Dimensional recurrence relations and Laplace's method

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Contents

Introduction

Feynman integrals are needed for physicists but also they are very interesting objects from a mathematical point of view.

They are:

- functions of continuous variables - scalar products of momenta and masses
- functions of discrete parameters- powers of propagators
- function of the space - time dimension parameter d

All these parameters can be used to set up an equation (differential or difference) and to obtain the result for integral as a solution of these equations.

Papers about dimensional recurrence relations:

O. V. T.,
Connection between Feynman integrals having different values of the space-time dimension, Phys. Rev. D 54 (1996) 6479.
O. V. T.
Application and explicit solution of recurrence relations with respect to space-time dimension, Null. Phys. Prof. Suppl. 89 (2000) 237
J. Fleischer, F. Jegerlehner and O. V. T., *A New Hypergeometric Representation of One-Loop Scalar Integrals in d Dimensions* , Nucl. Phys. B 672 (2003) 303;
O.V. T., *Hypergeometric representation of the two-loop equal mass sunrise diagram* , Phys. Lett. B638 (2006) 195;
B. Kniehl, O.V.T.,
Analytic result for the one-loop scalar pentagon integral with massless propagators , DESY-09-218(2010),

Results obtained by dimensional recurrences

With the help of these algorithms new analytic results were obtained:

- hypergeometric representation for the one - loop integrals corresponding to diagrams with three- and four external legs with arbitrary kinematics
- analytic formula for the on-shell one-loop massless pentagon type integral
- hypergeometric representation for the two-loop 'sunrise' propagator type integral

Method of dimensional recurrences is well suited for integrals with many kinematic variables!

The dimensional recurrence relation is not an IBP relation. It comes from observation of the particular structure of the integrand for Feynman integral $G^{(d)}$ written in α parametric representation:

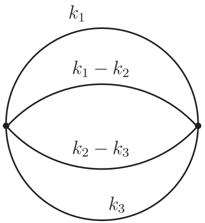
$$G^{(d)} = N \int_0^\infty d\alpha_1 ... \int_0^\infty d\alpha_L \prod_k \left(\alpha_k^{\nu_k - 1} \right) \frac{\exp[i \frac{Q}{D(\alpha)} - i \sum m_j^2 \alpha_j]}{D(\alpha)^{d/2}}.$$

From this representation one can obtain "master equation":

$$G^{(d)} = (-1)^L \, D(\partial) \, G^{(d+2)} \Big|_{m=m_0}$$

where $D(\alpha)$ is Symanzik polynomial and

$$\partial_k = \frac{\partial}{\partial m_k^2}$$



Example: 3-loop vacuum type integral

$$D(\partial) = \partial_2\partial_3\partial_4 + \partial_1\partial_3\partial_4 + \partial_1\partial_2\partial_4 + \partial_1\partial_2\partial_3,$$

The term $D(\partial)\ G^{(d+2)}$ is a sum of integrals with increased powers of propagators. They can be reduced to basis integrals by using generalized recurrence relations or IBP relations.

Dimensional recurrence relations include integral with fixed powers of propagators but with different shifts of the space - time dimension d and simpler integrals considered as inhomogeneous part of the equation:

$$P_r(d)\ I_k^{(d+r)} + P_{r-2}(d)\ I_k^{(d+r-2)} + ... + P_0(d)\ I_k^{(d)} = R(d)$$

where $P_r(d)$ are polynomials in d, masses and scalar products of external momenta.

General solution of dimensional recurrences can be written in the form:

$$I_k^{(d)}(\{s_j\}) = \sum_r \Phi_r(d, \{s_j\})\, w_s(d, \{s_j\}) + \Phi_{pt.sol}(d, \{s_j\})$$

where Φ_r are functions from the fundamental set of solutions for homogeneous part of the equation, $\Phi_{pt.sol}$ is a particular solution of the equation and $w_s(d, \{s_j\})$ are the so-called 'periodics' satisfying the following condition:

$$w_s(d+2, \{s_j\}) = w_s(d, \{s_j\})$$

Evaluation of the periodic functions $w_s(d, \{s_j\})$ turns out to be very difficult problem.

In my papers it was proposed to find these periodic functions from

- the comparison of the obtained solution at $|d| \to \infty$ with the asymptotic expansion of this integral derived by Laplace method or by method of steepest descent directly from the integral written in a parametric form
- by setting up differential equation for $w_s(d, \{s_j\})$ w.r.t. masses or scalar products of external momenta if $w_s(d, \{s_j\})$ has nontrivial dependence on these parameters.

R. Lee presentations on "Loops and Legs 2010" (Wörlitz, Germany), "Acat - 2011" (London, England):

Why not use?

The homogeneous part of the solution depends on several (or one) periodic functions. Their determination appears extremely difficult!
Initial Tarasov's idea to fix them from the large-$\mathscr{D}$ does not work for multiloop integrals.

Today I will talk about "extremely difficult" problem, and about idea which "does not work for multiloop integrals".

Because of these extreme difficulties today I will restrict myself to the Laplace's method.

Laplace's method is:

- General
- Accurate
- Intuitive
- Extended to the complex plane
- Extended to multifold integrals

Pierre-Simon,
marquis de Laplace
(1749-1827)

Laplace method:

Laplace, P.S. (1774)

Memoir on the probability of causes of events Mèmories de Mathématique et de Physique, Tome Sixiéme

Integrals of Laplace's type are integrals of the form:

$$F(\lambda) = \int_a^b f(\alpha) \exp[\lambda S(\alpha)] \, d\alpha$$

where $f(\alpha)$, $S(\alpha)$ are real infinitely differentable functions, and λ assumed to be large.
If $S(\alpha)$ has nondegenerated maximum inside the integration region located at $\alpha = \alpha^0$ then at $\lambda \to \infty$ the integral $F(\lambda)$ has the following asymptotic expansion

$$F(\lambda) \sim \exp[\lambda S(\alpha^0)] \sum_{k=0}^{\infty} c_k \lambda^{-k-1/2},$$

where

$$c_k = \frac{\Gamma\left(k+\frac{1}{2}\right)}{(2k)!} \left(\frac{d}{d\alpha}\right)^k \left[f(\alpha) \left(\frac{2(S(\alpha^0) - S(\alpha))}{(\alpha-\alpha^0)^2} \right)^{-k-\frac{1}{2}} \right] \Bigg|_{\alpha=\alpha^0}$$

The leading term in the expansion reads:

$$F(\lambda) \approx \sqrt{-\frac{2\pi}{\lambda S''(\alpha^0)}}\, f(\alpha^0)\, e^{\lambda S(\alpha^0)}, \qquad (\lambda \to +\infty).$$

Famous example: Stirling's approximation

Laplace's method can be used to derive Stirling's approximation

$$N! \approx \sqrt{2\pi N}\, N^N e^{-N}, \qquad N >> 1.$$

From the definition of the Gamma function

$$N! = \Gamma(N+1) = \int_0^\infty e^{-x} x^N dx.$$

Now we change variables, letting

$$x = Nz, \qquad \text{so that} \;\; dx = Ndz$$

Plug these values back in to obtain

Example: Stirling's approximation

$$\begin{aligned} N! &= \int_0^\infty e^{-Nz}(Nz)^N N dz \\ &= N^{N+1}\int_0^\infty e^{-Nz} z^N dz \\ &= N^{N+1}\int_0^\infty e^{-Nz} e^{N\ln z} dz \\ &= N^{N+1}\int_0^\infty e^{N(\ln z - z)} dz \end{aligned}$$

This integral has the form necessary for Laplace's method with

$$S(z) = \ln z - z,$$

which is twice-differentiable:

$$S'(z) = \frac{1}{z} - 1, \qquad S''(z) = -\frac{1}{z^2}.$$

Example: Stirling's approximation

The find maximum of $S(z)$ we must solve equation

$$S'(z) = 0,$$

and to determine sign of the second derivative $S''(z)$ at the point $z = z_0$ where $S'(z) = 0$. The solution of the equation $S'(z) = 0$ is

$$z_0 = 1,$$

and the second derivative at this point is

$$S''(z)\Big|_{z=z_0} = -1.$$

The negative value of $S''(z_0)$ means that at this point function $S(z)$ has maximum. See, for example,
D.V. Widder, *Advanced Calculus*

Laplace's method was extended to n -fold integrals

$$F(\lambda) = \int_\Omega d\alpha\ f(\alpha)\ \exp[\lambda S(\alpha)], \qquad \alpha = (\alpha_1, ..., \alpha_n),$$

if $S(\alpha)$ has maximum inside integration region at $\alpha = \alpha^{(0)}$ then at $\lambda \to \infty$,

$$F(\lambda) \sim \exp[\lambda S(\alpha^{(0)})]\lambda^{-n/2} \sum_{k=0}^{\infty} a_k \lambda^{-k},$$

The main term of the asymptotic expansion reads:

$$F(\lambda) \sim \exp[\lambda S(\alpha^0)] \left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}} \frac{f(\alpha^0) + O(\lambda^{-1})}{\sqrt{|\det S''_{\alpha\alpha}(\alpha^0)|}},$$

Coefficients a_k can be found by expanding integrand at $\alpha = \alpha^0$.

Let's consider now three-loop vacuum type Feynman integral

$$J_4^{(d)} = \frac{1}{(i\pi^{d/2})^3} \int \int \int d^dk_1 \; d^dk_2 \; d^dk_3$$
$$\times \frac{1}{(k_1^2 - m^2)((k_1-k_2)^2 - m^2)((k_2-k_3)^2 - m^2)(k_3^2 - m^2)}$$

Dimensional recurrence relation for this integral reads:

$$J_4^{(d+2)} = -\frac{128m^6(d-2)}{3d(d-1)(3d-4)(3d-2)} J_4^{(d)}$$
$$-\frac{16m^4(11d-16)}{3d(d-1)(3d-4)(3d-2)} \left(T_1^d(m^2)\right)^3 .$$

We introduce another function $\overline{J}_4^{(d)}$, defined by equation :

$$J_4^{(d)} = \frac{(64m^6)^{\frac{d}{2}}\,\Gamma(2-d)\Gamma\left(4-\frac{3d}{2}\right)}{m^8\Gamma^2\left(2-\frac{d}{2}\right)}\,\overline{J}_4^{d}$$

satisfying simpler recurrence relation

$$\overline{J}_4^{(-2\lambda+2)} = \overline{J}_4^{(-2\lambda)} - \frac{\pi^{3/2}}{216\sqrt{3}}\,\frac{(11\lambda+8)\,\Gamma^3(\lambda+1)}{\Gamma\left(\lambda+\frac{3}{2}\right)\Gamma\left(\lambda+\frac{4}{3}\right)\Gamma\left(\lambda+\frac{5}{3}\right)}\left(\frac{16}{27}\right)^{\lambda}$$

Setting $\lambda = k + \varepsilon,$ and denoting $\overline{J}_4^{(-2\lambda)} = \widetilde{J}_4^{(k)}$ gives equation:

$$\widetilde{J}_4^{(k-1)} = \widetilde{J}_4^{(k)} + \overline{R}_4^{(k+\varepsilon)},$$

where

$$\overline{R}_4^{(\lambda)} = -\frac{\pi^{3/2}}{216\sqrt{3}} \frac{(11\lambda+8)\,\Gamma^3(\lambda+1)}{\Gamma\left(\lambda+\frac{3}{2}\right)\Gamma\left(\lambda+\frac{4}{3}\right)\Gamma\left(\lambda+\frac{5}{3}\right)} \left(\frac{16}{27}\right)^{\lambda}.$$

Solution of this equation reads:

$$\tilde{J}_4^{(k)} = -\sum_{r=k}^{\infty} \overline{R}_4^{(r+\varepsilon)} + C(k+\varepsilon) = -\sum_{r=0}^{\infty} \overline{R}_4^{(r+k+\varepsilon)} + \overline{C}(k+\varepsilon),$$

where $\overline{C}(k+\varepsilon)$ is a periodic function

$$\overline{C}(k+\varepsilon) = \overline{C}(\varepsilon).$$

Substituting explicit expression for $\overline{R}_4^{(\lambda)}$, after some algebra leads to

$$\overline{J}_4^{(d)} = \frac{\pi^{3/2}}{216\sqrt{3}} \left(\frac{16}{27}\right)^{\lambda+1} \frac{3\,\Gamma^3(\lambda+2)}{\Gamma\left(\lambda+\frac{5}{2}\right)\Gamma\left(\lambda+\frac{7}{3}\right)\Gamma\left(\lambda+\frac{8}{3}\right)}$$

$$\times \left\{ {}_4F_3\left[\begin{array}{c} 1, 2+\lambda, 2+\lambda, 2+\lambda;\ \\ \lambda+\frac{5}{2}, \lambda+\frac{7}{3}, \lambda+\frac{8}{3};\ \end{array} \frac{16}{27}\right] \right.$$

$$\left. -\frac{11}{3}(\lambda+2)\,{}_4F_3\left[\begin{array}{c} 1, 2+\lambda, 2+\lambda, 3+\lambda;\ \\ \lambda+\frac{5}{2}, \lambda+\frac{7}{3}, \lambda+\frac{8}{3};\ \end{array} \frac{16}{27}\right] \right\} + \overline{C}(\lambda).$$

At $\lambda \to \infty$ the terms with hypergeometric functions are exponentially decreasing. The functions ${}_4F_3$ are not increasing. For example,

$$
{}_4F_3\left[\begin{matrix} 1, 2+\lambda, 2+\lambda, 2+\lambda\,; \\ \lambda+\frac{5}{2}, \lambda+\frac{7}{3}, \lambda+\frac{8}{3}\,; \end{matrix}\ \frac{16}{27}\right]
= \sum_{r=0}^{\infty} \frac{(\lambda+2)_r}{(\lambda+\frac{5}{2})_r} \frac{(\lambda+2)_r}{(\lambda+\frac{7}{3})_r} \frac{(\lambda+2)_r}{(\lambda+\frac{8}{3})_r} \left(\frac{16}{27}\right)^r < \frac{27}{11},
$$

because

$$
\frac{(\lambda+2)_r}{(\lambda+\frac{5}{2})_r} \le \frac{(\lambda+\frac{5}{2})_r}{(\lambda+\frac{5}{2})_r} = 1, \quad \frac{(\lambda+2)_r}{(\lambda+\frac{7}{3})_r} \le \frac{(\lambda+\frac{7}{3})_r}{(\lambda+\frac{7}{3})_r} = 1,
$$

$$
\frac{(\lambda+2)_r}{(\lambda+\frac{8}{3})_r} \le \frac{(\lambda+2)_r}{(\lambda+2)_r} = 1, \quad \sum_{r=0}^{\infty} \left(\frac{16}{27}\right)^r = \frac{27}{11}
$$

Almost the same way one can show that the limit of the second ${}_4F_3$ function at $\lambda \to \infty$ is finite.

Applying Stirling's formula one can show that

$$\frac{\Gamma^3(\lambda+2)}{\Gamma\left(\lambda+\frac{5}{2}\right)\Gamma\left(\lambda+\frac{7}{3}\right)\Gamma\left(\lambda+\frac{8}{3}\right)} \to \frac{1}{\lambda^{3/2}},$$

and therefore at $\lambda \to \infty$ due to the factor $(16/27)^{\lambda+1}$ the terms with ${}_4F_3$ functions drops out and we get:

$$\overline{J}_4^{(d)} \sim \overline{C}(\lambda)$$

At $\lambda \to \infty$ the leading term for our initial integral is:

$$J_4^{(d)} \sim \frac{\Gamma(2+2\lambda)\Gamma\left(4+3\lambda\right)}{m^8(64m^6)^{\lambda}\Gamma^2\left(2+\lambda\right)}\,\overline{C}(\lambda) \sim \frac{54\sqrt{6}}{m^8}\left(\frac{27}{16m^6}\right)^{\lambda}\lambda^{3\lambda+2}e^{-3\lambda}\,\overline{C}(\lambda).$$

Let's turn now to the evaluation of the leading asymptotic term starting from the parametric representation for $J_4^{(d)}$. Transforming integral from Minkowski to Euclidean momenta leads to:

$$(i\pi^{d/2})^3 J_4^{(d)} = \int \frac{d^d k_1 \; d^d k_2 \; d^d k_3}{(k_1^2 - m^2)((k_1-k_2)^2 - m^2)((k_2-k_3)^2 - m^2)(k_3^2 - m^2)}$$

$$= \int \frac{(-i) d^d k_{1E} \; d^d k_{2E} \; d^d k_{3E}}{(k_{1E}^2 + m^2)((k_{1E}-k_{2E})^2 + m^2)((k_{2E}-k_{3E})^2 + m^2)(k_{3E}^2 + m^2)}.$$

By using Laplace transformation for the propagators

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha e^{-\alpha(k^2+m^2)},$$

and Gaussian integration formula for momenta

$$\int d^dk\ e^{[-Ak^2+2pk]} = \left(\frac{\pi}{A}\right)^{\frac{d}{2}} e^{\left[\frac{p^2}{A}\right]},$$

leads to the following α parametric representation of the integral:

$$J_4^{(d)} = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \frac{\exp[-m^2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)]}{D^{d/2}},$$

where

$$D = \alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_2\alpha_3.$$

Changing variables

$$\alpha_k \to \lambda \alpha_k, \qquad \lambda = -\frac{d}{2} > 0$$

and exponentiating D:

$$D^\lambda = e^{\lambda \ \ln D}$$

leads to the integral needed for application of the Laplace's method:

$$J_4^{(d)} = \lambda^{3\lambda+4} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \exp[\lambda S],$$

where

$$S = \ln D - m^2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4).$$

Extremum of $S(\alpha)$ will be defined as a solution of the system of equations:

$$\frac{\partial S}{\partial \alpha_1} = -m^2 + \frac{\alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4}{D} = 0,$$

$$\frac{\partial S}{\partial \alpha_2} = -m^2 + \frac{\alpha_3\alpha_4 + \alpha_1\alpha_4 + \alpha_1\alpha_3}{D} = 0,$$

$$\frac{\partial S}{\partial \alpha_3} = -m^2 + \frac{\alpha_2\alpha_4 + \alpha_1\alpha_4 + \alpha_1\alpha_2}{D} = 0.$$

$$\frac{\partial S}{\partial \alpha_4} = -m^2 + \frac{\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3}{D} = 0.$$

The solution of this system can be obtained with the help of Maple or Mathematica. Below is the code for Maple.

```
###########################################################################
#                                                                         #
#   Solving system of equation for the three - loop bubble integral      #
#                                                                         #
#                                                                         #
# `Copyright (C) 2010-2012 by Oleg V.  Tarasov. All rights reserved.`; #
#                                                                         #
###########################################################################
>
> kernelopts(gcfreq=100*10^6):
>
> D_form := a2*a3*a4 + a1*a3*a4 + a1*a2*a4 + a1*a2*a3:
>
> S_function :=  ln(D_form)-(a1+a2+a3+a4)*mm:
>
> A:= [a1,a2,a3,a4]:
>
# making system of equations
>
> for j in A do
>  equ||j := diff(S_function,j);
> end do:
>
> Equations := {equa1,equa2,equa3,equa4}:
>
```

```
> Solutions := [solve(Equations,{a1,a2,a3,a4})];

                                   3           3           3           3
        Solutions := [{a3 = ----, a4 = ----, a2 = ----, a1 = ----}]
                                  4  mm       4  mm       4  mm       4  mm

>
> S_function_0 := simplify(subs(Solutions[1],S_function));
                                                              1
              S_function_0 := 3 ln(3) - 4 ln(2) + ln(---) - 3
                                                               3
                                                             mm

>
> quit;
bytes used=8693280, alloc=5569540, time=0.56
```

The system has unique solution at $\alpha_k = \alpha_k^0$:

$$\alpha_1^0 = \alpha_2^0 = \alpha_3^0 = \alpha_4^0 = \frac{3}{4m^2}.$$

One can check that at this point for the second derivatives:

$$S_{ij} = \left.\frac{\partial^2 S}{\partial\alpha_i \partial\alpha_j}\right|_{\alpha_k=\alpha_k^0},$$

the following relations are true:

$$S_{11} = -m^4 < 0, \quad \begin{vmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{vmatrix} = \frac{80}{81} m^8 > 0,$$

$$\begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{vmatrix} = -\frac{704}{729} m^{12} < 0,$$

$$\begin{vmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{vmatrix} = \frac{2048}{2187} m^{16} > 0.$$

The fulfillment of these conditions means that at $\alpha = \alpha^0$ the function $S(\alpha)$ has absolute maximum and therefore Laplace's method for finding asymptotic expansion of the integral at $\lambda \to \infty$ can be used. Substituting solution into expression for $S(\alpha)$ gives:

$$S(\alpha^{(0)}) = -3 + \ln\left(\frac{27}{16m^6}\right).$$

The Hessian needed for the Laplace formula is:

$$|\det S''_{\alpha\alpha}| = \left\| \begin{array}{cccc} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{array} \right\| = \frac{2048}{2187} m^{16}.$$

Applying Laplace's formula

$$F(\lambda) \sim \exp[\lambda S(\alpha^0)] \left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}} \frac{f(\alpha^0) + O(\lambda^{-1})}{\sqrt{|\det S''_{\alpha\alpha}(\alpha^0)|}},$$

for $n = 4$ to our α parametric integral at $\lambda \to \infty$ we get

$$J_4^{(d)} \sim \frac{81\pi^2}{8\sqrt{6}m^8} \lambda^{3\lambda+2} \left(\frac{27}{16m^6}\right)^{\lambda} e^{-3\lambda}.$$

Comparing this formula with the leading term of the obtained solution of difference equation we come to the relation:

$$\frac{54\sqrt{6}}{m^8}\left(\frac{27}{16m^6}\right)^{\lambda}\lambda^{3\lambda+2}\,e^{-3\lambda}\,\overline{C}=\frac{81\pi^2}{8\sqrt{6}m^8}\lambda^{3\lambda+2}\left(\frac{27}{16m^6}\right)^{\lambda}\,e^{-3\lambda},$$

and therefore:

$$\overline{C}(\lambda)=\frac{\pi^2}{32}.$$

The final result for the integral reads:

$$\bar{J}_4^{(d)} = \frac{\pi^{3/2}}{216\sqrt{3}} \left(\frac{16}{27}\right)^{\lambda+1} \frac{3\,\Gamma^3(\lambda+2)}{\Gamma\left(\lambda+\frac{5}{2}\right)\Gamma\left(\lambda+\frac{7}{3}\right)\Gamma\left(\lambda+\frac{8}{3}\right)}$$

$$\times \left\{ {}_4F_3\left[\begin{matrix} 1, 2+\lambda, 2+\lambda, 2+\lambda; \\ \lambda+\frac{5}{2}, \lambda+\frac{7}{3}, \lambda+\frac{8}{3}; \end{matrix} \; \frac{16}{27}\right]\right.$$

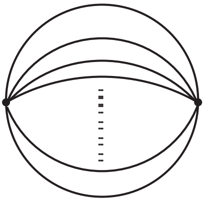
$$\left. -\frac{11}{3}(\lambda+2)\,{}_4F_3\left[\begin{matrix} 1, 2+\lambda, 2+\lambda, 3+\lambda; \\ \lambda+\frac{5}{2}, \lambda+\frac{7}{3}, \lambda+\frac{8}{3}; \end{matrix} \; \frac{16}{27}\right]\right\} + \frac{\pi^2}{32}.$$

$$J_4^{(d)} = \frac{(64m^6)^{\frac{d}{2}}\,\Gamma(2-d)\Gamma\left(4-\frac{3d}{2}\right)}{m^8\Gamma^2\left(2-\frac{d}{2}\right)}\,\bar{J}_4^d$$

Summarizing all the steps we can formulate an algorithm.

Algorithm 1:

- Read carefully books about Laplace's method and difference equations
- Solve difference equation, obtain result in terms of periodic functions
- Find the limit of the obtained solution at $|d| \to \infty$.
- Derive parametric representation for the integral and transform the integrand to the form appropriate to the Laplace's method
- Differentiate $S(\alpha)$ with respect to α parameters, make a system of equations out of the first derivatives
- Solve the system of equations. Do not overload yourself: use MAPLE or MATHEMATICA.
- The rule of thumb: USE BOTH !!! Sometime MAPLE and MATHEMATICA give wrong results!
- Find second derivatives for S, prove that the solution correspond to the absolute maximum. Evaluate $\det |S_{\alpha\alpha}''(\alpha^0)|$.
- Substitute the obtained solution into the function $S(\alpha)$
- Substitute $S(\alpha^0)$ and $\det |S_{\alpha\alpha}''(\alpha^0)|$ into Laplace's formula,
- By careful comparison of the result obtained from Laplace's formula with the asymptotic term of the obtained solution find periodic function.



Example: L-loop vacuum type integral

Asymptotic expansion at $|d| \to \infty$ for the mellon - type vacuum diagrams can be obtained for any number of loops. For the L-loop integral

$$S(\alpha) = \ln\left[\alpha_1 ... \alpha_{L+1}\left(\frac{1}{\alpha_1} + ... + \frac{1}{\alpha_{L+1}}\right)\right] + m^2(\alpha_1 + ... + \alpha_{L+1}),$$

is totally symmetric in all α variables.

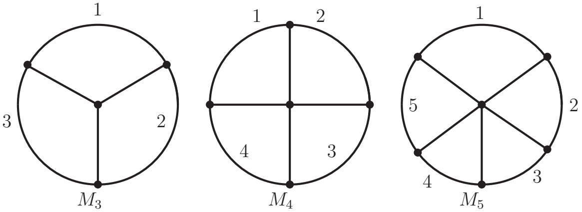
I checked explicitly up to 10 loops that the only maximum of this function is located at the symmetric point

$$\alpha_1^{(0)} = \ldots = \alpha_{L+1}^{(0)} = \frac{L}{2(L+1)m^2}.$$

At this point:

$$S(\alpha^{(0)}) = \ln\left[(L+1)\left(\frac{L}{m^2(L+1)}\right)^L\right] - L$$

My conjecture is that for any number of loops the only maximum of $S(\alpha)$ is located at the symmetric point and therefore we can always apply Laplace's formula and obtain expansion of these integrals in $1/d$.



Example: 3-, 4- and 5- loop vacuum type integrals

Data needed for Laplace's method for these integrals:

$$M_3: \qquad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \frac{1}{2},$$

$$S_3(\alpha^0) = -3 + \ln\left(\frac{2}{m^6}\right), \quad \det\left|S''_{\alpha\alpha}(\alpha^0)\right| = \frac{1}{2m^{12}},$$

$$M_4: \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1 - \frac{1}{\sqrt{3}}, \quad \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \frac{1}{\sqrt{3}},$$

$$S_4(\alpha^0) = -4 + \ln\left(\frac{4(2\sqrt{3}+3)}{9m^8}\right),$$

$$\det\left|S''_{\alpha\alpha}(\alpha^0)\right| = \frac{3^6}{m^{16}}\left(\frac{91}{8} - 7\sqrt{3}\right),$$

$$M_5: \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \frac{3-\sqrt{5}}{2},$$

$$\alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = \frac{\sqrt{5}-1}{2},$$

$$S_5(\alpha^0) = -5 + \ln\left(\frac{4}{m^{10}}\right).$$

$$\det\left|S''_{\alpha\alpha}(\alpha^0)\right| = \frac{25}{32m^{20}}\left(5 - 2\sqrt{5}\right)$$

Concluding remark

The Laplace's method can be used if the maximum of the $S(\alpha)$ function is located inside the integration region.

If the the stationary point is located outside the integration region then the method of steepest descend must be applied.

This is VERY COMPLICATED ISSUE !!! Not for this talk!
Algorithm 2 will be reported somewhere else.