Renormdynamics, negative binomial distribution and Riemann zeta function

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Renormdynamic equations of motion and their solutions are given. New equation for NBD distribution and Riemann zeta function invented.

1. RENORMDYNAMICS

In quantum field theory (QFT) [1] existence of a given theory means, that we can control its behavior at some scales (short or large distances) by renormalization theory [1, 2]. If the theory exists, than we want to solve it, which means to determine what happens on other (large or short) scales. This is the problem (and content) of Renormdynamics. The result of the Renormdynamics, the solution of its discrete or continual motion equations, is the effective QFT on a given scale (different from the initial one). We can invent scale variable λ and consider QFT on $D+1+1$ dimensional space-time-scale. For the scale variable $\lambda \in (0,1]$ it is natural to consider q-discretization, $0<q<1$, $\lambda_n = q^n$, $n=0,1,2,...$ and p-adic, nonarchimedian metric, with $q^{-1}=p$ – prime integer number. The field variable $\varphi(x,t,\lambda)$ is complex function of the real, x, t, and p-adic, λ, variables. The solution of the UV renormdynamic problem means, to find evolution from finite to small scales with respect to the scale (time) $\tau = \ln \lambda/\lambda_0 \in (0,-\infty)$. Solution of the IR renormdynamic problem means to find evolution from finite to the large scales, $\tau = \ln \lambda/\lambda_0 \in (0,\infty)$. This evolution is determined by Renormdynamic motion equations with respect to the scale-time.

2. RENORMDYNAMICS OF QCD

The Renormdynamic (RD) equations play an important role in our understanding of Quantum Chromodynamics (QCD) and the strong interactions. The beta function is the

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most prominent object for QCD RD equations. The calculation of the one-loop β-function in QCD has lead to the discovery of asymptotic freedom in this model and to the establishment of QCD as the theory of strong interactions [3–5].

RD equation for the coupling constant of QCD belongs to the following class of equations,

$$\dot{a} = \beta_1 a + \beta_2 a^2 + ... = \sum_{n\geq 1} \beta_n a^n. \tag{1}$$

The Eq. (1) can be reparametrized,

$$\begin{aligned}
&a(t) = f(A(t)) = A + f_2 A^2 + ... + f_n A^n + ... = \sum_{n\geq 1} f_n A^n,\\
&\dot{A} = b_1 A + b_2 A^2 + ... = \sum_{n\geq 1} b_n A^n, \qquad (2)\\
&\dot{a} = \dot{A} f'(A) = (b_1 A + b_2 A^2 + ...)(1 + 2 f_2 A + ... + n f_n A^{n-1} + ...)\\
&= \beta_1 (A + f_2 A^2 + ... + f_n A^n + ...) + \beta_2 (A^2 + 2 f_2 A^3 + ...) + ... + \beta_n (A^n + n f_2 A^{n+1} + ...) + ...\\
&= \beta_1 A + (\beta_2 + \beta_1 f_2) A^2 + (\beta_3 + 2\beta_2 f_2 + \beta_1 f_3) A^3 + ... + (\beta_n + (n-1)\beta_{n-1} f_2 + ... + \beta_1 f_n) A^n + ...\\
&= \sum_{n,n_1,n_2\geq 1} A^n b_{n_1} n_2 f_{n_2} \delta_{n,n_1+n_2-1} = \sum_{n,m\geq 1; m_1,...,m_k\geq 0} A^n \beta_m f_1^{m_1} ... f_k^{m_k} f(n,m,m_1,...,m_k),\\
&f(n,m,m_1,...,m_k) = \frac{m!}{m_1!...m_k!} \delta_{n,m_1+2m_2+...+km_k} \delta_{m,m_1+m_2+...+m_k}, \qquad (3)
\end{aligned}$$

$$\begin{aligned}
&b_1 = \beta_1,\ b_2 = \beta_2 + f_2\beta_1 - 2 f_2 b_1 = \beta_2 - f_2 \beta_1,\\
&b_3 = \beta_3 + 2 f_2 \beta_2 + f_3 \beta_1 - 2 f_2 b_2 - 3 f_3 b_1 = \beta_3 + 2(f_2^2 - f_3)\beta_1,\\
&b_4 = \beta_4 + 3 f_2 \beta_3 + f_2^2 \beta_2 + 2 f_3 \beta_2 - 3 f_4 b_1 - 3 f_3 b_2 - 2 f_2 b_3, ...\\
&b_n = \beta_n + ... + \beta_1 f_n - 2 f_2 b_{n-1} - ... - n f_n b_1, ...
\end{aligned} \tag{4}$$

So, by reparametrization, beyond the critical dimension ($\beta_1 \neq 0$) we can change any coefficient but β_1. We can fix any higher coefficient with zero value, if we take

$$f_2 = \frac{\beta_2}{\beta_1},\ f_3 = \frac{\beta_3}{2\beta_1} + f_2^2,\ ... , f_n = \frac{\beta_n + ...}{(n-1)\beta_1}, ... \tag{5}$$

In this case we have exact classical dynamics in the (external) space-time and simple scale dynamics,

$$g = \left(\frac{\mu}{\mu_0}\right)^{\frac{D-4}{2}} g_0 = e^{-\varepsilon\tau} g_0; \varphi(\tau,t,x) = e^{-(D-2)/2\tau} \varphi_0(t,x), \psi(\tau,t,x) = e^{-(D-1)/2\tau} \psi_0(t,x) \tag{6}$$

We will consider in applications also the case when only one of the higher coefficients is nonzero.

In the critical dimension of space-time $\beta_1 = 0$ and we can change by reparametrization any coefficient but β_2 and β_3. From the relations (4), we can define the minimal form of the RD equation

$$\dot{A} = \beta_2 A^2 + \beta_3 A^3, \tag{7}$$

e.g. $b_4 = 0$ when

$$f_3 = \frac{\beta_4}{\beta_2} + \frac{\beta_3}{\beta_2} f_2 + f_2^2, \tag{8}$$

f_2 remains arbitrary and we can take, e.g. $f_2 = 0$.

We can solve (7) as implicit function,

$$u^{\beta_3/\beta_2} e^{-u} = c e^{\beta_2 t}, \; u = \frac{1}{A} + \frac{\beta_3}{\beta_2}, \tag{9}$$

than, as in the noncritical case, explicit solution will be given by reparametrization representation (2). If we know somehow the coefficients β_n, e.g. for first several exact and for others asymptotic values (see, e.g. [6]) than we can construct reparametrization function (2) and find the dynamics of the running coupling constant.

3. NBD AND RIEMANN ZETA FUNCTION

Negative binomial distribution (NBD)

$$P(n) = \frac{\Gamma(r+n)}{\Gamma(r) n!}(1-p)^r p^n = \frac{\Gamma(r+n)}{\Gamma(r) n!}\left(\frac{r}{<n>+r}\right)^r \left(\frac{<n>}{<n>+r}\right)^n, \; \sum_{n\geq 0} P(n) = 1, \tag{10}$$

provides a very good parametrization for multiplicity distributions in e^+e^- annihilation; in deep inelastic lepton scattering; in proton-proton collisions; in proton-nucleus scattering. Hadronic collisions at high energies (LHC) lead to charged multiplicity distributions whose shapes are well fitted by a single NBD in fixed intervals of central (pseudo)rapidity η [7].

The generating function for NBD is

$$F(h) = \left(1 + \frac{<n>}{r}(1-h)\right)^{-r} = \left(\frac{r}{<n>+r}\right)^r (1-ph))^{-r} = \sum_{n\geq 0} P(n) h^n. \tag{11}$$

An useful property of NBD with parameters $<n>$ and r is that it is the distribution of a sum of r independent random variables with a Bose-Einstein distribution[1] and mean

[1] A Bose-Einstein, or geometrical, distribution is a thermal distribution for single state systems.

$< n > /r$,

$$p(n) = \frac{1}{< n > +1}(\frac{< n >}{< n > +1})^n = (e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2})e^{-\beta\hbar\omega(n+1/2)},$$
$$\sum np(n) =< n >= \frac{1}{e^{\beta\hbar\omega} - 1}, \; f(x) = \sum_n x^n p(n) = (1+ < n > (1-x))^{-1}$$
$$T = \frac{\hbar\omega}{\ln(1 + \frac{1}{<n>})} \simeq \hbar\omega < n >, \; < n >\gg 1. \tag{12}$$

Temperature defined in (12) gives an estimation of the Glukvar temperature when it radiates hadrons. We see that universality of NBD in hadron-production is similar to the universality of black body radiation. We can put

$$F(r, < n >)^m = F(mr, m < n >) \tag{13}$$

in the closed nonlocal form

$$Q_q F = F^q, \; Q_q = q^D, \quad D = \frac{rd}{dr} + \frac{< n > d}{d < n >} = \frac{x_1 d}{dx_1} + \frac{x_2 d}{dx_2}. \tag{14}$$

Note that $F(x_1, x_2)$ may be any function of the type

$$F(x_1, x_2) = f\left(\frac{x_1}{x_2}\right)^{x_2} = \varphi\left(\frac{x_1}{x_2}\right)^{x_1}, \; \varphi(x) = f(x)^{\frac{1}{x}}. \tag{15}$$

We can consider also n-dimensional generalization

$$D_n = \frac{x_1 d}{dx_1} + ... + \frac{x_n d}{dx_n}, \; F(x_1, ..., x_n) = f\left(\frac{x_i}{x_j}\right)^{x_k}. \tag{16}$$

By construction we know the solution of the nice equation (14) as GF of NBD, F. We obtain corresponding differential equations, if we consider $q = 1 + \varepsilon$, for small ε,

$$(D(D-1)...(D-m+1) - (lnF)^m)\Psi = 0, \; (\frac{\Gamma(D+1)}{\Gamma(D+1-m)} - (\ln F)^m)\Psi = 0,$$
$$(D_m - \Phi^m)\Psi = 0, \; m = 1, 2, 3, ..., \; D_m = \frac{\Gamma(D+1)}{\Gamma(D+1-m)}, \Phi = \ln F, \tag{17}$$

with the solution $\Psi = F = \exp(\Phi)$.

Let us consider the values $q = n, n = 1, 2, 3, ...$ and take sum of the corresponding equations (14), we find

$$\zeta(-D)F = \frac{F}{1-F}. \tag{18}$$

Now we invent a Hamiltonian H with spectrum corresponding to the set of nontrivial zeros of the zeta function, in correspondence with Riemann hypothesis,

$$-D_n = \frac{n}{2} + iH_n, \; H_n = i(\frac{n}{2} + D_n), \; D_n = x_1\partial_1 + x_2\partial_2 + ... + x_n\partial_n,$$

$$H_n^+ = H_n = \sum_{m=1}^{n} H_1(x_m),\ H_1(x) = i(\frac{1}{2} + x\partial_x) = -\frac{1}{2}(x\hat{p} + \hat{p}x),\ \hat{p} = -i\partial_x\,. \tag{19}$$

The Hamiltonian $H = H_n$ is hermitian, its spectrum is real. The case $n = 1$ corresponds to the Riemann hypothesis. The case $n = 2$ corresponds to NBD,

$$\zeta(1 + iH_2)F = \frac{F}{1-F},\ F(x_1, x_2; h) = (1 + \frac{x_1}{x_2}(1-h))^{-x_2}. \tag{20}$$

Let us scale $x_2 \to \lambda x_2$ and take $\lambda \to \infty$ in (20), we obtain

$$\begin{aligned}
&\zeta(\frac{1}{2} + iH(x))e^{-(1-h)x} = \frac{1}{e^{(1-h)x} - 1},\ H(x) = i(\frac{1}{2} + x\partial_x) = -\frac{1}{2}(x\hat{p} + \hat{p}x),\\
&H(x)\psi_E = E\psi_E,\ \psi_E = cx^{-s},\ s = \frac{1}{2} + iE,\ c = 1/\sqrt{2\pi},\\
&\int_0^\infty dx\psi_E(x)^*\psi_{E'}(x) = \delta(E - E'),
\end{aligned} \tag{21}$$

$$\zeta(-D)e^{-x} = \zeta(\frac{1}{2} + iH(x))e^{-x} = \frac{1}{e^x - 1}, \tag{22}$$

$$\begin{aligned}
&\int_0^\infty dx x^{s-1}\zeta(\frac{1}{2} + iH(x))e^{-x} =< x^{s-1}|\zeta(\frac{1}{2} + iH(x))e^{-x} >= \int_0^\infty dx x^{s-1}\frac{1}{e^x - 1} = \Gamma(s)\zeta(s),\\
&< x^{s-1}|\zeta(\frac{1}{2} + iH(x))e^{-x} >=< \zeta(\frac{1}{2} - iH(x))x^{s-1}|e^{-x} >\\
&= \zeta(\frac{1}{2} - iE) < x^{s-1}|e^{-x} >= \zeta(\frac{1}{2} - iE)\Gamma(s),\ \zeta(\frac{1}{2} - iE) = \zeta(s).
\end{aligned} \tag{23}$$

A slightly different consideration is the following. If we rescale $x \to xy$ in (22), multiply by y^{s-1} and integrate by y, we obtain usual integral formula for zeta-function

$$\begin{aligned}
&\zeta(-D)\int_0^\infty y^{s-1}e^{-xy}dy = \int_0^\infty dy\frac{y^{s-1}}{e^{xy} - 1},\\
&\zeta(-D)x^{-s}\Gamma(s) = x^{-s}\int_0^\infty dy\frac{y^{s-1}}{e^y - 1},\\
&\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty dy\frac{y^{s-1}}{e^y - 1}\,.
\end{aligned} \tag{24}$$

The equation (22) can be obtained by the same consideration without reduction,

$$F(nx) = F^n(x) \Rightarrow \zeta(-D)F(x) = (F^{-1}(x) - 1)^{-1},\ F(x) = e^{ax}. \tag{25}$$

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