INCLUSIVE TAU LEPTON DECAY: 
THE EFFECTS DUE TO HADRONIZATION

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The $\tau$ lepton is the only lepton which is heavy enough ($M_\tau \simeq 1.777 \text{ GeV}$) to decay into hadrons. The interest to this process is primarily due to

- Tests of QCD and Standard Model
- Constraints on “New physics”
- Precise experimental data
- No need in phenomenological models
- Probes infrared hadron dynamics
The experimentally measurable quantity here is

\[ R_\tau = \frac{\Gamma(\tau^- \to \text{hadrons}^- \nu_\tau)}{\Gamma(\tau^- \to e^- \bar{\nu}_e \nu_\tau)} \]

\[ = R_{\tau, V} + R_{\tau, A} + R_{\tau, S} \]

\[ = 3.640 \pm 0.010, \]

\[ R_{\tau, V} = R_{\tau, V}^{J=0} + R_{\tau, V}^{J=1} \]

\[ = 1.783 \pm 0.011 \pm 0.002, \]

\[ R_{\tau, A} = R_{\tau, A}^{J=0} + R_{\tau, A}^{J=1} \]

\[ = 1.695 \pm 0.011 \pm 0.002. \]

The theoretical prediction for the quantities on hand reads

\[ R_{J=1}^{V/A,\tau} = \frac{N_c}{2} |V_{ud}|^2 S_{EW} \left( \Delta_{QCD}^{V/A} + \delta'_{EW} \right), \]

\[ N_c = 3, \quad |V_{ud}| = 0.9738 \pm 0.0005, \quad S_{EW} = 1.0194 \pm 0.0050, \quad \delta'_{EW} = 0.0010, \]

\[ \Delta_{QCD}^{V/A} = 2 \int_{m_{V/A}^2}^{M_{\tau}^2} f \left( \frac{s}{M_{\tau}^2} \right) R_{V/A}^{s} \left( s \right) \frac{ds}{M_{\tau}^2}, \]

where \( M_{\tau} = 1.777 \text{ GeV}, \ f(x) = (1 - x)^2 (1 + 2x), \)

\[ R_{V/A}^{s} = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \left[ \Pi_{V/A}^{s+i\varepsilon} - \Pi_{V/A}^{s-i\varepsilon} \right] = \frac{1}{\pi} \text{Im} \lim_{\varepsilon \to 0^+} \Pi_{V/A}^{s+i\varepsilon}, \]

\[ \Pi_{\mu\nu}(q^2) = i \int d^4x \ e^{iqx} \langle 0 | T \left\{ J_\mu(x) J_\nu(0) \right\} | 0 \rangle \equiv i \frac{1}{12\pi^2} (g_{\mu\nu} q^2 - q_\mu q_\nu) \Pi(q^2). \]

**Braaten, Narison, Pich, NPB373(1992); Le Diberder, Pich, PLB289(1992).**
For practical purposes it is convenient to use Adler function

\[ D(Q^2) = - \frac{d \Pi(-Q^2)}{d \ln Q^2}, \quad Q^2 = -q^2 = -s. \]


Its ultraviolet behavior can be approximated by

\[ D(Q^2) \simeq D^{(\ell)}_{\text{pert}}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[ \alpha^{(\ell)}_s(Q^2) \right]^j, \quad Q^2 \to \infty, \]

where \( \alpha^{(\ell)}_s(Q^2) \) is the \( \ell \)-loop perturbative running coupling.

One–loop: \( \alpha^{(1)}_{\text{pert}}(Q^2) = 4\pi / \left[ \beta_0 \ln(Q^2/\Lambda^2) \right] \), \( \beta_0 = 11 - 2n_f/3 \), \( d_1 = 1/\pi \).


For functions \( R(s) \) and \( D(Q^2) \) the following relations hold:

\[ R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0+} \int_{s-i\varepsilon}^{s+i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta} \quad \text{leftrightarrow} \quad D(Q^2) = Q^2 \int_{m^2}^{\infty} \frac{R(s)}{(s+Q^2)^2} ds. \]

Adler, PRD10(1974); Radyushkin (1982); Krasnikov, Pivovarov, PLB116(1982).
The masses of all final state particles are neglected \((m = 0)\).

In this analysis, it is convenient to handle the leading–order terms separately from the strong corrections:

\[
R(s) = r^{(0)}(s) + r^{(\ell)}(s), \quad D(Q^2) = d^{(0)}(Q^2) + d^{(\ell)}(Q^2).
\]

The one–loop level \((\ell = 1)\) with three active flavors \((n_f = 3)\) is assumed in what follows.

There are two equally justified methods of description \(\Delta_{QCD}\).
Method I: Use of definitions of $R(s)$ and $D(Q^2)$ only

The QCD contribution to $R_{J=1}^{V/A}$

$$\Delta_{QCD} = 2 \int_0^{M^2_\tau} f\left(\frac{s}{M^2_\tau}\right) R(s) \frac{ds}{M^2_\tau}$$

can be represented as

$$\Delta_{QCD} = \frac{1}{\pi i} \int_{0+i\varepsilon}^{M^2_\tau+i\varepsilon} f\left(\frac{s}{M^2_\tau}\right) \Pi(s) \frac{ds}{M^2_\tau}$$

$$+ \frac{1}{\pi i} \int_{M^2_\tau-i\varepsilon}^{0-i\varepsilon} f\left(\frac{s}{M^2_\tau}\right) \Pi(s) \frac{ds}{M^2_\tau}.$$

Integration by parts eventually results in

$$\Delta_{QCD} = g(1) R(M^2_\tau) + \int_0^{M^2_\tau} g\left(\frac{\sigma}{M^2_\tau}\right) \varrho(\sigma) \frac{d\sigma}{\sigma}, \quad g(x) = x(2 - 2x^2 + x^3),$$

$$\varrho(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \left[ d_s(-\sigma - i\varepsilon) - d_s(-\sigma + i\varepsilon) \right].$$
Method II: Method I + deformation of integration contour

Identically to “Method I”, $\Delta_{QCD}$ is rewritten here as the sum of two integrals along the edges of physical cut of $\Pi(q^2)$:

$$\Delta_{QCD} = \frac{1}{\pi i} \int_{M_T^2+i\varepsilon}^{M_T^2-i\varepsilon} f\left(\frac{s}{M_T^2}\right) \Pi(s) \frac{ds}{M_T^2}$$

$$+ \frac{1}{\pi i} \int_{M_T^2-i\varepsilon}^{0-i\varepsilon} f\left(\frac{s}{M_T^2}\right) \Pi(s) \frac{ds}{M_T^2}.$$

Additional deformation of contour: $C_1 + C_2 \rightarrow - (C_0 + C_M)$

$$\Delta_{QCD} = \frac{i}{\pi} \left[ \int_{C_0} f\left(\frac{s}{M_T^2}\right) \Pi(s) \frac{ds}{M_T^2} + \int_{C_M} f\left(\frac{s}{M_T^2}\right) \Pi(s) \frac{ds}{M_T^2} \right]$$

$$= \frac{1}{2\pi^2} \lim_{\varepsilon \to 0} \left[ \int_{\pi - \varepsilon}^{\pi + \varepsilon} D\left(M_T^2 e^{i\theta}\right) \left(1 + 2e^{i\theta} - 2e^{3i\theta} - e^{4i\theta}\right) d\theta \right].$$
All the presented above is valid only for “genuine physical” functions $\Pi_{\text{phys}}(q^2)$ and $D_{\text{phys}}(Q^2)$ in the massless limit.

However, one has to deal with their perturbative approximations, which are inconsistent with dispersion relations for these functions.

Hadronic decay of $\tau$ lepton within perturbative approach: the direct use of perturbative approximations $\Pi_{\text{pert}}(q^2)$ and $D_{\text{pert}}(Q^2)$ in the aforementioned expressions for $\Delta_{\text{QCD}}$. 
Method I + one–loop pQCD:

\[ \Delta_{\text{QCD}} = 1 + \int_0^{\infty} h \left( \frac{\sigma}{M_T^2} \right) \rho_{\text{pert}}^{(1)}(\sigma) \frac{d\sigma}{\sigma}, \]

\[ h(x) = g(x) \theta(1 - x) + g(1) \theta(x - 1), \]

\[ \rho_{\text{pert}}^{(1)}(\sigma) = \frac{4}{\beta_0} \frac{1}{\ln^2(\sigma/\Lambda^2) + \pi^2}. \]

Method II + one–loop pQCD:

\[ \Delta_{\text{QCD}} = 1 + \frac{4}{\beta_0} \int_0^{\pi} \frac{\lambda A_1(\theta) + \theta A_2(\theta)}{\pi (\lambda^2 + \theta^2)} d\theta, \]

\[ A_1(\theta) = 1 + 2 \cos(\theta) - 2 \cos(3\theta) - \cos(4\theta), \]

\[ A_2(\theta) = 2 \sin(\theta) - 2 \sin(3\theta) - \sin(4\theta), \]

\[ \lambda = \ln \left( \frac{M_T^2}{\Lambda^2} \right). \]
Unknown “genuine physical” Adler function $D_{phys}(Q^2)$:

$$C_1 + C_2 = -(C_0 + C_M)$$

The use of either of two integration contours would have led to the same result.

One–loop perturbative Adler function $D^{(1)}_{pert}(Q^2)$:

$$C_1 + C_2 \neq -(C_0 + C_M)$$

The integration contours used within methods I and II are not equivalent.
The leading–order perturbative term:

The massless parton model prediction

\[ \Pi^{(0)}_{\text{pert}}(q^2) = -\ln\left(\frac{-q^2}{\mu^2}\right) \]

\[ \rightarrow \left\{ \begin{array}{l}
   d^{(0)}_{\text{pert}}(Q^2) = 1, \\
   r^{(0)}_{\text{pert}}(s) = 1
\end{array} \right\}, \quad |q^2| \to \infty \]

gives the same result for \( \Delta_{\text{QCD}} \) within either of two methods:

\[ \Delta^{(0)}_{\text{QCD}} = 1. \]

The one–loop perturbative correction:

In Method II the residue term

\[ \Delta^{(1)}_{\text{res}} = \frac{4}{\beta_0} h_1\left(\frac{\Lambda^2}{M_T^2}\right), \quad \text{where} \]

\[ h_1(x) = h_2(x) \theta(1-x) + h_2(1) \theta(x-1), \]

\[ h_2(x) = x(2 - 2x^2 - x^3), \]

is additionally accounted for.
**COMPARISON WITH EXPERIMENTAL DATA**

**ALEPH–2008 data:** $\Delta^V_{\text{exp}} = 1.224 \pm 0.050$, $\Delta^A_{\text{exp}} = 0.748 \pm 0.034$.

However, within perturbative approach $\Delta^V_{\text{pert}} \equiv \Delta^A_{\text{pert}}$.

**Method I: One solution for V-channel, none for A-channel**

\[
\Lambda = \left( 844^{+726}_{-393} \right) \text{ MeV}
\]

**no solution**
Method II: Two solutions for V-channel, none for A-channel

\[ \Lambda = (434^{+117}_{-127}) \text{ MeV} \]

\[ \Lambda = (1652^{+21}_{-23}) \text{ MeV} \]

no solution
Perturbative approach, one-loop level, $n_f = 3$ active flavors:

<table>
<thead>
<tr>
<th>Method</th>
<th>V-channel</th>
<th>A-channel</th>
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</thead>
<tbody>
<tr>
<td>Method I</td>
<td>$\Lambda = (844^{+726}_{-393})$ MeV</td>
<td>no solution</td>
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V-channel: perturbative approach gives three equally justified solutions, but only highlighted one is usually retained.

A-channel: perturbative approach fails to describe experimental data on inclusive $\tau$ lepton hadronic decay.
Dispersion relations impose stringent physical nonperturbative constraints on the quantities on hand:

\[
D(Q^2) = Q^2 \int_{m^2}^{\infty} \frac{R(s)}{(s + Q^2)^2} ds
\]

\[
\begin{cases}
D(Q^2) = 0 \text{ at } Q^2 = 0; \\
\text{the only cut } Q^2 \leq -m^2
\end{cases}
\]

**BASIC IDEA:** merge perturbative approximation for Adler function with such nonperturbative constraints

\[
R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{s + i\varepsilon}^{s - i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta}
\]

\[
D(Q^2) = Q^2 \int_{m^2}^{\infty} \frac{R(s)}{(s + Q^2)^2} ds
\]
\[ D(Q^2) = d^{(0)}(Q^2) + d_s(Q^2) \]

\[ R(s) = r^{(0)}(s) + \theta \left(1 - \frac{m^2}{s}\right) \int_{s}^{\infty} \varrho(\sigma) \frac{d\sigma}{\sigma}, \]

\[ \varrho(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0^+} \left[ d_s(-\sigma - i\varepsilon) - d_s(-\sigma + i\varepsilon) \right] \]

\[ D(Q^2) = d^{(0)}(Q^2) + \frac{Q^2}{Q^2 + m^2} \int_{m^2}^{\infty} \varrho(\sigma) \frac{\sigma - m^2}{\sigma + Q^2} \frac{d\sigma}{\sigma} \]


This derivation requires only dispersion relations for \( D(Q^2) \) and \( R(s) \) and the fact that \( d_s(Q^2) \to 0 \) for \( Q^2 \to \infty \).

Neither additional approximations nor model–dependent assumptions were involved.
The derived expression for $D(Q^2)$:

- no unphysical singularities
- correct analytic properties in $Q^2$
- applicable in entire $0 \leq Q^2 < \infty$


Obtained $D(Q^2)$ leads to the same result for $\Delta_{QCD}$ for either choice of the integration contour.

In the massless limit ($m = 0$) derived representations become identical to those of the “Analytic Perturbation Theory” (APT).


But it is essential to keep $m \neq 0$ within approach on hand.
Some attempts to improve massless APT behavior of $D(Q^2)$:

**APT + relativistic quark mass threshold resummation:**

- Analytic QCD
- Experimental

**APT + vector meson dominance assumption:**

- Cvetic, Valenzuela, Schmidt (2005)–(2007)

[ plots taken from MPLA21(2006) and NPBPS164(2007) ]
TAU DECAY WITHIN DISPERSIVE APPROACH

• The effects due to hadronization are retained \((m \neq 0)\)

• Smooth kinematic threshold for the leading term of \(R(s)\):

\[
r_{V/A}^{(0)}(s) = \left(1 - \frac{m_{V/A}^2}{s}\right)^{3/2} \quad d_{V/A}^{(0)}(Q^2) = 1 + \frac{3}{\xi} \left\{1 + \frac{u(\xi)}{2} \ln \left[1 + 2\xi (1 - u(\xi))\right]\right\}
\]

where \(u(\xi) = \sqrt{1 + \xi^{-1}}, \ \xi = Q^2/m_{V/A}^2\)

• Nonperturbative model for one–loop spectral density:

\[
\rho(\sigma) = \frac{4}{\beta_0} \frac{1}{\ln^2(\sigma/\Lambda^2) + \pi^2} + \frac{\Lambda^2}{\sigma}
\]

In turn, expression for $\Delta_{QCD}^{V/A}$ has taken the following form:

$$
\Delta_{QCD}^{V/A} = \sqrt{1 - \zeta_{V/A}} \left(1 + 6\zeta_{V/A} - \frac{5}{8}\zeta_{V/A}^2 + \frac{3}{16}\zeta_{V/A}^3\right) \\
- 3\zeta_{V/A} \left(1 + \frac{1}{8}\zeta_{V/A}^2 - \frac{1}{32}\zeta_{V/A}^3\right) \ln \left[\frac{2}{\zeta_{V/A}} \left(1 + \sqrt{1 - \zeta_{V/A}}\right) - 1\right] \\
+ \int_{m_{V/A}^2}^{\infty} H\left(\frac{\sigma}{M_T^2}\right) \rho(\sigma) \frac{d\sigma}{\sigma},
$$

where $\zeta_{V/A} = \frac{m_{V/A}^2}{M_T^2}$,

$$
H(x) = g(x) \theta(1 - x) + g(1) \theta(x - 1) - g(\zeta_{V/A}),
$$

$$
g(x) = x(2 - 2x^2 + x^3)
$$

This results in nearly identical solutions for QCD scale parameter $\Lambda$ in both vector and axial–vector channels:

\[ \Lambda = (408 \pm 30) \text{ MeV} \quad \text{and} \quad \Lambda = (437 \pm 34) \text{ MeV} \]
The obtained solutions also agree with perturbative (V) one:

<table>
<thead>
<tr>
<th>V-channel</th>
<th>Perturbative approach</th>
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</tr>
</thead>
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<td>A-channel</td>
<td>( \Lambda = (434^{+117}_{-127}) \text{ MeV} )</td>
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\[ \text{no solution} \]

\[ \Lambda, \text{GeV} \]

\[ \Delta_{\text{QCD}} \]
Theoretical description of $\tau$ lepton hadronic decay is performed in the framework of Dispersive approach to QCD.

The significance of effects due to hadronization is argued.

The approach on hand is capable of describing experimental data on inclusive $\tau$ lepton hadronic decay in both vector and axial–vector channels.

The vicinity of obtained values of QCD scale parameter $\Lambda$ testifies to the self–consistency of developed approach.