Conformal group: R-matrix and star-triangle relations.

D. Chicherin [2], A. P. Isaev[1] and S. Derkachov [3]

[1]Bogoliubov Laboratory of Theoretical Physics,
JINR, Dubna, Russia

[2]Chebyshev Laboratory, St.-Petersburg State University,
14th Line, 29b, Saint-Petersburg, 199178 Russia

[3]St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences, Fontanka 27, 191023 St. Petersburg, Russia

CALC – 2012
July 23 – August 2, 2012, Dubna, Russia

Plan

1. **Introduction**
   - Star-triangle relations (STR) and multiloop calculations
   - Definition of conformal algebra $conf(\mathbb{R}^{p,q})$
   - Spinor representations of $conf(\mathbb{R}^{p,q}) = so(p+1, q+1)$
   - Differential representation of $conf(\mathbb{R}^{p,q}) = so(p+1, q+1)$
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Star-triangle relation

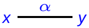
Firstly considered in CFT and the "Method Of Uniqueness" (E.S.Fradkin, M.Ya.Palchik, 1978; A.N.Vasilev,Yu.M.Pis'mak,Yu.R.Khonkonen, 1981; D.Kazakov, 1983)

$$\int \frac{d^D z}{(x-z)^{2\alpha'}\, z^{2(\alpha+\beta)}\, (z-y)^{2\beta'}} = \frac{G(\alpha,\beta)}{(x)^{2\beta}\, (x-y)^{2(\frac{D}{2}-\alpha-\beta)}\, (y)^{2\alpha}} \,,$$

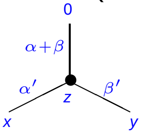
where $x, y, z \in \mathbb{R}^D$, $\boxed{\alpha' := \frac{D}{2} - \alpha}$,

$$(x)^{2\beta} = (x_\mu x^\mu)^\beta \,, \quad G(\alpha,\beta) = \frac{a(\alpha+\beta)}{a(\alpha)a(\beta)} \,, \quad a(\beta) = \frac{\Gamma(\beta')}{\pi^{D/2}\, 2^{2\beta}\, \Gamma(\beta)} \,.$$

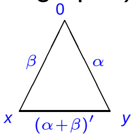
Graphic representation of STR (reconstruction of graphs):



$= \frac{1}{(x-y)^{2\alpha}} \quad \Rightarrow$



$= G(\alpha,\beta)\cdot$



Operator version of STR: (API, 2003)

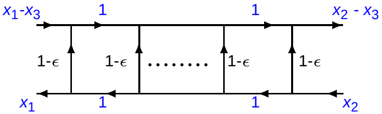
$$\boxed{\hat{p}^{-2\alpha} \cdot \hat{q}^{-2(\alpha+\beta)} \cdot \hat{p}^{-2\beta} = \hat{q}^{-2\beta} \cdot \hat{p}^{-2(\alpha+\beta)} \cdot \hat{q}^{-2\alpha}}$$!!!

where we have used Heisenberg algebra: $[\hat{q}_\mu, \, \hat{p}_\nu] = \delta_{\mu\nu}$.

L-loop ladder diagrams for ϕ^3 FT $\Leftrightarrow$ D-dimensional conformal QM

Consider dimensionally regularized ($D = 4 - 2\epsilon$) massless integrals which correspond to the diagrams:

$$D_L(x_a) \;\; =$$



The operator version is:

$$D_L(x_a) \sim \langle x_1 | \hat{p}^{-2} \cdot \left(\hat{q}^{-2} \cdot (\hat{q} - x_3)^{-2} \cdot \hat{p}^{-2} \right)^L | x_2 \rangle \; .$$

where we have introduced the states $\hat{q}|x_2\rangle = x_2|x_2\rangle$. The generating function for D_L (API, 2003,2007):

$$D_g(x_a) = \sum_{L=0}^{\infty} g^L \, D_L(x_a) \sim \frac{1}{(x_1^2 x_2^2)^{(D/2-1)}} \langle u | \left(\hat{p}^2 - \frac{g_x}{\hat{q}^2} \right)^{-1} | v \rangle \; ,$$

where $g_x = g/(a(1)(x_3)^2)$ is a new coupling constant and $u_i = \frac{(x_1)_i}{(x_1)^2} - \frac{(x_3)_i}{(x_3)^2} = \frac{1}{x_1} - \frac{1}{x_3}$, $v_i = \frac{(x_2)_i}{(x_2)^2} - \frac{(x_3)_i}{(x_3)^2} = \frac{1}{x_2} - \frac{1}{x_3}$.

Conformal invariance gives the identity

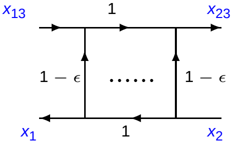
$$u^{2(D/2-1)} \langle u| \left(\hat{p}^2 - \frac{g}{\hat{q}^2}\right)^{-1} |v\rangle = (u')^{2(D/2-1)} \langle u'| \left(\hat{p}^2 - \frac{g}{\hat{q}^2}\right)^{-1} |v'\rangle \, ,$$

where two new vectors (u', v') are related to vectors (u, v):

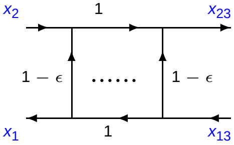
$$\frac{v^2}{u^2} = \frac{(v')^2}{(u')^2} \, , \quad \frac{(uv)}{u^2} = \frac{(u'v')}{(u')^2} \, .$$

One can choose $u' = \frac{1}{x_1} - \frac{1}{x_{12}}$, $v' = \frac{1}{x_{13}} - \frac{1}{x_{12}}$, where $x_{ab} = x_a - x_b$, $(\frac{1}{x})_\mu = \frac{x_\mu}{x^2}$ and the expansion over g leads to the identities for ladder diagrams (API, 2007)

$x_3^{2(1+L-\frac{D}{2})} \times$



$= (x_{12})^{2(1+L-\frac{D}{2})} \times$



This is main eq. which is needed for prooving of the magic identities for $D = 4(\epsilon = 0)$ J.Drummond, J. Henn, V.A.Smirnov and E.Sokatchev, 2006

$\mathbb{R}^{p,q}$ — pseudoeuclidean space with the metric

$$g_{\mu\nu} = \mathrm{diag}(\underbrace{1,\ldots,1}_{p},\underbrace{-1,\ldots,-1}_{q}) \, .$$

$\mathrm{conf}(\mathbb{R}^{p,q})$ — Lie algebra of the conformal group in $\mathbb{R}^{p,q}$ generated by $\{L_{\mu\nu}, P_\mu, K_\mu, D\}$ $(\mu,\nu = 0,1,\ldots,p+q-1)$:

$$[D\,,\,P_\mu] = i\,P_\mu \;\;,\quad [D\,,\,K_\mu] = -i\,K_\mu \;\;,$$

$$[L_{\mu\nu}\,,\,L_{\rho\sigma}] = i\,(g_{\nu\rho}\,L_{\mu\sigma} + g_{\mu\sigma}\,L_{\nu\rho} - g_{\mu\rho}\,L_{\nu\sigma} - g_{\nu\sigma}\,L_{\mu\rho})$$

$$[K_\rho\,,\,L_{\mu\nu}] = i\,(g_{\rho\mu}\,K_\nu - g_{\rho\nu}\,K_\mu)\,,\quad [P_\rho\,,\,L_{\mu\nu}] = i\,(g_{\rho\mu}\,P_\nu - g_{\rho\nu}\,P_\mu)\,,$$

$$[K_\mu\,,\,P_\nu] = 2i\,(g_{\mu\nu}\,D - L_{\mu\nu})\,,\quad [P_\mu\,,\,P_\nu] = 0\,,$$

$$[K_\mu\,,\,K_\nu] = 0\,,\quad [L_{\mu\nu}\,,\,D] = 0\,.$$

$L_{\mu\nu}$ – generators for the rotation group $SO(p,q)$ in $\mathbb{R}^{p,q}$,
P_ν – shift generators in $\mathbb{R}^{p,q}$,
D – dilatation operator,
K_ν – conformal boost generators.

We have the isomorphism:

$$\mathrm{conf}(\mathbb{R}^{p,q}) = so(p+1, q+1)$$

$$L_{\mu\nu} = M_{\mu\nu} \ , \quad K_\mu = M_{n,\mu} - M_{n+1,\mu} \ ,$$

$$P_\mu = M_{n,\mu} + M_{n+1,\mu} \ , \quad D = -M_{n,n+1} \ , \quad (n = p+q) \ .$$

where M_{ab} $(a, b = 0, 1, \ldots, n+1)$ are generators of $so(p+1, q+1)$:

$$[M_{ab}, M_{dc}] = i(g_{bd}M_{ac} + g_{ac}M_{bd} - g_{ad}M_{bc} - g_{bc}M_{ad}) \ ,$$

$$g_{ab} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}, 1, -1) \ .$$

Quadratic Casimir operator

$$C_2 = \frac{1}{2} M_{ab} M^{ab} = \frac{1}{2}(L_{\mu\nu}L^{\mu\nu} + P_\mu K^\mu + K_\mu P^\mu) - D^2 \ .$$

Let $n = p + q (= D)$ be even integer and γ_μ $(\mu = 0, ..., n-1)$ be $2^{\frac{n}{2}}$-dimensional gamma-matrices in $\mathbb{R}^{p,q}$:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} I \, ,$$

$$\gamma_{n+1} \equiv \alpha \, \gamma_0 \cdot \gamma_1 \cdots \gamma_{n-1} \, , \quad \alpha^2 = (-1)^{q + n(n-1)/2} \, ,$$

where α is such that $\gamma_{n+1}^2 = I$. Using gamma-matrices γ_μ in $\mathbb{R}^{p,q}$ one can construct representation T_1 of $\mathrm{conf}(\mathbb{R}^{p,q}) = so(p+1, q+1)$

$$T_1(L_{\mu\nu}) = \frac{i}{4} \, [\gamma_\mu , \gamma_\nu] \equiv \ell_{\mu\nu} \, , \quad T_1(K_\mu) = \gamma_\mu \, \frac{(1-\gamma_{n+1})}{2} \equiv k_\mu \, ,$$

$$T_1(P_\mu) = \gamma_\mu \, \frac{(1+\gamma_{n+1})}{2} \equiv p_\mu \, , \quad T_1(D) = -\frac{i}{2} \, \gamma_{n+1} \equiv d \, .$$

The common representation for γ_μ in $\mathbb{R}^{p,q}$ is:

$$\gamma_\mu = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}_\mu \\ \overline{\boldsymbol{\sigma}}_\mu & \mathbf{0} \end{pmatrix} , \quad \gamma_{n+1} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} ,$$

where $\boldsymbol{\sigma}_\mu \overline{\boldsymbol{\sigma}}_\nu + \boldsymbol{\sigma}_\nu \overline{\boldsymbol{\sigma}}_\mu = 2\, g_{\mu\nu} \mathbf{1}$, $\quad \overline{\boldsymbol{\sigma}}_\mu \boldsymbol{\sigma}_\nu + \overline{\boldsymbol{\sigma}}_\nu \boldsymbol{\sigma}_\mu = 2\, g_{\mu\nu} \mathbf{1}$.
The representation T_1 of $conf(\mathbb{R}^{p,q})$ is

$$\ell_{\mu\nu} = \begin{pmatrix} \frac{i}{4}(\boldsymbol{\sigma}_\mu \overline{\boldsymbol{\sigma}}_\nu - \boldsymbol{\sigma}_\nu \overline{\boldsymbol{\sigma}}_\mu) & \mathbf{0} \\ \mathbf{0} & \frac{i}{4}(\overline{\boldsymbol{\sigma}}_\mu \boldsymbol{\sigma}_\nu - \overline{\boldsymbol{\sigma}}_\nu \boldsymbol{\sigma}_\mu) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}_{\mu\nu} & \mathbf{0} \\ \mathbf{0} & \overline{\boldsymbol{\sigma}}_{\mu\nu} \end{pmatrix} ,$$

$$p^\mu = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \overline{\boldsymbol{\sigma}}^\mu & \mathbf{0} \end{pmatrix} , \quad k^\mu = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^\mu \\ \mathbf{0} & \mathbf{0} \end{pmatrix} , \quad d = -\frac{i}{2} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} .$$

Recall that

$$\boldsymbol{\sigma}_{\mu\nu} = ||(\boldsymbol{\sigma}_{\mu\nu})_\alpha{}^\beta|| , \quad \overline{\boldsymbol{\sigma}}_{\mu\nu} = ||(\overline{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}|| ,$$

are inequivalent spinor representations of $so(p,q) = spin(p,q)$.

Any element of $conf(\mathbb{R}^{p,q})$ in the representation T_1 is

$$A = i(\omega^{\mu\nu}\,\ell_{\mu\nu} + a^{\mu}\,p_{\mu} + b^{\mu}\,k_{\mu} + \beta\,d) =$$

$$= \begin{pmatrix} \frac{\beta}{2}\mathbf{1} + i\omega^{\mu\nu}\boldsymbol{\sigma}_{\mu\nu} & ib^{\mu}\boldsymbol{\sigma}_{\mu} \\ ia^{\mu}\overline{\boldsymbol{\sigma}}_{\mu} & -\frac{\beta}{2}\mathbf{1} + i\omega^{\mu\nu}\overline{\boldsymbol{\sigma}}_{\mu\nu} \end{pmatrix} \equiv \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} .$$

It can be considered as the matrix of parameters $\omega^{\mu\nu}$, a^{μ}, b^{μ}, $\beta \in \mathbb{R}$.

The standard differential representation ρ of $conf(\mathbb{R}^{p,q})$ can be obtained by the method of induced representations (G. Mack and A. Salam (1969))

$$\rho(P_\mu) = -i\partial_{x_\mu} \equiv \hat{p}_\mu \ , \quad \rho(D) = x^\mu \hat{p}_\mu - i\Delta \ ,$$

$$\rho(L_{\mu\nu}) = \hat{\ell}_{\mu\nu} + S_{\mu\nu} \ , \quad \rho(K_\mu) = 2x^\nu\,(\hat{\ell}_{\nu\mu} + S_{\nu\mu}) + (x^\nu x_\nu)\hat{p}_\mu - 2i\Delta x_\mu \ ,$$

$$\hat{\ell}_{\mu\nu} \equiv (x_\nu \hat{p}_\mu - x_\mu \hat{p}_\nu) \ ,$$

where $x_\mu \equiv \hat{q}_\mu$ are coordinates in $\mathbb{R}^{p,q}$, $\Delta \in \mathbb{R}$ – conformal parameter, $S_{\mu\nu} = -S_{\nu\mu}$ are spin generators with commutation relations as for $\hat{\ell}_{\mu\nu}$ and $[S_{\mu\nu}, x_\rho] = 0 = [S_{\mu\nu}, \hat{p}_\rho]$. For the quadratic Casimir operator we have:

$$\rho(C_2) = \frac{1}{2}\left(S_{\mu\nu}\,S^{\mu\nu} - \hat{\ell}_{\mu\nu}\,\hat{\ell}^{\mu\nu}\right) + \Delta(\Delta - n) \ .$$

The representations ρ_Δ and $\rho_{n-\Delta}$ are contragradient to each other and in particular we have $\rho_\Delta(C_2) = \rho_{n-\Delta}(C_2)$.

In the representation ρ elements of $conf(\mathbb{R}^{p,q})$ act on the fields $\Phi(\mathbf{x})$:

$$\rho(\omega^{\mu\nu} L_{\mu\nu} + a^\mu P_\mu + b^\mu K_\mu + \beta D)\, \Phi(\mathbf{x}) =$$

$$= \mathrm{Tr}\left[\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \left(T_1(M^{ab}) \cdot \rho(M_{ab})\right)\right] \Phi(\mathbf{x}) \; .$$

where $\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}$ is the matrix of parameters,

$$\frac{1}{2} T_1(M^{ab}) \cdot \rho(M_{ab}) = (T_1 \otimes \rho) \left(\frac{1}{2} M^{ab} \otimes M_{ab} \right) =$$

$$= \begin{pmatrix} \frac{\Delta - n}{2} \cdot \mathbf{1} + \mathbf{S} - \mathbf{p} \cdot \mathbf{x} \, , & \mathbf{p} \\ \mathbf{x} \cdot \mathbf{S} - \overline{\mathbf{S}} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{p} \cdot \mathbf{x} + (\Delta - \frac{n}{2}) \cdot \mathbf{x} \, , & -\frac{\Delta}{2} \cdot \mathbf{1} + \overline{\mathbf{S}} + \mathbf{x} \cdot \mathbf{p} \end{pmatrix} ,$$

Here we introduced

$$\mathbf{p} = \tfrac{1}{2} \boldsymbol{\sigma}^\mu \hat{p}_\mu = -\tfrac{i}{2} \boldsymbol{\sigma}^\mu \partial_{x_\mu} \; , \quad \mathbf{x} = -i\, \overline{\boldsymbol{\sigma}}^\mu x_\mu \; ,$$

$$\overline{\mathbf{S}} = \tfrac{1}{2} \overline{\boldsymbol{\sigma}}^{\mu\nu} S_{\mu\nu} \; , \quad \mathbf{S} = \tfrac{1}{2} \boldsymbol{\sigma}^{\mu\nu} S_{\mu\nu} \; .$$

The action of spin generators $S_{\mu\nu}$ on tensor fields of the type $(\ell, \dot{\ell})$ is

$$[S_{\mu\nu}\Phi]^{\dot\alpha_1\cdots\dot\alpha_{2\dot\ell}}_{\alpha_1\cdots\alpha_{2\ell}} = (\boldsymbol{\sigma}_{\mu\nu})_{\alpha_1}^{\ \ \alpha}\, \Phi^{\dot\alpha_1\cdots\dot\alpha_{2\dot\ell}}_{\alpha\alpha_2\cdots\alpha_{2\ell}} + \cdots + (\boldsymbol{\sigma}_{\mu\nu})_{\alpha_{2\ell}}^{\ \ \alpha}\, \Phi^{\dot\alpha_1\cdots\dot\alpha_{2\dot\ell}}_{\alpha_1\cdots\alpha_{2\ell-1}\alpha} + \\ + (\bar{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha_1}_{\ \ \dot\alpha}\, \Phi^{\dot\alpha\dot\alpha_2\cdots\dot\alpha_{2\dot\ell}}_{\alpha_1\cdots\alpha_{2\ell}} + \cdots + (\bar{\boldsymbol{\sigma}}_{\mu\nu})^{\dot\alpha_{2\ell}}_{\ \ \dot\alpha}\, \Phi^{\dot\alpha_1\cdots\dot\alpha_{2\ell-1}\dot\alpha}_{\alpha_1\cdots\alpha_{2\ell}} .$$

For symmetric representations it is convenient to work with the generating functions

$$\Phi(x,\lambda,\tilde\lambda) = \Phi^{\dot\alpha_1\cdots\dot\alpha_{2\dot\ell}}_{\alpha_1\cdots\alpha_{2\ell}}(x)\,\lambda^{\alpha_1}\cdots\lambda^{\alpha_{2\ell}}\,\tilde\lambda_{\dot\alpha_1}\cdots\tilde\lambda_{\dot\alpha_{2\dot\ell}} ,$$

where λ and $\tilde\lambda$ are auxiliary spinors and the action of $S_{\mu\nu}$ is

$$[S_{\mu\nu}\Phi](x,\lambda,\tilde\lambda) = \left[\lambda\,\boldsymbol{\sigma}_{\mu\nu}\partial_\lambda + \tilde\lambda\,\bar{\boldsymbol{\sigma}}_{\mu\nu}\partial_{\tilde\lambda}\right]\Phi(x,\lambda,\tilde\lambda) ,$$

where $\lambda\,\boldsymbol{\sigma}_{\mu\nu}\partial_\lambda = \lambda_\alpha\,(\boldsymbol{\sigma}_{\mu\nu})^\alpha_{\ \ \beta}\,\partial_{\lambda_\beta}$, $\tilde\lambda\,\bar{\boldsymbol{\sigma}}_{\mu\nu}\partial_{\tilde\lambda} = \tilde\lambda^{\dot\alpha}\,(\bar{\boldsymbol{\sigma}}_{\mu\nu})_{\dot\alpha}^{\ \ \dot\beta}\,\partial_{\tilde\lambda^{\dot\beta}}$. The generators $S_{\mu\nu}$ are represented as differential operators over spinors

$$S_{\mu\nu} = \lambda\,\boldsymbol{\sigma}_{\mu\nu}\partial_\lambda + \tilde\lambda\,\overline{\boldsymbol{\sigma}}_{\mu\nu}\partial_{\tilde\lambda} .$$

For 4-dimensional case $\mathbb{R}^{p,q} = \mathbb{R}^{1,3}$ we have 2-component Weyl spinors $\lambda, \tilde{\lambda}$ and tensor fields $\Phi^{\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{\ell}}}_{\alpha_1\cdots\alpha_{2\ell}}(x)$ are automatically symmetric under permutations of dotted and undotted indices separately.
Then for $n = 4$ we have

$$\boldsymbol{\sigma}_\mu = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)\ , \qquad \overline{\boldsymbol{\sigma}}_\mu = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3)\ ,$$

where $\sigma_0 = \mathrm{I}_2$ and $\sigma_1, \sigma_2, \sigma_3$ are standard Pauli matrices.
Consequently we obtain for the self-dual components of $S_{\mu\nu}$

$$\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}^{\mu\nu}\, S_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\,\lambda_1\partial_{\lambda_1} - \frac{1}{2}\,\lambda_2\partial_{\lambda_2} & \lambda_2\partial_{\lambda_1} \\ \lambda_1\partial_{\lambda_2} & -\frac{1}{2}\,\lambda_1\partial_{\lambda_1} + \frac{1}{2}\,\lambda_2\partial_{\lambda_2} \end{pmatrix}$$

and for anti-self-dual components of $S_{\mu\nu}$

$$\overline{\mathbf{S}} = \frac{1}{2}\overline{\boldsymbol{\sigma}}^{\mu\nu}\, S_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\,\tilde{\lambda}^{\dot{1}}\partial_{\tilde{\lambda}^{\dot{1}}} - \frac{1}{2}\,\tilde{\lambda}^{\dot{2}}\partial_{\tilde{\lambda}^{\dot{2}}} & \tilde{\lambda}^{\dot{2}}\partial_{\tilde{\lambda}^{\dot{1}}} \\ \tilde{\lambda}^{\dot{1}}\partial_{\tilde{\lambda}^{\dot{2}}} & -\frac{1}{2}\,\tilde{\lambda}^{\dot{1}}\partial_{\tilde{\lambda}^{\dot{1}}} + \frac{1}{2}\,\tilde{\lambda}^{\dot{2}}\partial_{\tilde{\lambda}^{\dot{2}}} \end{pmatrix}$$

In fact, the operator $\mathbf{S}$ is restricted to the space of homogeneous polynomials in components of the spinor λ of degree 2ℓ so that one can choose new variables $\chi_1 = -\frac{\lambda_1}{\lambda_2}$, $t = -\lambda_2$ and obtain that $\mathbf{S}$ coincides with the following matrix $\mathbf{S}^{(\ell)}$ which contains parameter ℓ (the eigenvalue of the operator $\frac{1}{2}t\partial_t$):

$$\mathbf{S}^{(\ell)} = \begin{pmatrix} \chi_1\partial_{\chi_1} - \ell\,, & -\partial_{\chi_1} \\ \chi_1^2\,\partial_{\chi_1} - 2\ell\,\chi_1\,, & -\chi_1\,\partial_{\chi_1} + \ell \end{pmatrix} \equiv \begin{pmatrix} S_3 & S_- \\ S_+ & -S_3 \end{pmatrix}\,,$$

Similarly the operator $\bar{\mathbf{S}}$ is restricted to the space of homogeneous polynomials in components of the spinor $\tilde{\lambda}$ of degree $2\dot{\ell}$ so that for the the choice $\chi_2 = -\frac{\tilde{\lambda}^{\dot{1}}}{\tilde{\lambda}^{\dot{2}}}$ one obtains $\overline{\mathbf{S}} = \overline{\mathbf{S}}^{(\dot{\ell})}$, where

$$\overline{\mathbf{S}}^{(\dot{\ell})} = \begin{pmatrix} \chi_2\partial_{\chi_2} - \dot{\ell}\,, & -\partial_{\chi_2} \\ \chi_2^2\,\partial_{\chi_2} - 2\dot{\ell}\,\chi_2\,, & -\chi_2\,\partial_{\chi_2} + \dot{\ell} \end{pmatrix} \equiv \begin{pmatrix} \overline{S}_3 & \overline{S}_- \\ \overline{S}_+ & -\overline{S}_3 \end{pmatrix}\,.$$

L-operators

Let V be a vector space and I is the identity operator in V. Consider an operator $\mathrm{R}(u) \in \mathrm{End}(V \otimes V)$ which is a function of spectral parameter u and satisfies Yang-Baxter equation in the braid form

$$R^{i_1 i_2}_{j_1 j_2}(u-v)\, R^{j_2 i_3}_{\ell_2 k_3}(u)\, R^{j_1 \ell_2}_{k_1 k_2}(v) = R^{i_2 i_3}_{j_2 j_3}(v)\, R^{i_1 j_2}_{j_1 \ell_2}(u)\, R^{j_1 \ell_2}_{k_1 k_2}(u-v) \, ,$$

$$\mathrm{R}_{12}(u-v)\, \mathrm{R}_{23}(u)\, \mathrm{R}_{12}(v) = \mathrm{R}_{23}(v)\, \mathrm{R}_{12}(u)\, \mathrm{R}_{23}(u-v) \;\; \in \;\; \mathrm{End}(V \otimes V \otimes V) \, .$$

Here we use standard matrix notations: $\mathrm{R}_{12}(u) = \mathrm{R}(u) \otimes I$, etc. Let V' be another vector space and I' is the identity operator in V'. We call operator $\mathrm{L}(u) \in \mathrm{End}(V \otimes V')$ the <u>L-operator</u> in the spaces V and V' if

$$\mathrm{R}_{12}(u-v)\, \mathrm{L}_{13}(u)\, \mathrm{L}_{23}(v) = \mathrm{L}_{13}(v)\, \mathrm{L}_{23}(u)\, \mathrm{R}_{12}(u-v) \;\; \in \;\; \mathrm{End}(V \otimes V \otimes V') \, .$$

Here $\mathrm{L}_{23}(v) = I \otimes \mathrm{L}(v)$, $\mathrm{R}_{12}(u) = \mathrm{R}(u) \otimes I'$, etc.

L-operators for $g\ell(N)$

Consider Lie algebra $g\ell(N,\mathbb{C})$ with generators E_{ij} $(i,j=1,\dots,N)$:

$$[E_{ij},\ E_{k\ell}] = \delta_{jk}E_{i\ell} - \delta_{i\ell}E_{kj}\ .$$

Defining repr. T is: $T(E_{ij}) = e_{ij}$, where e_{ij} are matrix units. Introduce permutation matrix P:

$$P = \sum_{ij} e_{ij}\otimes e_{ji} \quad \Rightarrow \quad P\,w_1\otimes w_2 = w_2\otimes w_1 \quad (\forall\, w_1, w_2 \in \mathbb{C}^N)\ .$$

E.g. the commutativity $v^i v^j = v^j v^i$ is written as

$$v\otimes v - P\,v\otimes v = 0 \quad \Leftrightarrow \quad (\delta^i_k\delta^j_r - \delta^i_r\delta^j_k)v^k v^r = 0\ .$$

Define Yang R-matrix (which satisfies Yang-Baxter equation)

$$\mathrm{R}_{12}(u) := I_N\otimes I_N + u\,P \;\Rightarrow\; \mathrm{R}^{ij}_{kr}(u) := \delta^i_k\delta^j_r + u\,\delta^i_r\delta^j_k\ .$$

Consider the matrix $||\mathrm{L}^k_j(u)||$ with operator coefficients

$$\mathrm{L}^k_j(u) = \delta^k_j\mathbf{1} + \frac{1}{u}\left(E^{(0)}_{jk} + \frac{1}{u}E^{(1)}_{jk} + \frac{1}{u^2}E^{(2)}_{jk} + \dots\right)\ ,$$

which satisfies *RLL* relations with Yang R-matrix

$$\mathrm{R}^{i_1 i_2}_{k_1 k_2}(u-v)\,\mathrm{L}^{k_1}_{j_1}(u)\,\mathrm{L}^{k_2}_{j_2}(v) = \mathrm{L}^{i_1}_{k_1}(v)\,\mathrm{L}^{i_2}_{k_2}(u)\,\mathrm{R}^{k_1 k_2}_{j_1 j_2}(u-v)\ .$$

Substitute here $||\mathrm{L}^k_j(u)||$ and expand over $1/u$ and $1/v$:

$$[E^{(0)}_{ij},\ E^{(0)}_{k\ell}] = \delta_{jk} E^{(0)}_{i\ell} - \delta_{i\ell} E^{(0)}_{kj}\ ,$$

$$[E^{(0)}_{ij},\ E^{(1)}_{k\ell}] = \delta_{i\ell} E^{(1)}_{kj} - \delta_{jk} E^{(1)}_{i\ell}\ ,$$

$$[E^{(1)}_{ij},\ E^{(1)}_{k\ell}] = [E^{(2)}_{ij},\ E^{(0)}_{k\ell}] + E^{(0)}_{ij} \cdot E^{(1)}_{k\ell} - E^{(1)}_{ij} \cdot E^{(0)}_{k\ell}\ , \dots\ .$$

The infinite dim. algebra generated by elements $E^{(\alpha)}_{kj}$ is called Yangian $Y(g\ell(N,\mathbb{C}))$.

There is the evaluation homomorphism: $Y(g\ell(N,\mathbb{C})) \to g\ell(N,\mathbb{C})$

$$E^{(0)}_{ij} \to E_{ij}\ , \quad E^{(k)}_{ij} \to 0\ , \quad k \geq 1\ .$$

It gives L-operator

$$L(u) \to \mathrm{L}^k_j(u) = u\,\delta^k_j \mathbf{1} + \rho(E_{jk})\ .$$

which is related to $s\ell(N,\mathbb{C})$-type spin chains with spins in repr. ρ.

Now we consider $so(p+1,q+1)$-type operator:
$\mathrm{L}^{(\rho)}(u) = u\,\mathrm{I} + \frac{1}{2}T_1(M^{ab}) \otimes \rho(M_{ab}) =$

$$= \begin{pmatrix} u_+ \cdot \mathbf{1} + \mathbf{S} - \mathbf{p}\cdot\mathbf{x}\,, & \mathbf{p} \\ \mathbf{x}\cdot\mathbf{S} - \overline{\mathbf{S}}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{p}\cdot\mathbf{x} + (\Delta - \frac{n}{2})\cdot\mathbf{x}\,, & u_-\cdot\mathbf{1} + \overline{\mathbf{S}} + \mathbf{x}\cdot\mathbf{p} \end{pmatrix},$$

where T_1 is the spinor representation and ρ is the differential representation of the conformal algebra $so(p+1,q+1)$;

$$u_+ = u + \frac{\Delta - n}{2}\,, \quad u_- = u - \frac{\Delta}{2}\,, \quad n = p + q\,,$$

We have used the expression for the polarized Casimir operator $\frac{1}{2}T_1(M^{ab}) \otimes \rho(M_{ab})$ which was appeared in the discussion of the differential representation of the conformal algebra.
Our aim is to prove the *RLL* relation

$$\mathrm{R}_{12}(u-v)\,\mathrm{L}_1^{(\rho)}(u)\,\mathrm{L}_2^{(\rho)}(v) = \mathrm{L}_1^{(\rho)}(v)\,\mathrm{L}_2^{(\rho)}(u)\,\mathrm{R}_{12}(u-v)$$

L-operators for $so(p+1, q+1)$

Let Γ_a $(a = 0, \dots, n+1)$ be $2^{\frac{n}{2}+1}$-dim. gamma-matrices in $\mathbb{R}^{p+1,q+1}$ $(n = p + q)$ which generate the Clifford algebra with the basis

$$\Gamma_{a_1 \dots a_k} = \frac{1}{k!} \sum_{s \in \mathcal{S}_k} (-1)^{\mathrm{p}(s)} \Gamma_{s(a_1)} \cdots \Gamma_{s(a_k)} \;\; (k \leq n+2) \, , \;\; \Gamma_{A_k} = 0 \;\; (k > n+2) \, ,$$

where $\mathrm{p}(s)$ denote the parity of s.
The $SO(p+1, q+1)$-invariant R-matrix is

$$\mathrm{R}(u) = \sum_{k=0}^{n+2} \frac{\mathrm{R}_k(u)}{k!} \cdot \Gamma_{a_1 \dots a_k} \otimes \Gamma^{a_1 \dots a_k} \quad \in \quad \mathrm{End}(V \otimes V) \, ,$$

where V is the $2^{\frac{n}{2}+1}$-dimensional space of spinor representation T of $SO(p+1, q+1)$. This R-matrix satisfies Yang-Baxter equation for special choice of $\mathrm{R}_k(u)$.

Functions $\mathrm{R}_k(u)$ have to obey the recurrent relation (R.Shankar and E.Witten (1978), M.Karowsky and H.Thun (1981))

$$\mathrm{R}_{k+2}(u) = -\frac{u+k}{u+n-k}\,\mathrm{R}_k(u)\,.$$

We itemize some cases of ρ when the condition is fulfilled

- The differential representation ρ:

$$M_{ab} \to \rho(M_{ab}) = i(y_a\,\partial_b - y_b\,\partial_a)\,, \tag{1}$$

where $\partial_a = \frac{\partial}{\partial y^a}$ and y_a are coordinates in the space $\mathbb{R}^{p+1,q+1}$.

- Fundamental (defining) $(n+2)$-dimensional representation ρ: (R.Shankar and E.Witten (1978), M.Karowsky and H.Thun (1981))

$$M_{ab} \to \rho(M_{ab}) = ig(e_{ab} - e_{ba})\,, \tag{2}$$

where e_{ab} are matrix units and $g = ||g_{ab}||$.

- The differential representation ρ for $S_{\mu\nu} = 0$ and arbitrary Δ:

$$M_{ab} \to \rho(M_{ab})\,, \quad S_{\mu\nu} = 0\,. \tag{3}$$

One can check that (2) and (3) can be extracted from the differential representation (1).

Now we construct R-operator as solution of the defining RLL-equation

$$\mathcal{R}_{12}(u-v)\,\mathrm{L}_1(u)\,\mathrm{L}_2(v) = \mathrm{L}_1(v)\,\mathrm{L}_2(u)\,\mathcal{R}_{12}(u-v)$$

with conformal L-operator. Here indices $1,2$ correspond to two infinite-dimensional spaces of differential representation ρ of conf$(\mathbb{R}^{p,q})$ and we consider two cases:

- Dimension $n = p + q$ of the space $\mathbb{R}^{p,q}$ is arbitrary and representation of the conformal group is special and corresponds to the scalars: $\mathbf{S} = 0$ and $\bar{\mathbf{S}} = 0$.
- Dimension $n = p + q$ of the space $\mathbb{R}^{p,q}$ is fixed by $n = 4$ and representation of the conformal group is arbitrary: $\mathbf{S} \neq 0$ and $\bar{\mathbf{S}} \neq 0$.

The n-dimensional scalar case

In this case the defining RLL-equation has the form

$$\mathcal{R}_{12}(u-v)\,\mathrm{L}_1(u_+,u_-)\,\mathrm{L}_2(v_+,v_-) = \mathrm{L}_1(v_+,v_-)\,\mathrm{L}_2(u_+,u_-)\,\mathcal{R}_{12}(u-v)\,, \tag{4}$$

where

$$\mathrm{L}_1(u_+,u_-) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x}_1 & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} u_+\cdot\mathbf{1} & \mathbf{p}_1 \\ \mathbf{0} & u_-\cdot\mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{x}_1 & \mathbf{1} \end{pmatrix},$$

$$\mathrm{L}_2(v_+,v_-) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x}_2 & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} v_+\cdot\mathbf{1} & \mathbf{p}_2 \\ \mathbf{0} & v_-\cdot\mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{x}_2 & \mathbf{1} \end{pmatrix},$$

and $u_+ = u + \frac{\Delta_1 - n}{2}$, $u_- = u - \frac{\Delta_1}{2}$, $v_+ = v + \frac{\Delta_2 - n}{2}$, $v_- = v - \frac{\Delta_2}{2}$.
The $\mathcal{R}$-operator in (4) interchanges a pair of parameters (u_+,u_-) in the first L-operator with a pair (v_+,v_-) from the second L-operator. It seems to be reasonable to consider also operators which perform the other interchanges of four parameters. In order to carry out it systematically we joint them in the set $\mathbf{u} = (v_+,v_-,u_+,u_-)$. Then $\mathcal{R}$-operator represents the permutation s such that

$$s \mapsto \mathcal{R}(u-v)\ ;\ s\,\mathbf{u} = (\underline{u_+,u_-},\underline{v_+,v_-}). \tag{5}$$

An arbitrary permutation can be builded from elementary transpositions s_1, s_2 and s_3

$$s_1\mathbf{u} = (\underline{v_-}, \underline{v_+}, u_+, u_-)\ ;\ s_2\mathbf{u} = (v_+, \underline{u_+}, \underline{v_-}, u_-)\ ;\ s_3\mathbf{u} = (v_+, v_-, \underline{u_-}, \underline{u_+})\,.$$

In particular: $s = s_2 s_1 s_3 s_2$. Thus we reduce the problem to construction of operators $\mathrm{S}_i(\mathbf{u})$ $(i = 1, 2, 3)$ which represent elementary transpositions

$$(\underline{v_+\ ,\ v_-}, u_+, u_-)\ :\ \mathrm{S}_1(\mathbf{u})\,\mathrm{L}_2(v_+, v_-) = \mathrm{L}_2(v_-, v_+)\,\mathrm{S}_1(\mathbf{u}) \tag{6}$$

$$\begin{gathered}(v_+, \underline{v_-\ ,\ u_+}, u_-)\ : \\ \mathrm{S}_2(\mathbf{u})\,\mathrm{L}_1(u_+, u_-)\,\mathrm{L}_2(v_+, v_-) = \mathrm{L}_1(v_-, u_-)\,\mathrm{L}_2(v_+, u_+)\,\mathrm{S}_2(\mathbf{u})\end{gathered} \tag{7}$$

$$(v_+, v_-, \underline{u_+\ ,\ u_-})\ :\ \mathrm{S}_3(\mathbf{u})\,\mathrm{L}_1(u_+, u_-) = \mathrm{L}_1(u_-, u_+)\,\mathrm{S}_3(\mathbf{u}) \tag{8}$$

Then $\mathcal{R}$-operator can be constructed form these building blocks:

$$\mathcal{R}(\mathbf{u}) = \mathrm{S}_2(s_1 s_3 s_2 \mathbf{u})\,\mathrm{S}_1(s_3 s_2 \mathbf{u})\,\mathrm{S}_3(s_2 \mathbf{u})\,\mathrm{S}_2(\mathbf{u}) \tag{9}$$

The Yang-Baxter relation for this $\mathcal{R}$-operator is the direct consequence of the Coxeter relations for the building blocks $\mathrm{S}_i(\mathbf{u})$.

We are going to the construct $\mathrm{S}_i(\mathbf{u})$ and at the first stage we consider operators S_1 and S_3 which are examples of the operator S:

$$\mathrm{S}\,\mathrm{L}(u_+, u_-) = \mathrm{L}(u_-, u_+)\,\mathrm{S} \tag{10}$$

For the scalar case the differential representation $\rho^{(\Delta)}$ of the conformal algebra is parameterized only by conf. dimension – Δ. Taking in mind the definition of u_+ and u_- we see that their transposition corresponds to $\Delta \to n - \Delta$ which preserve the Casimir operator C_2. Thus, S intertwines equivalent representations: $\rho^{(\Delta)} \sim \rho^{(n-\Delta)}$. From (10) we obtain the following equations:

- translation: $[\hat{p}_\mu, \, \mathrm{S}] = 0$,
- Lorentz rotation: $[(x_\mu \hat{p}_\nu - x_\nu \hat{p}_\mu)\,, \, \mathrm{S}] = 0$,
- dilatation: $[x_\mu \, \hat{p}^\mu, \, \mathrm{S}] = i\,(n - 2\Delta)\,\mathrm{S}$,
- conformal boost
  $\left[\left(2x_\mu (x \cdot \hat{p}) - x^2 \hat{p}_\mu\right)\,, \, \mathrm{S}\right] = 2i(n-\Delta)\,x_\mu \mathrm{S} - 2i\Delta \mathrm{S} x_\mu$.
  Note that in the scalar case $S_{\mu\nu} = 0$ the conformal boost equation is dispensable since it can be derived from first three equations.

Thus, we deduce

$$\mathrm{S} = \hat{p}^{\,2\left(\frac{n}{2}-\Delta\right)} .$$

and explicit expressions for S_1 and S_3 are the following

$$\mathrm{S}_1(v_- - v_+) = \hat{p}_2^{\,2(v_- - v_+)} \;\; ; \;\; \mathrm{S}_3(u_- - u_+) = \hat{p}_1^{\,2(u_- - u_+)}$$

It remains to construct the last building block for $\mathcal{R}$-operator – operator S_2. It happens that it can be produced directly from the operator S by using some kind of duality transformation

$$p \to x_2 - x_1 \equiv x_{21} \; ; \; u_+ \to v_- \; ; \; u_- \to u_+ \, ,$$

so that S_2 is the operator of multiplication by the function

$$\mathrm{S}_2(u_+ - v_-) = x_{12}^{2(u_+ - v_-)} .$$

Coxeter relations are evident and have the following explicit forms

$$\hat{p}_2^{2a}\, x_{12}^{2(a+b)}\, \hat{p}_2^{2b} = x_{12}^{2b}\, \hat{p}_2^{2(a+b)}\, x_{12}^{2b} \;\; ; \;\; \hat{p}_1^{2a}\, x_{12}^{2(a+b)}\, \hat{p}_1^{2b} = x_{12}^{2b}\, \hat{p}_1^{2(a+b)}\, x_{12}^{2a} \, , \tag{11}$$

and are both equivalent to the operator identity (A.P.Isaev (2003))

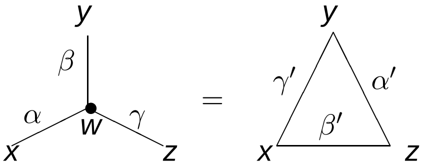
$$\hat{p}^{\,2a}\, x^{2(a+b)}\, \hat{p}^{\,2b} = x^{2b}\, \hat{p}^{\,2(a+b)}\, x^{2a} \, , \tag{12}$$

which can be rewritten in the form of well-known star-triangle relation

$$\int \frac{\mathrm{d}^n w}{(x-w)^{2\alpha}(y-w)^{2\beta}(z-w)^{2\gamma}} = \frac{\mathrm{V}(\alpha,\beta,\gamma)}{(y-z)^{2\alpha'}(x-z)^{2\beta'}(x-y)^{2\gamma'}} \, , \tag{13}$$

where $\alpha+\beta+\gamma = n$ and

$$\mathrm{V}(\alpha,\beta,\gamma) = \pi^{\frac{n}{2}} \frac{\Gamma(\alpha')\,\Gamma(\beta')\,\Gamma(\gamma')}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)} \;\; ; \;\; \alpha' = \frac{n}{2} - \alpha \, , \; \beta' = \frac{n}{2} - \beta \, , \; \gamma' = \frac{n}{2} - \gamma$$



Finally we find explicit expression for $\mathcal{R}$-operator

$$\mathcal{R}_{12}(u-v) = x_{12}^{2(u_- - v_+)}\,\hat{p}_2^{2(u_+ - v_+)}\,\hat{p}_1^{2(u_- - v_-)}\,x_{12}^{2(u_+ - v_-)}\,, \qquad (14)$$

which satisfies the YB equation. For $\Delta_1 = \Delta_2 = \Delta$ the $\mathcal{R}_{12}$ is

$$R_{ab}(\alpha;\xi) := (\hat{q}_{(ab)})^{2(\alpha+\xi)}(\hat{p}_{(a)})^{2\alpha}(\hat{p}_{(b)})^{2\alpha}(\hat{q}_{(ab)})^{2(\alpha-\xi)} =$$
$$= 1 + \alpha\, h_{(ab)}(\xi) + \alpha^2 \dots\,,$$

where $\alpha = u - v$, $\xi = \frac{n}{2} - \Delta$ and Hamiltonian densities $h_{(ab)}(x)$ are

$$h_{(ab)}(\xi) = 2\,\ln(\hat{q}_{(ab)})^2 + (\hat{q}_{(ab)})^{2\xi}\,\ln(\hat{p}_{(a)}^{\,2}\,\hat{p}_{(b)}^{\,2})\,(\hat{q}_{(ab)})^{-2\xi} =$$
$$= \hat{p}_{(a)}^{-2\xi}\,\ln(\hat{q}_{(ab)})^2\,\hat{p}_{(a)}^{2\xi} + \hat{p}_{(b)}^{-2\xi}\,\ln(\hat{q}_{(ab)})^2\,\hat{p}_{(b)}^{2\xi} + \ln(\hat{p}_{(a)}^{\,2}\,\hat{p}_{(b)}^{\,2})\,.$$

Using the standard procedure one can construct an integrable system with Hamiltonian $H(\xi) = \sum_{a=1}^{N-1} h_{(a,a+1)}(\xi)$. For $n = p + q = 1$ and $\xi = 1/2$ this Hamiltonian reproduces the Hamiltonian for the Lipatov's integrable model.

General R-operator in the case $so(2,4) = su(2,2)$

The differential representation ρ is characterized by three parameters – scaling dimension Δ and two spins ℓ, $\dot{\ell}$ and now the operator $\mathrm{L}(u)$ contains four parameters u and Δ, ℓ, $\dot{\ell}$. These parameters are combined in a pairs $\mathbf{u}_+$ and $\mathbf{u}_-$ which are analogs of u_+ and u_-

$$\mathbf{u}_+ \equiv (u_+\,,\,\ell) = \left(u + \frac{\Delta - n}{2}\,,\,\ell\right) \quad ; \quad \mathbf{u}_- \equiv (u_-\,,\,\dot{\ell}) = \left(u - \frac{\Delta}{2}\,,\,\dot{\ell}\right) .$$

We have the following expression for the operator $\mathrm{L}(u)$

$$\mathrm{L}(\mathbf{u}_+\,,\,\mathbf{u}_-) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{x} & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} u_+ \cdot \mathbf{1} + \mathbf{S}^{(\ell)} & \mathbf{p} \\ \mathbf{0} & u_- \cdot \mathbf{1} + \overline{\mathbf{S}}^{(\dot{\ell})} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\mathbf{x} & \mathbf{1} \end{pmatrix} ,$$

and the defining RLL-relation has the form

$$\mathcal{R}_{12}(u-v)\,\mathrm{L}_1(\mathbf{u}_+,\mathbf{u}_-)\,\mathrm{L}_2(\mathbf{v}_+,\mathbf{v}_-) = \mathrm{L}_1(\mathbf{v}_+,\mathbf{v}_-)\,\mathrm{L}_2(\mathbf{u}_+,\mathbf{u}_-)\,\mathcal{R}_{12}(u-v) \tag{15}$$

We start with construction of operators S_1 and S_3 which are two copies of the operator S defined by the equation

$$\mathrm{S} \cdot \mathrm{L}(\mathbf{u}_+, \mathbf{u}_-) = \mathrm{L}(\mathbf{u}_-, \mathbf{u}_+) \cdot \mathrm{S} \tag{16}$$

The exchange $\mathbf{u}_+ \leftrightarrow \mathbf{u}_-$ is equivalent to $u_+ \leftrightarrow u_-$ and $\ell \leftrightarrow \dot{\ell}$, i.e. $\Delta \leftrightarrow 4 - \Delta$ and $\ell \leftrightarrow \dot{\ell}$. Differential representation of the conformal algebra conf$(\mathbb{R}^{1,3})$ is parameterized by three numbers $\Delta,\ \ell,\ \dot{\ell}$ and we denote it by $\rho^{\Delta,\ell,\dot{\ell}}$. Thus operator S intertwines equivalent representations $\rho^{\Delta,\ell,\dot{\ell}} \sim \rho^{4-\Delta,\dot{\ell},\ell}$. As in the previous case the operator S has transparent representation theory meaning.

In this situation it is convenient to work with the generating functions

$$\Phi(x,\lambda,\tilde{\lambda}) = \Phi^{\dot{\alpha}_1\cdots\dot{\alpha}_{2\dot{\ell}}}_{\alpha_1\cdots\alpha_{2\ell}}(x)\,\lambda^{\alpha_1}\cdots\lambda^{\alpha_{2\ell}}\,\tilde{\lambda}_{\dot{\alpha}_1}\cdots\tilde{\lambda}_{\dot{\alpha}_{2\dot{\ell}}}\ ,$$

where λ and $\tilde{\lambda}$ are auxiliary spinors. Let us introduce the convolution

$$\mathrm{F}(\lambda,\tilde{\lambda}) * \mathrm{G}(\lambda,\tilde{\lambda}) = \mathrm{F}(\partial_\lambda,\partial_{\tilde{\lambda}})\,\mathrm{G}(\lambda,\tilde{\lambda})\Big|_{\lambda=0,\tilde{\lambda}=0}$$

and use it to represent the intertwining operator as an integral operator acting on generating functions

$$[\mathrm{S}\,\Phi]\,(X) = \int \mathrm{d}^4 y\,\mathrm{S}(X,Y) * \Phi(Y)$$

where we combine space-time coordinates and two spinors in one compact notation $X = (x,\lambda,\tilde{\lambda})$, $Y = (y,\eta,\tilde{\eta})$ and denote generating function by $\Phi(X)$.

The set of equations (Translation, Lorentz rotations, Dilatations, Conformal boosts) for the kernel of S coincides with the set of equations for a Green function for two fields of the types $(\ell, \dot{\ell})$ and $(\dot{\ell}, \ell)$ in conformal field theory and the solution is well known (V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova, I.T. Todorov, G.M. Sotkov, R.P. Zaikov, (1977,1978))

$$\mathrm{S}(X, Y) = \frac{1}{(2\ell)!} \frac{1}{(2\dot{\ell})!} \frac{\left(\tilde{\lambda}\,(\overline{\mathbf{x}-\mathbf{y}})\,\eta\right)^{2\ell} \left(\lambda\,(\mathbf{x}-\mathbf{y})\,\tilde{\eta}\right)^{2\dot{\ell}}}{(x-y)^{2(4-\Delta)}} \,.$$

where we use compact notation

$$\mathbf{x} = \sigma_\mu \frac{x^\mu}{|x|} \;;\; \overline{\mathbf{x}} = \overline{\sigma}_\mu \frac{x^\mu}{|x|} \,. \qquad (17)$$

Formula for the kernel $\mathrm{S}(X, Y)$ leads to the following explicit expression for the action of operator S on the generating function

$$[\mathrm{S}\,\Phi](X) = \int \frac{\mathrm{d}^4 y\;\Phi\left(y\,,\,\tilde{\lambda}\,\overline{(\mathbf{x}-\mathbf{y})}\,,\,\lambda\,(\mathbf{x}-\mathbf{y})\right)}{(x-y)^{2(4-\Delta)}}$$

The operators S_1 and S_3 act on the function $\Phi(X_1; X_2)$ in a similar manner ($\hat{p}_2 = i\partial_{x_2}, \hat{p}_1 = i\partial_{x_1}$)

$$[\mathrm{S}_1(v_- - v_+)\,\Phi](X_1; X_2) = \int \frac{\mathrm{d}^4 y \; e^{iy\hat{p}_2}}{y^{2(v_- - v_+ + 2)}} \; \Phi(X_1; x_2\,,\, \tilde{\lambda}_2\,\bar{\mathbf{y}}\,,\, \lambda_2\,\mathbf{y})$$

$$[\mathrm{S}_3(u_- - u_+)\,\Phi](X_1; X_2) = \int \frac{\mathrm{d}^4 y \; e^{iy\hat{p}_1}}{y^{2(u_- - u_+ + 2)}} \; \Phi(x_1\,,\, \tilde{\lambda}_1\,\bar{\mathbf{y}}\,,\, \lambda_1\,\mathbf{y}\,; X_2) \quad (18)$$

In order to construct operator S_2 we take into account the same observation as in a scalar case: it can be produced directly from the operator S using duality transformation

$$y \to p\ ;\ p \to x_2 - x_1 \equiv x_{21}\ ;\ \mathbf{u}_+ \to \mathbf{v}_-\ ;\ \mathbf{u}_- \to \mathbf{u}_+\,,$$

The change $\mathbf{u}_+ \to \mathbf{v}_-\ ;\ \mathbf{u}_- \to \mathbf{u}_+$ implies the corresponding change of spinors so that the expression for the action of operator S_2 on the generating function $\Phi\big(X_1\,;\,X_2\big)$ is

$$[\mathrm{S}_2(u_+ - v_-)\,\Phi]\big(X_1\,;\,X_2\big) = \int \frac{\mathrm{d}^4 p \; e^{ip\,x_{21}}}{p^{2(u_+ - v_- + 2)}}\,\Phi\big(x_1\,,\, \tilde{\lambda}_2\,\bar{\mathbf{p}}\,,\, \tilde{\lambda}_1\,;\, x_2\,,\, \lambda_2\,,\, \lambda_1\,\mathbf{p}\big)\,. \quad (19)$$

The corresponding Coxeter relations have the more complicated form than in the scalar case. The first triple relation

$$\mathrm{S}_1(a)\,\mathrm{S}_2(a+b)\,\mathrm{S}_1(b) = \mathrm{S}_2(b)\,\mathrm{S}_1(a+b)\,\mathrm{S}_2(a)$$

in explicit form looks as follows

$$\int \frac{\mathrm{d}^4z\,\mathrm{d}^4k\,\mathrm{d}^4y\;\, e^{i\,z\,\hat{p}_2}e^{i\,k\,x_{21}}e^{i\,y\,\hat{p}_2}}{z^{2(a+2)}k^{2(a+b+2)}y^{2(b+2)}}\cdot\Phi(x_1\,,\,\lambda_2\,\mathbf{z}\,\bar{\mathbf{k}}\,,\,\tilde{\lambda}_1\,;\,x_2\,,\,\lambda_1\,\mathbf{k}\,\bar{\mathbf{y}}\,,\,\tilde{\lambda}_2\,\bar{\mathbf{z}}\,\mathbf{y}\,) =$$

$$= \int \frac{\mathrm{d}^4q\,\mathrm{d}^4y\,\mathrm{d}^4k\;\, e^{i\,q\,x_{21}}e^{i\,y\,\hat{p}_2}e^{i\,k\,x_{21}}}{q^{2(b+2)}y^{2(a+b+2)}k^{2(a+2)}}\cdot\Phi(x_1\,,\,\lambda_2\,\mathbf{y}\,\bar{\mathbf{k}}\,,\,\tilde{\lambda}_1\,;\,x_2\,,\,\lambda_1\,\mathbf{q}\,\bar{\mathbf{y}}\,,\,\tilde{\lambda}_2\bar{\mathbf{q}}\,\mathbf{k}\,)\,, \tag{20}$$

and the second triple relation

$$\mathrm{S}_3(a)\,\mathrm{S}_2(a+b)\,\mathrm{S}_3(b) = \mathrm{S}_2(b)\,\mathrm{S}_3(a+b)\,\mathrm{S}_2(a)$$

is equivalent to the similar integral relation

$$\int \frac{\mathrm{d}^4 z\, \mathrm{d}^4 k\, \mathrm{d}^4 y\, e^{i\,z\,\hat{p}_1} e^{i\,k\,x_{21}} e^{i\,y\,\hat{p}_1}}{z^{2(a+2)} k^{2(a+b+2)} y^{2(b+2)}}\, \Phi(x_1\,,\,\lambda_1\, \mathbf{z}\,\bar{\mathbf{y}}\,,\,\tilde{\lambda}_2\,\bar{\mathbf{k}}\,\mathbf{y}\,;\,x_2\,,\,\lambda_2\,,\,\tilde{\lambda}_1\,\bar{\mathbf{z}}\,\mathbf{k}\,) =$$

$$= \int \frac{\mathrm{d}^4 q\, \mathrm{d}^4 y\, \mathrm{d}^4 k\, e^{i\,q\,x_{21}} e^{i\,y\,\hat{p}_1} e^{i\,k\,x_{21}}}{q^{2(b+2)} y^{2(a+b+2)} k^{2(a+2)}}\, \Phi(x_1\,,\,\lambda_1\, \mathbf{q}\,\bar{\mathbf{k}}\,,\,\tilde{\lambda}_2\,\bar{\mathbf{q}}\,\mathbf{y}\,;\,x_2\,,\,\lambda_2\,,\,\tilde{\lambda}_1\,\bar{\mathbf{y}}\,\mathbf{k}\,). \tag{21}$$

These relations are equivalent to the following generalization of the scalar star-triangle relation

$$\frac{\hat{p}^{\mu_1}\cdots\hat{p}^{\mu_m}}{\hat{p}^{\,2(a+m)}}\, \frac{\mathrm{A}_{\mu_1\nu_1}\cdots\mathrm{A}_{\mu_m\nu_m}}{x^{2(a+b+m)}}\, \frac{\hat{p}^{\nu_1}\cdots\hat{p}^{\nu_m}}{\hat{p}^{\,2(b+m)}} =$$

$$= \frac{x^{\mu_1}\cdots x^{\mu_m}}{x^{\,2(b+m)}}\, \frac{\mathrm{A}_{\mu_1\nu_1}\cdots\mathrm{A}_{\mu_m\nu_m}}{\hat{p}^{2(a+b+m)}}\, \frac{x^{\nu_1}\cdots x^{\nu_m}}{x^{\,2(a+m)}}$$

The Yang-Baxter R-matrix is

$$[\mathcal{R}_{12}\,\Phi](X_1;X_2) = \int \frac{\mathrm{d}^4q\,\mathrm{d}^4k\,\mathrm{d}^4y\,\mathrm{d}^4z\;\; e^{i\,(q+k)\,x_{21}}\,e^{i\,k\,(y-z)}}{q^{2(u_- - v_+ + 2)} z^{2(u_+ - v_+ + 2)} y^{2(u_- - v_- + 2)} k^{2(u_+ - v_- + 2)}} \cdot \Phi(x_1 - y\,,\,\lambda_2\,\mathbf{z}\,\bar{\mathbf{k}}\,,\,\tilde{\lambda}_2\,\bar{\mathbf{q}}\,\mathbf{y}\,;\,x_2 - z\,,\,\lambda_1\,\mathbf{q}\,\bar{\mathbf{z}}\,,\,\tilde{\lambda}_1\,\bar{\mathbf{y}}\,\mathbf{k}\,)\,. \tag{22}$$

It seems that the integrable model of the type (Zamolodchikov's "Fishnet" diagram Integrable System) related to this spinorial R-matrix and spinorial star-triangle relation is not known.