DRA Method

Review and status

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Calculations for Modern and Future Colliders

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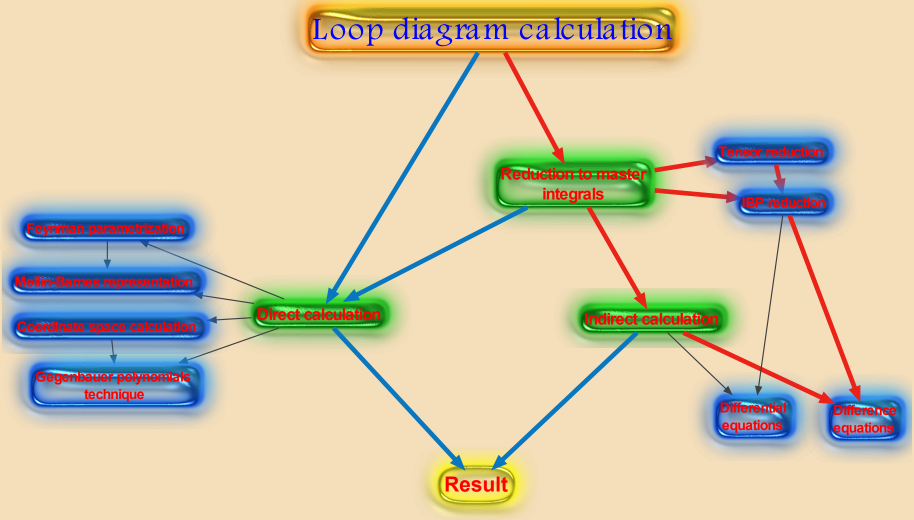
Outline

DRA Method: What is it about?



DRA Method

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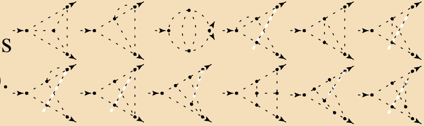

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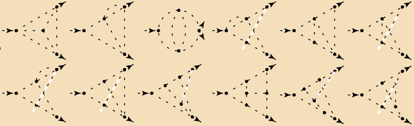
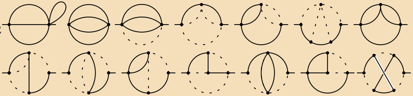


- 3-loop onshell mass operator type integrals (Lee and Smirnov 2011).



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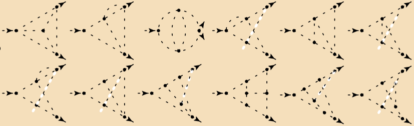

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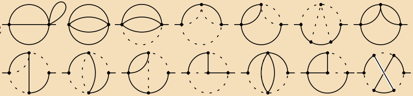
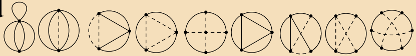


- 4-loop QED-type tadpoles(Lee and Terekhov 2011).



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- 3-loop onshell massless vertices (Lee, Smirnov and Smirnov 2010).
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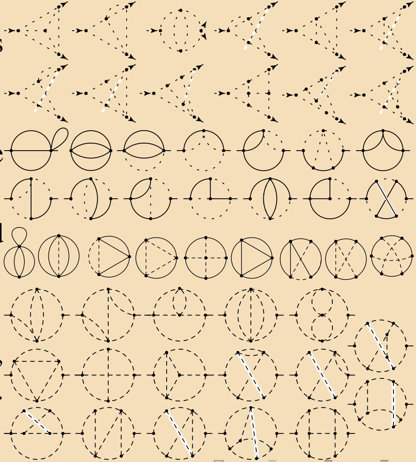

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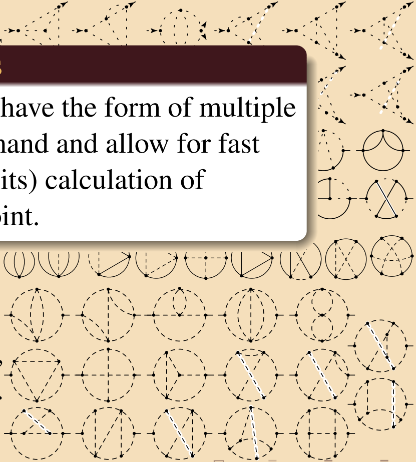
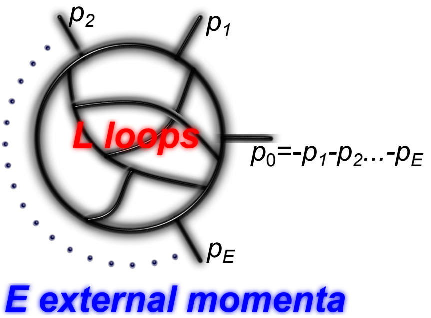


Form of the DRA results

Results are exact in $\mathscr{D}$ and have the form of multiple sums with factorized summand and allow for fast high-precision (e.g. 10^3digits) calculation of ϵ-expansion around any point.

Loop Integral

L loop, $E+1$ legs



Loop integral

$$J(\mathbf{n}) = \int \frac{d^{\mathscr{D}} l_1}{\pi^{\mathscr{D}/2}} \ldots \frac{d^{\mathscr{D}} l_L}{\pi^{\mathscr{D}/2}} j(\mathbf{n})$$

$$= \int \frac{d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L}{\pi^{\frac{L\mathscr{D}}{2}} D_1^{n_1} \ldots D_N^{n_N}}$$

$D_1, \ldots, D_M$ — denominators of the diagram,
$D_{M+1}, \ldots, D_N$ conveniently chosen numerators.

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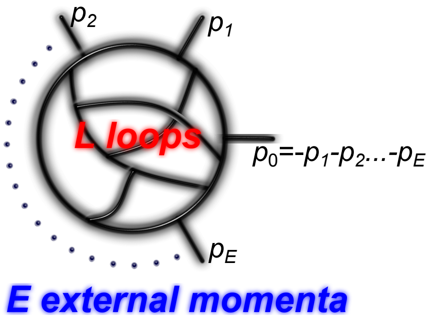

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$$J(\mathbf{n}) = \int \frac{d^{\mathcal{D}} l_1}{\pi^{\mathcal{D}/2}} \dots \frac{d^{\mathcal{D}} l_L}{\pi^{\mathcal{D}/2}} j(\mathbf{n}) = \int \frac{d^{\mathcal{D}} l_1 \dots d^{\mathcal{D}} l_L}{\pi^{\frac{L\mathcal{D}}{2}} D_1^{n_1} \dots D_N^{n_N}}$$

$D_1, \dots, D_M$ — denominators of the diagram,
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Prerequisites

All D_k linearly depend on $s_{ij} = l_i \cdot q_j$, any s_{ij} can be expressed via D_k. $\Longrightarrow N = \# s_{ij} = L(L+1)/2 + LE$

Notation

$$q_{1,..L} = l_{1,..L}$$

$$q_{L+1,..L+E} = p_{1,..E}$$

IBP reduction

IBP identities (Tkachov 1981, Chetyrkin and Tkachov 1981)

$$\int d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L \, \frac{\partial}{\partial l_i} \cdot q_j j(\mathbf{n}) = 0$$

Explicitely making a differentiation, we obtain identities betwen the integrals with shifted indices.

Reduction

Using IBP Identities, it is possible to reduce all integrals to a finite set of them, called masters. For a given subset of $\{D_1, \ldots, D_M\}$ there can be

- No masters $\Longrightarrow$ The corresponding topology is reducible
- One master $\Longrightarrow$ The corresponding topology is said to have simple master
- Several masters $\Longrightarrow$ The corresponding topology is said to have multimaster — a column of masters.

Operator representation

Operators $A_1,\ldots,A_N,B_1,\ldots,B_N$

In order to write identities between integrals with different indices, it is convenient to introduce the operators:

$$(A_\alpha f)(n_1,\ldots,n_N) = n_\alpha f(n_1,\ldots,n_\alpha+1,\ldots,n_N),$$
$$(B_\alpha f)(n_1,\ldots,n_N) = f(n_1,\ldots,n_\alpha-1,\ldots,n_N).$$

Commutator

$[A_\alpha, B_\beta] = \delta_{\alpha\beta}$

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Compact form of identities

$$n_1 J(n_1+1,n_2) = J(n_1,n_2-1) + J(n_1,n_2) \implies A_1 J = B_2 J + J$$

Dimensional recurrence relation

Dimensional recurrence relation (Tarasov 1996)

Original Tarasov's formula is derived from the parametric representation. For no numerators it has a nicely-looking form

$$J^{(\mathscr{D}-2)}(\mathbf{n}) = \mu^L \sum_{\text{trees}} A_{i_1} \ldots A_{i_L} J^{(\mathscr{D})}(\mathbf{n}),$$

$i_1, \ldots, i_L$ enumerate tree chords; $\mu = |g| = \pm 1$.

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Baikov's approach to reduction (Baikov 1997)

Change of variables $d^{\mathscr{D}} l_1 \dots d^{\mathscr{D}} l_L \longrightarrow ds_{11} ds_{12} \dots ds_{L,L+E}$.
Jacobian is expressed via Gram determinant

$$V(l_1, \dots l_L, p_1, \dots, p_E) = \det\{q_i \cdot q_j\} = P(D_1, \dots D_N)$$

$P(D_1, \dots D_N)$ is polynomial of $L+E$-th order.

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$$\int \frac{d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L}{\pi^{L\mathscr{D}/2} D_1^{n_1} \ldots D_N^{n_N}} = \frac{\mu^L \pi^{-LE/2-L(L-1)/4}}{\Gamma\left[(\mathscr{D}-E-L+1)/2, \ldots, (\mathscr{D}-E)/2\right]} \times \int \left(\prod_{i=1}^{L} \prod_{j=i}^{L+E} ds_{ij}\right) \frac{\left[P(D_1, \ldots D_N)\right]^{(\mathscr{D}-E-L-1)/2}}{\left[V(p_1, \ldots, p_E)\right]^{(\mathscr{D}-E-1)/2} D_1^{n_1} \ldots D_N^{n_N}}$$

Dimensional recurrence relation

DRR from Baikov's formula

Lowering&Raising DRR from Baikov's formula (Lee 2010b)

$$J^{(\mathscr{D}+2)}(\mathbf{n}) = \frac{(2\mu)^L [V(p_1,\ldots,p_E)]^{-1}}{\pi^{L\mathscr{D}/2}(\mathscr{D}-E-L+1)_L} \int d^{\mathscr{D}}l_1 \ldots d^{\mathscr{D}}l_L P(D_1,\ldots,D_N) j(\mathbf{n}) . \qquad \text{(LDRR)}$$

$$J^{(\mathscr{D}-2)}(\mathbf{n}) = \frac{(-\mu)^L}{\pi^{L\mathscr{D}/2}} \int d^{\mathscr{D}}l_1 \ldots d^{\mathscr{D}}l_L \begin{vmatrix} \partial_{s_{11}} & \cdots & \frac{1}{2}\partial_{s_{1L}} \\ \vdots & \ddots & \vdots \\ \frac{1}{2}\partial_{s_{1L}} & \cdots & \partial_{s_{LL}} \end{vmatrix} j(\mathbf{n}) . \qquad \text{(RDRR)}$$

Dimensional recurrence relation

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$$J^{(\mathscr{D}+2)}(\mathbf{n}) = \frac{(2\mu)^L [V(p_1,\ldots,p_E)]^{-1}}{(\mathscr{D}-E-L+1)_L} P(B_1,\ldots,B_N) J^{(\mathscr{D})}(\mathbf{n}) . \qquad \text{(LDRR)}$$

$$J^{(\mathscr{D}-2)}(\mathbf{n}) = \mu^L \det\left[\sum_k \frac{\partial D_k}{\partial s_{ij}} A_k \bigg|_{i,j=1,\ldots L}\right] J^{(\mathscr{D})}(\mathbf{n}) . \qquad \text{(RDRR)}$$

Automatization

These formulae have no reference to the graph and therefore can be easily implemented.

Dimensional recurrence relation

Example:Obtaining DRRs is very easy

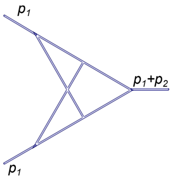
Integral

$$J(\mathbf{n}) = \int \frac{d^{\mathscr{D}} l_1 d^{\mathscr{D}} l_2 \, D_7^{n_7}}{\pi^{\mathscr{D}} D_1^{n_1} \ldots D_6^{n_6}}$$

$$D_1 = l_1^2, D_2 = l_2^2, D_3 = (p_1 - l_1)^2, D_4 = (p_2 - l_2)_1^2,$$

$$D_5 = (p_1 - l_1 + l_2)^2, D_6 = (p_2 - l_2 + l_1)^2, D_7 = (l_1 - l_2)^2$$

2 loop vertex



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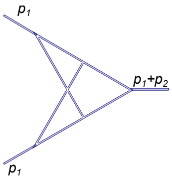

D_k are linear functions of s_{ij}

$$D_1 = s_{11}, D_2 = s_{22}, D_3 = s_{11} - 2s_{13} + p_1^2,$$

$$D_4 = s_{22} - 2s_{24} + p_2^2, D_5 = s_{11} + s_{22} - 2s_{12} - 2s_{13} + 2s_{23} + p_1^2,$$

$$D_6 = D_5 = s_{11} + s_{22} - 2s_{12} - 2s_{24} + 2s_{14} + p_2^2, D_7 = s_{11} + s_{22} - 2s_{12}$$

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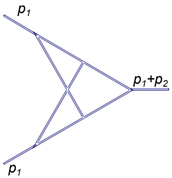

Expressing in terms of Ds

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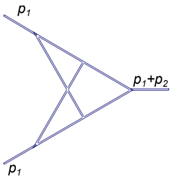

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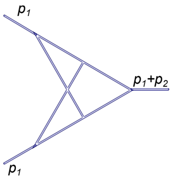

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$$= \partial_{D_1}\partial_{D_2} + \partial_{D_2}\partial_{D_3} + \partial_{D_1}\partial_{D_4} + \partial_{D_3}\partial_{D_4} + \partial_{D_1}\partial_{D_5} + \partial_{D_2}\partial_{D_5} + \partial_{D_3}\partial_{D_5} + \partial_{D_4}\partial_{D_5}$$
$$+ \partial_{D_1}\partial_{D_6} + \partial_{D_2}\partial_{D_6} + \partial_{D_3}\partial_{D_6} + \partial_{D_4}\partial_{D_6} + \partial_{D_1}\partial_{D_7} + \partial_{D_2}\partial_{D_7} + \partial_{D_3}\partial_{D_7} + \partial_{D_4}\partial_{D_7}$$

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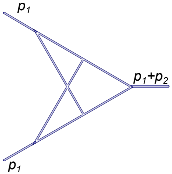

Expressing in terms of As

Replace $\partial_{D_i} \to -A_i$, add $\mu^L = \pm 1$ factor

$$= \partial_{D_1}\partial_{D_2} + \partial_{D_2}\partial_{D_3} + \partial_{D_1}\partial_{D_4} + \partial_{D_3}\partial_{D_4} + \partial_{D_1}\partial_{D_5} + \partial_{D_2}\partial_{D_5} + \partial_{D_3}\partial_{D_5} + \partial_{D_4}\partial_{D_5}$$
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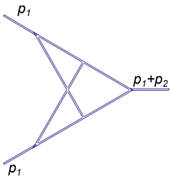

Expressing in terms of As

Replace $\partial_{D_i} \to -A_i$, add $\mu^L = \pm 1$ factor and voila:

$$J^{(\mathscr{D}-2)}(\mathbf{n}) = \left(A_1A_2 + A_2A_3 + A_1A_4 + A_3A_4 + A_1A_5 + A_2A_5 + A_3A_5 + A_4A_5 + A_1A_6 + A_2A_6 + A_3A_6 + A_4A_6 + A_1A_7 + A_2A_7 + A_3A_7 + A_4A_7\right) J^{(\mathscr{D})}(\mathbf{n})$$

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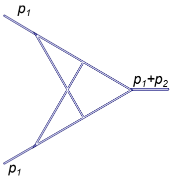

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$$+ A_1A_6 + A_2A_6 + A_3A_6 + A_4A_6 + A_1A_7 + A_2A_7 + A_3A_7 + A_4A_7\big) J^{(\mathscr{D})}(\mathbf{n})$$

If we take master on the l.h.s. and reduce r.h.s., we obtain an equation for the masters.

Solution of DRR

Inhomogeneous part

General form of DRR for master

$$J(\nu+1) = C(\nu)J(\nu) + R(\nu),$$

J can be either simple, or multi- master. $R(\nu)$ contains simpler integrals, which are assumed to be known.

Notation

$\nu = \mathscr{D}/2$

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$$J(\nu) = R(\nu-1) + C(\nu-1)J(\nu-1)$$

Solution of DRR

Inhomogeneous part

General form of DRR for master

$$J(\nu+1) = C(\nu)J(\nu) + R(\nu),$$

J can be either simple, or multi- master. $R(\nu)$ contains simpler integrals, which are assumed to be known.

Notation

$\nu = \mathscr{D}/2$

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Notation

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Solution of DRR

$$J(\nu) = R(\nu-1) + C(\nu-1)R(\nu-2) + C(\nu-1)C(\nu-2)R(\nu-3)$$
$$+ C(\nu-1)C(\nu-2)C(\nu-3)J(\nu-3)$$

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$$+ C(\nu-1)C(\nu-2)C(\nu-3)R(\nu-4)$$
$$+ C(\nu-1)C(\nu-2)C(\nu-3)C(\nu-4)J(\nu-4)$$

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$$+ C(\nu-1)C(\nu-2)C(\nu-3)C(\nu-4)R(\nu-5)$$
$$+ C(\nu-1)C(\nu-2)C(\nu-3)C(\nu-4)C(\nu-5)J(\nu-5)$$

Solution of DRR

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General form of DRR for master

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Notation

$\nu = \mathscr{D}/2$

Solution of DRR

$$J^{\mathrm{ih}}(\nu) = \sum_{k=1}^{\infty}\prod_{l=1}^{k-1} C(\nu-l)\,R(\nu-k) \text{ or } J^{\mathrm{ih}}(\nu) = -\sum_{k=0}^{\infty}\prod_{l=0}^{k} C^{-1}(\nu+l)\,R(\nu+k)$$

Solution of DRR

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We should add a general solution J^0 of the homogeneous equation

$$J(\nu) = J^{\mathrm{ih}}(\nu) + J^0(\nu)$$

Solution of DRR

Homogeneous part

Homogeneous equation

$$J^0(\nu+1) = C(\nu)\, J^0(\nu)$$

Summing factor

Summing factor: some solution of

$$S(\nu) = S(\nu+1)\, C(\nu)$$

Solution of DRR

Homogeneous part

Homogeneous equation

$$J^0(\nu+1) = C(\nu) J^0(\nu)$$

Simple master

$C(\nu)$ is a rational function which we represent as

$$C(\nu) = c\frac{\prod_{i=1}^{A}(\nu-\alpha_i)}{\prod_{j=1}^{B}(\nu-\beta_j)}$$

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E.g., we can take

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General solution vs specific solution

$$J^0(\nu) = S^{-1}(\nu)\,\omega(z),$$

$\omega(z) = \omega(\exp[2i\pi\nu])$ is an arbitrary periodic function of ν. Obviously, we have to use some information not contained in DRR to fix $\omega(z)$.

Mittag-Leffler's&Liouville's theorems

Informal formulation of Mittag-Leffler's&Liouville's theorems

If we know about a function $f(z)$ **on the complex plane** z

1. that it has only poles, no branching sungularities
2. the position of the poles and singular terms of expansion of $f(z)$ in each (including the possible pole at $z = \infty$)
3. One zeroth order term of function expansion in any point

then we know $f(z)$.

Complex vs Real analysis

" f is analytic function falling off at infinity" $\Longrightarrow \begin{cases} f = 0 & \text{(complex)} \\ f = e^{-x^2}, \frac{1}{1+x^2}, \ldots & \text{(real)} \end{cases}$

Key idea of DRA method

Complex $\mathscr{D}$ to fix $\omega(z)$

General solution reads

$$J(\nu) = J^{\mathrm{ih}}(\nu) + S^{-1}(\nu)\,\omega(z)$$

Let us express $\omega(z)$ as

$$\omega(z) = S(\nu)\left[J(\nu) - J^{\mathrm{ih}}(\nu)\right]$$

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Suppose that we know all singularities of $S(\nu)J(\nu)$ on some **basic stripe** $\{\nu, \operatorname{Re}\nu \in (\nu_0, \nu_0+1]\}$ and its behaviour at $\operatorname{Im}\nu \to \pm\infty$. Then we can use Mittag-Leffler's&Liouville's theorems to fix $\omega(z)$.

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Complex $\mathscr{D}$ to fix $\omega(z)$

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Important observation

$S(\nu)$ allows for the mutiplication by periodic function. We can use this freedom to get rid of some poles in $S(\nu)J(\nu)$ and/or improve its behavior at $\operatorname{Im}\nu \to \pm\infty$. The same concerns the choice of the basic stripe.

Analytical properties from parametric representation

Parametric representation

If I is the number of internal lines of the integral, parametric representation reads

$$J(\nu) = \Gamma(I - L\nu) \int dx_1 \dots dx_I \delta\left(1 - \sum x_i\right) \frac{[Q(x)]^{\nu L - I}}{[P(x)]^{\nu(L+1) - I}}$$

$P(x) > 0$ and $Q(x) > 0$ are determined in terms of trees and 2-trees of the graph.

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$P(x) > 0$ and $Q(x) > 0$ are determined in terms of trees and 2-trees of the graph.

Analytical properties

- The integral is a meromorphic function (Bernshtein and Gelfand 1969)
- When $\operatorname{Im} \nu \to \pm\infty$ the integral can be estimated as

$$J(\nu) \lesssim \text{const} \times |\Gamma(I - L\nu)| \sim \text{const} \times e^{-\pi L |\operatorname{Im} \mathscr{D}|/4}$$

Fixing ω

In real life

- Use FIESTA (Smirnov et al. 2009) to determine the position and order of singularities of $J(\nu)$ on the basic stripe. (Very rarely it is possible to manually analyze parametric representation).
- Try to multiply $S(\nu)$ by some periodic factors of $\sin(\pi(\nu-\nu_0))$ to make $S(\nu)J(\nu)$ regular on the basic stripe.
- Don't go too far in that because $\sin(\pi(\nu-\nu_0))$ makes $\nu\to\pm i\infty$ behaviour of $S(\nu)J(\nu)$ worse. If $S(\nu)J(\nu)$ vanishes at some points, instead, divide by $\sin(\pi(\nu-\nu_0))$ to improve behaviour at infinity.
- If it was not possible to cancel all singularities of $J(\nu)$ on the basic stripe, use Mellin-Barnes (or other techniques) to fix the singular coefficients of $S(\nu)J(\nu)$.
- Finally, use Mittag-Leffler's&Liouville's theorems to fix ω.

Numerical issues

Form of the DRA results

The DRA results are expressed in terms of the multiple sums

$$\sum_{\infty > k_1 \geqslant \ldots \geqslant k_n} f_1(k_1) \ldots f_n(k_n)$$

The summand has a factorized form.

Form of the MB results

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$$\sum_{k_1} \ldots \sum_{k_n} f(k_1 \ldots k_n)$$

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Complexity scales **linearly** with n.

for $k = 0..k_{max}$ **do**
  for $i = 0..n$ **do**
    $S_i = S_i + S_{s-1} f_i(k)$
  end
end
return S_n

Complexity scales **exponentially**.

for $k_1 = 0..k_{max}$ **do** ...//n-fold
  for $k_n = 0..n$ **do**
    $S = S + f(k_1, \ldots)$
  end
end
return S

Numerical issues

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The DRA results are expressed in terms of the multiple sums

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Convergence acceleration

- Mostly, the summands in DRA results fall off exponentially. Evaluation time scales as $\#_{\text{digits}}$.
- Sometimes, the sums in DRA results fall off as a power. **Convergence acceleration**: Iterative transformation (Broadhurst 1996)

$$S_k \to w_k S_k + (1 - w_k) S_{k+1},$$

where w_k is some properly chosen weight. Evaluation time scales as $(\#_{\text{digits}})^2$.

Some results

$$\frac{\quad}{\Gamma^4(-1+\varepsilon)} = \frac{(1-\varepsilon)^2}{(1-3\varepsilon)(2-3\varepsilon)(1-2\varepsilon)}\Bigg(-4+\frac{44\varepsilon}{3}-\frac{224\zeta_3\varepsilon^4}{3}+\left(-\frac{64a_1^4}{3}+\frac{64}{3}\pi^2a_1^2-512a_4+\frac{272\pi^4}{45}\right)\varepsilon^5+\left(\frac{128a_1^5}{5}-\frac{128}{3}\pi^2a_1^3-\frac{544\pi^4a_1}{15}-3072a_5+2480\zeta_5\right)\varepsilon^6+\left(-\frac{128a_1^6}{5}+64\pi^2a_1^4+\frac{544}{5}\pi^4a_1^2-18432a_6+\frac{9760\zeta_3^2}{3}-7680s_6+\frac{64\pi^6}{5}\right)\varepsilon^7+\left(-\frac{148480s_{7a}}{7}+\frac{174080s_{7b}}{7}-44640a_1^2\zeta_5-\frac{185600}{7}a_1\zeta_3^2+\frac{148480a_1s_6}{7}+\frac{768a_1^7}{35}-\frac{384}{5}\pi^2a_1^5-\frac{1088}{5}\pi^4a_1^3-\frac{7648\pi^6a_1}{135}-110592a_7-\frac{1440\pi^4\zeta_3}{7}-\frac{260720\pi^2\zeta_5}{21}+\frac{1545736\zeta_7}{7}\right)\varepsilon^8+O\left(\varepsilon^9\right)\Bigg)$$

$$\frac{\quad}{\Gamma^4(-1+\varepsilon)} = \frac{(1-\varepsilon)^3}{(1-4\varepsilon)(3-4\varepsilon)(1-3\varepsilon)(2-3\varepsilon)(1-2\varepsilon)}\Bigg(-6+50\varepsilon-\frac{344\varepsilon^2}{3}+\frac{3584\zeta_3\varepsilon^5}{3}+\left(\frac{2048a_1^4}{3}-\frac{2048}{3}\pi^2a_1^2+16384a_4-\frac{8704\pi^4}{45}\right)\varepsilon^6+\left(-\frac{8192a_1^5}{5}+\frac{8192}{3}\pi^2a_1^3+\frac{34816\pi^4a_1}{15}+196608a_5-174592\zeta_5\right)\varepsilon^7+\left(\frac{16384a_1^6}{5}-8192\pi^2a_1^4-\frac{69632}{5}\pi^4a_1^2+2359296a_6-\frac{1266688\zeta_3^2}{3}+1081344s_6-\frac{13312\pi^6}{9}\right)\varepsilon^8+\left(\frac{39845888s_{7a}}{7}-\frac{50987008s_{7b}}{7}+12570624a_1^2\zeta_5+\frac{49807360}{7}a_1\zeta_3^2-\frac{39845888a_1s_6}{7}-\frac{196608a_1^7}{35}+\frac{98304}{5}\pi^2a_1^5+\frac{278528}{5}\pi^4a_1^3+\frac{1599488\pi^6a_1}{135}+28311552a_7+\frac{411648\pi^4\zeta_3}{7}+\frac{76306432\pi^2\zeta_5}{21}-\frac{433559040\zeta_7}{7}\right)\varepsilon^9+O\left(\varepsilon^{10}\right)\Bigg)$$

Some results

$$\frac{}{\Gamma^4(-1+\varepsilon)} = \frac{2}{3} + \frac{4\varepsilon}{3} + \frac{2\varepsilon^2}{3} + \left(\frac{16\zeta_3}{3} - \frac{44}{3}\right)\varepsilon^3 + \left(\frac{200\zeta_3}{3} - \frac{4\pi^4}{15} - 116\right)\varepsilon^4 + \left(\frac{64a_1^4}{3} - \frac{64}{3}\pi^2 a_1^2 + 512a_4 + \frac{1192\zeta_3}{3} + 96\zeta_5 - \frac{326\pi^4}{45} - \frac{1928}{3}\right)\varepsilon^5 + \left(-\frac{512a_1^5}{15} + \frac{448a_1^4}{3} + \frac{512}{9}\pi^2 a_1^3 - \frac{448}{3}\pi^2 a_1^2 + \frac{2416\pi^4 a_1}{45} + 3584a_4 + 4096a_5 + \frac{64\zeta_3^2}{3} + \frac{5864\zeta_3}{3} - 2784\zeta_5 - \frac{8\pi^6}{21} - \frac{2126\pi^4}{45} - \frac{9328}{3}\right)\varepsilon^6 + \left(\frac{2048a_1^6}{45} - \frac{3584a_1^5}{15} - \frac{1024}{9}\pi^2 a_1^4 + \frac{2368a_1^4}{3} + \frac{3584}{9}\pi^2 a_1^3 - \frac{9664}{45}\pi^4 a_1^2 - \frac{2368}{3}\pi^2 a_1^2 + \frac{16912\pi^4 a_1}{45} + 18944a_4 + 28672a_5 + 32768a_6 - \frac{14872\zeta_3^2}{3} - \frac{32\pi^4\zeta_3}{15} + 8760\zeta_3 - 20736\zeta_5 + 1328\zeta_7 + 12288s_6 - \frac{25408\pi^6}{945} - \frac{2182\pi^4}{9} - 14032\right)\varepsilon^7 + O\left(\varepsilon^8\right)$$

$$\frac{}{\Gamma^4(-1+\varepsilon)} = \frac{1}{4} + \frac{\varepsilon}{2} + \left(\frac{13\zeta_3}{2} - 8\right)\varepsilon^3 + \left(4\zeta_3 - \frac{5\pi^4}{8} - \frac{241}{4}\right)\varepsilon^4 + \left(-36\zeta_3 + \frac{693\zeta_5}{2} - \frac{\pi^4}{5} - \frac{669}{2}\right)\varepsilon^5 + \left(\frac{241\zeta_3^2}{2} - 289\zeta_3 + 72\zeta_5 - \frac{44\pi^6}{21} + \frac{21\pi^4}{5} - 1636\right)\varepsilon^6 + \left(16\zeta_3^2 - \frac{493\pi^4\zeta_3}{20} - \frac{3061\zeta_3}{2} - 2484\zeta_5 + \frac{45921\zeta_7}{4} - \frac{2\pi^6}{7} + \frac{589\pi^4}{20} - 7472\right)\varepsilon^7 + O\left(\varepsilon^8\right)$$

Some results

$$\frac{[\text{diagram}]}{\Gamma^4(-1+\varepsilon)} = \frac{(1-\varepsilon)^3}{1-2\varepsilon}\left(5\zeta_5\varepsilon^3 + \left(-7\zeta_3^2 - \frac{11\pi^6}{378}\right)\varepsilon^4 + \left(\frac{\pi^4\zeta_3}{30} + 212\zeta_7\right)\varepsilon^5 + O\left(\varepsilon^6\right)\right)$$

$$\frac{[\text{diagram}]}{\Gamma^4(-1+\varepsilon)} = \frac{3}{2} + \frac{7\varepsilon}{2} + \frac{9\varepsilon^2}{2} + \left(-3\zeta_3 - \frac{39}{2}\right)\varepsilon^3 + \left(109\zeta_3 - \frac{\pi^4}{20} - 208\right)\varepsilon^4 + \left(32a_1^4 - 32\pi^2a_1^2 + 768a_4 + 855\zeta_3\right.$$
$$\left. + 189\zeta_5 - \frac{547\pi^4}{60} - 1254\right)\varepsilon^5 + \left(-\frac{192a_1^5}{5} + 240a_1^4 + 64\pi^2a_1^3 - 240\pi^2a_1^2 + \frac{272\pi^4a_1}{5} + 5760a_4\right.$$
$$\left. + 4608a_5 - 498\zeta_3^2 + 4851\zeta_3 - 3531\zeta_5 - \frac{17\pi^6}{21} - \frac{271\pi^4}{4} - 6336\right)\varepsilon^6 + \left(3456s_{7a} + 3456s_{7b}\right.$$
$$+ 252a_1^4\zeta_3 - 252\pi^2a_1^2\zeta_3 + 4320a_1\zeta_3^2 + 6048a_4\zeta_3 - 3456a_1s_6 + \frac{192a_1^6}{5} - 288a_1^5 - 96\pi^2a_1^4$$
$$+ 1344a_1^4 + 480\pi^2a_1^3 - \frac{816}{5}\pi^4a_1^2 - 1344\pi^2a_1^2 + \frac{14\pi^6a_1}{5} + 408\pi^4a_1 + 32256a_4 + 34560a_5$$
$$+ 27648a_6 - 5378\zeta_3^2 - \frac{841\pi^4\zeta_3}{10} + 23968\zeta_3 - 1692\pi^2\zeta_5 - 28845\zeta_5 + 28953\zeta_7 + 11520s_6$$
$$\left. - \frac{2101\pi^6}{105} - \frac{7567\pi^4}{20} - 29384\right)\varepsilon^7 + O\left(\varepsilon^8\right)$$

Some results

$$\frac{\;}{\Gamma^4(-1+\varepsilon)} = -\frac{1}{6} - \frac{5\varepsilon}{6} + \left(-\zeta_3 - \frac{11}{3}\right)\varepsilon^2 + \left(\frac{2\zeta_3}{3} - \frac{\pi^4}{60} - \frac{44}{3}\right)\varepsilon^3 + \left(\frac{31\zeta_3}{3} + 53\zeta_5 - \frac{\pi^4}{6} - \frac{166}{3}\right)\varepsilon^4 + \left(\frac{16a_1^4}{3} - \frac{16}{3}\pi^2 a_1^2 + 128a_4 - 128\zeta_3^2 + \frac{38\zeta_3}{3} + 154\zeta_5 - \frac{44\pi^6}{189} - \frac{85\pi^4}{36} - \frac{602}{3}\right)\varepsilon^5 + \left(1920s_{7a} + 1920s_{7b} + 140a_1^4\zeta_3 - 140\pi^2 a_1^2\zeta_3 + 2400a_1\zeta_3^2 + 3360a_4\zeta_3 - 1920a_1 s_6 - \frac{32a_1^5}{3} + \frac{16a_1^4}{3} + \frac{160}{9}\pi^2 a_1^3 - \frac{16}{3}\pi^2 a_1^2 + \frac{14\pi^6 a_1}{9} + \frac{160\pi^4 a_1}{9} + 128a_4 + 1280a_5 - \frac{736\zeta_3^2}{3} - \frac{1429\pi^4\zeta_3}{30} - \frac{784\zeta_3}{3} - 940\pi^2\zeta_5 - 353\zeta_5 + \frac{27591\zeta_7}{2} - \frac{124\pi^6}{189} - \frac{481\pi^4}{90} - \frac{2122}{3}\right)\varepsilon^6 + O\left(\varepsilon^7\right)$$

$$\frac{\;}{\Gamma^4(-1+\varepsilon)} = -\frac{1}{6} - \frac{5\varepsilon}{6} + \left(-\frac{\zeta_3}{2} - \frac{11}{3}\right)\varepsilon^2 + \left(\frac{13\zeta_3}{6} - \frac{\pi^4}{120} - \frac{44}{3}\right)\varepsilon^3 + \left(\frac{29\zeta_3}{6} + \frac{43\zeta_5}{2} - \frac{5\pi^4}{24} - \frac{166}{3}\right)\varepsilon^4 + \left(-\frac{105\zeta_3^2}{2} - \frac{197\zeta_3}{6} + \frac{231\zeta_5}{2} - \frac{17\pi^6}{189} - \frac{41\pi^4}{120} - \frac{602}{3}\right)\varepsilon^5 + \left(1024s_{7a} + 1024s_{7b} + \frac{224}{3}a_1^4\zeta_3 - \frac{224}{3}\pi^2 a_1^2\zeta_3 + 1280a_1\zeta_3^2 + 1792a_4\zeta_3 - 1024a_1 s_6 - \frac{64a_1^4}{3} + \frac{64}{3}\pi^2 a_1^2 + \frac{112\pi^6 a_1}{135} - 512a_4 + \frac{241\zeta_3^2}{6} - \frac{4699\pi^4\zeta_3}{180} - \frac{1363\zeta_3}{3} - \frac{1504\pi^2\zeta_5}{3} + \frac{307\zeta_5}{2} + \frac{28499\zeta_7}{4} - \frac{44\pi^6}{63} + \frac{2347\pi^4}{360} - \frac{2122}{3}\right)\varepsilon^6 + O\left(\varepsilon^7\right)$$

Some results

$$\frac{[\text{diagram}]}{\Gamma^4(-1+\varepsilon)} = \Big(\frac{16a_1^5}{5} - \frac{16}{3}\pi^2 a_1^3 - \frac{53\pi^4 a_1}{15} - 384a_5 + \frac{873\zeta_5}{2}\Big)\varepsilon^4 + \Big(-\frac{32a_1^6}{3} - \frac{112a_1^5}{5} + \frac{80}{3}\pi^2 a_1^4 + \frac{112}{3}\pi^2 a_1^3$$
$$+ \frac{124}{3}\pi^4 a_1^2 + \frac{371\pi^4 a_1}{15} + 2688a_5 - 7680a_6 + \frac{2859\zeta_3^2}{2} - \frac{6111\zeta_5}{2} - 4032s_6 + \frac{7457\pi^6}{1890}\Big)\varepsilon^5 + \Big(-\frac{160320s_{7a}}{7}$$
$$+ \frac{242880s_{7b}}{7} + 158a_1^4\zeta_3 - 158\pi^2 a_1^2\zeta_3 - 55800a_1^2\zeta_5 - \frac{200400}{7}a_1\zeta_3^2 + 3792a_4\zeta_3 + \frac{160320a_1 s_6}{7} + \frac{2432a_1^7}{105}$$
$$+ \frac{224a_1^6}{3} - \frac{1216}{15}\pi^2 a_1^5 + \frac{688a_1^5}{5} - \frac{560}{3}\pi^2 a_1^4 - \frac{9856}{45}\pi^4 a_1^3 - \frac{688}{3}\pi^2 a_1^3 - \frac{868}{3}\pi^4 a_1^2 - \frac{11561\pi^6 a_1}{315} - \frac{2279\pi^4 a_1}{15}$$
$$-16512a_5 + 53760a_6 - 116736a_7 - \frac{20013\zeta_3^2}{2} - \frac{13451\pi^4\zeta_3}{42} - \frac{121010\pi^2\zeta_5}{7} + \frac{37539\zeta_5}{2} + \frac{3977181\zeta_7}{14}$$
$$+28224s_6 - \frac{7457\pi^6}{270}\Big)\varepsilon^6 + O\big(\varepsilon^7\big)$$

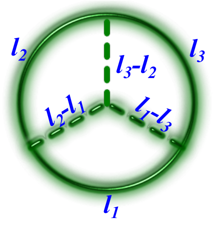
Cheating?

Not a fair play of course: DRA method gives **exact in $\mathscr{D}$** results, so obtaining new ε-terms is easy. Exact results for tadpoles above are taken from (Lee and Terekhov 2011).

Example

Twisted 3-loop tadpole

tadpole



Example

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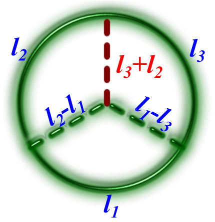

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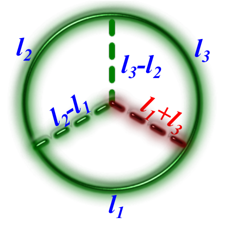

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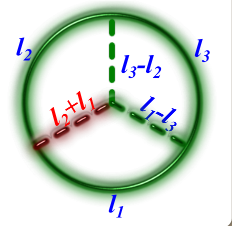

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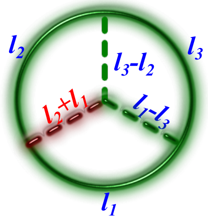

Integral

$$J(\mathbf{n}) = \int \frac{d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_3}{\pi^{\frac{3\mathscr{D}}{2}} D_1^{n_1} \ldots D_6^{n_6}},$$

$$D_{1\ldots 3} = l_{1\ldots 3}^2 + 1,\, D_4 = (l_3 - l_2)^2,$$

$$D_5 = (l_1 - l_3)^2,\, D_6 = (l_2 + l_1)^2$$

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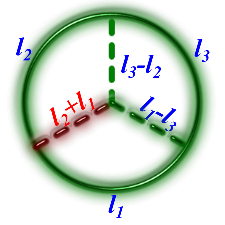
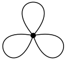
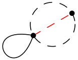
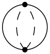
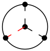
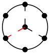


Masters

$J_1 =$ , $J_2 =$  $= 4^{D-3}$ , $J_3 =$  $\Longleftarrow$Trivial

 $\Longleftarrow$for $\mathscr{D} \leqslant 2$ IR divs, for $\mathscr{D} \geqslant 4$ UV divs

$J_4 = J^{(\mathscr{D}+2)}_{222111} =$  $\Longleftarrow$New master, finite on $\Re\mathscr{D} \in [3,5)$

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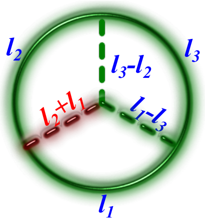

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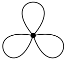
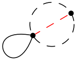
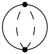
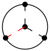
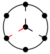
$J_1 =$ , $J_2 =$  $= 4^{D-3}$ , $J_3 =$  $\Longleftarrow$Trivial

 $\Longleftarrow$for $\mathscr{D} \leqslant 2$ IR divs, for $\mathscr{D} \geqslant 4$ UV divs

$J_4 = J_{222111}^{(\mathscr{D}+2)} =$  $\Longleftarrow$New master, finite on $\Re\mathscr{D} \in [3,5)$

DRR

$$J^{(\mathscr{D}+2)}(\mathbf{n}) = \frac{(2\mu)^L [V(p_1,\ldots,p_E)]^{-1}}{(\mathscr{D}-E-L+1)_L} P(B_1,\ldots,B_N) J^{(\mathscr{D})}(\mathbf{n}).$$

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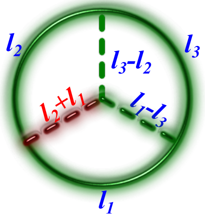
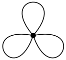
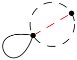
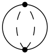
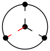
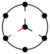


Masters

$J_1 =$ , $J_2 =$  $= 4^{D-3}$ , $J_3 =$  $\Longleftarrow$Trivial

 $\Longleftarrow$for $\mathscr{D} \leqslant 2$ IR divs, for $\mathscr{D} \geqslant 4$ UV divs

$J_4 = J_{222111}^{(\mathscr{D}+2)} =$  $\Longleftarrow$New master, finite on $\Re\mathscr{D} \in [3,5)$

DRR

$$J^{(\mathscr{D}+2)}(\mathbf{n}) = \frac{4}{(\mathscr{D}-2)(\mathscr{D}-1)} P(B_1,\ldots,B_6) J^{(\mathscr{D})}(\mathbf{n}).$$

$$P(B_1,\ldots,B_6) = \det[s_{ij}]$$

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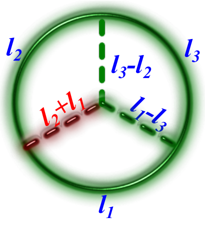

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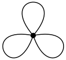
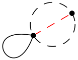
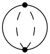
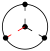
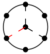
$J_1 =$ , $J_2 =$  $= 4^{D-3}$ , $J_3 =$  $\Longleftarrow$Trivial

 $\Longleftarrow$for $\mathscr{D} \leqslant 2$ IR divs, for $\mathscr{D} \geqslant 4$ UV divs

$J_4 = J_{222111}^{(\mathscr{D}+2)} =$  $\Longleftarrow$New master, finite on $\Re\mathscr{D} \in [3,5)$

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$$J^{(\mathscr{D}+2)}(\mathbf{n}) = \frac{4}{(\mathscr{D}-2)(\mathscr{D}-1)} P(B_1,\ldots,B_6)\, J^{(\mathscr{D})}(\mathbf{n})\,.$$

$$P(B_1,\ldots,B_6) = \det[s_{ij}] = 4 - 4B_1 + B_1^2 - 4B_2 + 3B_1B_2 - \frac{1}{2}B_1^2B_2 + \ldots$$

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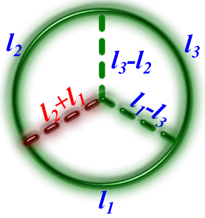

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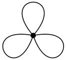
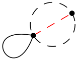
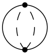
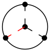
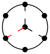
$J_1 =$ , $J_2 =$  $= 4^{D-3}$ , $J_3 =$  $\Longleftarrow$ Trivial

 $\Longleftarrow$ for $\mathscr{D} \leqslant 2$ IR divs, for $\mathscr{D} \geqslant 4$ UV divs

$J_4 = J^{(\mathscr{D}+2)}_{222111} =$  $\Longleftarrow$ New master, finite on $\Re\mathscr{D} \in [3,5)$

DRR

$$J^{(\mathscr{D}+2)}_{222111} = \frac{4J^{(\mathscr{D})}_{222111} - 4J^{(\mathscr{D})}_{122111} + J^{(\mathscr{D})}_{022111} - 4J^{(\mathscr{D})}_{212111} + 3J^{(\mathscr{D})}_{112111} - \frac{1}{2}J^{(\mathscr{D})}_{012111} + \ldots}{(\mathscr{D}-2)(\mathscr{D}-1)/4}$$

$$P(B_1,\ldots,B_6) = \det[s_{ij}] = 4 - 4B_1 + B_1^2 - 4B_2 + 3B_1B_2 - \frac{1}{2}B_1^2B_2 + \ldots$$

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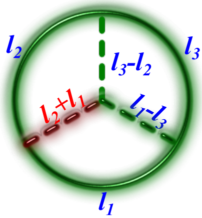

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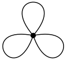
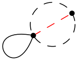
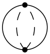
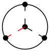
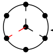
$J_1 =$ , $J_2 =$  $= 4^{D-3}$ , $J_3 =$  $\Longleftarrow$Trivial

 $\Longleftarrow$for $\mathscr{D} \leqslant 2$ IR divs, for $\mathscr{D} \geqslant 4$ UV divs

$J_4 = J^{(\mathscr{D}+2)}_{222111} =$  $\Longleftarrow$New master, finite on $\Re\mathscr{D} \in [3,5)$

DRR

$$J_4(\nu+1) = \frac{4(\nu-1)(\nu-7/4)}{(\nu-2)^2\nu(\nu-1/2)(\nu-11/4)} J_4(\nu) + \frac{3(\nu-1)(3\nu-5)(3\nu-4)\left(184\nu^3-1224\nu^2+2708\nu-1989\right)}{4(\nu-2)\nu(2\nu-5)(2\nu-3)(2\nu-1)(4\nu-11)(4\nu-9)} J_3(\nu)$$
$$+ \frac{6(\nu-1)(3\nu-5)(3\nu-4)(7\nu-13)}{(\nu-2)\nu(2\nu-5)(2\nu-3)(2\nu-1)(4\nu-11)} J_2(\nu) - \frac{(\nu-1)^2\left(80\nu^5-724\nu^4+2602\nu^3-4544\nu^2+3759\nu-1131\right)}{2(\nu-2)\nu(2\nu-5)(2\nu-3)^2(2\nu-1)(4\nu-11)} J_1(\nu)$$

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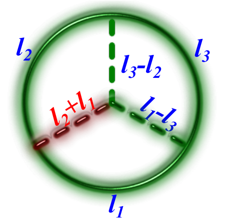

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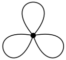
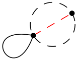
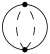
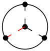
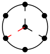
$J_1 =$ , $J_2 =$  $= 4^{D-3}$ , $J_3 =$  $\Longleftarrow$ Trivial

 $\Longleftarrow$ for $\mathscr{D} \leqslant 2$ IR divs, for $\mathscr{D} \geqslant 4$ UV divs

$J_4 = J^{(\mathscr{D}+2)}_{222111} =$  $\Longleftarrow$ New master, finite on $\Re \mathscr{D} \in [3,5)$

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$$J_4(\nu+1) = \frac{4(\nu-1)(\nu-7/4)}{(\nu-2)^2\nu(\nu-1/2)(\nu-11/4)} J_4(\nu) + R(\nu)$$

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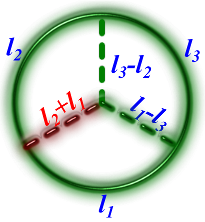

Summing factor

$$S(\nu)=S(\nu+1)\,\frac{4(\nu-1)(\nu-7/4)}{(\nu-2)^2\nu(\nu-1/2)(\nu-11/4)}$$

DRR

$$J_4(\nu+1)=\frac{4(\nu-1)(\nu-7/4)}{(\nu-2)^2\nu(\nu-1/2)(\nu-11/4)}J_4(\nu)+R(\nu)$$

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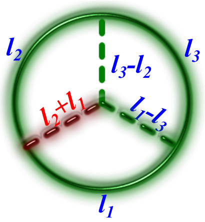

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DRR

$$J_4(\nu+1) = \frac{4(\nu-1)(\nu-7/4)}{(\nu-2)^2\nu(\nu-1/2)(\nu-11/4)} J_4(\nu) + R(\nu)$$

$$S(\nu+1)J_4(\nu+1) = S(\nu)J_4(\nu) + S(\nu+1)R(\nu)$$

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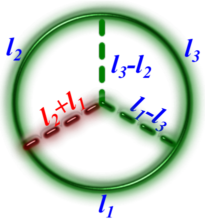

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DRR

$$J_4(\nu+1) = \frac{4(\nu-1)(\nu-7/4)}{(\nu-2)^2\nu(\nu-1/2)(\nu-11/4)}J_4(\nu) + R(\nu)$$

$$S(\nu)J_4(\nu) = \omega(z) - \sum_{k=0}^{\infty} S(\nu+1+k)R(\nu+k)$$

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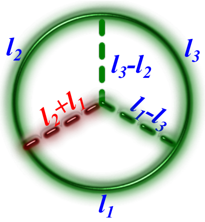

Summing factor

$$S(\nu) = S(\nu+1)\,\frac{4(\nu-1)(\nu-7/4)}{(\nu-2)^2\nu(\nu-1/2)(\nu-11/4)}$$

$$S(\nu) = \left\{\frac{4^\nu\Gamma(\nu-1)\Gamma(\nu-7/4)}{\Gamma(\nu-2)^2\Gamma(\nu)\Gamma(\nu-1/2)\Gamma(\nu-11/4)}\right\}^{-1}$$

DRR

$$J_4(\nu+1) = \frac{4(\nu-1)(\nu-7/4)}{(\nu-2)^2\nu(\nu-1/2)(\nu-11/4)} J_4(\nu) + R(\nu)$$

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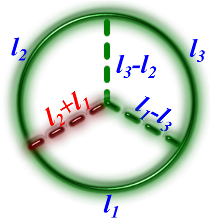

Summing factor

$$S(\nu) = S(\nu+1)\,\frac{4(\nu-1)(\nu-7/4)}{(\nu-2)^2\nu(\nu-1/2)(\nu-11/4)}$$

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DRR

$$J_4(\nu+1) = \frac{4(\nu-1)(\nu-7/4)}{(\nu-2)^2\nu(\nu-1/2)(\nu-11/4)}J_4(\nu) + R(\nu)$$

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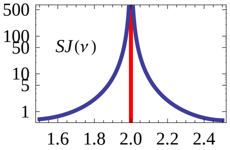
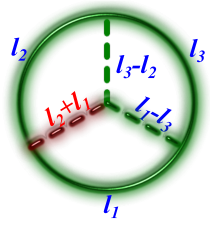


Example

Twisted 3-loop tadpole

twisted tadpole



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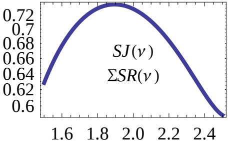

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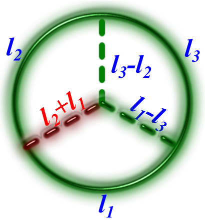

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$$|S(\nu)J(\nu)| \overset{\nu\to\pm i\infty}{\sim} |S(\nu)\Gamma(3\nu)| \to 0$$

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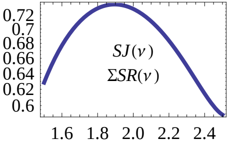

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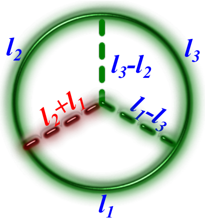

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DRR

Conclusion: $\omega(z)$ **has nosingularities and falls off at** $z \to 0,\infty \implies \omega = 0$

$$S(\nu)J_4(\nu) = \omega(z) - \sum_{k=0}^{\infty} S(\nu+1+k)\,R(\nu+k)$$

Example

Twisted 3-loop tadpole

In 12 sec. we obtain with 10^3-digits precision

$$
\begin{aligned}
J_4(2-\varepsilon) = {} & 0.1137056388801093811655357570836468638489997312 8\ldots \\
& + 0.432572085331584079008271914837761976110650124 3\ldots\varepsilon \\
& + 1.234780884637645769680928205178898681790397505\ldots\varepsilon^2 \\
& + 2.641785606423661330799226158724959195582238 73\ldots\varepsilon^3 \\
& + 4.977221799634845628146248581353044070801636 79\ldots\varepsilon^4 \\
& + 8.276947330345444480023346101118910792770640 2\ldots\varepsilon^5 + \ldots
\end{aligned}
$$

Example

Twisted 3-loop tadpole

Using PSLQ we can convert it to nice analytic form

$$
\begin{aligned}
J_4(2-\varepsilon) = {} & \frac{3}{2} - 2\ln 2 \\
& + \left(\frac{21\zeta_3}{8} - \frac{15}{2} + \frac{\pi^2}{2} + 4\ln^2 2 - 3\ln 2\right)\varepsilon \\
& + \left(\frac{21a_1\zeta_3}{2} + a_1^4 - \frac{16a_1^3}{3} - \pi^2 a_1^2 + 6a_1^2 - \frac{5\pi^2 a_1}{2} - 11a_1 + 24a_4 - \frac{45\zeta_3}{8} - \frac{151\pi^4}{480} + \frac{9\pi^2}{8} + \frac{69}{2}\right)\varepsilon^2 \\
& + \left(-21a_1^2\zeta_3 + \frac{135a_1\zeta_3}{2} - \frac{4a_1^5}{5} + \frac{29a_1^4}{3} + \frac{4}{3}\pi^2 a_1^3 - 8a_1^3 + \frac{2}{3}\pi^2 a_1^2 + 22a_1^2 + \frac{151\pi^4 a_1}{120} - \frac{15\pi^2 a_1}{4} - 15a_1 + 104a_4\right. \\
& \left. + 96a_5 - \frac{63\pi^2\zeta_3}{32} + \frac{111\zeta_3}{8} - \frac{651\zeta_5}{16} - \frac{367\pi^4}{1440} + \frac{7\pi^2}{8} - \frac{291}{2}\right)\varepsilon^3 \\
& + \ldots\varepsilon^4 \\
& + \ldots\varepsilon^5 + \ldots
\end{aligned}
$$

Solution of DRR for Multimasters

Specific solution of inhomogeneous equation

General form of DRR(Multimasters)

$$\mathbf{J}(\nu+1)=\mathbb{C}(\nu)\,\mathbf{J}(\nu)+\mathbf{R}(\nu)\,,$$

Now $\mathbf{J}$ and $\mathbf{R}$ are columns, $\mathbb{C}$ is a square matrix with elements being rational functions.

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Specific solution of inhomogeneous equation: **no problem**

$$\mathbf{J}^{\mathrm{ih}}(\nu)=\sum_{k=1}^{\infty}\prod_{l=1}^{k-1}\mathbb{C}(\nu-l)\,\mathbf{R}(\nu-k)$$

$$\mathbf{J}^{\mathrm{ih}}(\nu)=-\sum_{k=0}^{\infty}\prod_{l=0}^{k}\mathbb{C}^{-1}(\nu+l)\,\mathbf{R}(\nu+k)$$

Solution of DRR for Multimasters

General solution of the homogeneous equation

Homogeneous equation

Now a twisted system of equations

$$\mathbf{J}^0(\nu+1) = \mathbb{C}(\nu)\mathbf{J}^0(\nu)$$

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Hope

Maybe we can reduce the system to triangular form by passing to

$$\tilde{\mathbf{J}}(\nu) = \mathbb{T}(\nu)\mathbf{J}(\nu),$$

where $\mathbb{T}(\nu)$ is some smartly chosen rational matrix? **Whether/how can we check this?**

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There is a tool!

Petkovšek's algorithm `Hyper` (Petkovšek et al. 1996) checks whether a given n-th order difference equation (n>1) has a hypergeometric solution, i.e., a solution $f(\nu)$ such that $\frac{f(\nu+1)}{f(\nu)}$ is a rational function.

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Disappointment

`Hyper` tells us that the equations for multimaster are really twisted $\Longrightarrow$ No general way to obtain the solution.

Solution of DRR for Multimasters

General solution of the homogeneous equation

Guessing the solution

If we would not know the origin of the homogeneous equation, we would fail. Fortunately, we can use some additional methods to guess the solution (Lee and Smirnov, to be published). The result has the form of hypergeometric sums.

Is there a way to check the guess?

- Numerically
- Using Zeiberger's algorithm `Zeil` (Petkovšek et al. 1996)

Solution of DRR for Multimasters

Construction of the summing factor

Sums in the denominators

If we know a fundamental matrix $\mathbb{J}^0(\nu)$ of the homogeneous solutions, the summing factor is

$$\mathbb{S}(\nu) = \left[\mathbb{J}^0(\nu)\right]^{-1}.$$

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Observation

$$\left[\mathbb{J}^0(\nu)\right]^{-1} = \left[\mathbb{J}^0(\nu)\right]^A / \left|\mathbb{J}^0(\nu)\right|$$

$\left|\mathbb{J}^0(\nu)\right|$ satisfies

$$\left|\mathbb{J}^0(\nu+1)\right| = \left|\mathbb{C}(\nu)\right| \left|\mathbb{J}^0(\nu)\right|$$

Solution of this equation is the product of Γ-functions (as for the case of simple master). Arbitrary periodic factor is not a problem.

Fixing ω for multimasters

In real life

- Find summing factor $\mathbb{S}(\nu)$, using some additional techniques.
- Use FIESTA to determine the singularities of $\mathbf{J}(\nu)$ on the basic stripe.
- Try to multiply $\mathbb{S}(\nu)$ from the left by some periodic matrices, constructed of $\sin(\pi(\nu-\nu_0))$, to make $\mathbb{S}(\nu)\mathbf{J}(\nu)$ regular on b.s.
- Don't go too far in that because $\sin(\pi(\nu-\nu_0))$ makes $\nu \to \pm i\infty$ behaviour of $\mathbb{S}(\nu)\mathbf{J}(\nu)$ worse. If $\mathbb{S}(\nu)\mathbf{J}(\nu)$ vanishes at some points, divide by $\sin(\pi(\nu-\nu_0))$ to improve behaviour at infinity. Find also "hidden" zeros — the points where $|\mathbb{S}(\nu)| = 0$, but $\mathbb{S}(\nu) \neq 0$.
- If it was not possible to cancel all singularities of $J(\nu)$ on the basic stripe, use Mellin-Barnes (or other techniques) to fix the singular coefficients of $\mathbb{S}(\nu)\mathbf{J}(\nu)$.
- Finally, use Mittag-Leffler's&Liouville's theorems to fix $\omega(z)$.

Tools

- Finding DRR for masters: manually or automatically, using formulae presented
- IBP reduction of the right-hand side: `FIRE` and many private versions
- Determining the position and order of singularities: `FIESTA`
- Finding missing constants: Mellin-Barnes technique
- Finding summing factor for multimasters: Mellin-Barnes technique
- Checking summing factor for multimasters `Zeil`
- DRA application&High-precision numerics: `DRAMA` is being developed
- Expressing results in terms of conventional transcendental constants: `PSLQ`

Summary

- DRA method is being very successful in the calculation of multiloop integrals. Since the previous workshop CALC09 it was successfully applied to a number of problems:
  - Calculation of master integrals for 3-loop onshell massless vertices.
  - Calculation of master integrals for 3-loop onshell mass operator type integrals.
  - Calculation of master integrals for 4-loop QED-type tadpoles.
  - Calculation of master integrals for 4-loop massless propagators.
- The application of the DRA method to multimasters is being currently developed: `DRAMA` in its early stage already successfully applied to the masters for 3-loop static quark potential (work in progress with V. Smirnov).
- Future: The application of the DRA method to the problems with several scales.

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