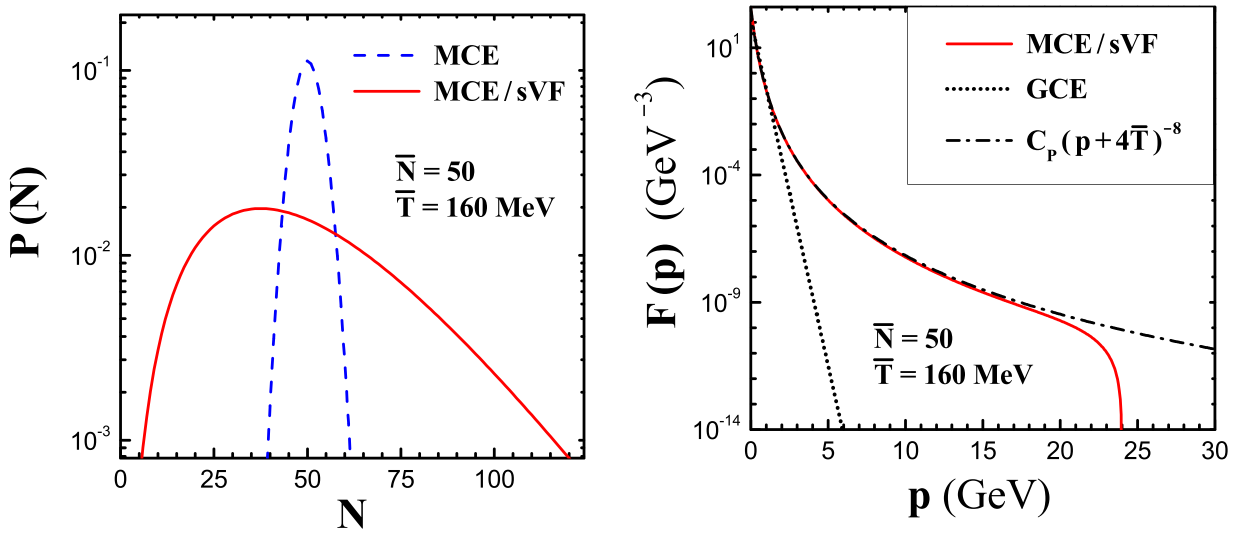

Figure 3. The system energy is $E = 3\overline{N}\,\overline{T} = 24$ GeV. (a) A comparison of the multiplicity distributions of massless neutral particles calculated with the MCE/sVF (solid line) and the MCE (dashed line). (b) The momentum spectrum of massless neutral particles calculated within the MCE/sVF (7), solid line, and the GCE (4), dotted line. The approximation (8) of the MCE/sVF spectrum is shown by the dashed-dotted line.

FIGURE CAPTIONS

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