A novel strong coupling expansion of the QCD Hamiltonian

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Introducing an infinite spatial lattice with box length a, a systematic expansion of the physical QCD Hamiltonian in $\lambda = g^{-2/3}$ can be obtained, with the free part being the sum of the Hamiltonians of the quantum mechanics of spatially constant fields for each box, and interaction terms proportional to λ^n with n spatial derivatives connecting different boxes. As an example, the energy of the vacuum and the lowest scalar glueball is calculated up to order λ^2 for the case of $SU(2)$ Yang-Mills theory.

1. INTRODUCTION

The quantum Hamiltonian of $SU(2)$ Yang-Mills theory, to which we limit ourselves here for simplicity, can be obtained [1] by exploiting the time-dependence of the gauge transformations to put $A_{a0}(x) = 0,\ a = 1, 2, 3$, and quantizing the spatial fields in the Schrödinger representation, $\Pi_{ai}(\mathbf{x}) = -E_{ai}(\mathbf{x}) = -i\delta/\delta A_{ai}(\mathbf{x})$. The physical states Ψ have to satisfy the Schrödinger eq. and the three non-Abelian Gauss law constraints

$$H\Psi = E\Psi \ , \qquad H = \int d^3\mathbf{x}\, \frac{1}{2} \sum_{a,i} \left[\left(\frac{\delta}{\delta A_{ai}(\mathbf{x})} \right)^2 + B_{ai}^2(A(\mathbf{x})) \right] , \tag{1}$$

$$G_a(\mathbf{x})\Psi = 0 \ , \qquad G_a(\mathbf{x}) = -i \left(\delta_{ac}\partial_i + g\epsilon_{abc}A_{bi}(\mathbf{x}) \right) \frac{\delta}{\delta A_{ci}(\mathbf{x})} \tag{2}$$

with the chromo-magnetic fields $B_{ai}(A) = \epsilon_{ijk} \left(\partial_j A_{ak} + \frac{1}{2} g\epsilon_{abc}A_{bj}A_{ck} \right)$ and the generators $G_a(\mathbf{x})$ of the residual time-independent gauge transformations, satisfying $[G_a(\mathbf{x}), H] = 0$, and $[G_a(\mathbf{x}), G_b(\mathbf{y})] = ig\delta^3(\mathbf{x} - \mathbf{y})\epsilon_{abc}G_c(\mathbf{x}$. The matrix elements have Cartesian measure

$$\langle \Phi_1 | \mathcal{O} | \Phi_2 \rangle = \int \prod_{\mathbf{x}} \prod_{ik} dA_{ik}(\mathbf{x})\ \Phi_1^* \mathcal{O} \Phi_2 \ . \tag{3}$$

In order to implement the Gauss laws (2) into (1) to obtain the physical Hamiltonian, it is very useful to Abelianise them by a suitable point transformation of the gauge fields.

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2. THE PHYSICAL HAMILTONIAN OF SU(2) YANG-MILLS THEORY AND ITS STRONG COUPLING EXPANSION

Point transformation to the new set of adapted coordinates [2], the 3 q_j $(j=1,2,3)$ and the 6 elements $S_{ik}=S_{ki}$ $(i,k=1,2,3)$ of the positive definite symmetric 3×3 matrix S,

$$A_{ai}\left(q,S\right) = O_{ak}\left(q\right)S_{ki} - \frac{1}{2g}\epsilon_{abc}\left(O\left(q\right)\partial_i O^T\left(q\right)\right)_{bc} \,, \tag{4}$$

where $O(q)$ is an orthogonal 3×3 matrix parametrised by the q_i, leads to an Abelianisation of the Gauss law constraints

$$G_a\Phi = 0 \quad \Leftrightarrow \quad \frac{\delta}{\delta q_i}\Phi = 0 \quad \text{(Abelianisation)}.$$

Eq. (4) corresponds to the symmetric gauge $\chi_i(A)=\epsilon_{ijk}A_{jk}=0$. It has been proven in [3] that, at least for strong coupling, the symmetric gauge exists, i.e. any time-independent gauge field can be carried over uniquely into the symmetric gauge, and in [2] that both indices of the tensor field S are spatial indices, leaving S a colorless local field.

According to the general scheme of [1], the correctly ordered physical quantum Hamiltonian in the symmetric gauge in terms of the colorless physical variables $S_{ik}(\mathbf{x})$ and the corresponding canonically conjugate momenta $P_{ik}(\mathbf{x})\equiv -i\delta/\delta S_{ik}(\mathbf{x})$ reads [4]

$$\begin{aligned} H(S,P) \;=\; & \frac{1}{2}\mathcal{J}^{-1}\int d^3\mathbf{x}\; P_{ai}\;\mathcal{J}P_{ai} + \frac{1}{2}\int d^3\mathbf{x}\left(B_{ai}(S)\right)^2 \\ & -\mathcal{J}^{-1}\int d^3\mathbf{x}\int d^3\mathbf{y}\Big\{\Big(D_i(S)_{ma}P_{im}\Big)(\mathbf{x})\mathcal{J}\langle\mathbf{x}\; a|{}^*D^{-2}(S)|\mathbf{y}\; b\rangle\Big(D_j(S)_{bn}P_{nj}\Big)(\mathbf{y})\Big\} \end{aligned} \tag{5}$$

with the covariant derivative $D_i(S)_{kl}\equiv\delta_{kl}\partial_i - g\epsilon_{klm}S_{mi}$, the Faddeev-Popov (FP) operator

$${}^*D_{kl}(S)\equiv\epsilon_{kmi}D_i(S)_{ml} = \epsilon_{kli}\partial_i - g\gamma_{kl}(S)\,, \qquad \gamma_{kl}(S)\equiv S_{kl}-\delta_{kl}\mathrm{tr}S, \tag{6}$$

and the Jacobian $\mathcal{J}\equiv\det|{}^*D|$. The matrix element of a physical operator O is given by

$$\langle\Psi'|O|\Psi\rangle \;\propto \int\prod_{\mathbf{x}}\Big[dS(\mathbf{x})\Big]\mathcal{J}\Psi'^*[S]O\Psi[S]\;. \tag{7}$$

The inverse of the FP operator can be expanded in the number of spatial derivatives

$$\begin{aligned} \langle\mathbf{x}\; k|{}^*D^{-1}(S)|\mathbf{y}\; l\rangle \;=\; & -\frac{1}{g}\gamma^{-1}_{kl}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) + \frac{1}{g^2}\gamma^{-1}_{ka}(\mathbf{x})\epsilon_{abc}\partial^{(\mathbf{x})}_c\left[\gamma^{-1}_{bl}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\right] \\ & -\frac{1}{g^3}\gamma^{-1}_{ka}(\mathbf{x})\epsilon_{abc}\partial^{(\mathbf{x})}_c\Big[\gamma^{-1}_{bi}(\mathbf{x})\epsilon_{ijk}\partial^{(\mathbf{x})}_k\big[\gamma^{-1}_{jl}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\big]\Big] + \ldots\;. \end{aligned} \tag{8}$$

In order to perform a consistent expansion, also the non-locality in the Jacobian $\mathcal{J}$ has to be taken into account [4]. The Jacobian $\mathcal{J}$ factorizes $\mathcal{J} = \mathcal{J}_0 \widetilde{\mathcal{J}}$ with the local

$$\mathcal{J}_0 \equiv \det|\gamma| = \prod_{\mathbf{x}} \prod_{i<j} (\phi_i(\mathbf{x}) + \phi_j(\mathbf{x})) \qquad (\phi_i = \text{eigenvalues of } S), \tag{9}$$

and the non-local $\widetilde{\mathcal{J}}$, which can be included into the wave functional $\widetilde{\Psi}(S) := \widetilde{\mathcal{J}}^{-1/2}\Psi(S)$ leading to the corresponding transformed Hamiltonian

$$\widetilde{H}(S,P) := \widetilde{\mathcal{J}}^{1/2} H(S,P) \widetilde{\mathcal{J}}^{-1/2} = H(S,P)\Big|_{J\to J_0} + V_{\text{measure}}(S)\ . \tag{10}$$

It is Hermitean with respect to the local measure $\mathcal{J}_0$ on the cost of extra terms V_{measure} and can be expanded in the number of spatial derivatives using (8).

Next, an ultraviolet cutoff a is put by introducing an infinite spatial lattice of granulas $G(\mathbf{n}, a)$, here cubes of length a, situated at sites $\mathbf{x} = a\mathbf{n}$ ($\mathbf{n} \in Z^3$), and considering the averaged variables

$$S(\mathbf{n}) := \frac{1}{a^3} \int_{G(\mathbf{n},a)} d\mathbf{x}\ S(\mathbf{x})$$

and discretised spatial derivatives relating the $S(\mathbf{n})$ of different granulas (see [4] for details).

After an appropriate rescaling of the dynamical fields a novel strong coupling expansion of the Hamiltonian in $\lambda = g^{-2/3}$ can be obtained [4]

$$\widetilde{H} = \frac{g^{2/3}}{a}\left[\mathcal{H}_0 + \lambda \sum_\alpha \mathcal{V}_\alpha^{(\partial)} + \lambda^2 \left(\sum_\beta \mathcal{V}_\beta^{(\Delta)} + \sum_\gamma \mathcal{V}_\gamma^{(\partial\partial\neq\Delta)}\right) + \mathcal{O}(\lambda^3)\right] \tag{11}$$

as an alternative to existing strong coupling expansions [5] and [6] based on Wilsonian lattice QCD. The "free part" in (11) is just the sum of Hamiltonians $\mathcal{H}_0 = \sum_{\mathbf{n}} \mathcal{H}_0^{QM}(\mathbf{n})$ of Yang-Mills quantum mechanics of spatially constant fields [7–10] at each site, and the $\mathcal{V}_\alpha^{(\partial)}$ and $\mathcal{V}_\beta^{(\Delta)}$ are interaction parts, relating different sites. The local measure $\mathcal{J}_0 = \prod_{\mathbf{n}} \mathcal{J}_0^{QM}(\mathbf{n})$ is correspondingly the product of the quantum mechanical measures at each site. In terms of the principal-axes variables of the positive definite symmetric 3×3 matrix field S

$$S = R^T(\alpha,\beta,\gamma)\ \text{diag}\ (\phi_1,\phi_2,\phi_3)\ R(\alpha,\beta,\gamma)\ , \tag{12}$$

with the $SO(3)$ matrix R parametrized by the three Euler angles $\chi = (\alpha,\beta,\gamma)$, we find

$$\mathcal{J}_0^{QM} \to \sin\beta \prod_{i<j}\left(\phi_i^2 - \phi_j^2\right) \quad \to \quad 0 < \phi_1 < \phi_2 < \phi_3 \quad \text{(principle orbits)} \tag{13}$$

and (with the intrinsic spin angular momenta ξ_i)

$$\mathcal{H}_0^{QM} = \frac{1}{2}\sum_{ijk}^{\text{cyclic}} \left[\pi_i^2 - \frac{2i}{\phi_j^2-\phi_k^2}\left(\phi_j\pi_j - \phi_k\pi_k\right) + \xi_i^2\frac{\phi_j^2+\phi_k^2}{(\phi_j^2-\phi_k^2)^2} + \phi_j^2\phi_k^2\right] . \tag{14}$$

Its low energy spectrum and eigenstates at any site $\mathbf{n}$

$$\mathcal{H}_0^{QM}(\mathbf{n})|\Phi^{(S)\pm}_{i,M}\rangle_{\mathbf{n}} = \epsilon_i^{(S)\pm}(\mathbf{n})|\Phi^{(S)\pm}_{i,M}\rangle_{\mathbf{n}} , \tag{15}$$

characterised by the quantum numbers of spin S, M, and parity P, are known with high accuracy [10]. Hence the eigenstates of $\mathcal{H}_0$ in (11) are free glueball excitations of the lattice. The interactions $\mathcal{V}$ (11) can be included using perturbation theory in λ.

3. CALCULATION OF THE GLUEBALL SPECTRUM UP TO ORDER λ^2

Using 1st and 2nd order perturbation theory in λ give the results [4]

$$E^+_{\text{vac}} = \mathcal{N}\frac{g^{2/3}}{a}\left[4.1167 + 29.894\lambda^2 + \mathcal{O}(\lambda^3)\right] \tag{16}$$

for the energy of the interacting glueball vacuum and

$$E_1^{(0)+}(k) - E^+_{\text{vac}} = \left[\, 2.270 + 13.511\lambda^2 + \mathcal{O}(\lambda^3)\right]\frac{g^{2/3}}{a} + 0.488\frac{a}{g^{2/3}}k^2 + \mathcal{O}((a^2k^2)^2) \tag{17}$$

for the energy spectrum of the interacting spin-0 glueball, up to λ^2 for the (+) b.c. and similar results for the (−) b.c. The first, zeroth order numbers, correspond to the result of Yang-Mills quantum mechanics. Note that Lorentz invariance asks for energy momentum relation $E = \sqrt{M^2+k^2} \simeq M + (2M)^{-1}\ k^2$. The result, shown here for the scalar glueball, which limits itself to the terms in the Hamiltonian containing the Laplace-operator Δ as a first step, violates this condition by about a factor of two. Including all spin-orbit coupling terms in the Hamiltonian dropped in this first approach and considering all possible $J = L+S = 0$ states is expected restore Lorentz invariance.

To study the coupling constant renormalisation in the IR, consider the physical glueball mass

$$M = \frac{g_0^{2/3}}{a}\left[\mu + c g_0^{-4/3}\right] . \tag{18}$$

Independence of the box size a is given for the two cases, $g_0 = 0$ or $g_0^{4/3} = -c/\mu$. The first solution corresponds to the perturbative fixed point, and the second, if it exists ($c < 0$), to an infrared fixed point. My result for the lowest spin-0 glueball $c_1^{(0)}/\mu_1^{(0)} = 5.95$ suggests, that no infrared fixed points exist, in accordance with the corresponding result of Wilsonian lattice QCD [11]. Solving the above equation (18) for positive ($c > 0$) one obtains

$$g_0^{2/3}(Ma) = \frac{Ma}{2\mu} + \sqrt{\left(\frac{Ma}{2\mu}\right)^2 - \frac{c}{\mu}}\ , \qquad a > a_c := 2\sqrt{c\mu}/M \tag{19}$$

with the physical glueball mass M. For a typical $M \sim 1.6$ GeV [12] we find $a_c \sim 1.4$ fm. Comparing the behaviour of the bare coupling constant (19), obtained for boxes of large size a, with those obtained for small boxes in [7, 8], should lead to information about the intermediate region, including the possibility of the existence of phase transitions.

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