Modified frequentist determination of confidence intervals for Poisson distribution

S.I.Bitioukov and N.V.Krasnikov

INR RAS, Moscow 117312

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Abstract

We propose modified frequentist definition for the determination of confidence intervals for the case of Poisson statistics. Namely, we require that $1 - \beta' \geq \sum_{n=o}^{n=n_{obs}+k} P(n|\lambda) \geq \alpha'$. We show that this definition is equivalent to the Bayesian method with prior $\pi(\lambda) \sim \lambda^k$. We also propose modified frequentist definition for the case of nonzero background.

In high energy physics one of the standard problems [1] is the determination of the confidence intervals for the parameter λ in Poisson distribution

$$P(n|\lambda) = \frac{\lambda^n}{n!}\exp(-\lambda).\tag{1}$$

There are two methods to solve this problem - the frequentist and the Bayesian.

In this paper we propose the modified frequentist definition of the confidence interval for the case of Poisson distribution. We show that the modified frequentist distribution is equivalent to the Bayesian approach.

In Bayesian method [1, 2] due to Bayes theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

the probability density for the λ parameter is determined as

$$p(\lambda|n_{obs}) = \frac{P(n_{obs}|\lambda)\pi(\lambda)}{\int_0^\infty (P(n_{obs}|\lambda')\pi(\lambda')d\lambda'}.\tag{2}$$

Here $\pi(\lambda)$ is the prior function and in general it is not known that is the main problem of the Bayesian method. Formula (2) reduces the statistics problem to the probability problem. At the $(1-\alpha)$ probability level the parameters λ_{up} and λ_{down} are determined from the equation [1]

$$\int_{\lambda_{down}}^{\lambda_{up}} p(\lambda|n_{obs})d\lambda = 1 - \alpha\tag{3}$$

and the unknown parameter λ lies between λ_{down} and λ_{up} with the probability $1-\alpha$. The solution of the equation (3) is not unique. One can define

$$\int_{\lambda_{up}}^{\infty} p(\lambda|n_{obs})d\lambda = \alpha',\tag{4}$$

$$\int_0^{\lambda_{down}} p(\lambda|n_{obs})d\lambda = \beta'.\tag{5}$$

In general the parameters α' and β' are arbitrary except the evident equality

$$\alpha' + \beta' = \alpha.\tag{6}$$

The most popular are the following options [1]:

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[1]Usually α is taken equal to 0.05.

1. $\lambda_{down} = 0$ - upper limit.

2. $\lambda_{up} = \infty$ - lower limit.

3. $\int_0^{\lambda_{up}} p(\lambda|n_{obs})d\lambda = \int_{\lambda_{down}}^{\infty} p(\lambda|n_{obs})d\lambda = \frac{\alpha}{2}$ - symmetric interval.

4. The shortest interval - $p(\lambda|n_{obs})$ inside the interval is bigger or equal to $p(\lambda|n_{obs})$ outside the interval.

In frequentist approach the Neyman belt construction [3](see Fig. 1) is used for the determination of the confidence intervals.
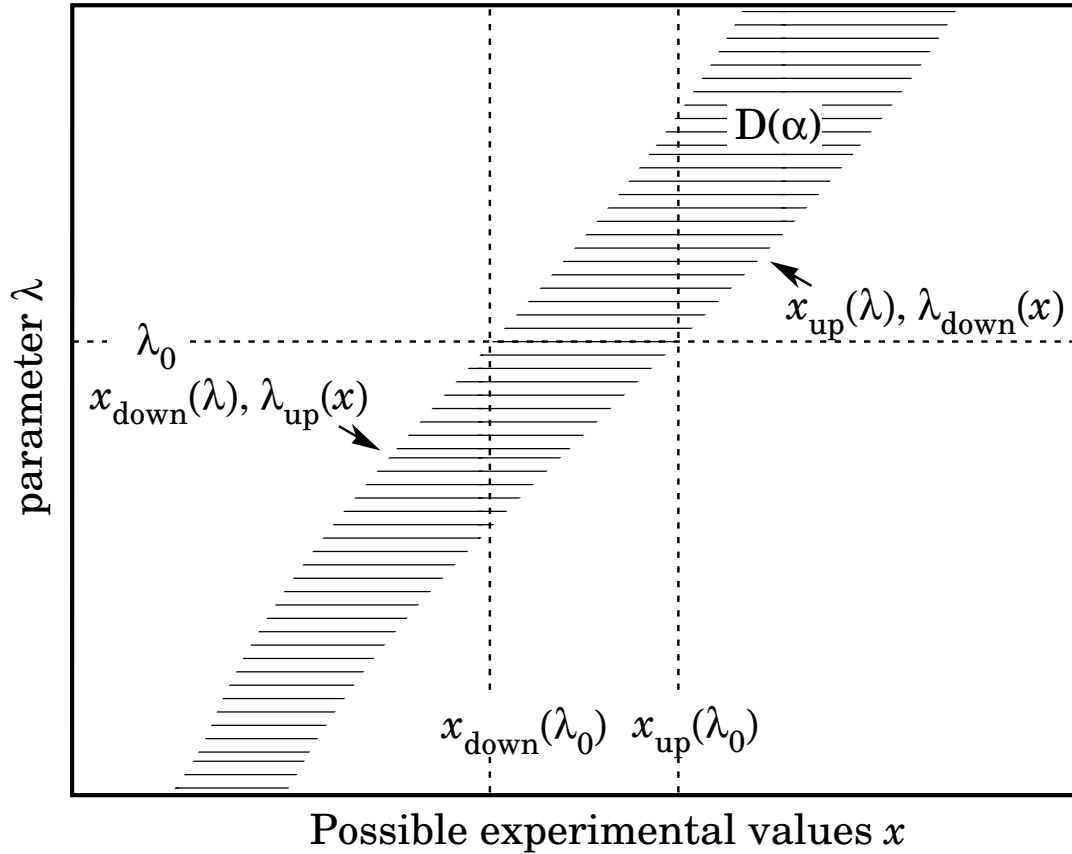

Figure 1: Neyman belt construction

For the continuous observable $-\infty < x < \infty$ with the probability density $f(x, \lambda)$ [2] we require that

$$\int_{x_{down}}^{x_{up}} f(x, \lambda)dx = 1 - \alpha \,, \tag{7}$$

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[2]Here λ is some unknown parameter and $\int_{-\infty}^{\infty} f(x, \lambda)dx = 1$.

or

$$\int_{x_{up}}^{\infty} f(x, \lambda)dx = \beta^{`} , \tag{8}$$

$$\int_{-\infty}^{x_{down}} f(x, \lambda)dx = \alpha^{`} , \tag{9}$$

$$\alpha^{`} + \beta^{`} = \alpha . \tag{10}$$

The equations[3]

$$\int_{x_{obs}}^{\infty} f(x, \lambda_{down})dx = \beta^{`} , \tag{11}$$

$$\int_{-\infty}^{x_{obs}} f(x, \lambda_{up})dx = \alpha^{`} \tag{12}$$

determine the interval of possible values $\lambda_{down} \leq \lambda \leq \lambda_{up}$ of the parameter λ at the $(1-\alpha)$ confidence level. The equations (11,12) are equivalent to the equations

$$\int_{x_{obs}}^{\infty} f(x, \lambda_{down})dx = \beta^{`} , \tag{13}$$

$$\int_{x_{obs}}^{\infty} f(x, \lambda_{up})dx = 1 - \alpha^{`} \tag{14}$$

or to the equations

$$\int_{-\infty}^{x_{obs}} f(x, \lambda_{down})dx = \alpha^{`} , \tag{15}$$

$$\int_{-\infty}^{x_{obs}} f(x, \lambda_{down})dx = 1 - \beta^{`} . \tag{16}$$

For Poisson distribution $P(n|\lambda)$ the analog of the equation (7) has the form

$$\sum_{n_{down}(\lambda)}^{n_{up}(\lambda)} P(n|\lambda) \geq 1 - \alpha . \tag{17}$$

The equations for the determination of λ_{down} and λ_{up} (analogs of the equations (11, 12)) have the form [4, 5, 6]

$$\sum_{n=n_{obs}}^{\infty} P(n|\lambda_{down}) = \beta^{'} , \tag{18}$$

$$\sum_{n=0}^{n_{obs}} P(n|\lambda_{up}) = \alpha^{'} . \tag{19}$$

The analogs of the equations (13, 14) and (15, 16) for Poisson distribution are

$$\sum_{n=n_{obs}}^{\infty} P(n|\lambda_{down}) = \beta^{'} , \tag{20}$$

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[3]Here x_{obs} is the observed value of random variable x.

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$$\sum_{n=n_{obs}}^{\infty} P(n|\lambda_{up}) = 1 - \alpha' \tag{21}$$

and

$$\sum_{n=0}^{n_{obs}} P(n|\lambda_{down}) = 1 - \beta' , \tag{22}$$

$$\sum_{n=0}^{n_{obs}} P(n|\lambda_{up}) = \alpha' \tag{23}$$

correspondingly. Unlike to the case of continuous variable the equations (18, 19), (20, 21) and (22, 23) are not equivalent for the discrete variable n and they differ in the presence or absence of $P(n_{obs}|\lambda_{up,down})$ in some equations. For instance, for $\beta' = 0$, $\alpha' = \alpha$ (upper limit case) the equations (19, 23) coincide and read as

$$\sum_{n=0}^{n_{obs}} P(n|\lambda_{up}) = \alpha , \tag{24}$$

while the equation (21) is equivalent to

$$\sum_{n=0}^{n_{obs}-1} P(n|\lambda_{up}) = \alpha , \tag{25}$$

For $n_{obs} = 3$ and $\alpha = 0.05$ we find that

$$\lambda \leq 7.75 , (Eq.\ 24) , \tag{26}$$

$$\lambda \leq 6.30 , (Eq.\ 25) . \tag{27}$$

Consider the probability to observe the number of events $n \leq n_{obs}$

$$P_-(n_{obs}|\lambda) = \sum_{n=0}^{n_{obs}} P(n|\lambda) . \tag{28}$$

To determine possible values λ_{down} and λ_{up} of the confidence interval we require that

$$1 - \beta' \geq P_-(n_{obs}|\lambda) \geq \alpha' , \tag{29}$$

where $\alpha' + \beta' = \alpha$. The equations for the determination λ_{up} and λ_{down} have the form

$$P_-(n_{obs}|\lambda_{up}) = \alpha' , \tag{30}$$

$$P_-(n_{obs}|\lambda_{down}) = 1 - \beta' . \tag{31}$$

Note that as in the case of Bayesian approach the choice of α' and β' is not unique. Due to the identity [6]

$$P_-(n_{obs}|\lambda) = \int_\lambda^\infty P(n_{obs}|\lambda')d\lambda' \qquad (32)$$

the confidence interval $[\lambda_{down}, \lambda_{up}]$ is determined from the equations

$$\alpha' = \int_{\lambda_{up}}^\infty P(n_{obs}|\lambda')d\lambda' , \qquad (33)$$

$$\beta' = \int_0^{\lambda_{down}} P(n_{obs}|\lambda')d\lambda' . \qquad (34)$$

The parameter λ lies in the interval

$$\lambda_{down} \leq \lambda \leq \lambda_{up} \qquad (35)$$

with the probability $(1 - \alpha' - \beta')$. Due to equations (33, 34) our modified frequentist definition (29) is equivalent to Bayes definitions (3, 4, 5) with flat prior $\pi(\lambda) = 1$, namely:

$$\int_{\lambda_{down}}^{\lambda_{up}} P(n_{obs}|\lambda')d\lambda' = 1 - \alpha' - \beta' . \qquad (36)$$

As an alternative to the definition (29) we can require that

$$1 - \alpha' \geq \sum_{n=n_{obs}}^\infty P(n, \lambda) = -P_-(n_{obs}, \lambda) + 1 + P(n_{obs}, \lambda) \geq \beta' . \qquad (37)$$

The definition (37) leads to the equations (20, 21) for the determination of λ_{down} and λ_{up}. The equations (20, 21) are equivalent to the Bayes equations with the prior function $\pi(\lambda) \sim \frac{1}{\lambda}$.

The coverage of the definition (29) means the following. For a hypothetical ensemble of similar experiments the probability to observe the number of events $n \leq n_{obs}$ satisfies the inequalities (29). As we noted before the choice of λ_{down} and λ_{up} is not unique. Probably the most natural choice is the use of the ordering principle. According to this principle we require that the probability density $P(n_{obs}|\lambda)$ inside the confidence interval $[\lambda_{down}, \lambda_{up}]$ is bigger or equal to the probability density outside this interval. For Poisson distribution this requirement leads to the formula

$$P(n_{obs}|\lambda_{down}) = P(n_{obs}|\lambda_{up}) \qquad (38)$$

for the determination of λ_{up} and λ_{down}. For such ordering principle α' and β' are not independent quantities. It is natural to use $\alpha = \alpha' + \beta'$ as a single free parameter. Note that the equations (19) and (23) for the determination of an upper limit λ_{up} in frequentist and modified frequentist approach coincide whereas the equations (19) and (21) are different. Namely, the equation (21) is equivalent to the equation

$$\sum_{n=0}^{n_{obs}-1} P(n|\lambda_{up}) = \alpha'. \tag{39}$$

Classical frequentist equation (18) is equivalent to the Bayes equation (5) with prior $\pi(\lambda) \sim \frac{1}{\lambda}$.

It is possible to generalize our modified frequentist definition (29), namely:

$$1 - \beta' \geq P_-(n_{obs}|\lambda; k) \geq \alpha', \tag{40}$$

where

$$P_-(n_{obs}|\lambda; k) \equiv \sum_{n=0}^{n_{obs}+k} P(n|\lambda) \tag{41}$$

and $k = 0, \pm 1, \pm 2, ...$

One can find that definitions (40, 41) lead to Bayes equations (4, 5) with the prior function $\pi(\lambda) \sim \lambda^k$. Upper limits for three values of k are shown in Table 1 ($\alpha = 0.1$), in Table 2 ($\alpha = 0.05$) and, correspondingly, in Fig. 2 and Fig. 3.

Table 1: Upper limits (λ_{up}) for confidence level 90% ($\alpha = 0.1$).

n_{obs}	k=-1	k=0	k=+1
0	-	2.30	3.89
1	2.30	3.89	5.32
2	3.89	5.32	6.68
3	5.32	6.68	7.99
4	6.68	7.99	9.27
5	7.99	9.27	10.53
6	9.27	10.53	11.77
7	10.53	11.77	12.99
8	11.77	12.99	14.21
9	12.99	14.21	15.41
10	14.21	15.41	16.60

Table 2: Upper limits (λ_{up}) for confidence level 95% ($\alpha = 0.05$).

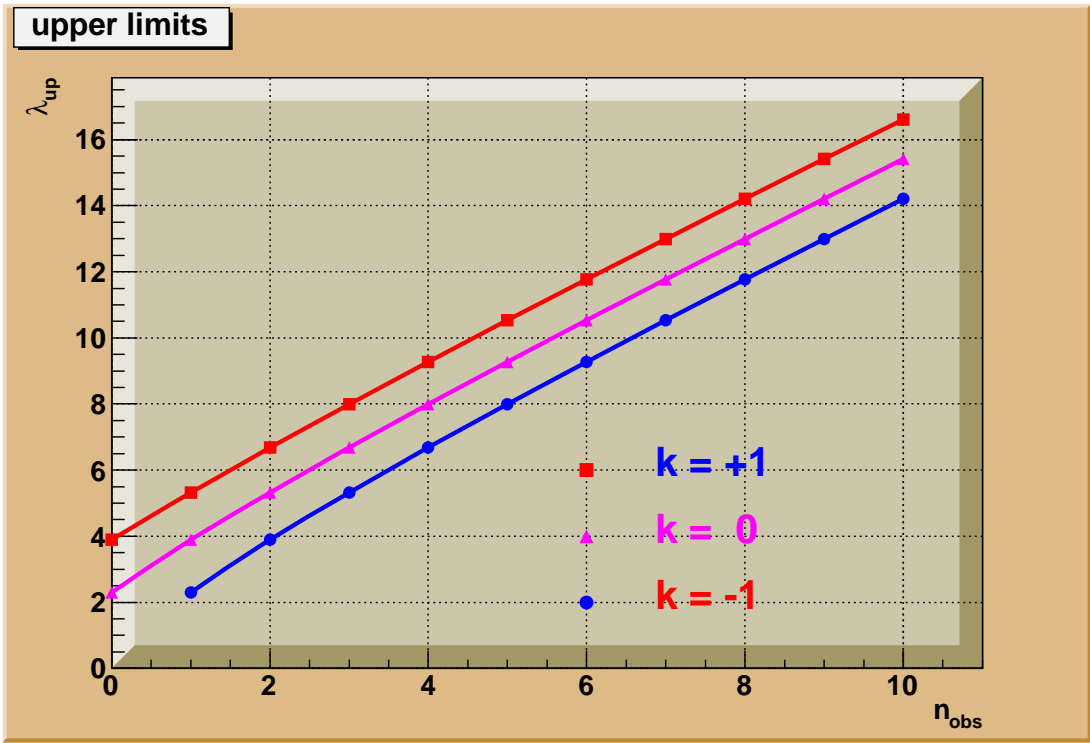| n_{obs}	k=-1	k=0	k=+1
0	-	3.00	4.74
1	3.00	4.74	6.30
2	4.74	6.30	7.75
3	6.30	7.75	9.15
4	7.75	9.15	10.51
5	9.15	10.51	11.84
6	10.51	11.84	13.15
7	11.84	13.15	14.43
8	13.15	14.43	15.71
9	14.43	15.71	16.96
10	15.71	16.96	18.21

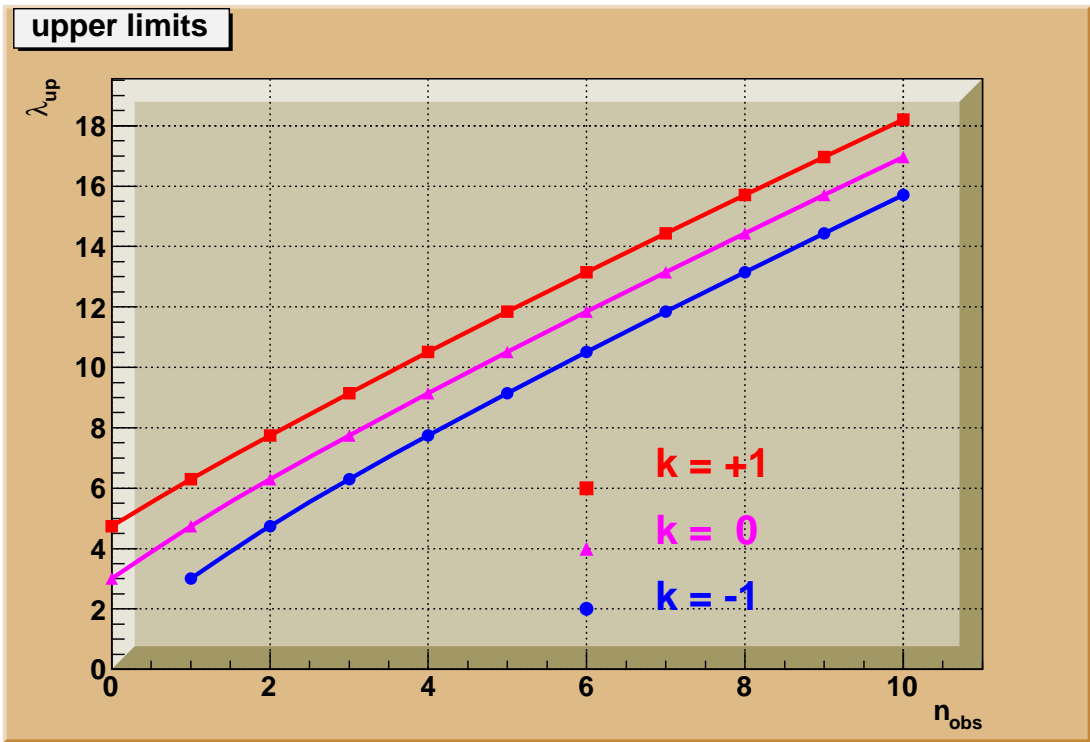Figure 2: Upper limits (λ_{up}) for confidence level 90% ($\alpha = 0.1$), $k = -1,\ 0, +1$.

Figure 3: Upper limits (λ_{up}) for confidence level 95% ($\alpha = 0.05$), $k = -1,\ 0, +1$.

We can further generalize definitions (40, 41) by the introduction

$$P_-(n_{obs}|\lambda; c_k) \equiv \sum_k c_k^2 P_-(n_{obs}|\lambda; k) \,, \tag{42}$$

where $\sum_k c_k^2 = 1$. Again we require that

$$1 - \beta' \geq P_-(n_{obs}|\lambda; c_k) \geq \alpha' \,. \tag{43}$$

One can find that our definition (43) is equivalent to Bayes approach with prior function

$$\pi(\lambda) = \sum_k c_k^2 l_k \lambda^k \,, \tag{44}$$

where

$$l_k = \frac{n!}{(n+k)!} \,. \tag{45}$$

For the case when we have nonzero background the parameter λ is represented in the form

$$\lambda = b + s \,. \tag{46}$$

Here $b \geq 0$ is known background and s is unknown signal. In Bayes approach the generalization of the formula (2) reads

$$p(s|n_{obs}, b) = \frac{P(n_{obs}|b+s)\pi(b,s)}{\int_0^\infty P(n_{obs}|b+s')\pi(b,s')ds'} \,. \tag{47}$$

For flat prior we have

$$p(s|n_{obs}, b) = \frac{P(n_{obs}|b+s)}{\int_b^\infty P(n_{obs}|\lambda')d\lambda'} \,. \tag{48}$$

So we see that the main effect of nonzero background is the appearance of the factor

$$K(n_{obs}, b) = \int_b^\infty P(n_{obs}|\lambda')d\lambda' \tag{49}$$

in the denominator of formula (48). For zero background $K(n_{obs}, b = 0) = 1$. One can interpret the appearance of additional factor $K(n_{obs}, b)$ in terms of conditional probability. Really, for flat prior the $P(n_{obs}, \lambda)d\lambda$ is the probability that parameter λ lies in the interval $[\lambda, \lambda+d\lambda]$. For the case of nonzero background b parameter $\lambda = b+s \geq b$. The probability that $\lambda \geq b$ is equal to $p(\lambda \geq b|n_{obs}) = K(n_{obs}, b)$. The conditional probability that λ lies

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in the interval $[\lambda, \lambda + d\lambda]$ provided $\lambda \geq b$ is determined by the standard formula of the conditional probability

$$p(\lambda, n_{obs}|\lambda \geq b)d\lambda = \frac{p(\lambda, n_{obs})}{p(\lambda \geq b)}d\lambda = \frac{p(\lambda, n_{obs})}{K(n_{obs}s)}d\lambda \qquad (50)$$

and it coincides with the Bayes formula (48).

In the frequentist approach the naive generalization of the inequality (29) is

$$1 - \beta' \geq P_-(n_{obs}|s + b) \geq \alpha' . \qquad (51)$$

One can show that

$$1 - \alpha' - \beta' = \int_{b+s_{down}}^{b+s_{up}} P(n_{obs}|\lambda')d\lambda' \leq \int_b^\infty P(n_{obs}|\lambda')d\lambda' . \qquad (52)$$

However the main drawback of the definition (51) is that the probability that the signal s lies in the interval $0 \leq s \leq \infty$ is equal to $\int_b^\infty P(n_{obs}|\lambda')d\lambda'$ and it is less than unity for nonzero background $s > 0$ that contradicts to the intuition that the full probability that the signal s lies between zero and infinity must be equal to unity. To cure this drawback let us require that

$$1 - \beta' \geq \frac{P_-(n_{obs}|s + b)}{P_-(n_{obs}|b)} \geq \alpha' . \qquad (53)$$

The inequality (53) leads to the equations for the determination of s_{down} and s_{up} which coincide with the corresponding Bayes equations. The generalization of the inequalities (53) is straightforward, for instance the inequality (43) reads

$$1 - \beta' \geq \frac{P_-(n_{obs}|b + s; c_k)}{P_-(n_{obs}|b; c_k)} \geq \alpha' . \qquad (54)$$

Upper limit on the signal s derived from the inequality (53) coincides with the upper limit in CL_s method [7, 8].

Note that frequentist equations (18, 19) for $\lambda_{up} = \lambda_{down}$ don't satisfy the evident equality $\alpha' + \beta' = 1$. One of the possible generalizations of the equations (18, 19) looks as follows

$$P_{-1}(n_{obs}|\lambda_{up}) = \alpha' , \qquad (55)$$

$$P_{+1}(n_{obs}|\lambda_{down}) = \beta' , \qquad (56)$$

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where

$$P_{-1}(n_{obs}|\lambda) = \sum_{n=0}^{n_{obs}-1} P(n|\lambda) \ + \frac{1}{2} P(n_{obs}|\lambda) \,, \tag{57}$$

$$P_{+1}(n_{obs}|\lambda) = \sum_{n_{obs}+1}^{n=\infty} P(n|\lambda) \ + \frac{1}{2} P(n_{obs}|\lambda) \,. \tag{58}$$

Note that

$$P_{-1}(n_{obs}|\lambda) + P_{+1}(n_{obs}|\lambda) = 1 \tag{59}$$

and $\alpha' + \beta' = 1$ for $\lambda_{up} = \lambda_{down}$. The equations (55 - 58) are equivalent to the Bayes equations (4, 5) with prior

$$\pi(\lambda) = \frac{1}{2}(1 + \frac{n_{obs}}{\lambda})\,. \tag{60}$$

The modified frequentist definition (29) takes the form

$$1 - \beta' \geq P_{-1}(n_{obs}|\lambda) \geq \alpha' \,. \tag{61}$$

To conclude let us stress our main result. For Poisson distribution we have proposed modified frequentist definition of the confidence interval and have shown the equivalence of the modified frequentist approach and Bayes approach. It means that frequentist approach is in fact nonunique.

References

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