Multiloop calculations in normalization point scheme

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Outline

- Motivation
- Theoretical background
  - normalization point scheme
  - free of divergences representation of RG-functions ("Theory without divergences")
  - 5-loop results for ϕ^4 model
- Technical details
  - some ideas used to improve sector decomposition

Motivation

Main objects of interest:

- models of critical dynamics
- theory of turbulence

Our main goal is to perform numerical calculation of β-function and anomalous dimensions for this models.

- ϵ-expansion
- dimensional regularization
- Euclidean space

Motivation. Why numerical calculations?

Models of critical dynamics[1] (A,B,...,H) are nonrelativistic dissipative models

$$\Delta(k,\omega) \propto \frac{1}{i\omega + \nu k^2}$$

Most models are calculated only at 2-loop order.

Simpliest model "A"is calculated at 3-loop order.

3-loop corrections → dilogarithm (Spence's integral)

4-loop corrections → a number of unsuccessfull attempts[2]

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[1] see **A.N.Vasil'ev** The Field Theoretic Renormalization Group in Critical Behaviour Theory and Stochastic Dynamics

[2] by A.N.Vasil'ev, S.E.Derkachov and J.Honkonen.

Motivation.

Solution: To develop a numerical approach that can be applied to a wide range of models with minimal modifications

The main testing ground for our approch is model ϕ^4 ($D = 4 - \epsilon$, Euclidean space)

- analitical results up to 5-loop order[3] [4]

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[3] **Chetyrkin K.G, Kataev A.L., Tkachev F.V.**, Phys.Lett., B99, 147 (1981); B101,457(E) (1981)

[4] **Kleinert H., Neu J., Shulte-Frohlinde V., Chetyrkin K.G., Larin S.A.**, Phys.Lett., B272,39 (1991); Erratum: B319, 545 (1993)

Theory without divergences

A common scheme used for calculations in the framework of ε-expansion:

- calculation of renormalization constants Z (singular functions containing poles on ε)
- calculation of anomalous dimensions (γ), β-function, fixed point and other quantities using the obtained Z (all poles on ε cancel each other)

$$\gamma_i = \beta \partial_g \ln Z_i$$

$\swarrow$ $\searrow$

effecively extract pole residues on ϵ from diagrams	find representation for β and γ where there are no poles on ϵ
$\downarrow$	$\downarrow$
Sector decomposition	Theory without divergences

Theory without divergences

We need

- to construct representation of β and γ through renormalized Green functions
- to represent renormalized quantities as a unified integrals
- No subtractions like "large"−"large" should arise in this integrals

The key point for all this tasks is subtraction scheme.

We use subtraction scheme that can be termed "zero-momentum subtraction at normalization point $m = \mu$"

μ – renormalization mass

Subtraction scheme

Renormalized action for model ϕ^4:

$$S = -\frac{1}{2}(m^2 Z_1 + p^2 Z_2 + \delta m^2)\phi^2 - \frac{1}{4!} g \mu^\varepsilon Z_3 \phi^4 \,, \qquad (1)$$

To fix subtraction scheme we need to specify 4 normalization conditions for renormalization constants Z_1, Z_2, Z_3, and parameter δm^2 (shift of critical temperature):

$$\Gamma_2^R \,|_{p=0, m=\mu} = -\mu^2 \,, \quad \partial_{p^2} \Gamma_2^R \,|_{p=0, m=\mu} = -1 \,, \qquad (2)$$

$$\Gamma_4^R \,|_{p=0, m=\mu} = -g\mu^\varepsilon \,, \qquad \Gamma_2^R \,|_{p=0, m=0} = 0 \,. \qquad (3)$$

Compare to subtraction at zero momentum:

$$\Gamma_2^R|_{p=0} = -m^2 \,, \quad \partial_{p^2} \Gamma_2^R|_{p=0} = -1 \,, \quad \Gamma_4^R|_{p=0} = -g\mu^\epsilon$$

Contribution of counter terms that comes from renormalization constants may be replaced by action of R-operation

R-operation

$$\Gamma^R = R\Gamma = (1-K)R'\Gamma, \tag{4}$$

Where R' - incomplete R operation (eliminating divergences in subgraphs)

$1-K$ - eliminates remaining superficial divergence

The choice of a particular operator K amounts to the choice of subtraction scheme

$$K\Gamma_4 = \Gamma_4|_{p=0,m=\mu}\,, \tag{5}$$

$$K\Gamma_2 = \left[\Gamma_2|_{m=0} + \frac{m^2}{\mu^2}\left(\Gamma_2|_{m=\mu} - \Gamma_2|_{m=0}\right)\right]_{p=0} + p^2\partial_{p^2}\Gamma_2|_{p=0,m=\mu}\,. \tag{6}$$

For quantities in which substitution $m=\mu$ is performed, operation (6) becomes:

$$K\Gamma_2|_{m=\mu} = \left(\Gamma_2|_{p=0} + p^2\partial_{p^2}\Gamma_2|_{p=0}\right)|_{m=\mu}\,. \tag{7}$$

Theory without divergences

What was planned:

- to construct representation of β and γ through renormalized Green functions
- to represent renormalized quantities as a unified integral
  Combination of representation[5]
  $$\chi^R = R\chi = \prod_i (1 - K_i)\chi, \quad ^6$$
  and our subtraction scheme solves this problem
- No subtractions like “large”−“large” should arise in this integrals
  Using $F(k) - \sum_{m=0}^{n} \frac{k^m}{m!} F^{(m)}|_{k=0} = \frac{1}{n!} \int_0^1 da(1-a)^n \partial_a^{n+1} F(ak)$
  $$R(\chi)|_{m=\mu} = \prod_i \frac{1}{n_i!} \int_0^1 da_i (1-a_i)^{n_i} \partial_{a_i}^{n_i+1} \chi(\{a\}), \qquad ^{7\ 8}$$

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[5] **O.I.Zav'yalov**, Renormalized Quantum Field Theory, 1979,1990

[6] i enumerates subgraphs and diagram χ as a whole

[7] a_i - a parameters that strech moments flowing into subgraph χ_i

[8] n_i- subgraph dimension

Representation of β and γ through renormalized Green functions

In our subtraction scheme renormalization group equations are the same as in MS:

$$\left(\mu\partial_\mu + \beta\partial_g - \gamma_{m^2} m^2 \partial_{m^2}\right) \Gamma_n^R = n\gamma_\varphi \Gamma_n^R, \tag{8}$$

Usualy β and γ_i are calculated from Z_i using:

$$\beta = -g(\varepsilon + \gamma_g), \qquad \gamma_i = -\varepsilon \cdot \frac{g\partial_g \ln Z_i}{1 + g\partial_g \ln Z_g}\,. \tag{9}$$

We will calculate quantities (9) using RG-equations at normalization point ($p = 0$, $m = \mu$)

Representation of β and γ through renormalized Green functions.

RG-equations at normalization point

It is enough to choose the following set of Green functions:

$$\bar{\Gamma}_1 = -\left(\frac{\Gamma_2 - \Gamma_2|_{m=0}}{m^2}\right)\Big|_{p=0}, \; \bar{\Gamma}_2 = -\partial_{p^2}\Gamma_2|_{p=0}, \; \bar{\Gamma}_4 = \frac{\Gamma_4|_{p=0}}{-g\mu^{2\varepsilon}}, \tag{10}$$

$$R\bar{\Gamma}_i\,|_{m=\mu} = 1, \quad \bar{\Gamma}_i(m,\mu) \equiv \bar{\Gamma}_i(\tau), \quad \tau \equiv \frac{m^2}{\mu^2}, \; i = 1, 2, 4.$$

RG-equations have the following form:

$$\Big(\beta\partial_g - (2+\gamma_{m^2})\tau\partial_\tau - \gamma_i\Big)\bar{\Gamma}_i^R = 0, \quad i = 1, 2, 4. \tag{11}$$

At normalization point ($\tau = 1$):

$$(2+\gamma_{m^2})\,F_i = \gamma_i, \qquad F_i \equiv \hat{\partial}_\tau \bar{\Gamma}_i^R, \qquad i = 1, 2, 4, \tag{12}$$

$$\hat{\partial}_\tau \ldots \equiv -\partial_\tau \ldots\,|_{\tau=1}\ .$$

Representation of β and γ through renormalized Green functions.

$$(2+\gamma_{m^2})\,F_i = \gamma_i\,, \qquad F_i \equiv \hat{\partial}_\tau \bar{\Gamma}_i^R\,, \qquad i = 1,\,2,\,4$$

Solving this system:

$$\gamma_i = \frac{2F_i}{1+F_2-F_1}\,, \qquad i = 1,\,2,\,4. \tag{13}$$

F_i are finite quantities, but they are not suitable for calculations due to complicated subtraction operation.

$$K\Gamma_2 = \Gamma_2\,|_{p=0,m=0} + \frac{m^2}{\mu^2}\,(\Gamma_2\,|_{p=0,m=\mu} - \Gamma_2\,|_{p=0,m=0}) + p^2\cdot(\partial_{p^2}\Gamma_2)\,|_{p=0,m=\mu}$$

Thus we can't use representation
$R\chi = \prod_i \frac{1}{n_i!}\int_0^1 da_i(1-a_i)^{n_i}\partial_{a_i}^{n_i+1}\chi(\{a\})$,

More suitable quantities are $f_i = R(-m^2\partial_{m^2}\bar{\Gamma}_i)|_{m=\mu}$, for which subtraction has the simple form:

$$(K\Gamma_2)\,|_{m=\mu} = \Gamma_2\,|_{p=0,m=\mu} + p^2\cdot(\partial_{p^2}\Gamma_2)\,|_{p=0,m=\mu}\,, \tag{14}$$

and representation $R\chi = \prod_i \frac{1}{n_i!}\int_0^1 da_i(1-a_i)^{n_i}\partial_{a_i}^{n_i+1}\chi(\{a\})$, is possible

Representation of β and γ through renormalized Green functions.

It is possible to express quantities f_i in terms of F_i

$$f_i = \frac{F_i}{1 - F_1}\,, \qquad i = 2,\, 4\,, \tag{15}$$

and obtain representation:

$$\gamma_i = \frac{2f_i}{1 + f_2}\,, \quad f_i = R(-m^2 \partial_{m^2} \bar{\Gamma}_i)|_{m=\mu}\,, \qquad i = 2,\, 4\,. \tag{16}$$

this is desired representation of anomalous dimensions that satisfies our requirements.
Although relation (15) looks very simple, it took much effort to prove it.

Thus,

- anomalous dimensions are calculated **directly from diagrams** of 1-irreducible functions
- the contribution of each diagram is presented in **a form of UV-finite integral**

Current results

We've calculated[9] 5-loop corrections for index η and ω (compared to exact results[10]):

$$\eta = 0.0256566\epsilon^5 - 0.00832874\epsilon^4 + 0.018689988\epsilon^3 + 0.0(185)\epsilon^2$$

$$\eta_{exact} = 0.025656451\epsilon^5 - 0.008328770\epsilon^4 + 0.0186899862\epsilon^3 + 0.0(185)\epsilon^2$$

$$\omega = 20.741\epsilon^5 - 5.23517\epsilon^4 + 1.618219\epsilon^3 - 0.(629)\epsilon^2 + \epsilon$$

$$\omega_{exact} = 20.74984\epsilon^5 - 5.2351359\epsilon^4 + 1.61822067\epsilon^3 - 0.(629)\epsilon^2 + \epsilon$$

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[9] about 80 hours on 64-core cluster

[10] **Chetyrkin K.G, Kataev A.L., Tkachev F.V.**, Phys.Lett., B99, 147 (1981); B101,457(E) (1981)
Kleinert H., Neu J., Shulte-Frohlinde V., Chetyrkin K.G., Larin S.A., Phys.Lett., B272,39 (1991); Erratum: B319, 545 (1993)

Fast Sector Decomposition for diagrams at zero external momenta

Fast Sector Decomposition

Why we don't use FIESTA?

- in our approach integrand has the following form:

$$\int_0^1 da_1 \ldots da_n \int_0^1 du_1 \ldots du_m \delta(1-\sum_{i=1}^{m} u_i) f(u_1, \ldots, u_m, a_1, \ldots, a_n)$$

  which doesn't suites good to any known decomposition strategy
- we don't need to extract poles on ϵ
- we'd like to take into account graph symmetries

Fast Sector Decomposition

Main idea:

analyse graph not integrand

- no symbolic calculations (symbolic calculations are very slow)

  we construct sector decomposition for diagramm analysing it as graph (not as expression).
  (graph analysis can be written in a very efficient way)

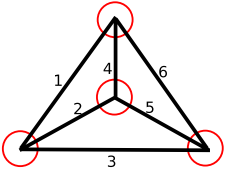
- do not calculate equal sectors

  graph analisys easily allows one to find nontrivial equal sectors using graph symmetries
  less sectors ⇒ faster calculations ⇒ higher accuracy

Fast Sector Decomposition

To construct Feynman representation for graph we use
"conservation laws"

$$\gamma = \frac{m^{-3\varepsilon}}{8}\,\Gamma(3\varepsilon/2)\,\Gamma^3(2-\varepsilon/2)\int_0^1 du_1 \ldots \int_0^1 du_6\,\frac{\delta(1-\sum_{i=1}^{6} u_i)}{\mathcal{D}^{2-\varepsilon/2}}.$$



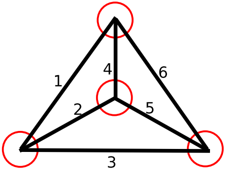
Conservation laws: $(1,2,3),(1,4,6),(2,4,5),(3,5,6)$

$$\begin{aligned}\mathcal{D} &= u_1u_2u_3 + u_1u_2u_4 + u_1u_2u_5 + u_1u_2u_6 + u_1u_3u_4 + u_1u_3u_5 + u_1u_3u_6 \\ &+ u_1u_4u_5 + u_1u_4u_6 + u_1u_5u_6 + u_2u_3u_4 + u_2u_3u_5 + u_2u_3u_6 \\ &+ u_2u_4u_5 + u_2u_4u_6 + u_2u_5u_6 + u_3u_4u_5 + u_3u_4u_6 + u_3u_5u_6 + u_4u_5u_6\end{aligned}$$

Fast Sector Decomposition

To construct Feynman representation for graph we use "conservation laws"

$$\gamma = \frac{m^{-3\varepsilon}}{8}\Gamma(3\varepsilon/2)\,\Gamma^3(2-\varepsilon/2)\int_0^1 du_1 \ldots \int_0^1 du_6\, \frac{\delta(1-\sum_{i=1}^{6} u_i)}{\mathcal{D}^{2-\varepsilon/2}}.$$



Conservation laws: $(1,2,3), (1,4,6), (2,4,5), (3,5,6)$

$$\begin{aligned}\mathcal{D} = {} & u_1u_2u_4 + u_1u_2u_5 + u_1u_2u_6 + u_1u_3u_4 + u_1u_3u_5 + u_1u_3u_6 \\ + {} & u_1u_4u_5 + u_1u_5u_6 + u_2u_3u_4 + u_2u_3u_5 + u_2u_3u_6 \\ & + u_2u_4u_6 + u_2u_5u_6 + u_3u_4u_5 + u_3u_4u_6 + u_4u_5u_6\end{aligned}$$

Completly equivalent to 1- and 2- tree approach (but more intuitive)

Fast Sector Decomposition

Main task of Sector Decomposition is to remove integrable divergences like

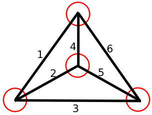
$$\int_0^1 dx \int_0^1 dy \frac{1}{x+y}$$

splitting on sectors ($x < y$ and $x > y$) and transforming expression to

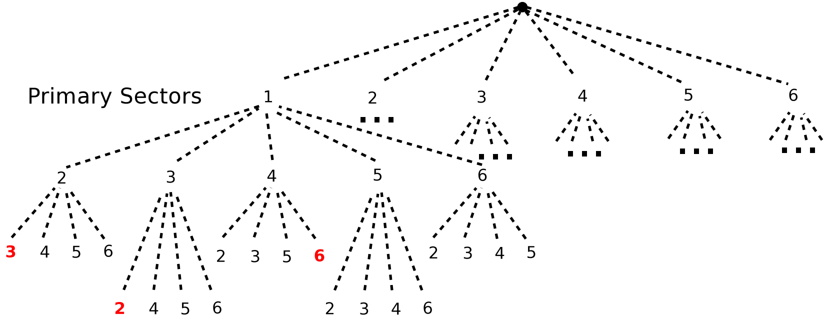
$$\int_0^1 dx \int_0^x dy \frac{1}{x+y} + \int_0^1 dy \int_0^y dx \frac{1}{x+y} =$$

$$= \int_0^1 d\tilde{x} \int_0^1 dy \frac{1}{1+\tilde{x}} + \int_0^1 dx \int_0^1 d\tilde{y} \frac{1}{1+\tilde{y}}$$

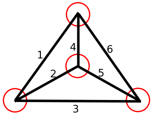
Fast Sector Decomposition



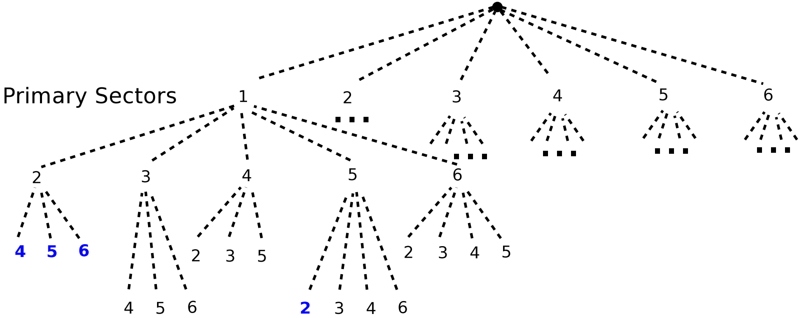
Conservation laws: $(1, 2, 3), (1, 4, 6), (2, 4, 5), (3, 5, 6)$



Fast Sector Decomposition



Conservation laws: $(1,2,3),(1,4,6),(2,4,5),(3,5,6)$



Total: 96 sectors, only 4 unique sectors [11]

(1,2,4), (1,2,5), (1,2,6), (1,5,2)

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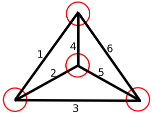
[11] can be easily found using Nickel index
Nikel B., Meiron D. Baker G. Compilation of 2-pt. and 4-pt. graphs for continous spin models. - University of Guelf Report,1977

Fast Sector Decomposition



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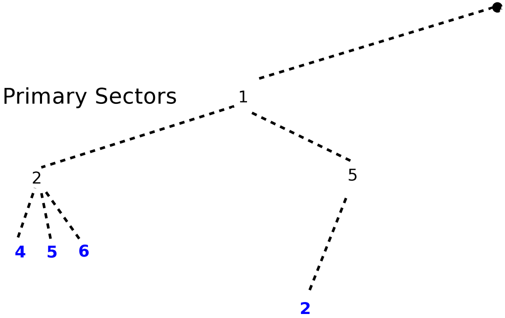
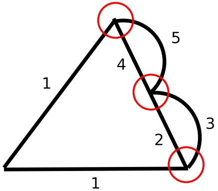


Using symmetries we are able to calculate this diagramm 24x faster.

Fast Sector Decomposition

$$\gamma = \frac{m^{-3\varepsilon}}{8}\,\Gamma(3\varepsilon/2)\,\Gamma^3(2-\varepsilon/2)\int_0^1 du_1 \ldots \int_0^1 du_5\, \frac{u_1\delta(1-\sum_{i=1}^5 u_i)}{\mathcal{D}^{2-\varepsilon/2}}.$$

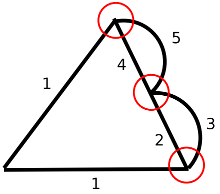


Conservation laws: $(1,2,3),(1,4,5),(2,3,4,5)$

$$\begin{aligned}\mathcal{D} &= {\color{red}u_1u_2u_3} + u_1u_2u_4 + u_1u_2u_5 + u_1u_3u_4 + u_1u_3u_5 \\ &+ {\color{red}u_1u_4u_5} + u_2u_3u_4 + u_2u_3u_5 + u_2u_4u_5 + u_3u_4u_5\end{aligned}$$

Fast Sector Decomposition

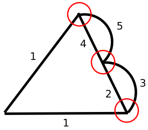
$$\gamma = \frac{m^{-3\varepsilon}}{8} \Gamma(3\varepsilon/2) \Gamma^3(2-\varepsilon/2) \int_0^1 du_1 \dots \int_0^1 du_5 \, \frac{u_1 \delta(1 - \sum_{i=1}^5 u_i)}{\mathcal{D}^{2-\varepsilon/2}}.$$



Conservation laws: $(1,2,3), (1,4,5), (2,3,4,5)$

$$\mathcal{D} = \begin{aligned} &+ u_1 u_2 u_4 + u_1 u_2 u_5 + u_1 u_3 u_4 + u_1 u_3 u_5 \\ &+ u_2 u_3 u_4 + u_2 u_3 u_5 + u_2 u_4 u_5 + u_3 u_4 u_5 \end{aligned}$$

Fast Sector Decomposition



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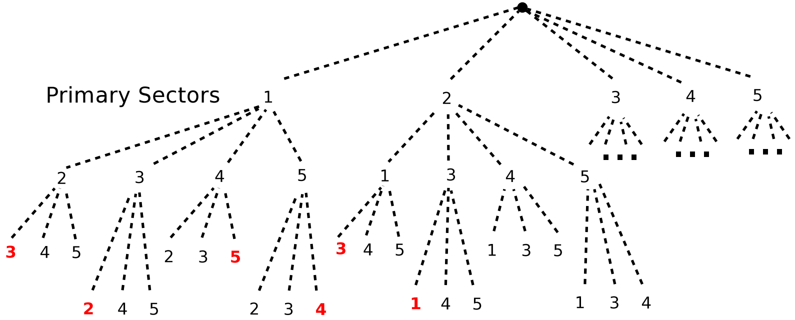

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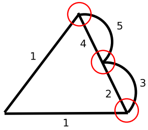
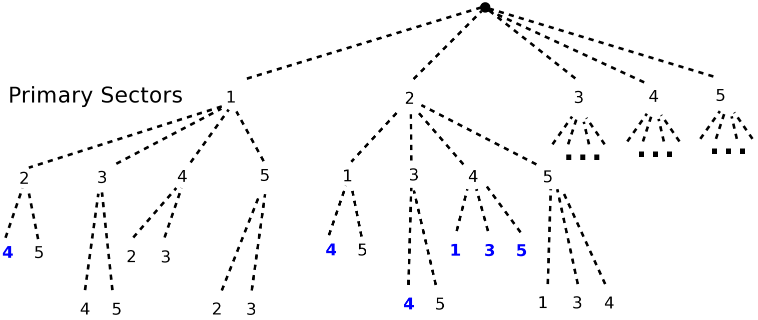


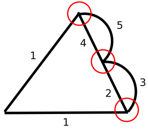
Conservation laws: $(1,2,3),(1,4,5),(2,3,4,5)$



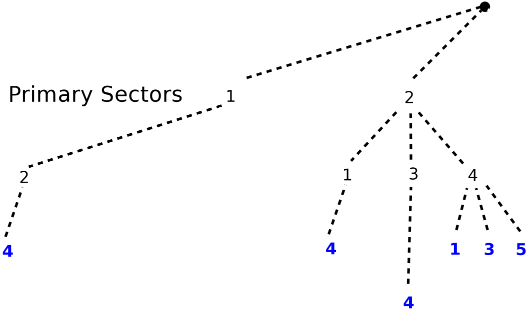
Total: 48 sectors, only 6 unique sectors

(1,2,4), (2,1,4), (2,3,4), (2,4,1), (2,4,3), (2,4,5)

Fast Sector Decomposition

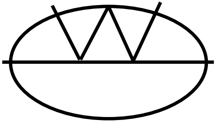


Conservation laws: $(1,2,3),(1,4,5),(2,3,4,5)$



Using symmetries we are able to calculate this diagramm 8x faster.

6 loop?



Decomposition time: 2 hours

Evaluation with accuracy 0.01%: 2 days on 48 core cluster

To calculate 6 loop corrections we need

- either to use realy huge cluster
- or to use combined calculation scheme
  (evaluate as much diagramms as possible analiticaly using master integrals and rest integrals evaluate using normalization point scheme)

Thank you