Evaluating five-loop Konishi in $\mathcal{N} = 4$ SYM

V.A. Smirnov

Nuclear Physics Institute of Moscow State University

arXiv:1202.5733

B. Eden, P. Heslop, G.P. Korchemsky, V.A. Smirnov,
E. Sokatchev

arXiv:1202.5733
B. Eden, P. Heslop, G.P. Korchemsky, V.A. Smirnov,
E. Sokatchev
The Konishi operator

$$\mathcal{K} = \mathrm{tr}\left(\Phi^I \Phi^I\right)$$

with Φ^I (with $I = 1, \ldots, 6$) in the adjoint representation of $SU(N_c)$.

arXiv:1202.5733
B. Eden, P. Heslop, G.P. Korchemsky, V.A. Smirnov,
E. Sokatchev
The Konishi operator

$$\mathcal{K} = \mathrm{tr}\left(\Phi^I \Phi^I\right)$$

with Φ^I (with $I = 1, \ldots, 6$) in the adjoint representation of $SU(N_c)$.
Its anomalous dimension

$$\Delta_{\mathcal{K}} \;=\; 2 + \gamma_{\mathcal{K}}(a) = 2 + \sum_{\ell=1}^{\infty} a^\ell \gamma_{\mathcal{K}}^{(\ell)}$$

with $a = g^2 N_c/(4\pi^2)$

$$\gamma_{\mathcal{K}}(a) = 3\,a - 3\,a^2 + \frac{21}{4}\,a^3 - \left(\frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5\right)a^4$$

$$+\left(\frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}{\zeta_3}^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7\right)a^5 + O(a^6) + O(1/N_c^2)$$

$$\gamma_{\mathcal{K}}(a) = 3\,a - 3\,a^2 + \frac{21}{4}\,a^3 - \left(\frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5\right)a^4$$
$$+\left(\frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}{\zeta_3}^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7\right)a^5 + O(a^6) + O(1/N_c^2)$$

one loop

[L. Andrianopoli & S. Ferrara'96]

$$\gamma_{\mathcal{K}}(a) = 3\,a - 3\,a^2 + \frac{21}{4}\,a^3 - \left(\frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5\right)a^4$$
$$+\left(\frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}{\zeta_3}^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7\right)a^5 + O(a^6) + O(1/N_c^2)$$

one loop [L. Andrianopoli & S. Ferrara'96]

two loops

[B. Eden, C. Schubert & E. Sokatchev'00; M. Bianchi, S. Kovacs, G. Rossi & Y. S. Stanev]

$$\gamma_{\mathcal{K}}(a) = 3\,a - 3\,a^2 + \frac{21}{4}\,a^3 - \left(\frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5\right)a^4$$

$$+\left(\frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}{\zeta_3}^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7\right)a^5 + O(a^6) + O(1/N_c^2)$$

one loop [L. Andrianopoli & S. Ferrara'96]

two loops

[B. Eden, C. Schubert & E. Sokatchev'00; M. Bianchi, S. Kovacs, G. Rossi & Y. S. Stanev]

three loops [A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko & V. N. Velizhanin'04;

B. Eden, C. Jarczak and E. Sokatchev'04]

$$\gamma_K(a) = 3\,a - 3\,a^2 + \frac{21}{4}\,a^3 - \left(\frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5\right)a^4$$
$$+\left(\frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}{\zeta_3}^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7\right)a^5 + O(a^6) + O(1/N_c^2)$$

one loop [L. Andrianopoli & S. Ferrara'96]

two loops

[B. Eden, C. Schubert & E. Sokatchev'00; M. Bianchi, S. Kovacs, G. Rossi & Y. S. Stanev]

three loops [A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko & V. N. Velizhanin'04; B. Eden, C. Jarczak and E. Sokatchev'04]

four loops

[F. Fiamberti, A. Santambrogio, C. Sieg & D. Zanon'07,08; V. N. Velizhanin'08,09]

$$\gamma_{\mathcal{K}}(a) = 3\,a - 3\,a^2 + \frac{21}{4}\,a^3 - \left(\frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5\right)a^4$$

$$+\left(\frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}{\zeta_3}^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7\right)a^5 + O(a^6) + O(1/N_c^2)$$

one loop [L. Andrianopoli & S. Ferrara'96]

two loops

[B. Eden, C. Schubert & E. Sokatchev'00; M. Bianchi, S. Kovacs, G. Rossi & Y. S. Stanev]

three loops [A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko & V. N. Velizhanin'04; B. Eden, C. Jarczak and E. Sokatchev'04]

four loops

[F. Fiamberti, A. Santambrogio, C. Sieg & D. Zanon'07,08; V. N. Velizhanin'08,09]

five loops: a prediction based on integrability in AdS/CFT

[Z. Bajnok, A. Hegedus, R. A. Janik & T. Lukowski'09; G. Arutyunov, S. Frolov & R. Suzuki'10; J. Balog & A. Hegedus'10]

Evaluating $\gamma_{\mathcal{K}}^{(5)}$:

Evaluating $\gamma_{\mathcal{K}}^{(5)}$:

- $\gamma_{\mathcal{K}}^{(5)}$ in terms of five-loop integrals

Evaluating $\gamma_{\mathcal{K}}^{(5)}$:

- $\gamma_{\mathcal{K}}^{(5)}$ in terms of five-loop integrals
- $\gamma_{\mathcal{K}}^{(5)}$ in terms of four-loop integrals (using infrared rearrangement)

Evaluating $\gamma_{\mathcal{K}}^{(5)}$:

- $\gamma_{\mathcal{K}}^{(5)}$ in terms of five-loop integrals
- $\gamma_{\mathcal{K}}^{(5)}$ in terms of four-loop integrals (using infrared rearrangement)
- $\gamma_{\mathcal{K}}^{(5)}$ in terms of 24 master integrals (using IBP reduction)

Evaluating $\gamma_{\mathcal{K}}^{(5)}$:

- $\gamma_{\mathcal{K}}^{(5)}$ in terms of five-loop integrals
- $\gamma_{\mathcal{K}}^{(5)}$ in terms of four-loop integrals (using infrared rearrangement)
- $\gamma_{\mathcal{K}}^{(5)}$ in terms of 24 master integrals (using IBP reduction)
- evaluating unknown master integrals (using gluing)

Our tool to evaluate it is the OPE of the stress-tensor multiplet in $\mathcal{N} = 4$ SYM, with the superconformal primary state

$$\mathcal{O}^{IJ}_{\mathbf{20}'} = \mathrm{tr}\left(\Phi^I \Phi^J\right) - \frac{1}{6}\delta^{IJ}\,\mathrm{tr}\left(\Phi^K \Phi^K\right) \, .$$

Our tool to evaluate it is the OPE of the stress-tensor multiplet in $\mathcal{N}=4$ SYM, with the superconformal primary state

$$\mathcal{O}^{IJ}_{\mathbf{20'}} \;\; = \mathrm{tr}\left(\Phi^I\Phi^J\right) - \tfrac{1}{6}\delta^{IJ}\,\mathrm{tr}\left(\Phi^K\Phi^K\right) \;.$$

It proves convenient to introduce auxiliary $SO(6)$ harmonic variables Y_I, defined as a (complex) null vector, $Y^2 \equiv Y_I Y_I = 0$, and project the indices of $\mathcal{O}^{IJ}$ as

$$\mathcal{O}(x,y) \equiv Y_I\,Y_J\,\mathcal{O}^{IJ}_{\mathbf{20'}}(x) = Y_I\,Y_J\;\mathrm{tr}\left(\Phi^I(x)\Phi^J(x)\right)$$

Our tool to evaluate it is the OPE of the stress-tensor multiplet in $\mathcal{N} = 4$ SYM, with the superconformal primary state

$$\mathcal{O}^{IJ}_{\mathbf{20'}} = \mathrm{tr}\left(\Phi^I \Phi^J\right) - \frac{1}{6}\delta^{IJ}\,\mathrm{tr}\left(\Phi^K \Phi^K\right) .$$

It proves convenient to introduce auxiliary $SO(6)$ harmonic variables Y_I, defined as a (complex) null vector, $Y^2 \equiv Y_I Y_I = 0$, and project the indices of $\mathcal{O}^{IJ}$ as

$$\mathcal{O}(x,y) \equiv Y_I\, Y_J\, \mathcal{O}^{IJ}_{\mathbf{20'}}(x) = Y_I\, Y_J\; \mathrm{tr}\left(\Phi^I(x)\Phi^J(x)\right)$$

Its scaling dimension is protected.

The four-point correlation function is the first one to receive perturbative corrections:

$$G_4 = \langle \mathcal{O}(x_1, y_1)\mathcal{O}(x_2, y_2)\mathcal{O}(x_3, y_3)\mathcal{O}(x_4, y_4)\rangle = \sum_{\ell=0}^{\infty} a^\ell \, G_4^{(\ell)}(1, 2, 3, 4)$$

The four-point correlation function is the first one to receive perturbative corrections:

$$G_4 = \langle \mathcal{O}(x_1, y_1)\mathcal{O}(x_2, y_2)\mathcal{O}(x_3, y_3)\mathcal{O}(x_4, y_4)\rangle = \sum_{\ell=0}^{\infty} a^\ell \, G_4^{(\ell)}(1,2,3,4)$$

where (for $\ell \geq 1$)

$$G_4^{(\ell)}(1,2,3,4) = \frac{2\,(N_c^2-1)}{(4\pi^2)^4} R(1,2,3,4) F^{(\ell)}(x_i)$$

[B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev'11]

$$F^{(\ell)}(x_i) = \frac{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2}{\ell!\,(-4\pi^2)^\ell} \int d^4x_5 \ldots d^4x_{4+\ell}\, f^{(\ell)}(x_1, \ldots, x_{4+\ell})\,,$$

[B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev'11]

$$F^{(\ell)}(x_i) = \frac{x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2}{\ell!\,(-4\pi^2)^\ell} \int d^4x_5 \ldots d^4x_{4+\ell}\, f^{(\ell)}(x_1, \ldots, x_{4+\ell})\,,$$

where $x_{ij}^2 = (x_i - x_j)^2$,

$$f^{(\ell)}(x_1, \ldots, x_{4+\ell}) = \frac{P^{(\ell)}(x_1, \ldots, x_{4+\ell})}{\prod_{1 \le i < j \le 4+\ell} x_{ij}^2}$$

and $P^{(\ell)}$ is a homogeneous polynomial in x_{ij}^2 of degree $(\ell - 1)(\ell + 4)/2$.
It is symmetric under the exchange of any pair of points x_i and x_j (both external and internal).

$F^{(\ell)}(x_i)$ up to six loops in the planar sector

[B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev'12]

$F^{(\ell)}(x_i)$ up to six loops in the planar sector

[B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev'12]

For example,

$$P^{(1)} = 1\,, \qquad P^{(2)} = \tfrac{1}{48} \sum_{\sigma \in S_6} x^2_{\sigma_1\sigma_2} x^2_{\sigma_3\sigma_4} x^2_{\sigma_5\sigma_6} = x^2_{12} x^2_{34} x^2_{56} + \ldots$$

$F^{(\ell)}(x_i)$ up to six loops in the planar sector

[B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev'12]

For example,

$$P^{(1)} = 1\,, \qquad P^{(2)} = \tfrac{1}{48} \sum_{\sigma\in S_6} x^2_{\sigma_1\sigma_2} x^2_{\sigma_3\sigma_4} x^2_{\sigma_5\sigma_6} = x^2_{12} x^2_{34} x^2_{56} + \ldots$$

OPE ($x_2 \to x_1$)

$$\mathcal{O}(x_1, y_1)\mathcal{O}(x_2, y_2) = c_{\mathcal{I}} \frac{(Y_1 \cdot Y_2)^2}{x^4_{12}} \mathcal{I} + c_{\mathcal{K}}(a) \frac{(Y_1 \cdot Y_2)^2}{(x^2_{12})^{1-\gamma_{\mathcal{K}}/2}} \mathcal{K}(x_2) + c_{\mathcal{O}} \frac{(Y_1 \cdot Y_2)}{x^2_{12}} \mathcal{O}^{IJ}_{\mathbf{20'}}(x_2) + \ldots$$

The operators $\mathcal{I}$ and $\mathcal{O}_{\mathbf{20'}}$ are protected:

$$c_{\mathcal{I}} = (N_c^2 - 1)/(32\pi^4)\,, \qquad c_{\mathcal{O}} = 1/(2\pi^2)$$

The operators $\mathcal{I}$ and $\mathcal{O}_{\mathbf{20'}}$ are protected:

$$c_{\mathcal{I}} = (N_c^2 - 1)/(32\pi^4)\,, \qquad c_{\mathcal{O}} = 1/(2\pi^2)$$

Apply OPE to the first and the second pairs of the operators (in the limit $x_1 \to x_2$, $x_3 \to x_4$) using

$$\langle \mathcal{K}(x_2)\,\mathcal{K}(x_4)\rangle = \frac{d_{\mathcal{K}}}{(x_{24}^2)^{2+\gamma_{\mathcal{K}}}}\,,$$

$$\langle \mathcal{O}_{\mathbf{20'}}^{IJ}(x_2)\,\mathcal{O}_{\mathbf{20'}}^{KL}(x_4)\rangle = \frac{c_{\mathcal{I}}}{2x_{24}^4}\left(\delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK} - \frac{1}{3}\delta^{IJ}\delta^{KL}\right)$$

The operators $\mathcal{I}$ and $\mathcal{O}_{\mathbf{20'}}$ are protected:

$$c_{\mathcal{I}} = (N_c^2 - 1)/(32\pi^4)\,, \qquad c_{\mathcal{O}} = 1/(2\pi^2)$$

Apply OPE to the first and the second pairs of the operators (in the limit $x_1 \to x_2$, $x_3 \to x_4$) using

$$\begin{aligned}
\langle \mathcal{K}(x_2)\,\mathcal{K}(x_4)\rangle &= \frac{d_{\mathcal{K}}}{(x_{24}^2)^{2+\gamma_{\mathcal{K}}}}\,,\\
\langle \mathcal{O}_{\mathbf{20'}}^{IJ}(x_2)\,\mathcal{O}_{\mathbf{20'}}^{KL}(x_4)\rangle &= \frac{c_{\mathcal{I}}}{2x_{24}^4}\left(\delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK} - \frac{1}{3}\delta^{IJ}\delta^{KL}\right)
\end{aligned}$$

with the normalization choice

$$d_{\mathcal{K}} = 3\,\frac{N_c^2 - 1}{(4\pi^2)^2}\,, \qquad c_{\mathcal{K}}(a) = \frac{1}{12\pi^2} + O(a)$$

to obtain

$$G_4 \xrightarrow[x_4\to x_3]{x_2\to x_1} \frac{(N_c^2-1)^2}{4(4\pi^2)^4}\frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4} + \frac{N_c^2-1}{(4\pi^2)^4}\left[\frac{y_{12}^2 y_{34}^2 (y_{13}^2 y_{24}^2 + y_{14}^2 y_{23}^2)}{x_{12}^2 x_{34}^2 x_{13}^4}\right.$$

$$\left. + \frac{1}{3}\frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2 x_{13}^4}\left(c_{\mathcal{K}}^2(a) u^{\gamma_{\mathcal{K}}(a)/2} - 1\right)\right] + \ldots \, ,$$

to obtain

$$G_4 \quad \overset{x_2\to x_1}{\underset{x_4\to x_3}{\longrightarrow}} \quad \frac{(N_c^2-1)^2}{4(4\pi^2)^4}\frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4} + \frac{N_c^2-1}{(4\pi^2)^4}\left[\frac{y_{12}^2 y_{34}^2 (y_{13}^2 y_{24}^2 + y_{14}^2 y_{23}^2)}{x_{12}^2 x_{34}^2 x_{13}^4}\right.$$
$$\left. + \frac{1}{3}\frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2 x_{13}^4}\left(c_{\mathcal{K}}^2(a) u^{\gamma_{\mathcal{K}}(a)/2} - 1\right)\right] + \dots ,$$

where $y_{ij}^2 = (Y_i \cdot Y_j)$ and

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,,$$

to obtain

$$G_4 \xrightarrow{\substack{x_2\to x_1\\ x_4\to x_3}} \frac{(N_c^2-1)^2}{4(4\pi^2)^4}\frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4} + \frac{N_c^2-1}{(4\pi^2)^4}\left[\frac{y_{12}^2 y_{34}^2(y_{13}^2 y_{24}^2 + y_{14}^2 y_{23}^2)}{x_{12}^2 x_{34}^2 x_{13}^4}\right.$$
$$\left. +\frac{1}{3}\frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2 x_{13}^4}\left(c_{\mathcal{K}}^2(a)u^{\gamma_{\mathcal{K}}(a)/2}-1\right)\right] + \ldots\ ,$$

where $y_{ij}^2 = (Y_i \cdot Y_j)$ and

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,,$$

so that $u \to 0$, $v \to 1$ in the limit $x_2 \to x_1\,, x_4 \to x_3$.

Therefore,

$$\sum_{\ell \geq 1} a^{\ell} F^{(\ell)}(x_i) \overset{x_2 \to x_1}{\underset{x_4 \to x_3}{\longrightarrow}} \frac{1}{6 x_{13}^4} \left(c_{\mathcal{K}}^2(a) u^{\gamma_{\mathcal{K}}(a)/2} - 1 \right) \left[1 + O(u, 1-v) \right]$$

Therefore,

$$\sum_{\ell\geq 1} a^\ell F^{(\ell)}(x_i) \xrightarrow[x_4\to x_3]{x_2\to x_1} \frac{1}{6x_{13}^4}\left(c_{\mathcal{K}}^2(a)u^{\gamma_{\mathcal{K}}(a)/2}-1\right)\left[1+O(u,1-v)\right]$$

and

$$\ln\left(1+6\sum_{\ell\geq 1} a^\ell \widehat{F}^{(\ell)}(x_i)\right) \xrightarrow[v\to 1]{u\to 0} \frac{1}{2}\gamma_{\mathcal{K}}(a)\ln u + \ln\left(c_{\mathcal{K}}^2(a)\right) + O(u,1-$$

Therefore,

$$\sum_{\ell \geq 1} a^\ell F^{(\ell)}(x_i) \overset{\substack{x_2 \to x_1 \\ x_4 \to x_3}}{\longrightarrow} \frac{1}{6x_{13}^4} \left(c_{\mathcal{K}}^2(a) u^{\gamma_{\mathcal{K}}(a)/2} - 1 \right) \left[1 + O(u, 1-v) \right]$$

and

$$\ln \left(1 + 6 \sum_{\ell \geq 1} a^\ell \widehat{F}^{(\ell)}(x_i) \right) \overset{\substack{u \to 0 \\ v \to 1}}{\longrightarrow} \frac{1}{2} \gamma_{\mathcal{K}}(a) \ln u + \ln \left(c_{\mathcal{K}}^2(a) \right) + O(u, 1 -$$

where $x_{13}^4 F^{(\ell)}(x_i) \overset{\substack{x_2 \to x_1 \\ x_4 \to x_3}}{\longrightarrow} \widehat{F}(x_i)$ and

$$\gamma_{\mathcal{K}}(a) = \sum_{\ell \geq 1} a^\ell \gamma_{\mathcal{K}}^{(\ell)}, \quad (c_{\mathcal{K}}(a))^2 = 1 + 3 \sum_{\ell \geq 1} a^\ell \alpha^{(\ell)}$$

The $\ell-$loop correction to the logarithm of the correlation function is given by an $\ell-$folded integral over the internal coordinates $x_5, \ldots, x_{4+\ell}$:

$$\ln\left(1 + 6\sum_{\ell\geq 1} a^\ell \widehat{F}^{(\ell)}(x_i)\right)$$

$$= \sum_{\ell\geq 1} a^\ell \int d^4x_5 \ldots d^4x_{4+\ell}\, \mathcal{I}_\ell(x_1, \ldots, x_4|x_5, \ldots, x_{4+\ell}),$$

$$= \sum_{\ell\geq 1} a^\ell\, I^{(\ell)}$$

where $\mathcal{I}_\ell$ is symmetric under the $S_4 \times S_\ell$ permutations of the four external coordinates, $x_1, \ldots, x_4$ and the ℓ internal coordinates $x_5, \ldots, x_{4+\ell}$.

Up to five loops

$$
\begin{aligned}
I^{(1)} &= 6\,\widehat{F}^{(1)}\,, \\
I^{(2)} &= 6\left[\widehat{F}^{(2)} - 3(\widehat{F}^{(1)})^2\right], \\
I^{(3)} &= 6\left[\widehat{F}^{(3)} - 6\widehat{F}^{(1)}\widehat{F}^{(2)} + 12(\widehat{F}^{(1)})^3\right], \\
I^{(4)} &= 6\left[\widehat{F}^{(4)} - 6\widehat{F}^{(1)}\widehat{F}^{(3)} - 3(\widehat{F}^{(2)})^2 + 36\widehat{F}^{(2)}(\widehat{F}^{(1)})^2 - 54(\widehat{F}^{(1)})^4\right. \\
I^{(5)} &= 6\left[\widehat{F}^{(5)} - 6\widehat{F}^{(1)}\widehat{F}^{(4)} - 6\widehat{F}^{(3)}\widehat{F}^{(2)} + 36\widehat{F}^{(3)}(\widehat{F}^{(1)})^2\right. \\
&\left. +36\widehat{F}^{(1)}(\widehat{F}^{(2)})^2 - 216\widehat{F}^{(2)}(\widehat{F}^{(1)})^3 + \tfrac{1296}{5}(\widehat{F}^{(1)})^5\right].
\end{aligned}
$$

$l = 5$, the planar limit:

$$I^{(5)} = \int d^4x_5 \ldots d^4x_9\, \mathcal{I}_5(x_1,\ldots,x_4|x_5,\ldots,x_9)$$

with

$$\mathcal{I}_5 = -\frac{6}{5!(4\pi^2)^5}\frac{x_{13}^4}{\prod_{i=5}^9 x_{1i}^4 x_{3i}^4}\left[\frac{1}{5!}\frac{\widehat{P}_{5,6,7,8,9}}{x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2x_{68}^2x_{69}^2x_{78}^2x_{79}^2x_{89}^2}\right.$$

$$-\frac{1}{4}x_{12}^4\frac{\widehat{P}_{5,6,7,8}}{x_{56}^2x_{57}^2x_{58}^2x_{67}^2x_{68}^2x_{78}^2} - \frac{1}{2}x_{13}^4\frac{\widehat{P}_{5,6,7}}{x_{56}^2x_{57}^2x_{67}^2}$$

$$\frac{\widehat{P}_{8,9}}{x_{89}^2} + 6(x_{13}^4)^2\frac{\widehat{P}_{5,6,7}}{x_{56}^2x_{57}^2x_{67}^2} + 9(x_{13}^4)^2\frac{\widehat{P}_{5,6}}{x_{56}^2}\frac{\widehat{P}_{7,8}}{x_{78}^2}$$

$$\left.-108(x_{13}^4)^3\frac{\widehat{P}_{5,6}}{x_{56}^2} + \frac{1296}{5}(x_{13}^4)^4\right] + S_5 \text{ permutations}$$

where $\widehat{P}_{5,6,7,8,9} = P^{(5)}\Big|_{x_2=x_1,x_4=x_3}$ etc., and

$$
\begin{aligned}
P^{(5)} = & -\tfrac{1}{2} x_{13}^2 x_{16}^2 x_{18}^2 x_{19}^2 x_{24}^4 x_{26}^2 x_{29}^2 x_{37}^2 x_{38}^2 x_{39}^2 x_{47}^2 x_{48}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 \,, \\
& +\tfrac{1}{4} x_{13}^2 x_{16}^2 x_{18}^2 x_{19}^2 x_{24}^4 x_{26}^2 x_{29}^2 x_{37}^4 x_{39}^2 x_{48}^4 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 \\
& +\tfrac{1}{4} x_{13}^4 x_{17}^2 x_{19}^2 x_{24}^2 x_{26}^2 x_{27}^2 x_{29}^2 x_{36}^2 x_{39}^2 x_{48}^6 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 \\
& +\tfrac{1}{6} x_{13}^2 x_{16}^2 x_{19}^4 x_{24}^4 x_{28}^2 x_{29}^2 x_{37}^4 x_{38}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{68}^2 \\
& -\tfrac{1}{8} x_{13}^4 x_{16}^2 x_{18}^2 x_{24}^4 x_{28}^2 x_{29}^2 x_{37}^2 x_{39}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{69}^2 x_{78}^2 \\
& +\tfrac{1}{28} x_{13}^2 x_{17}^2 x_{18}^2 x_{19}^2 x_{24}^8 x_{36}^2 x_{38}^2 x_{39}^2 x_{56}^2 x_{57}^2 x_{58}^2 x_{59}^2 x_{67}^2 x_{69}^2 x_{78}^2 \\
& +\tfrac{1}{12} x_{13}^2 x_{16}^2 x_{17}^2 x_{19}^2 x_{26}^2 x_{27}^2 x_{28}^2 x_{29}^2 x_{35}^2 x_{38}^2 x_{39}^2 x_{45}^2 x_{46}^2 x_{47}^2 x_{49}^2 x_{57}^2 x_{58}^2 x_{68}^2 \\
& +S_9 \text{ permutations}
\end{aligned}
$$

To evaluate

$$\gamma_{\mathcal{K}}(a) = 2\frac{d}{d\ln u}\ln\left(1 + 6\sum_{\ell\geq 1} a^\ell \widehat{F}^{(\ell)}(x_i)\right)$$

we need the coefficient at $\ln u$ of this integral in the limit, $x_1 \to x_2$ and $x_3 \to x_4$, i.e. $u \to 0$.

To evaluate

$$\gamma_{\mathcal{K}}(a) = 2\frac{d}{d\ln u}\ln\left(1 + 6\sum_{\ell\geq 1} a^{\ell}\widehat{F}^{(\ell)}(x_i)\right)$$

we need the coefficient at $\ln u$ of this integral in the limit, $x_1 \to x_2$ and $x_3 \to x_4$, i.e. $u \to 0$.
Put $x_1 = x_2$ and $x_3 = x_4$ and introduce dimensional regularization (in coordinate space) with $D = 4 - 2\epsilon$

$$\mu^{l\epsilon}\int d^D x_5 \ldots d^D x_9\, \mathcal{I}_\ell(x_1, x_1, x_3, x_3|x_5, \ldots, x_9)\,.$$

To evaluate

$$\gamma_{\mathcal{K}}(a) = 2\frac{d}{d\ln u}\ln\left(1 + 6\sum_{\ell\geq 1} a^{\ell}\widehat{F}^{(\ell)}(x_i)\right)$$

we need the coefficient at $\ln u$ of this integral in the limit, $x_1 \to x_2$ and $x_3 \to x_4$, i.e. $u \to 0$.
Put $x_1 = x_2$ and $x_3 = x_4$ and introduce dimensional regularization (in coordinate space) with $D = 4 - 2\epsilon$

$$\mu^{l\epsilon}\int d^D x_5 \ldots d^D x_9\, \mathcal{I}_\ell(x_1, x_1, x_3, x_3 | x_5, \ldots, x_9)\,.$$

The integral has a simple pole in $\epsilon = (4 - D)/2$.

$$\gamma_{\mathcal{K}}(a) = \frac{d}{d\ln\mu^2} \ln\left(1 + 6\sum_{\ell\geq 1} a^\ell \widehat{F}_\epsilon^{(\ell)}(x_i)\right)$$

$$\gamma_{\mathcal{K}}(a) = \frac{d}{d\ln\mu^2}\ln\left(1 + 6\sum_{\ell\geq 1} a^\ell \widehat{F}_\epsilon^{(\ell)}(x_i)\right)$$

The problem reduces to evaluating the pole part of

$$\int d^D x_5 \ldots d^D x_9 \, \mathcal{I}_5(x_1, x_1, x_3, x_3 | x_5, \ldots, x_9)$$

in ϵ.

$$\gamma_{\mathcal{K}}(a) = \frac{d}{d\ln\mu^2}\ln\left(1 + 6\sum_{\ell\geq 1} a^\ell \widehat{F}_\epsilon^{(\ell)}(x_i)\right)$$

The problem reduces to evaluating the pole part of

$$\int d^D x_5 \ldots d^D x_9 \, \mathcal{I}_5(x_1, x_1, x_3, x_3 | x_5, \ldots, x_9)$$

in ϵ.

IRR (infrared rearrangement) [A.A. Vladimirov'80]

IRR

If the pole part of a given (incompletely renormalized) diagram is independent of momenta and masses, try to set to zero as many momenta and masses as possible without generating IR divergences.

IRR

If the pole part of a given (incompletely renormalized) diagram is independent of momenta and masses, try to set to zero as many momenta and masses as possible without generating IR divergences.

An advanced strategy: set *all* the momenta and masses to zero and introduce a mass or momentum.

IRR

If the pole part of a given (incompletely renormalized) diagram is independent of momenta and masses, try to set to zero as many momenta and masses as possible without generating IR divergences.

An advanced strategy: set *all* the momenta and masses to zero and introduce a mass or momentum.

(If IR divergences are still generated they can be removed immediately by the R^*-operation
[K.G. Chetyrkin & F.V. Tkachov'82, K.G. Chetyrkin & V.A. Smirnov'82])

If the pole part of a given (incompletely renormalized) diagram is independent of momenta and masses, try to set to zero as many momenta and masses as possible without generating IR divergences.

An advanced strategy: set *all* the momenta and masses to zero and introduce a mass or momentum.

(If IR divergences are still generated they can be removed immediately by the R^*-operation
[K.G. Chetyrkin & F.V. Tkachov'82, K.G. Chetyrkin & V.A. Smirnov'82])

A four-loop example:

$$I(x_1,x_3)=\frac{1}{\pi^{2D}}\int\frac{(x_{13}^2)^4\,d^Dx_5\ldots d^Dx_8}{x_{15}^2x_{16}^2x_{17}^2x_{18}^2x_{35}^2x_{36}^2x_{37}^2x_{38}^2x_{56}^2x_{68}^2x_{78}^2x_{57}^2}$$

There is an UV simple pole in ϵ

$$I(x_1,x_3) = (x_{13}^2)^{-4\epsilon}\left[\frac{C}{\epsilon} + O(\epsilon^0)\right]$$

from the integration over $x_5,\ldots,x_8$ close to x_1 and from the symmetrical region where $x_5,\ldots,x_8$ are all close to x_3.

There is an UV simple pole in ϵ

$$I(x_1, x_3) = (x_{13}^2)^{-4\epsilon} \left[\frac{C}{\epsilon} + O(\epsilon^0) \right]$$

from the integration over $x_5, \ldots, x_8$ close to x_1 and from the symmetrical region where $x_5, \ldots, x_8$ are all close to x_3.
UV divergences come from regions where the integrand considered as a generalized function of x_i is ill-defined.
The integrand is unintegrable in a vicinity of the two external points, x_1 and x_3

There is an UV simple pole in ϵ

$$I(x_1, x_3) = (x_{13}^2)^{-4\epsilon} \left[\frac{C}{\epsilon} + O(\epsilon^0) \right]$$

from the integration over $x_5, \ldots, x_8$ close to x_1 and from the symmetrical region where $x_5, \ldots, x_8$ are all close to x_3.
UV divergences come from regions where the integrand considered as a generalized function of x_i is ill-defined.
The integrand is unintegrable in a vicinity of the two external points, x_1 and x_3
In a vicinity of x_1 ($x_3 \to \infty$):

$$F(x_1, x_5, \ldots, x_8) = \frac{1}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2}$$

Its divergent part is described by an UV counterterm

$$\Delta(x_1, x_5, \ldots, x_8) = \frac{C}{2\epsilon}\delta(x_1 - x_5)\ldots\delta(x_1 - x_8)\,,$$

Its divergent part is described by an UV counterterm

$$\Delta(x_1, x_5, \ldots, x_8) = \frac{C}{2\epsilon}\delta(x_1 - x_5)\ldots\delta(x_1 - x_8)\,,$$

A similar counterterm $\Delta(x_3, x_5, \ldots, x_8)$ is connected with the vicinity of x_3.

Its divergent part is described by an UV counterterm

$$\Delta(x_1, x_5, \ldots, x_8) = \frac{C}{2\epsilon}\delta(x_1 - x_5)\ldots\delta(x_1 - x_8)\,,$$

A similar counterterm $\Delta(x_3, x_5, \ldots, x_8)$ is connected with the vicinity of x_3.
The pole part of $I(x_1, x_3)$ is C/ϵ.

Its divergent part is described by an UV counterterm

$$\Delta(x_1, x_5, \ldots, x_8) = \frac{C}{2\epsilon}\delta(x_1 - x_5)\ldots\delta(x_1 - x_8)\,,$$

A similar counterterm $\Delta(x_3, x_5, \ldots, x_8)$ is connected with the vicinity of x_3.
The pole part of $I(x_1, x_3)$ is C/ϵ.
We are not going to momentum space via Fourier transform because

- we would obtain four-loop integrals,
- exponents of the propagators would depend on ϵ.

IRR

Apply IRR in the coordinate space: treat the coordinates x_1, x_5 as external and x_6, x_7, x_8 as internal points.
(Setting an external momentum to zero $\sim$ integrating over the corresponding coordinate.)

IRR

Apply IRR in the coordinate space: treat the coordinates x_1, x_5 as external and x_6, x_7, x_8 as internal points.
(Setting an external momentum to zero $\sim$ integrating over the corresponding coordinate.)
Then C can be obtained from

$$F(x_1, x_5) = \int \frac{d^D x_6 d^D x_7 d^D x_8}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2} = \frac{C}{2\epsilon} \delta(x_1 - x_5) + O(\epsilon^0)$$

(No IR divergences have been generated.)

Apply IRR in the coordinate space: treat the coordinates x_1, x_5 as external and x_6, x_7, x_8 as internal points.
(Setting an external momentum to zero $\sim$ integrating over the corresponding coordinate.)
Then C can be obtained from

$$F(x_1, x_5) = \int \frac{d^D x_6 d^D x_7 d^D x_8}{x_{15}^2 x_{16}^2 x_{17}^2 x_{18}^2 x_{56}^2 x_{68}^2 x_{78}^2 x_{57}^2} = \frac{C}{2\epsilon} \delta(x_1 - x_5) + O(\epsilon^0)$$

(No IR divergences have been generated.)
This propagator integral is three-loop:

$$F(x_1, x_5) = f(\epsilon) \frac{1}{(x_{15}^2)^{2+3\epsilon}} \, .$$

Here the pole is hidden in $\frac{1}{(x_{15}^2)^{2+3\epsilon}}$ so that $f(\epsilon)$ is analytic at $\epsilon = 0$.

Here the pole is hidden in $\frac{1}{(x_{15}^2)^{2+3\epsilon}}$ so that $f(\epsilon)$ is analytic at $\epsilon = 0$. To reveal the pole take its D-dimensional Fourier transform:

$$\mathcal{F}\left[\frac{1}{(x^2)^\lambda}\right] = \frac{1}{\pi^{D/2}} \int d^D x \; \mathrm{e}^{ipx} \frac{1}{(x^2)^\lambda} = \frac{4^{D/2-\lambda}}{\Gamma(\lambda)} \frac{\Gamma(D/2-\lambda)}{(p^2)^{D/2-\lambda}} \,.$$

Here the pole is hidden in $\frac{1}{(x_{15}^2)^{2+3\epsilon}}$ so that $f(\epsilon)$ is analytic at $\epsilon = 0$. To reveal the pole take its D-dimensional Fourier transform:

$$\mathcal{F}\left[\frac{1}{(x^2)^\lambda}\right] = \frac{1}{\pi^{D/2}} \int d^D x \; \mathrm{e}^{ipx} \, \frac{1}{(x^2)^\lambda} = \frac{4^{D/2-\lambda}}{\Gamma(\lambda)} \frac{\Gamma(D/2-\lambda)}{(p^2)^{D/2-\lambda}} \, .$$

For $\lambda = 2 + 3\epsilon$ and for $x_5 = 0$:

$$\mathcal{F}\left[F(x_1, 0)\right] = f(\epsilon) \frac{4^{-4\epsilon}\Gamma(-4\epsilon)}{\Gamma(2+3\epsilon)} \frac{1}{(p^2)^{-4\epsilon}} = -\frac{f(0)}{4\epsilon} + O(\epsilon^0) \, .$$

Here the pole is hidden in $\frac{1}{(x_{15}^2)^{2+3\epsilon}}$ so that $f(\epsilon)$ is analytic at $\epsilon = 0$. To reveal the pole take its D-dimensional Fourier transform:

$$\mathcal{F}\left[\frac{1}{(x^2)^\lambda}\right] = \frac{1}{\pi^{D/2}} \int d^D x \; e^{ipx} \, \frac{1}{(x^2)^\lambda} = \frac{4^{D/2-\lambda}}{\Gamma(\lambda)} \frac{\Gamma(D/2-\lambda)}{(p^2)^{D/2-\lambda}} \, .$$

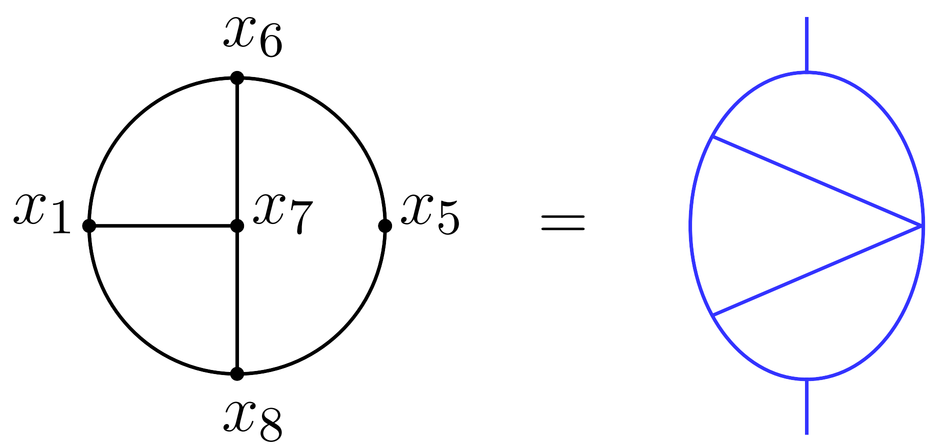
For $\lambda = 2 + 3\epsilon$ and for $x_5 = 0$:

$$\mathcal{F}\left[F(x_1, 0)\right] = f(\epsilon) \frac{4^{-4\epsilon}\Gamma(-4\epsilon)}{\Gamma(2+3\epsilon)} \frac{1}{(p^2)^{-4\epsilon}} = -\frac{f(0)}{4\epsilon} + O(\epsilon^0) \, .$$

We obtain

$$C = -\frac{1}{2} f(0) = -\frac{1}{2} F(x_1, 0) \Big|_{x_1^2=1, D=4}$$

The integral $F(x_1, x_5)$ corresponds to a planar graph.



Using a known result for the corresponding dual integral at $d = 4$ leads to

$$C = -10\,\zeta(5)$$

$$\int d^D x_5 \ldots d^D x_9 \, \mathcal{I}_5(x_1, x_1, x_3, x_3 | x_5, \ldots, x_9)$$

To evaluate the pole part (a simple pole) apply IRR.

$$\int d^D x_5 \ldots d^D x_9 \, \mathcal{I}_5(x_1, x_1, x_3, x_3 | x_5, \ldots, x_9)$$

To evaluate the pole part (a simple pole) apply IRR.
The pole is generated by the region where $x_5, \ldots, x_9$ are close either to x_1 or x_3.
There are no other sources of poles in ϵ.

$$\int d^D x_5 \ldots d^D x_9 \, \mathcal{I}_5(x_1, x_1, x_3, x_3 | x_5, \ldots, x_9)$$

To evaluate the pole part (a simple pole) apply IRR.
The pole is generated by the region where $x_5, \ldots, x_9$ are close either to x_1 or x_3.
There are no other sources of poles in ϵ.
Let $x_5, \ldots, x_9 \sim x_1$. As in our example, the problem reduces to the evaluation of the UV counterterm of

$$\hat{\mathcal{I}}_5(x_1, x_5, \ldots, x_9) = \lim_{x_3 \to \infty} \mathcal{I}_5(x_1, x_3; x_5, \ldots, x_9) \, .$$

$$\int d^D x_5 \ldots d^D x_9 \, \mathcal{I}_5(x_1, x_1, x_3, x_3 | x_5, \ldots, x_9)$$

To evaluate the pole part (a simple pole) apply IRR.
The pole is generated by the region where $x_5, \ldots, x_9$ are close either to x_1 or x_3.
There are no other sources of poles in ϵ.
Let $x_5, \ldots, x_9 \sim x_1$. As in our example, the problem reduces to the evaluation of the UV counterterm of

$$\hat{\mathcal{I}}_5(x_1, x_5, \ldots, x_9) = \lim_{x_3 \to \infty} \mathcal{I}_5(x_1, x_3; x_5, \ldots, x_9) \, .$$

Apply IRR:

consider x_1 and x_5 external and x_6, x_7, x_8, x_9 internal.

The problem reduces to the evaluation of the residue of

$$\frac{1}{(x_{15}^2)^{2+4\epsilon}} \int d^D x_6 d^D x_7 d^D x_8 d^D x_9 \, \hat{\mathcal{I}}_5(x_1, x_5, \ldots, x_9) \, .$$

The problem reduces to the evaluation of the residue of

$$\frac{1}{(x_{15}^2)^{2+4\epsilon}} \int d^D x_6 d^D x_7 d^D x_8 d^D x_9 \, \hat{\mathcal{I}}_5(x_1, x_5, \ldots, x_9) \, .$$

The pole comes from $\frac{1}{(x_{15}^2)^{2+4\epsilon}}$ so that we need to evaluate

$$\int d^D x_6 d^D x_7 d^D x_8 d^D x_9 \, \hat{\mathcal{I}}_5(x_1, x_5, \ldots, x_9) \, .$$

at $x_{15}^2 = 1$.

Around 17000 four-loop two-point Feynman integrals contributing to this integral and belonging to the family

$$G(a_1,\ldots,a_{14}) = \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2)^{a_1}(x_{17}^2)^{a_2}(x_{18}^2)^{a_3}(x_{19}^2)^{a_4}(x_6^2)^{a_5}(x_7^2)^{a_6}(x_8^2)^{a_7}}$$
$$\times \frac{1}{(x_9^2)^{a_8}(x_{67}^2)^{a_9}(x_{68}^2)^{a_{10}}(x_{69}^2)^{a_{11}}(x_{78}^2)^{a_{12}}(x_{79}^2)^{a_{13}}(x_{89}^2)^{a_{14}}} \;.$$

with various indices a_i.

Around 17000 four-loop two-point Feynman integrals contributing to this integral and belonging to the family

$$G(a_1,\ldots,a_{14}) = \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2)^{a_1}(x_{17}^2)^{a_2}(x_{18}^2)^{a_3}(x_{19}^2)^{a_4}(x_6^2)^{a_5}(x_7^2)^{a_6}(x_8^2)^{a_7}}$$
$$\times \frac{1}{(x_9^2)^{a_8}(x_{67}^2)^{a_9}(x_{68}^2)^{a_{10}}(x_{69}^2)^{a_{11}}(x_{78}^2)^{a_{12}}(x_{79}^2)^{a_{13}}(x_{89}^2)^{a_{14}}} \, .$$

with various indices a_i.

An IBP reduction to master integrals.

Solving IBP relations algorithmically:

- Laporta's algorithm

[Laporta & Remiddi'96; Laporta'00; Gehrmann & Remiddi'01]

Solving IBP relations algorithmically:

- Laporta's algorithm

[Laporta & Remiddi'96; Laporta'00; Gehrmann & Remiddi'01]

Implementations of the Laporta's algorithm

Solving IBP relations algorithmically:

- Laporta's algorithm

[Laporta & Remiddi'96; Laporta'00; Gehrmann & Remiddi'01]

Implementations of the Laporta's algorithm

Three public versions:

Solving IBP relations algorithmically:

- Laporta's algorithm

[Laporta & Remiddi'96; Laporta'00; Gehrmann & Remiddi'01]

Implementations of the Laporta's algorithm

Three public versions:

- AIR [Anastasiou & Lazopoulos'04]

Solving IBP relations algorithmically:

- Laporta's algorithm

[Laporta & Remiddi'96; Laporta'00; Gehrmann & Remiddi'01]

Implementations of the Laporta's algorithm

Three public versions:

- AIR [Anastasiou & Lazopoulos'04]
- FIRE [A. Smirnov'08]
  (in `Mathematica`; a `C++` version is private)

Solving IBP relations algorithmically:

- Laporta's algorithm

[Laporta & Remiddi'96; Laporta'00; Gehrmann & Remiddi'01]

Implementations of the Laporta's algorithm

Three public versions:

- AIR [Anastasiou & Lazopoulos'04]
- FIRE [A. Smirnov'08]
  (in `Mathematica`; a `C++` version is private)
- Reduze [C. Studerus'09, A. von Manteuffel & C. Studerus'11]

Solving IBP relations algorithmically:

- Laporta's algorithm

[Laporta & Remiddi'96; Laporta'00; Gehrmann & Remiddi'01]

Implementations of the Laporta's algorithm

Three public versions:

- AIR [Anastasiou & Lazopoulos'04]
- FIRE [A. Smirnov'08]
  (in `Mathematica`; a `C++` version is private)
- Reduze [C. Studerus'09, A. von Manteuffel & C. Studerus'11]

Private versions

[Gehrmann & Remiddi, Laporta, Czakon, Schröder, Pak, Sturm, Marquard & Seidel, Velizhanin, . . .]

C++ version of FIRE $\rightarrow$

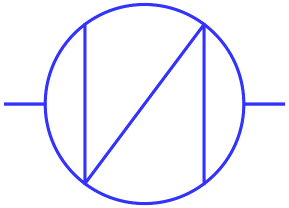
$$C_4 = w_{44}M_{44} + w_{61}M_{61} + w_{36}M_{36} + w_{31}M_{31} + w_{35}M_{35}$$
$$+w_{22}M_{22} + w_{32}M_{32} + w_{33}M_{33} + w_{34}M_{34} + w_{25}M_{25} + w_{23}M_{23}$$
$$+w_{27}M_{27} + w_{24}M_{24} + w_{26}M_{26} + w_{01}M_{01}$$
$$+w_{21}M_{21} + w_{12}M_{12} + w_{11}M_{11} + w_{14}M_{14} + w_{13}M_{13} + w_1 I_1 + w$$

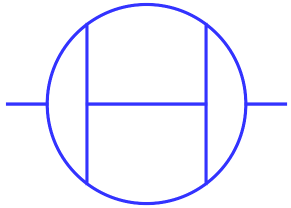
C++ version of FIRE $\rightarrow$

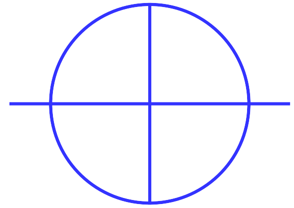
$$C_4 = w_{44}M_{44} + w_{61}M_{61} + w_{36}M_{36} + w_{31}M_{31} + w_{35}M_{35}$$
$$+w_{22}M_{22} + w_{32}M_{32} + w_{33}M_{33} + w_{34}M_{34} + w_{25}M_{25} + w_{23}M_{23}$$
$$+w_{27}M_{27} + w_{24}M_{24} + w_{26}M_{26} + w_{01}M_{01}$$
$$+w_{21}M_{21} + w_{12}M_{12} + w_{11}M_{11} + w_{14}M_{14} + w_{13}M_{13} + w_1 I_1 + w_2$$

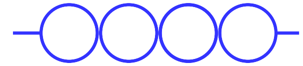
Only I_1 and I_2, are associated with non-planar graphs.

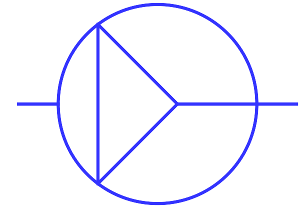
20 master integrals $M_{44}, \ldots, M_{13}$ correspond to planar graphs and can be represented, via duality, as four-loop propagator master (momentum) integrals.

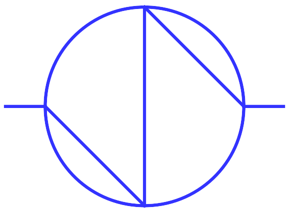

M_{44}

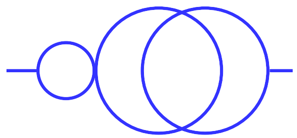

M_{61}

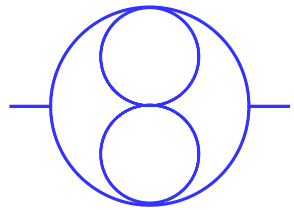

M_{36}

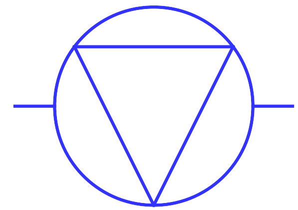

M_{31}

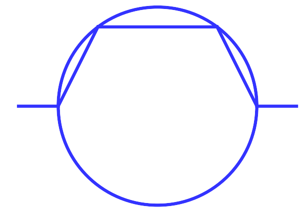

M_{35}

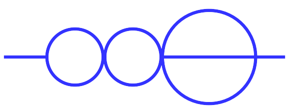

M_{22}

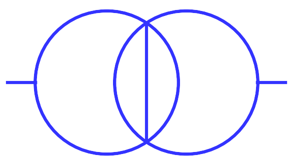

M_{32}

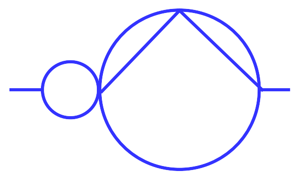

M_{33}

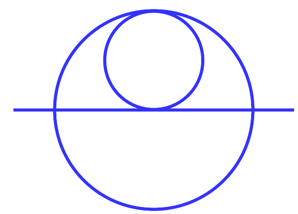

M_{34}

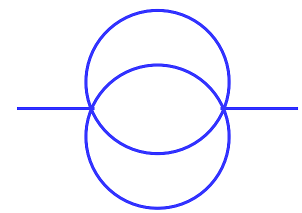

M_{25}

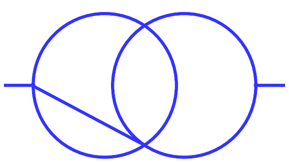

M_{23}

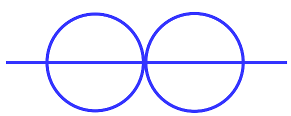

M_{27}

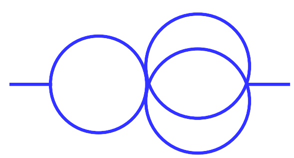

M_{24}

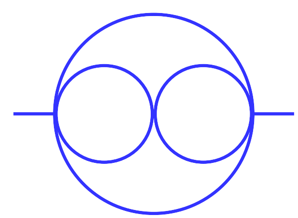

M_{26}

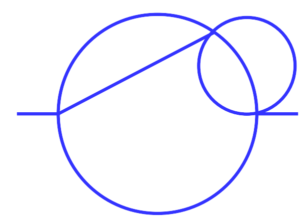

M_{01}

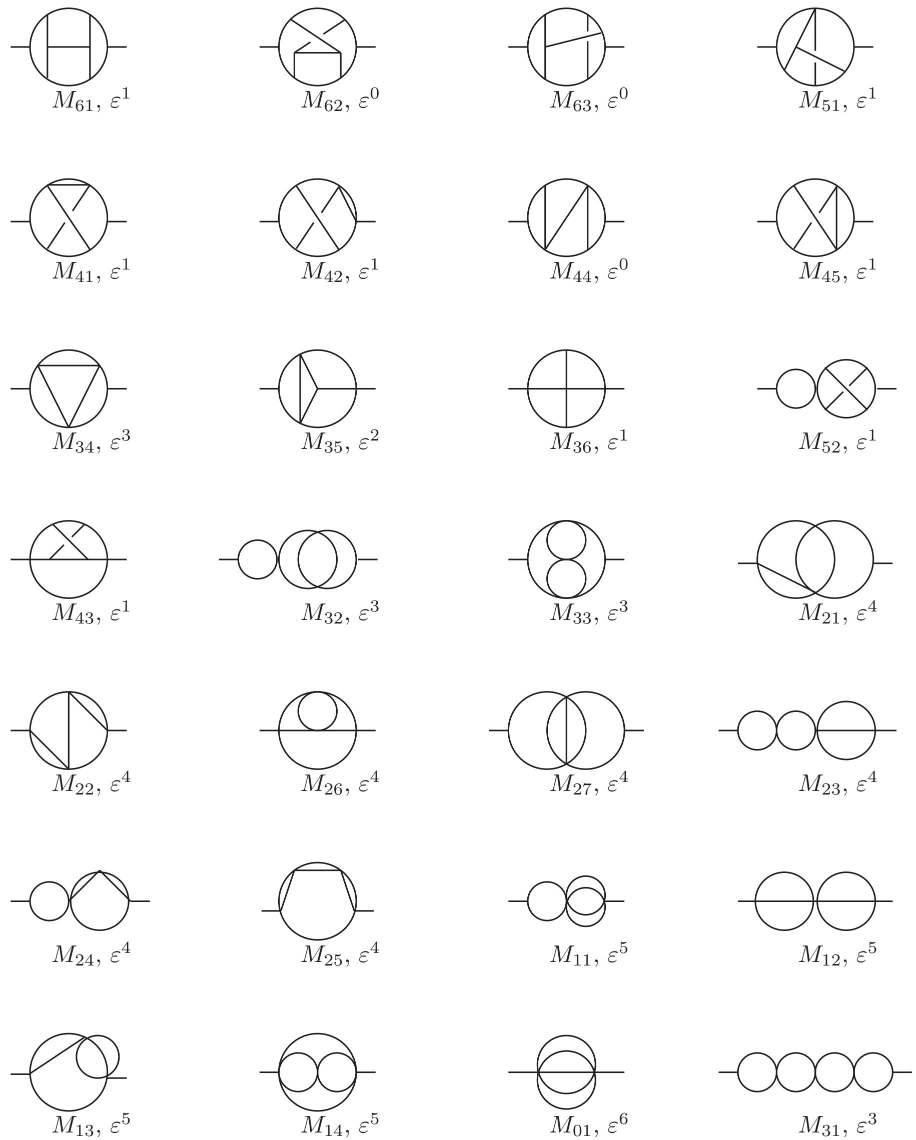

M_{21}


M_{12}


M_{11}


M_{14}

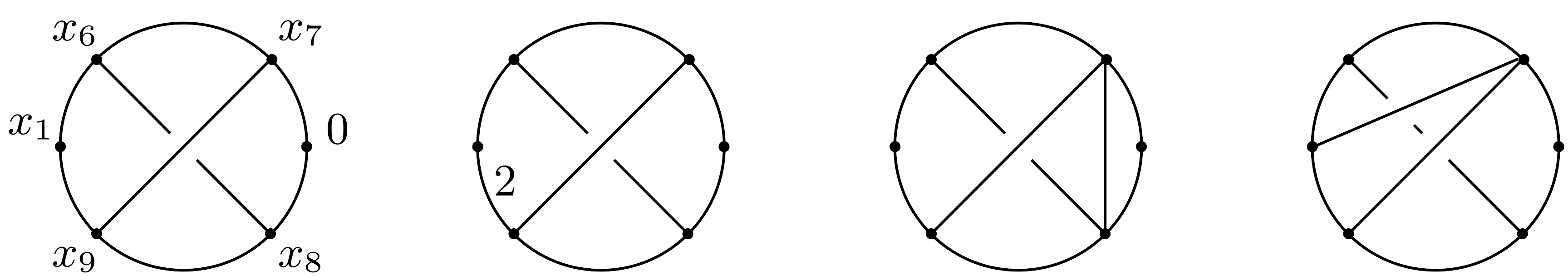

M_{13}


M_{61}, ε^1
M_{62}, ε^0
M_{63}, ε^0
M_{51}, ε^1
M_{41}, ε^1
M_{42}, ε^1
M_{44}, ε^0
M_{45}, ε^1
M_{34}, ε^3
M_{35}, ε^2
M_{36}, ε^1
M_{52}, ε^1
M_{43}, ε^1
M_{32}, ε^3
M_{33}, ε^3
M_{21}, ε^4
M_{22}, ε^4
M_{26}, ε^4
M_{27}, ε^4
M_{23}, ε^4
M_{24}, ε^4
M_{25}, ε^4
M_{11}, ε^5
M_{12}, ε^5
M_{13}, ε^5
M_{14}, ε^5
M_{01}, ε^6
M_{31}, ε^3

Results in an ϵ expansion up to transcendentality weight seven [P.A. Baikov & K.G. Chetyrkin'10]
and up to weight twelve [R.N. Lee, A.V. Smirnov & V.A. Smirnov]

Results in an ϵ expansion up to transcendentality weight seven [P.A. Baikov & K.G. Chetyrkin'10]
and up to weight twelve [R.N. Lee, A.V. Smirnov & V.A. Smirnov]

The two non-planar master integrals I_1 and I_2



I_1 I_2 $I_3(0)$ $I_4(0)$

We did not use the method by R. Lee [R. Lee'09]
based on dimensional recurrence relations.

Its applications [R. Lee, A. and V. Smirnovs'10,11]

$$I_1 = \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_7^2 x_{79}^2 x_8^2 x_{89}^2} = \frac{a_1}{\epsilon} + b_1 + c_1\epsilon + O(\epsilon^2)\,,$$

$$I_2 = \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{x_{16}^2 (x_{19}^2)^2 x_{67}^2 x_{68}^2 x_7^2 x_{79}^2 x_8^2 x_{89}^2} = \frac{a_2}{\epsilon} + b_2 + c_2\epsilon + O(\epsilon^2)$$

$$I_1 = \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_7^2 x_{79}^2 x_8^2 x_{89}^2} = \frac{a_1}{\epsilon} + b_1 + c_1\epsilon + O(\epsilon^2)\,,$$

$$I_2 = \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{x_{16}^2 (x_{19}^2)^2 x_{67}^2 x_{68}^2 x_7^2 x_{79}^2 x_8^2 x_{89}^2} = \frac{a_2}{\epsilon} + b_2 + c_2\epsilon + O(\epsilon^2)$$

$$\left(\frac{3a_1}{80} + \frac{9a_2}{160} + \frac{15\zeta_5}{16}\right)\epsilon^{-2}$$

$$+\left(-\frac{21a_1}{80} - \frac{9a_2}{80} + \frac{3b_1}{80} + \frac{9b_2}{160} + \frac{15{\zeta_3}^2}{16} + \frac{5\pi^6}{2016}\right)\epsilon^{-1}$$

$$+\left(\frac{741a_1}{640} + \frac{807a_2}{320} - \frac{21b_1}{80} - \frac{9b_2}{80} + \frac{3c_1}{80} + \frac{9c_2}{160} - \frac{225\zeta_7}{64} - \frac{5\pi^2\zeta_5}{16}\right.$$

$$\left.+\frac{7035\zeta_5}{128} + \frac{81{\zeta_3}^2}{16} + \frac{\pi^4\zeta_3}{32} - \frac{27\zeta_3}{4} - \frac{237}{16}\right) + O(\epsilon)$$

The absence of poles $\to$ two relations between coefficients $a_1, a_2, b_1, b_2, c_1, c_2$

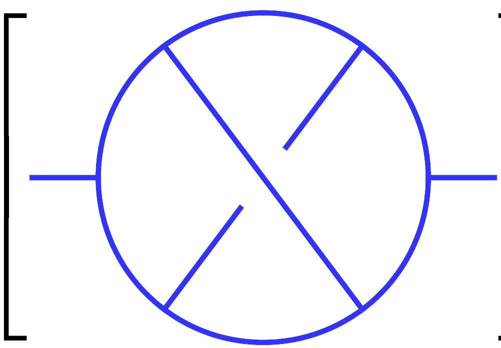
To evaluate a_1, a_2 take a Fourier transform:

$$\mathcal{F}\left[I_1\right] = \mathcal{F}\left[\frac{a_1}{\epsilon}(x_1^2)^{-4\epsilon} + O(\epsilon^0)\right] = \left(64\, a_1 + O(\epsilon)\right)\left(p^2\right)^{-2+5\epsilon}\,.$$

The absence of poles $\to$ two relations between coefficients $a_1, a_2, b_1, b_2, c_1, c_2$

To evaluate a_1, a_2 take a Fourier transform:

$$\mathcal{F}\left[I_1\right] = \mathcal{F}\left[\frac{a_1}{\epsilon}(x_1^2)^{-4\epsilon} + O(\epsilon^0)\right] = (64\, a_1 + O(\epsilon))\,(p^2)^{-2+5\epsilon}\,.$$

$\mathcal{F}\left[I_1\right] = 16 \Bigg[$  $\Bigg] = 16\big(20\zeta_5 + O(\epsilon)\big)(p^2)^{-2+5\epsilon}\,.$

so that $a_1 = 5\zeta_5$.

$$\mathcal{F}[I_2] = F\left[\frac{a_2}{\epsilon}(x_1^2)^{-1-4\epsilon} + O(\epsilon^0)\right] = 4\left(\frac{a_2}{\epsilon} + O(\epsilon)\right)(p^2)^{-1+5\epsilon}.$$

$$\mathcal{F}\left[\frac{1}{(x_{19}^2)^2}\right] = 2^{-2\epsilon}\Gamma(-\epsilon)(p^2)^{\epsilon} = -\frac{1}{\epsilon} + O(\epsilon^0).$$

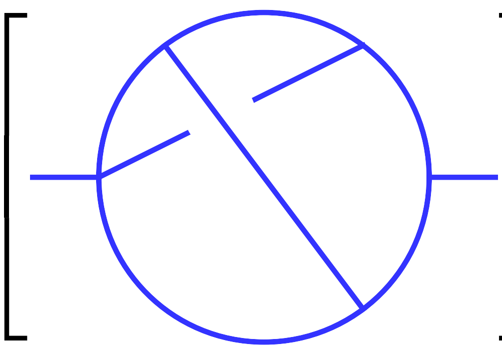
so that taking the residue at the pole reduces to shrinking the corresponding line to a point.

$$
\mathcal{F}[I_2] = F\left[\frac{a_2}{\epsilon}(x_1^2)^{-1-4\epsilon} + O(\epsilon^0)\right] = 4\left(\frac{a_2}{\epsilon} + O(\epsilon)\right)(p^2)^{-1+5\epsilon}\,.
$$

$$
\mathcal{F}\left[\frac{1}{(x_{19}^2)^2}\right] = 2^{-2\epsilon}\Gamma(-\epsilon)(p^2)^{\epsilon} = -\frac{1}{\epsilon} + O(\epsilon^0)\,.
$$

so that taking the residue at the pole reduces to shrinking the corresponding line to a point.

$$
\mathcal{F}[I_2] = -\frac{4}{\epsilon}\Bigg[
$$



$$
\Bigg] = -\frac{4}{\epsilon}(20\zeta_5 + O(\epsilon))(p^2)^{-1+5\epsilon}\,.
$$

We obtain $a_2 = -20\zeta_5\,.$

Introduce the following auxiliary integrals

$$I_3(\kappa) = \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^Dx_6 d^Dx_7 d^Dx_8 d^Dx_9}{(x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_7^2 x_8^2 x_{89}^2)^{1-\epsilon\kappa}} ,$$

$$I_4(\kappa) = \frac{e^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^Dx_6 d^Dx_7 d^Dx_8 d^Dx_9}{(x_{16}^2 x_{17}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{79}^2 x_7^2 x_8^2 x_{89}^2)^{1-\epsilon\kappa}} ,$$

with

$$I_i(\kappa) = b_i + \epsilon(c_i + \kappa\, d_i) + O(\epsilon^2) , \qquad i = 3, 4$$

Introduce the following auxiliary integrals

$$
\begin{aligned}
I_3(\kappa) &= \frac{\mathrm{e}^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_7^2 x_8^2 x_{89}^2)^{1-\epsilon\kappa}} \,, \\
I_4(\kappa) &= \frac{\mathrm{e}^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2 x_{17}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{79}^2 x_7^2 x_8^2 x_{89}^2)^{1-\epsilon\kappa}} \,,
\end{aligned}
$$

with

$$
I_i(\kappa) = b_i + \epsilon(c_i + \kappa\, d_i) + O(\epsilon^2)\,, \qquad i = 3,4
$$

Evaluate $I_3(0)$ and $I_4(0)$. They are not master integrals. We use `FIRE` to reduce them to master integrals,

in particular, I_1 and I_2.

$$b_3 = -\frac{2}{3}\, b_1 - \frac{7}{3}\, b_2 - 70\,\zeta_5 + \frac{26}{3}\zeta_3{}^2 - \frac{65}{567}\pi^6\,,$$

$$b_4 = -b_1 - 2b_2 - 45\zeta_5 + 7\zeta_3{}^2 - \frac{5}{54}\pi^6\,,$$

$$c_3 = \frac{14}{3}\, b_1 + \frac{14}{3}\, b_2 - \frac{2}{3}\, c_1 - \frac{7}{3}\, c_2 - \frac{4667}{6}\zeta_7 + \frac{130}{9}\pi^2\zeta_5 - \frac{100}{3}\zeta_5 + \frac{13}{45}\pi^4$$

$$c_4 = 2b_1 - 6b_2 - c_1 - 2c_2 - \frac{4193}{4}\zeta_7 + \frac{35}{3}\pi^2\zeta_5 - 275\zeta_5 + 35\zeta_3{}^2$$
$$+\frac{7}{30}\pi^4\zeta_3 - \frac{25}{54}\pi^6\,.$$

$$b_3 = -\frac{2}{3}\,b_1 - \frac{7}{3}\,b_2 - 70\,\zeta_5 + \frac{26}{3}{\zeta_3}^2 - \frac{65}{567}\pi^6\,,$$

$$b_4 = -b_1 - 2b_2 - 45\zeta_5 + 7{\zeta_3}^2 - \frac{5}{54}\pi^6\,,$$

$$c_3 = \frac{14}{3}\,b_1 + \frac{14}{3}\,b_2 - \frac{2}{3}\,c_1 - \frac{7}{3}\,c_2 - \frac{4667}{6}\zeta_7 + \frac{130}{9}\pi^2\zeta_5 - \frac{100}{3}\zeta_5 + \frac{13}{45}\pi^4$$

$$c_4 = 2b_1 - 6b_2 - c_1 - 2c_2 - \frac{4193}{4}\zeta_7 + \frac{35}{3}\pi^2\zeta_5 - 275\zeta_5 + 35{\zeta_3}^2 + \frac{7}{30}\pi^4\zeta_3 - \frac{25}{54}\pi^6\,.$$

Evaluate I_3 and I_4 at $\kappa = 1/2$ and $\kappa = 1$ and obtain I_1 and I_2, i.e. b_1, b_2 and c_1, c_2.

$I_i(\kappa)$, $i = 3, 4$ is a linear function of κ at $O(\epsilon) \to$

$$I_i(0) = 2I_i(1) - I_i(1/2) + O(\epsilon^2) = b_i + \epsilon c_i + O(\epsilon^2)\,.$$

$I_i(\kappa)$, $i = 3,4$ is a linear function of κ at $O(\epsilon) \to$

$$I_i(0) = 2I_i(1) - I_i(1/2) + O(\epsilon^2) = b_i + \epsilon c_i + O(\epsilon^2)\,.$$

Let $\kappa = 1$, i.e. with propagators $1/(x^2)^{1-\epsilon} \to$
$\mathcal{F}[I_3(1)]$ and $\mathcal{F}[I_4(1)]$ are given by conventional four-loop momentum Feynman integrals with propagators $1/p^2$.
$\mathcal{F}[I_3(1)] \to M_{45}$ of Baikov and Chetyrkin.

$$I_3(1) = 36{\zeta_3}^2 + \epsilon\Big(108\,\zeta_3\zeta_4 + 288{\zeta_3}^2 - 378\,\zeta_7\Big) + O(\epsilon^2)$$

$I_i(\kappa)$, $i = 3, 4$ is a linear function of κ at $O(\epsilon) \to$

$$I_i(0) = 2I_i(1) - I_i(1/2) + O(\epsilon^2) = b_i + \epsilon c_i + O(\epsilon^2)\,.$$

Let $\kappa = 1$, i.e. with propagators $1/(x^2)^{1-\epsilon} \to$
$\mathcal{F}[I_3(1)]$ and $\mathcal{F}[I_4(1)]$ are given by conventional four-loop momentum Feynman integrals with propagators $1/p^2$.
$\mathcal{F}[I_3(1)] \to M_{45}$ of Baikov and Chetyrkin.

$$I_3(1) = 36{\zeta_3}^2 + \epsilon\Big(108\,\zeta_3\zeta_4 + 288{\zeta_3}^2 - 378\,\zeta_7\Big) + O(\epsilon^2)$$

The second integral $\mathcal{F}[I_4(1)]$ is not a master integral.
We applied `FIRE` to reduce it to master integrals

$M_{01}, M_{11}, M_{35}, M_{13}, M_{36}, M_{12}, M_{21}$.

$$I_4(1) \quad = 36{\zeta_3}^2 + \epsilon \left(108\, \zeta_3\zeta_4 + 108{\zeta_3}^2 + \frac{189}{2}\zeta_7\right) + O(\epsilon^2)$$

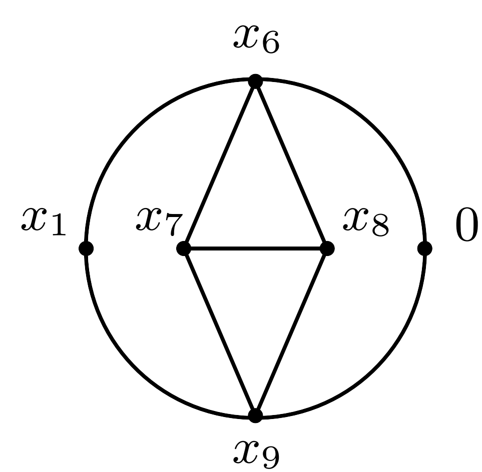
$$I_4(1) \quad = 36{\zeta_3}^2 + \epsilon\left(108\,\zeta_3\zeta_4 + 108{\zeta_3}^2 + \tfrac{189}{2}\zeta_7\right) + O(\epsilon^2)$$

Let now $\kappa = 1/2$.
For the integral $I_3(1/2)$, the conformal weight of the integrand at x_7 and x_8 equals the space-time dimension $4(1-\kappa\epsilon) = D$.

$$I_4(1) \;\; = 36{\zeta_3}^2 + \epsilon\left(108\,\zeta_3\zeta_4 + 108{\zeta_3}^2 + \tfrac{189}{2}\zeta_7\right) + O(\epsilon^2)$$

Let now $\kappa = 1/2$.
For the integral $I_3(1/2)$, the conformal weight of the integrand at x_7 and x_8 equals the space-time dimension $4(1-\kappa\epsilon) = D$.
Using inversion $x_i^\mu \to x_i^\mu / x_i^2$ we obtain

$$I_3(1/2) \;\; = \frac{\mathrm{e}^{4\gamma\epsilon}}{\pi^{2D}} \int \frac{d^D x_6 d^D x_7 d^D x_8 d^D x_9}{(x_{16}^2 x_{19}^2 x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_6^2 x_9^2 x_{89}^2)^{1-\epsilon/2}} =$$



The two-loop subintegral over x_7 and x_8 equals

$$\frac{\mathrm{e}^{2\gamma\epsilon}}{\pi^D} \int \frac{d^D x_7 d^D x_8}{(x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2)^{1-\epsilon/2}} = \frac{6\zeta_3 + (9\zeta_4 + 12\zeta_3)\epsilon + O(\epsilon^2)}{(x_{69}^2)^{1-\epsilon/2}}$$

The two-loop subintegral over x_7 and x_8 equals

$$\frac{\mathrm{e}^{2\gamma\epsilon}}{\pi^D} \int \frac{d^D x_7 d^D x_8}{(x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2)^{1-\epsilon/2}} = \frac{6\zeta_3 + (9\zeta_4 + 12\zeta_3)\epsilon + O(\epsilon^2)}{(x_{69}^2)^{1-\epsilon/2}}$$

Taking similarly the remaining integral over x_6 and x_9 $\rightarrow$

$$I_3(1/2) = \left[6\zeta_3 + (9\zeta_4 + 12\zeta_3)\epsilon + O(\epsilon^2)\right]^2 .$$

The two-loop subintegral over x_7 and x_8 equals

$$\frac{\mathrm{e}^{2\gamma\epsilon}}{\pi^D}\int \frac{d^D x_7 d^D x_8}{(x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2)^{1-\epsilon/2}} = \frac{6\zeta_3 + (9\zeta_4 + 12\zeta_3)\epsilon + O(\epsilon^2)}{(x_{69}^2)^{1-\epsilon/2}}$$

Taking similarly the remaining integral over x_6 and x_9 $\to$

$$I_3(1/2) = \left[6\zeta_3 + (9\zeta_4 + 12\zeta_3)\epsilon + O(\epsilon^2)\right]^2 .$$

$I_4(1/2) =$?

The two-loop subintegral over x_7 and x_8 equals

$$\frac{e^{2\gamma\epsilon}}{\pi^D}\int \frac{d^D x_7 d^D x_8}{(x_{67}^2 x_{68}^2 x_{78}^2 x_{79}^2 x_{89}^2)^{1-\epsilon/2}} = \frac{6\zeta_3 + (9\zeta_4 + 12\zeta_3)\epsilon + O(\epsilon^2)}{(x_{69}^2)^{1-\epsilon/2}}$$

Taking similarly the remaining integral over x_6 and x_9 $\to$

$$I_3(1/2) = \left[6\zeta_3 + (9\zeta_4 + 12\zeta_3)\epsilon + O(\epsilon^2)\right]^2 .$$

$I_4(1/2) =$?

$$I_4(1/2) = I_3(1/2) = 36{\zeta_3}^2 + \epsilon\left(108\zeta_3\zeta_4 + 144{\zeta_3}^2\right) + O(\epsilon^2) .$$

Method of gluing

[K.G. Chetyrkin & F.V. Tkachov'81, P.A. Baikov & K.G. Chetyrkin,10]

Method of gluing

[K.G. Chetyrkin & F.V. Tkachov'81, P.A. Baikov & K.G. Chetyrkin,10]

Gluing by vertex and gluing by line.

Method of gluing

[K.G. Chetyrkin & F.V. Tkachov'81, P.A. Baikov & K.G. Chetyrkin,10]

Gluing by vertex and gluing by line.

Let $F_\Gamma(q;d)$ be an l-loop dimensionally regularized scalar propagator massless Feynman integral corresponding to a graph Γ,

$$F_\Gamma(q;d) = C_\Gamma(\epsilon)(q^2)^{\omega/2-l\epsilon} \ ,$$

where $\omega = 4l - 2L$ is the degree of divergence and $C_\Gamma(\epsilon)$ is a meromorphic function which is finite at $\epsilon = 0$ if the integral is convergent.

Gluing

Let us denote by $\hat{\Gamma}$ the graph obtained from Γ by adding a new line which connects the two external vertices.

Gluing

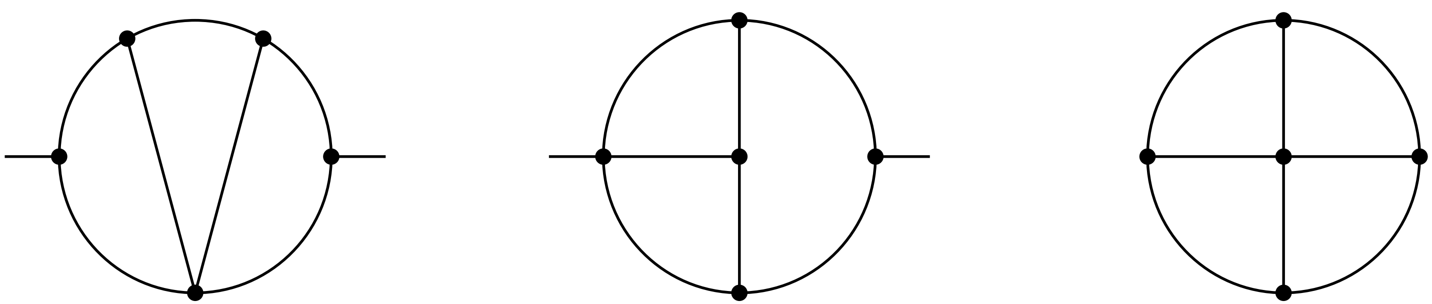
Let us denote by $\hat{\Gamma}$ the graph obtained from Γ by adding a new line which connects the two external vertices.

Gluing by line. Let us suppose that two UV- and IR-convergent graphs, Γ_1 and Γ_2, have degrees of divergence $\omega_1 = \omega_2 = -2$ and that $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are the same. Then $C_{\Gamma_1}(0) = C_{\Gamma_2}(0)$.

Gluing

Let us denote by $\hat{\Gamma}$ the graph obtained from Γ by adding a new line which connects the two external vertices.

Gluing by line. Let us suppose that two UV- and IR-convergent graphs, Γ_1 and Γ_2, have degrees of divergence $\omega_1 = \omega_2 = -2$ and that $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are the same. Then $C_{\Gamma_1}(0) = C_{\Gamma_2}(0)$.



$$C_{\Gamma_1}(0) = C_{\Gamma_2}(0) = 20\zeta_5$$

Let us prove (without calculation) that $I_3(0) = I_4(0)$.

$$I_i(0) = \frac{c_i(\epsilon)}{(x_1^2)^{1+4\epsilon}}, \qquad i = 3, 4\,.$$

Let us prove (without calculation) that $I_3(0) = I_4(0)$.

$$I_i(0) = \frac{c_i(\epsilon)}{(x_1^2)^{1+4\epsilon}}, \qquad i = 3, 4\,.$$

Add to each of these diagrams a new line with the usual propagator $1/x_1^2$, i.e. multiply $I_i(0)$ by $1/x_1^2$ (gluing by a line).

Let us prove (without calculation) that $I_3(0) = I_4(0)$.

$$I_i(0) = \frac{c_i(\epsilon)}{(x_1^2)^{1+4\epsilon}}, \qquad i = 3, 4\,.$$

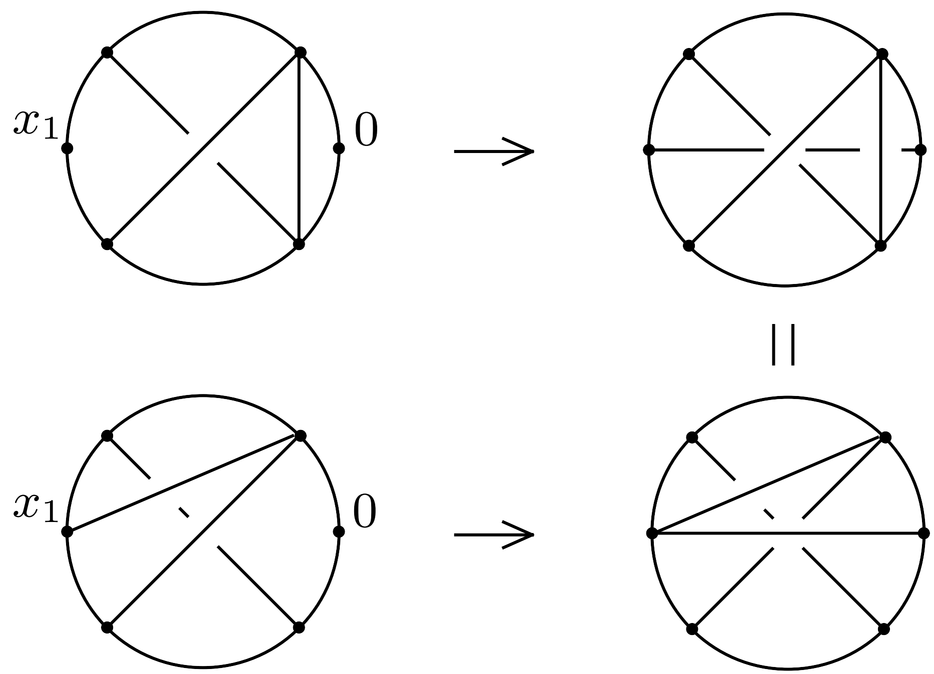
Add to each of these diagrams a new line with the usual propagator $1/x_1^2$, i.e. multiply $I_i(0)$ by $1/x_1^2$ (gluing by a line).

Take the Fourier transform

$$\mathcal{F}\left[\frac{I_i(0)}{x_1^2}\right] = \mathcal{F}\left[\frac{c_i(\epsilon)}{(x_1^2)^{2+4\epsilon}}\right] = c_i(\epsilon)\frac{2^{-10\epsilon}\Gamma(-5\epsilon)}{\Gamma(2+4\epsilon)}(p^2)^{5\epsilon}$$

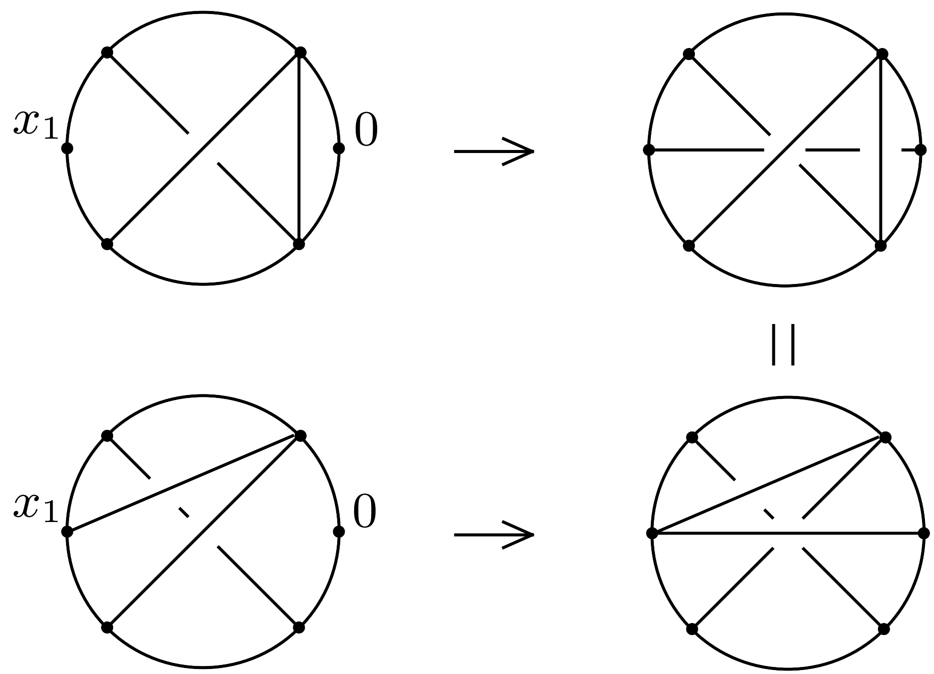
The pole part in ϵ is independent of p,
$\mathcal{F}\left[\frac{I_i(0)}{x_1^2}\right] = -\frac{c_i(0)}{5\epsilon} + O(\epsilon^0)$, i.e. independent of the way p flows through the diagram.

The pole part in ϵ is independent of p,
$\mathcal{F}\left[\frac{I_i(0)}{x_1^2}\right] = -\frac{c_i(0)}{5\epsilon} + O(\epsilon^0)$, i.e. independent of the way p flows through the diagram. This is the UV pole part of the vacuum graph (i.e. with $p=0$) obtained by gluing either from I_3 or I_4.



The pole part in ϵ is independent of p,

$\mathcal{F}\left[\frac{I_i(0)}{x_1^2}\right] = -\frac{c_i(0)}{5\epsilon} + O(\epsilon^0)$, i.e. independent of the way p flows through the diagram. This is the UV pole part of the vacuum graph (i.e. with $p = 0$) obtained by gluing either from I_3 or I_4.



So, $c_3(0) = c_4(0)$ and, therefore, $I_4(0) = I_3(0)$ at $\epsilon = 0$.

Consider I_3 and I_4 with all the indices equal to $1 - \epsilon/2 - \lambda/10$.
Formally, these are $I_3(\kappa)$ and $I_4(\kappa)$ at $\kappa = 1/2 - \lambda/(10\epsilon)$.

$$I_i(1/2 + \lambda/(10\epsilon)) = \frac{c_i(\epsilon, \lambda)}{(x_1^2)^{1-\epsilon/2-9\lambda/10}}, \qquad i = 3, 4\,.$$

Consider I_3 and I_4 with all the indices equal to $1-\epsilon/2-\lambda/10$.
Formally, these are $I_3(\kappa)$ and $I_4(\kappa)$ at $\kappa = 1/2-\lambda/(10\epsilon)$.

$$I_i(1/2+\lambda/(10\epsilon)) = \frac{c_i(\epsilon,\lambda)}{(x_1^2)^{1-\epsilon/2-9\lambda/10}}, \qquad i=3,4\,.$$

Add to these diagrams a new line with the propagator with the same exponent, i.e. multiply $I_i(\kappa)$ by $1/(x_1^2)^{1-\epsilon/2-\lambda/10}$ (gluing by a line)

Consider I_3 and I_4 with all the indices equal to $1-\epsilon/2-\lambda/10$.
Formally, these are $I_3(\kappa)$ and $I_4(\kappa)$ at $\kappa = 1/2-\lambda/(10\epsilon)$.

$$I_i(1/2+\lambda/(10\epsilon)) = \frac{c_i(\epsilon,\lambda)}{(x_1^2)^{1-\epsilon/2-9\lambda/10}}, \qquad i=3,4\,.$$

Add to these diagrams a new line with the propagator with the same exponent, i.e. multiply $I_i(\kappa)$ by $1/(x_1^2)^{1-\epsilon/2-\lambda/10}$ (gluing by a line)
Take the Fourier transform

$$\mathcal{F}\left[\frac{I_i(1/2+\lambda/(10\epsilon))}{(x_1^2)^{1-\epsilon/2-\lambda/10}}\right] = \mathcal{F}\left[\frac{c_i(\epsilon,\lambda)}{(x_1^2)^{2-\epsilon-\lambda}}\right] = c_i(\epsilon,\lambda)\frac{2^{2\lambda}\Gamma(\lambda)}{\Gamma(2-\epsilon-\lambda)}(p^2)$$

The pole part in λ is independent of p,

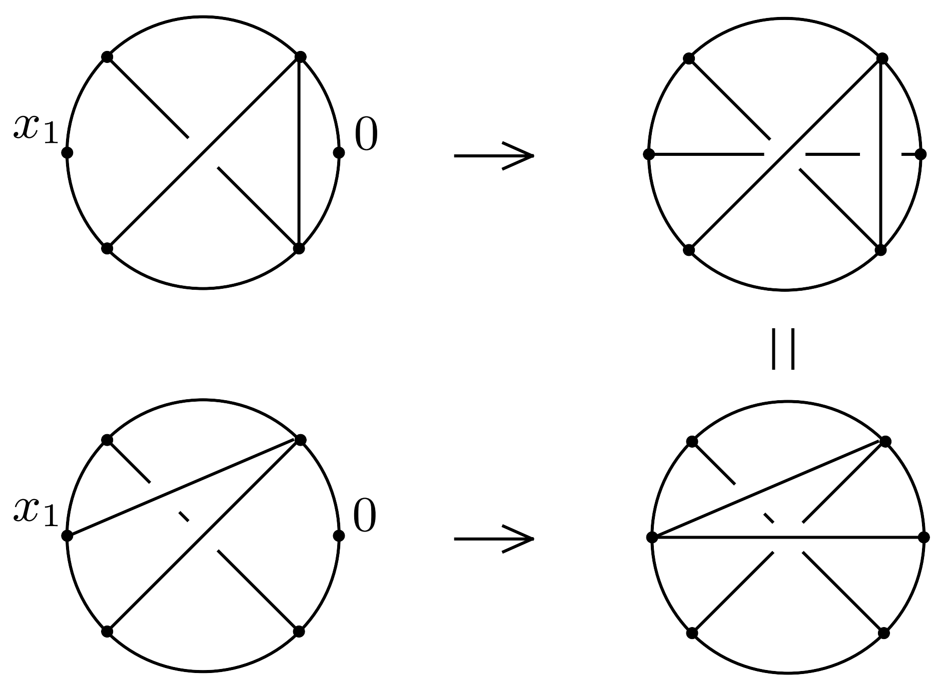
$$\mathcal{F}\left[\frac{I_i(1/2+\lambda/(10\epsilon))}{(x_1^2)^{1-\epsilon/2-\lambda/10}}\right] = \lambda^{-1}\frac{c_i(\epsilon,0)}{\Gamma(2-\epsilon)} + O(\lambda^0)\,,$$

i.e. independent of the way p flows through the diagram.

The pole part in λ is independent of p,

$$\mathcal{F}\left[\frac{I_i(1/2+\lambda/(10\epsilon))}{(x_1^2)^{1-\epsilon/2-\lambda/10}}\right] = \lambda^{-1}\frac{c_i(\epsilon,0)}{\Gamma(2-\epsilon)} + O(\lambda^0)\,,$$

i.e. independent of the way p flows through the diagram. This is the UV pole part of the vacuum graph (i.e. with $p=0$) which is the graph obtained by gluing either from I_3 and I_4.



So, $c_3(\epsilon, 0) = c_4(\epsilon, 0)$ and, therefore, $I_4(1/2) = I_3(1/2)$

So, $c_3(\epsilon, 0) = c_4(\epsilon, 0)$ and, therefore, $I_4(1/2) = I_3(1/2)$

$$I_3(\kappa) = 36{\zeta_3}^2 + \epsilon\Big(108\,\zeta_3\zeta_4 + 288\,\kappa{\zeta_3}^2 + (1-2\kappa)378\,\zeta_7\Big) + O(\epsilon^2)$$

$$I_4(\kappa) = 36{\zeta_3}^2 + \epsilon\Big(108\,\zeta_3\zeta_4 + (180-72\kappa){\zeta_3}^2 - \frac{189}{2}(1-2\kappa)\,\zeta_7\Big) + O($$

So, $c_3(\epsilon,0) = c_4(\epsilon,0)$ and, therefore, $I_4(1/2) = I_3(1/2)$

$$I_3(\kappa) = 36{\zeta_3}^2 + \epsilon\Big(108\,\zeta_3\zeta_4 + 288\,\kappa{\zeta_3}^2 + (1-2\kappa)378\,\zeta_7\Big) + O(\epsilon^2)$$

$$I_4(\kappa) = 36{\zeta_3}^2 + \epsilon\Big(108\,\zeta_3\zeta_4 + (180-72\kappa){\zeta_3}^2 - \tfrac{189}{2}(1-2\kappa)\,\zeta_7\Big) + O($$

We obtain

$$\begin{aligned} b_3 &= b_4 = 36{\zeta_3}^2\,, \\ c_3 &= 108\,\zeta_3\zeta_4 + 378\,\zeta_7\,, \\ c_4 &= 108\,\zeta_3\zeta_4 + 180{\zeta_3}^2 - \frac{189}{2}\,\zeta_7\,. \end{aligned}$$

This gives a system of linear relations for b_1, b_2 and c_1, c_2, with the solution

$$a_1 = 5\,\zeta_5\,, \quad b_1 = \frac{5}{378}\pi^6 - 13\,\zeta_3{}^2 + 35\,\zeta_5\,,$$

$$a_2 = -20\,\zeta_5\,, \quad b_2 = -\frac{10}{189}\pi^6 - 8\,\zeta_3{}^2 - 40\,\zeta_5\,,$$

$$c_1 = -\frac{13}{30}\pi^4\zeta_3 - 91\zeta_3{}^2 + 195\zeta_5 - \frac{5}{3}\pi^2\zeta_5 + \frac{345}{4}\zeta_7 + \frac{5}{54}\pi^6\,,$$

$$c_2 = -\frac{4}{15}\pi^4\zeta_3 - 16\zeta_3{}^2 - 80\zeta_5 + \frac{20}{3}\pi^2\zeta_5 - 520\zeta_7 - \frac{20}{189}\pi^6$$

Our results:

$$I_1 = \frac{5\,\zeta_5}{\epsilon} + \frac{5}{378}\pi^6 - 13\,{\zeta_3}^2 + 35\,\zeta_5$$

$$+\left(-\frac{13}{30}\pi^4\zeta_3 - 91{\zeta_3}^2 + 195\zeta_5 - \frac{5}{3}\pi^2\zeta_5 + \frac{345}{4}\zeta_7 + \frac{5}{54}\pi^6\right)\epsilon + \ldots$$

$$I_2 = -\frac{20\,\zeta_5}{\epsilon} - \frac{10}{189}\pi^6 - 8\,{\zeta_3}^2 - 40\,\zeta_5$$

$$+\left(-\frac{4}{15}\pi^4\zeta_3 - 16{\zeta_3}^2 - 80\zeta_5 + \frac{20}{3}\pi^2\zeta_5 - 520\zeta_7 - \frac{20}{189}\pi^6\right)\epsilon + \ldots$$

To check numerically our analytic results for these two non-planar integrals we used the code `FIESTA`

[A.V. Smirnov & M.N. Tentyukov'08]

which gave the precision of six digits.

To check numerically our analytic results for these two non-planar integrals we used the code `FIESTA`

[A.V. Smirnov & M.N. Tentyukov'08]

which gave the precision of six digits.

Modern sector decompositions

[T. Binoth & G. Heinrich'00; C. Bogner & S. Weinzierl'07; A.V. Smirnov & M.N. Tentyukov'08;

A.V. Smirnov, V.A. Smirnov, & M.N. Tentyukov'10; J. Carter & G. Heinrich'10]

$$\gamma_{\mathcal{K}}(a) = 3\,a - 3\,a^2 + \frac{21}{4}\,a^3 - \left(\frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5\right)a^4$$

$$+\left(\frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}{\zeta_3}^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7\right)a^5 + O(a^6) + O(1/N_c^2)$$

$$\gamma_{\mathcal{K}}(a) = 3\,a - 3\,a^2 + \frac{21}{4}\,a^3 - \left(\frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5\right)a^4$$

$$+\left(\frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}{\zeta_3}^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7\right)a^5 + O(a^6) + O(1/N_c^2)$$

$$\frac{81}{16}{\zeta_3}^2 = \left(\frac{9}{4}\zeta_3\right)^2$$

$$\gamma_K(a) = 3\,a - 3\,a^2 + \frac{21}{4}\,a^3 - \left(\frac{39}{4} - \frac{9}{4}\zeta_3 + \frac{45}{8}\zeta_5\right)a^4$$
$$+\left(\frac{237}{16} + \frac{27}{4}\zeta_3 - \frac{81}{16}{\zeta_3}^2 - \frac{135}{16}\zeta_5 + \frac{945}{32}\zeta_7\right)a^5 + O(a^6) + O(1/N_c^2)$$

$$\frac{81}{16}{\zeta_3}^2 = \left(\frac{9}{4}\zeta_3\right)^2$$

$$\left\{\frac{45}{8}\zeta_5, \quad \frac{945}{32}\zeta_7\right\} \leftrightarrow \left\{\frac{2^{-k}\pi^{2k-4}}{\zeta(2(k-2))}\right\}\Bigg|_{k=4,5,\ldots} = \left\{\frac{45}{8}, \frac{945}{32}, \frac{4725}{32}, \ldots\right\}$$

Conclusion

Why have we succeeded?

Why have we succeeded?

- We have used the expression for the correlation function G_4 constructed in

  [B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev'11]

Conclusion

Why have we succeeded?

- We have used the expression for the correlation function G_4 constructed in

  [B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev'11]

- We have reduced the problem to four-loop integrals (using IRR)

Conclusion

Why have we succeeded?

- We have used the expression for the correlation function G_4 constructed in

  [B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev'11]

- We have reduced the problem to four-loop integrals (using IRR)
- We were able to reduce all the integrals obtained to master integrals (using `FIRE`)

Conclusion

Why have we succeeded?

- We have used the expression for the correlation function G_4 constructed in

  [B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev'11]
- We have reduced the problem to four-loop integrals (using IRR)
- We were able to reduce all the integrals obtained to master integrals (using `FIRE`)
- We were able to evaluate the two unknown master integrals (using gluing)

Conclusion

Why have we succeeded?

- We have used the expression for the correlation function G_4 constructed in

  [B. Eden, P. Heslop, G.P. Korchemsky, E. Sokatchev'11]
- We have reduced the problem to four-loop integrals (using IRR)
- We were able to reduce all the integrals obtained to master integrals (using `FIRE`)
- We were able to evaluate the two unknown master integrals (using gluing)

An analytic six-loop calculation?