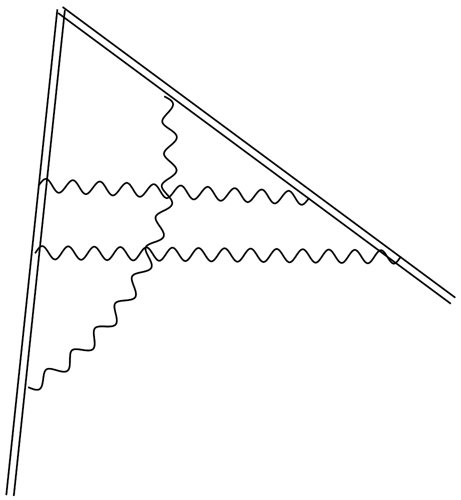
Analytic results for cusped Wilson loops



Dubna, July 2012

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in collaboration with

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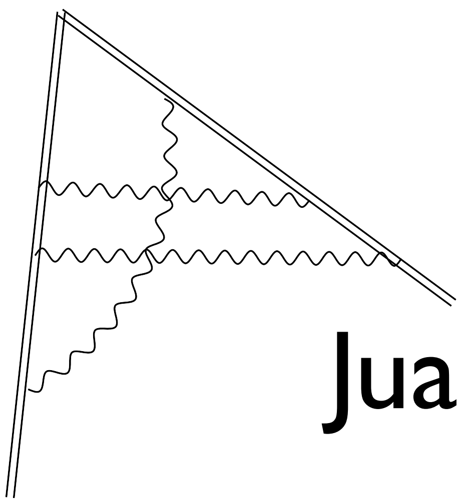
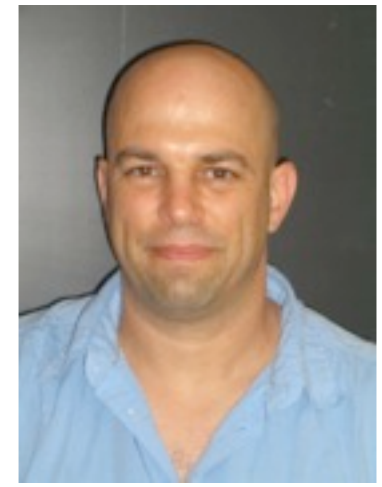


Juan Maldacena
(IAS)



Amit Sever
(IAS/Perimeter)

arXiv:1202.4455 [hep-th]

arXiv:1203.1019 [hep-th]

Tobias Huber
(Siegen)

arXiv: 1207.2161 [hep-th]

Outline of talk

- Introduction: cusped anomalous dimension $\Gamma_{\rm cusp}$
  and physical motivation

- Part 1: Exact result at small angles

- Part 2: Relation to Regge limit of massive scattering
  amplitudes
  full three-loop result

- Part 3: new scaling limit, Schrödinger problem
  solution to all orders

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}\int F_{\mu\nu}F^{\mu\nu}\,, \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu]$$

$$A^\mu = \sum_{a=1}^{N^2-1} A^\mu_a\, t^a_{ij}$$ gauge group SU(N)

Wilson loops:

required for gauge invariance of non-local objects

$\bar{\Psi}(x_2)$

$W_C[x_1, x_2]$

$\Psi(x_1)$

$$\mathcal{O} = \Psi(x_1) W_C[x_1, x_2] \bar{\Psi}(x_2)$$

$$W_C[x_1, x_2] = P e^{\int_C dx_\mu A^\mu}$$

P: path ordering

contain local operators

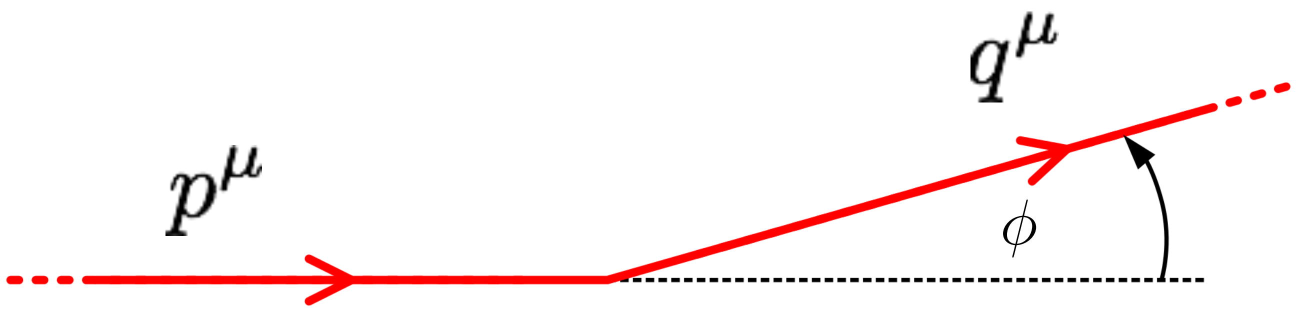
$$\sim 1 + \sigma^{\mu\nu} F_{\mu\nu} + \ldots$$

gauge dynamics - Wilson loops of arbitrary shapes

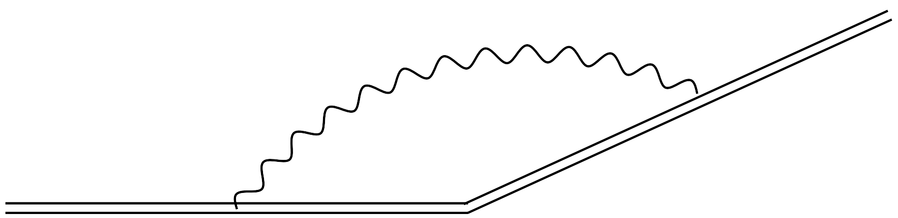
Cusp anomalous dimension

Wilson loop with cusp

$$\cos(\phi) = \frac{p \cdot q}{\sqrt{p^2 q^2}}$$



Γ_{cusp} governs ultraviolet (UV) divergences at cusp



Polyakov; Brandt, Neri, Sato
Korchemsky & Radyushkin '87

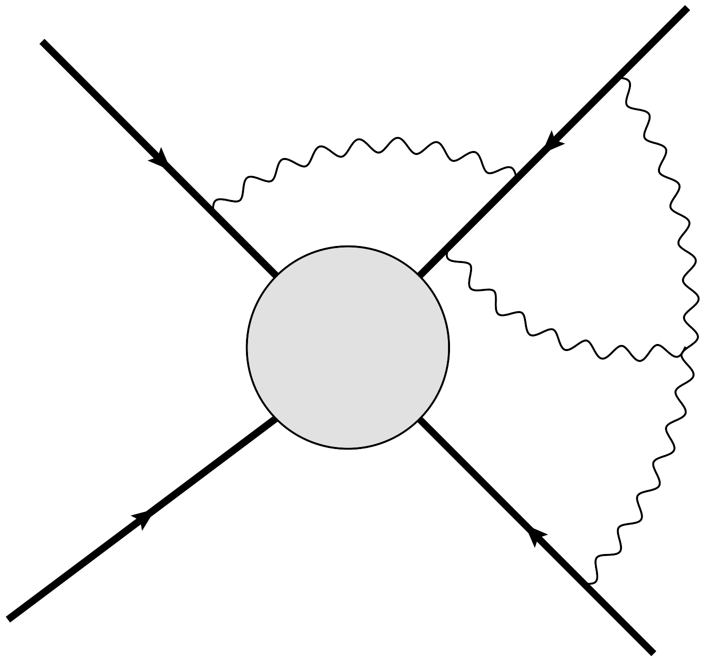
$$\langle W \rangle \sim e^{-|\ln \frac{\mu_{UV}}{\mu_{IR}}| \, \Gamma_{\mathrm{cusp}}}$$

similar to anomalous dimensions of composite operators

$$\Gamma_{\mathrm{cusp}}(\phi, \lambda, N) \qquad \lambda = g_{YM}^2 N$$

Physical relevance of $\Gamma_{\rm cusp}$

- IR divergences of massive amplitudes

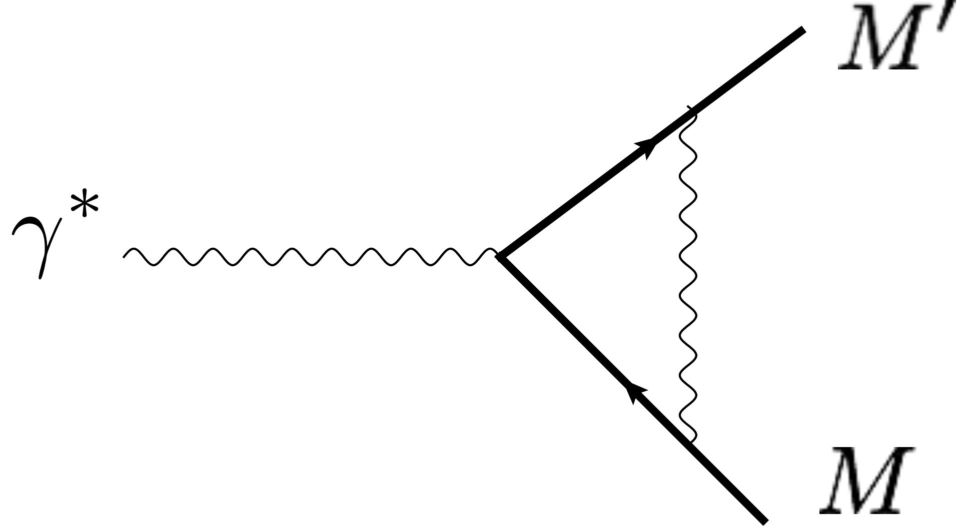


$$\mathcal{A} \sim e^{-|\log \mu_{IR}|\,\Gamma_{\rm cusp}}$$

Korchemsky, Radyushkin;
...

resummation of soft divergences

- similarly for massive form factors (e.g Isgur-Wise)



A lot of important work on Γ_{cusp} in Dubna!

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дубна

E2-85-779

G.P.Korchemsky*, A.V.Radyushkin

INFRARED ASYMPTOTICS OF PERTURBATIVE QCD

Renormalization Properties of the Wilson Loops in Higher Orders of Perturbation Theory

Submitted to "ЯФ"

* Rostov State University, USSR

1985

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E2-86-293

G.P.Korchemsky, A.V.Radyushkin

INFRARED ASYMPTOTICS OF PERTURBATIVE QCD. VERTEX FUNCTIONS

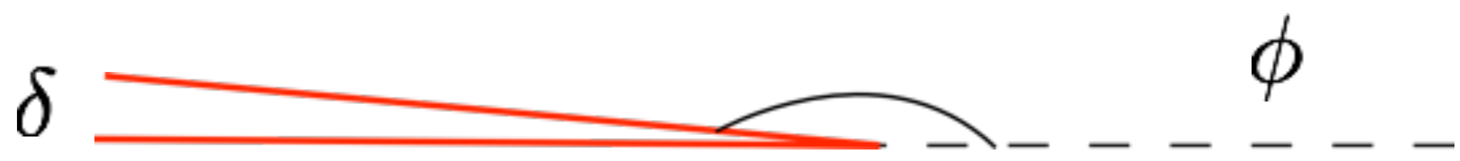
Submitted to "ЯФ"

1986

Limits and relations of $\Gamma_{\text{cusp}}(\phi)$

- vanishes at zero angle (straight line) $\quad \Gamma_{\text{cusp}}(\phi = 0, \lambda) = 0$

- related to quark-antiquark potential



$\delta = \pi - \phi \qquad \delta \ll 1$

- anomalous dimensions of large spin operators $\quad \lim_{\varphi \to \infty} \Gamma_{\text{cusp}}(i\varphi, \lambda) \sim \varphi\, \gamma_{\text{cusp}}(\lambda)$

known due to Beisert, Eden, Staudacher eq
integrability!

$$\mathcal{O}_J \sim \text{Tr}(Z\mathcal{D}^J Z)\,, \quad d = 2 + J + \gamma(J, \lambda)$$

$$\lim_{J\to\infty} \gamma(J, \lambda) \sim \log J\ \gamma_{\text{cusp}}(\lambda)$$

Korchemsky

Wilson loops in supersymmetric theories

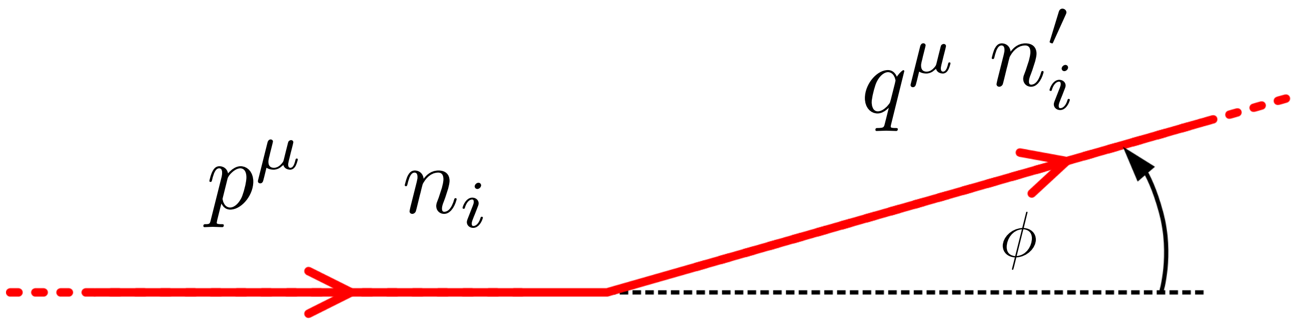
- loop couples to scalars $Tr(Pe^{\int ds A^\mu \dot{x}_\mu + ds\, n_i \Phi^i})$

  six scalars Φ^i

  Maldacena; Rey

- path-dependent coupling

$$\cos(\phi) = \frac{p\cdot q}{\sqrt{p^2 q^2}}$$



$$\cos(\theta) = n\cdot n'\,, \quad n^2 = n'^2 = 1$$

e.g. $n = (1,0,0,0,0,0)\,, \quad n' = (\cos(\theta), \sin(\theta), 0, 0, 0, 0)$

$$\Gamma_{\text{cusp}}(\phi, \theta, \lambda, N)$$

- θ dependence polynomial in $\xi = \dfrac{\cos\theta - \cos\phi}{\sin\phi}$

- supersymmetry $\Gamma_{\text{cusp}}(\phi = \pm\theta) = 0$

  Zarembo

QCD and supersymmetric Yang-Mills theories

- Γ_{cusp} known to two loops in QCD

Korchemsky, Radyushkin '87;
Kidonakis 2009

- Γ_{cusp} in N=4 SYM to two loops

Makeenko, Olesen, Semenoff 2006;
Drukker, Forini 2011

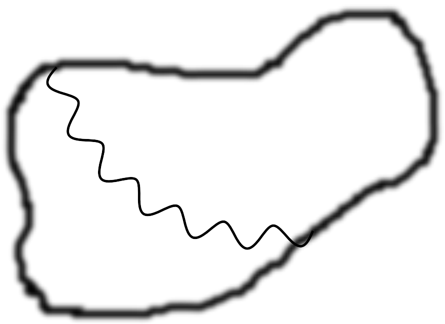
perturbative calculations very similar
QCD result only slightly more complicated to SYM

- certain structures more apparent in SYM
- insights can help to organize calculation even if there is no supersymmetry

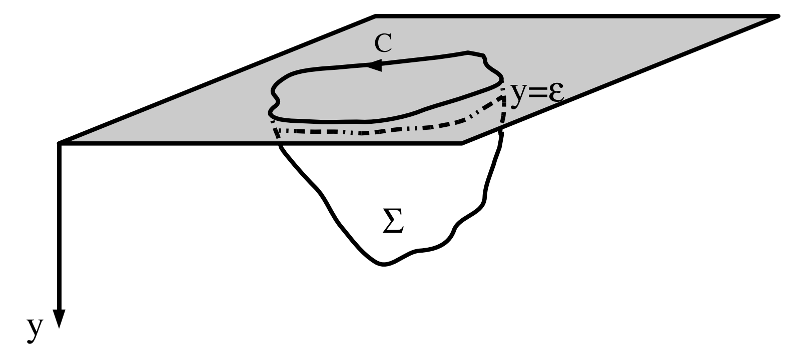
AdS/CFT correspondence

N=4 SYM
SU(N) gauge theory
scalars+fermions
conformal

dual string theory
description on AdS_5



Wilson loop $W[C]$

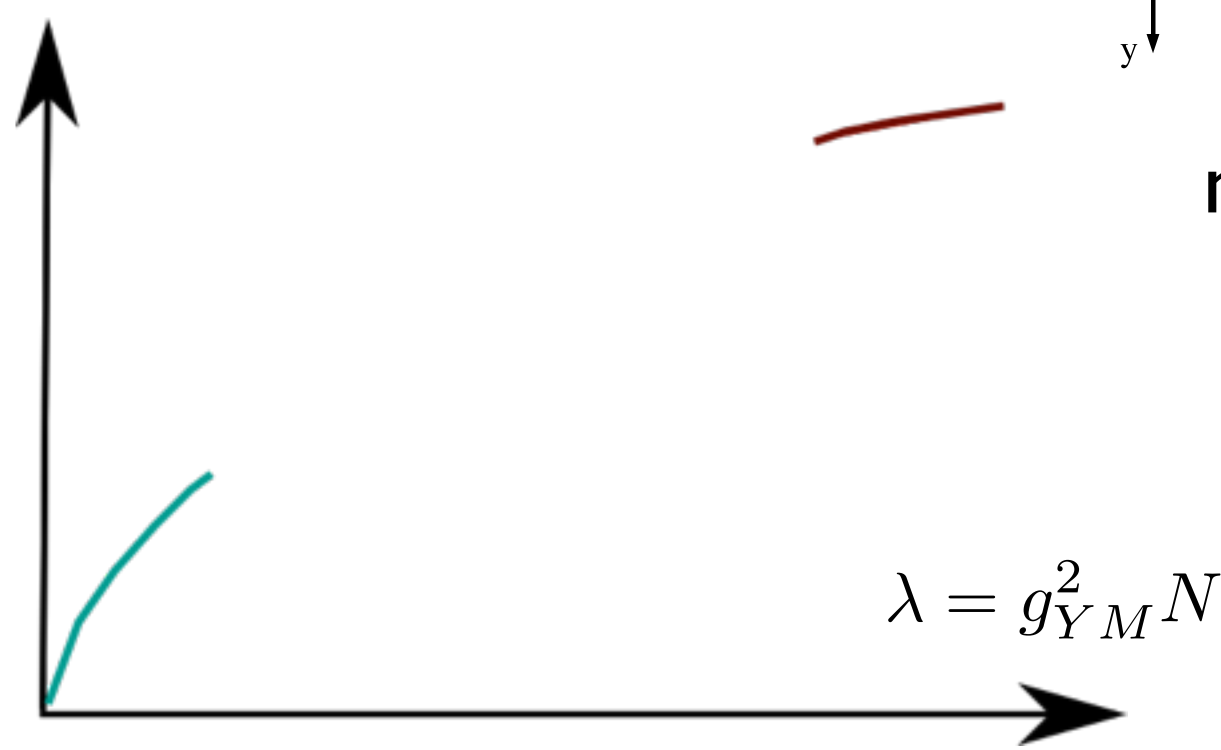


Feynman diagrams

$\lambda \ll 1$

minimal surface

$\lambda \gg 1$



Part 1:
exact result at small angles

First deviation from supersymmetric case can be computed exactly:

$$\Gamma_{\rm cusp} = (\phi^2 - \theta^2) H(\phi, \lambda, N) + \ldots$$

Correa, JMH, Maldacena, Sever

H obtained by relating it to Wilson loops on S^2

Drukker et al.
Pestun et al.

$$H(\phi, \lambda) = \frac{2\phi}{1 - \frac{\phi^2}{\pi^2}} B(\tilde{\lambda}) \,, \qquad \tilde{\lambda} = \lambda(1 - \frac{\phi^2}{\pi^2})$$

$$B = \frac{1}{4\pi^2} \sqrt{\lambda} \frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} + o(1/N^2)$$

I_a :modified Bessel function

non-planar part also known

Comments:

- perturbatively, H is a polynomial in ϕ, π

$$H = \phi \left[\left(\frac{\lambda}{8\pi^2} \right) (\pi^2 - \phi^2) + \frac{1}{3} \left(\frac{\lambda}{8\pi^2} \right)^2 (\pi^2 - \phi^2)^2 + \ldots \right]$$

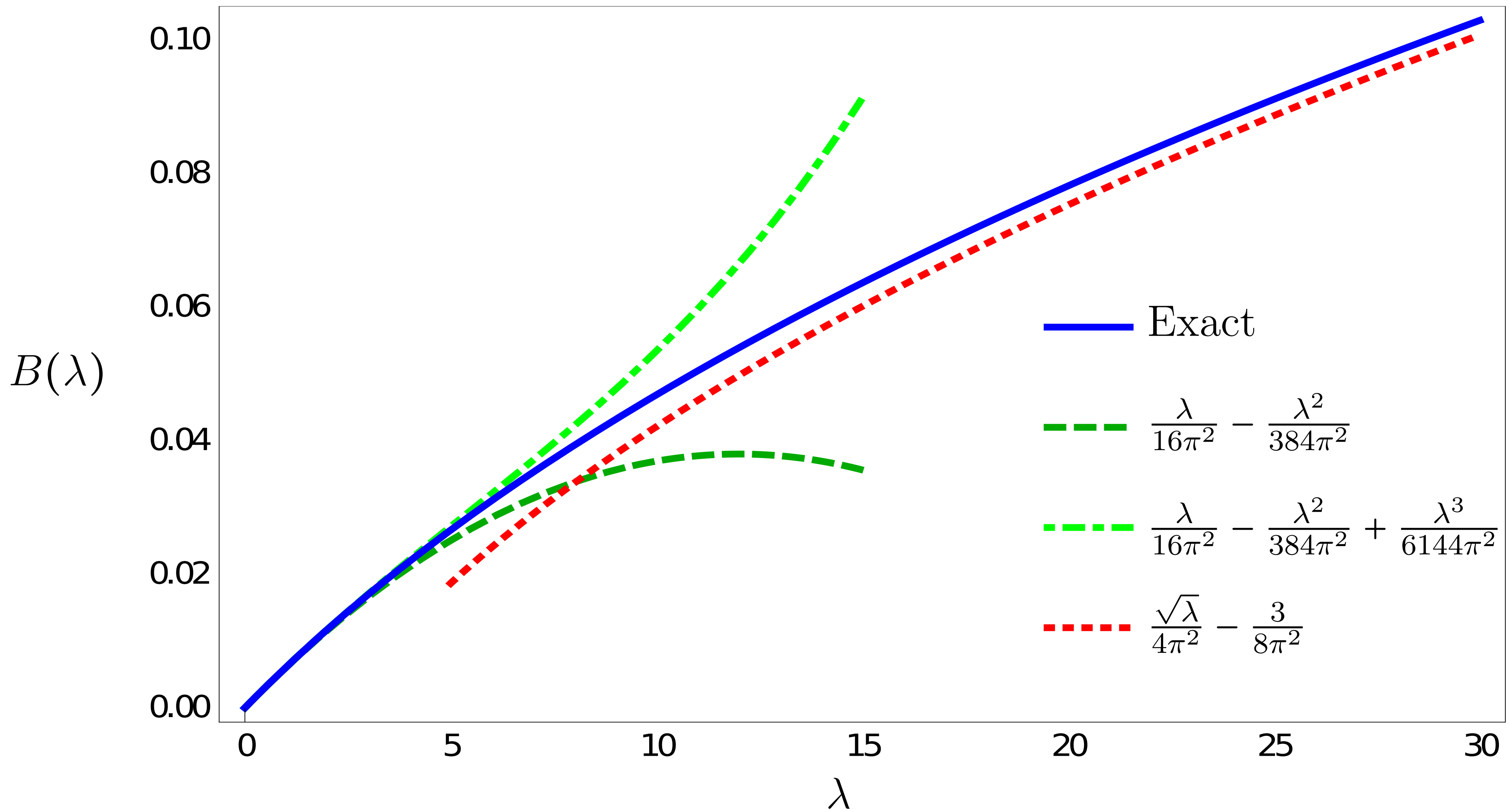
- strong coupling $H = \frac{\sqrt{\lambda}}{2} \frac{\phi}{\sqrt{1 - \frac{\phi^2}{\pi^2}}}$

agrees with formula extracted from
Drukker, Forini

Exact result interpolating between weak and strong coupling!

for small angle $\Gamma_{\text{cusp}} = \phi^2 B(\lambda) + o(\phi^4)\,, \quad \theta = 0$ Correa, JMH, Maldacena, Sever

``Bremsstrahlung function'' $B(\lambda)\,, \quad \lambda = g_{YM}^2 N$



(exact N dependence also known)

Part 2:
Regge limit of 4-pt amplitudes
full 3-loop result

Relation to Regge limit of massive amplitudes in N=4 SYM

- massive scattering amplitudes in N=4 SYM

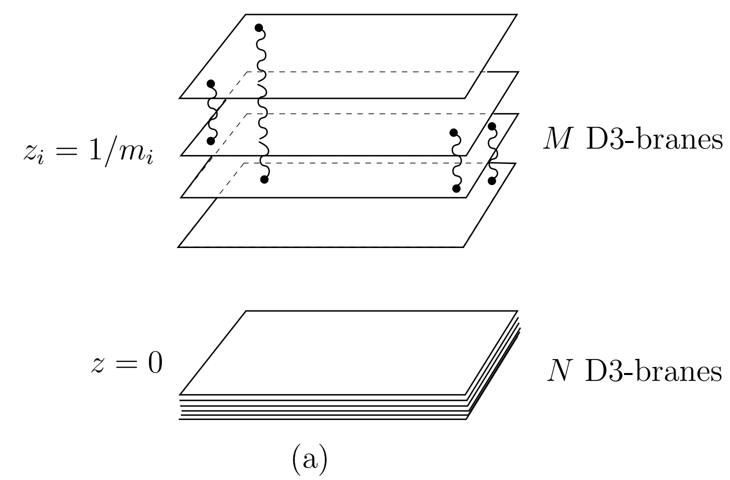
gauge theory

Higgs mechanism

$$\Phi \longrightarrow \langle \Phi \rangle + \varphi$$

$$U(N+M) \longrightarrow U(N) \times U(M) \longrightarrow U(N) \times U(1)^M$$

string theory



Alday, JMH, Plefka, Schuster

- dual conformal symmetry (planar)

$$y_i^A \to \frac{y_i^A}{y_i^2} \qquad y_i^A = (x_i^\mu, m_i)$$

isometries of AdS_5 space
Poincare coordinates

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu \qquad p_i^2 = -(m_i - m_{i+1})^2$$

- dual conformal symmetry $\quad s=(p_1+p_2)^2 \quad t=(p_2+p_3)^2$

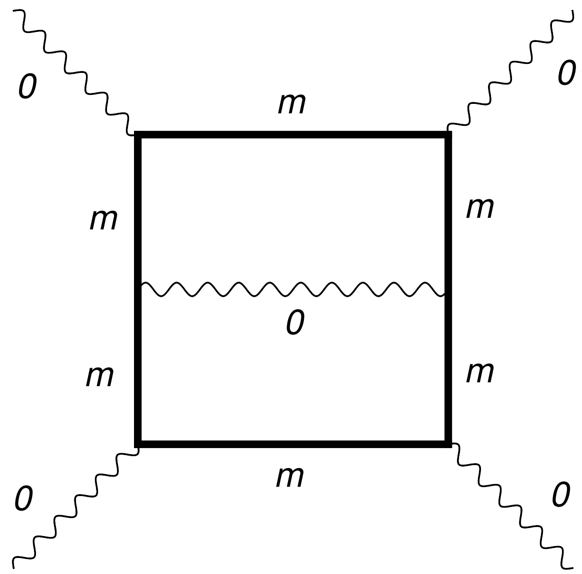
$$M_4(s,t;m_1,m_2,m_3,m_4)=M(u,v)$$

Alday, JMH, Plefka, Schuster

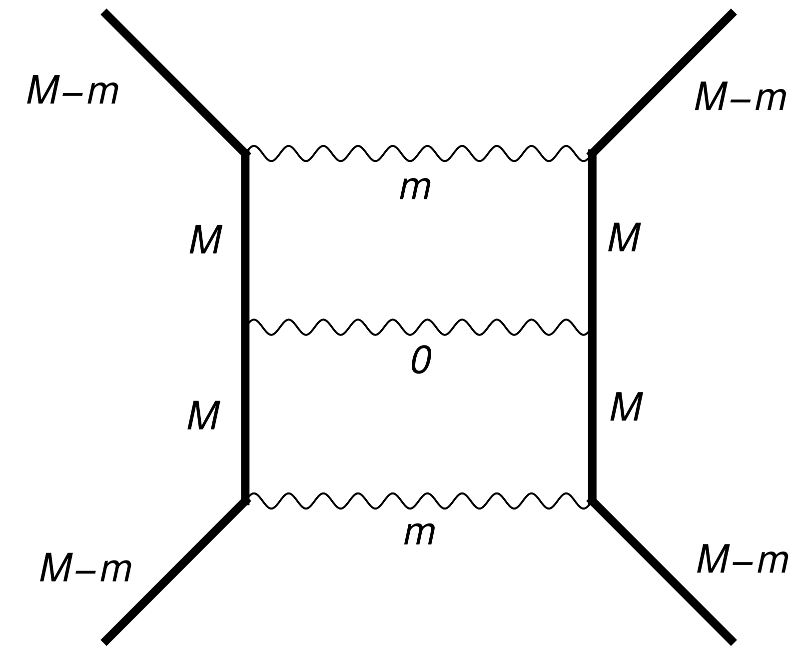
$$u=\frac{m_1 m_3}{s+(m_1-m_3)^2}\,,\qquad v=\frac{m_2 m_4}{t+(m_2-m_4)^2}$$

- relates two different physical pictures

JMH, Naculich, Spradlin, Schnitzer



$$(a):\quad u=\frac{m^2}{s}\,,\ v=\frac{m^2}{t}$$



$$(b):\quad u=\frac{m^2}{s}\,,\ v=\frac{M^2}{t}$$

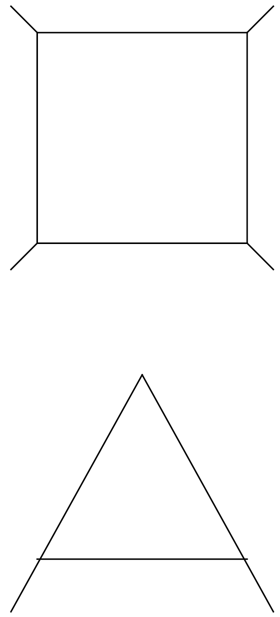
- $u\ll 1:$ Regge limit

soft IR divergences, ``Bhabha-scattering"

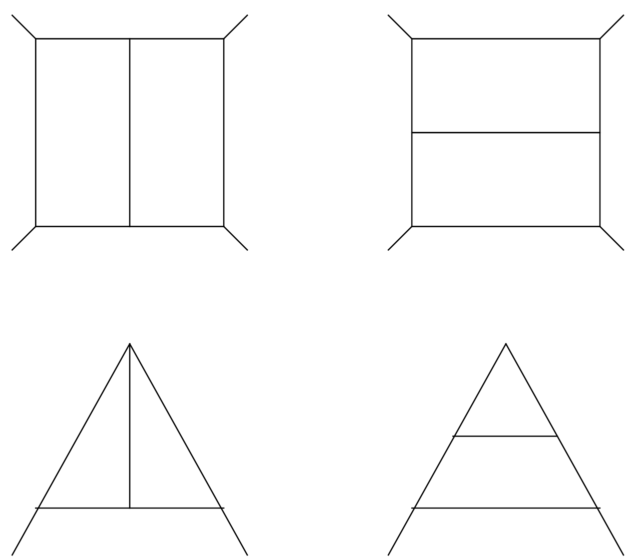
$$\boxed{M(u,v)\sim e^{\log u\,\Gamma_{\rm cusp}(\lambda,\phi)}}$$

- examples at integrand level

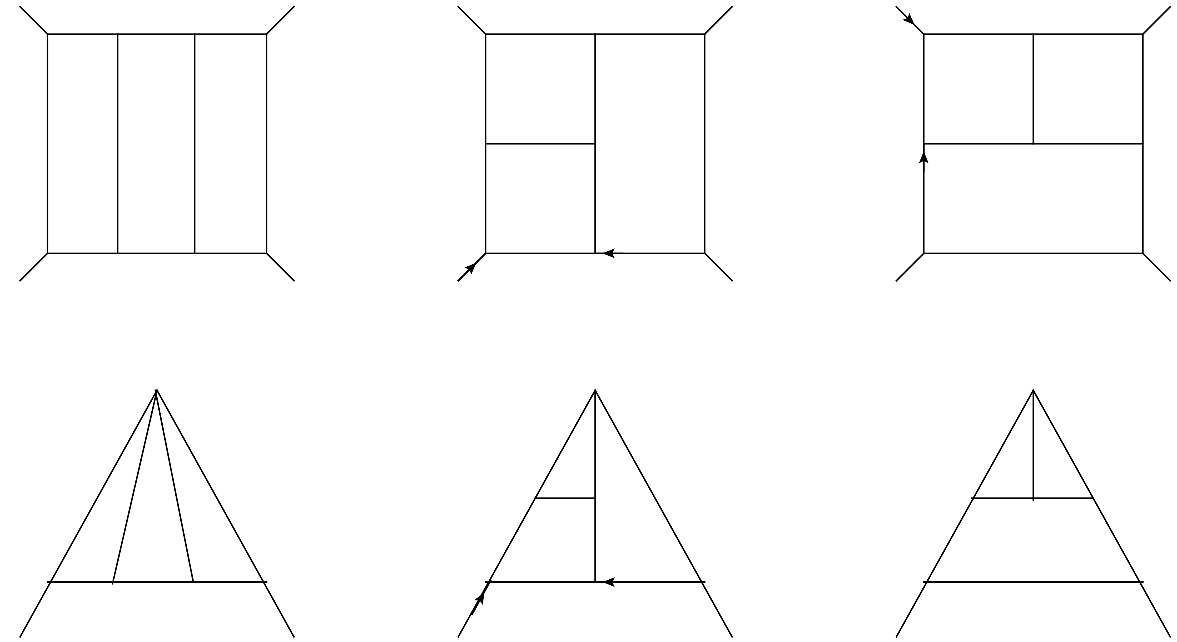
one loop:

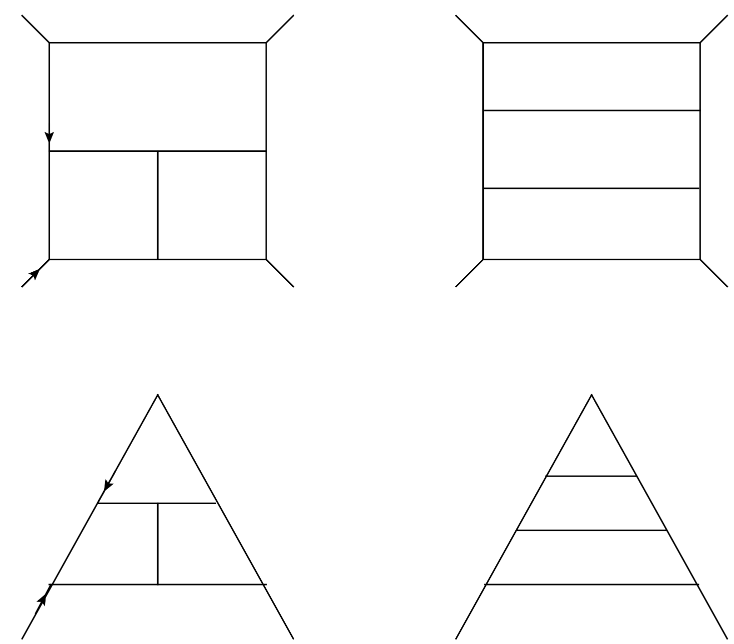


two loops:



three loops:





4-pt integrand 3-5 loops: Bern, Rozowsky, Yan + Dixon, Carrasco, Johansson

- planar integrand of 4-pt amplitude known

Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka; Bourjaily, DiRe, Shaikh, Volovich; Eden, Heslop, Sokatchev, Korchemsky

generalize to massive case

Alday, JMH, Plefka, Schuster

JMH, Naculich, Spradlin, Schnitzer

What functions are needed for $\Gamma_{\rm cusp}$?

- useful variable: $x = e^{i\phi}$
- logarithms, polylogarithms

$$\log(x) = \int_1^x \frac{dy}{y}$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{dy}{y} \mathrm{Li}_{n-1}(y)\,, \quad \mathrm{Li}_1(x) = -\log(1-x)$$

- harmonic polylogarithms (HPLs)

Gehrmann, Remiddi

$$H_1(x) = -\log(1-x)\,, \qquad H_0(x) = \log(x)\,, \qquad H_{-1}(x) = \log(1+x)\,.$$

$$H_{a_1,a_2,\ldots a_n}(x) = \int_0^x f_{a_1}(y) H_{a_2,\ldots,a_n}(y) dy$$

kernels $f_1(y) = \frac{1}{1-y}\,, \qquad f_0(y) = \frac{1}{y}\,, \qquad f_{-1}(y) = \frac{1}{1+y}$

- (for intermediate steps: Goncharov polylogarithms)

$$G(a_1,\ldots a_n; z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)\,, \qquad G(a_1;z) = \int_0^z \frac{dt}{t-a_1}\,.$$

structure of perturbative results

- full 3-loop result (schematically):

Correa, JMH, Maldacena, Sever

$$\xi = \frac{\cos\theta - \cos\phi}{\sin\phi}$$

$$\begin{aligned}\Gamma_{\rm cusp} &= \lambda\, \xi\, \phi \\ &+ \lambda^2\, \left[\xi\, \phi(\pi^2 - \phi^2) + \xi^2\, \left(\mathrm{Li}_3(e^{2i\phi}) + \ldots\right)\right] \\ &+ \lambda^3\, \left[\xi\phi(\pi^2-\phi^2)^2 + \xi^2\left(\mathrm{Li}_5(e^{2i\phi}) + \ldots\right) + \xi^3\left(\mathrm{HPL}(e^{2i\phi}) + \ldots\right)\right]\end{aligned}$$

- useful variable $x = e^{i\phi}$

$$\Gamma^{(1)}_{\rm cusp} = -\frac{1}{2}\xi\,[2\log x]$$

$$\begin{aligned}\Gamma^{(2)}_{\rm cusp} = &-\frac{1}{2}\xi\left[-\frac{2}{3}\log x\left(\log^2 x + \pi^2\right)\right] \\ &-\frac{1}{2}\xi^2\left[\frac{2}{3}\log^3 x + 2\log x\left(\zeta_2 + \mathrm{Li}_2(x^2)\right) - 2\mathrm{Li}_3(x^2) + 2\zeta_3\right]\end{aligned}$$

- uniform degree of integrals

cf. QCD/N=4 SYM transcendentality principle (KLOV)

- linear in ξ - exactly known
- highest term $\lambda^L \xi^L$ from new limit

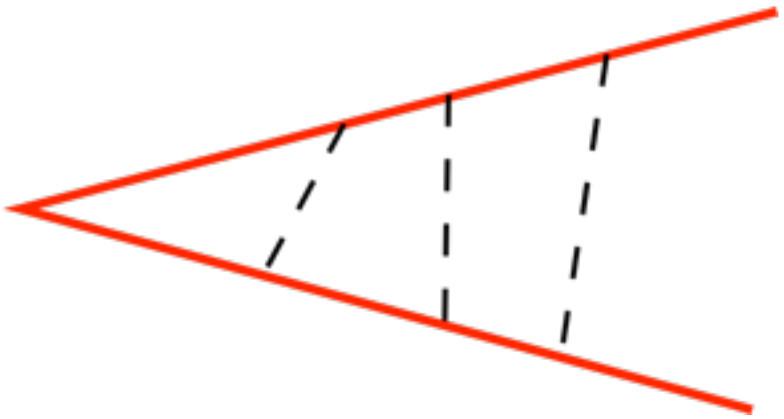
Part 3:
New scaling limit

New scaling limit

Correa, JMH, Maldacena, Sever

- highest term $\lambda^L \xi^L$ from new limit $\theta \to i\theta\,, \quad \theta \to \infty$

limit selects ladders



Bethe-Salpeter equation

$$\partial_\tau \partial_\sigma F(\sigma,\tau) = F(\sigma,\tau) P(\sigma,\tau)$$

ansatz $\quad F = \sum_n e^{-\Omega_n y_2} \Psi_n(y_1)$

$$y_1 = \tau - \sigma$$
$$y_2 = (\tau + \sigma)/2$$

- Γ_{cusp} from ground-state energy of Schrödinger problem

$$\left[-\partial_{y_1}^2 - \frac{\hat{\lambda}}{8\pi^2} \frac{1}{(\cosh y_1 + \cos\phi)} + \frac{\Omega^2(\phi)}{4} \right] \Psi(y_1,\phi) = 0$$

$$\hat{\lambda} \sim \lambda\,\xi$$
$$\Gamma_{\text{cusp}} = -\Omega_0$$

  - exactly solvable for zero angle (Pöschl-Teller)
  - iterative solution in coupling, or angle
  - numerical solution

$$\phi = 0: \;\; V = \frac{1}{\cosh^2 \frac{y_1}{2}}$$

leading order (LO)

Correa, JMH, Maldacena, Sever

solution at any loop order

JMH, Huber

$$\left[-\partial_{y_1}^2 - \frac{\hat{\lambda}}{8\pi^2}\frac{1}{(\cosh y_1 + \cos\phi)} + \frac{\Omega^2(\phi)}{4}\right]\Psi(y_1,\phi) = 0 \qquad \hat{\lambda} \sim \lambda\,\xi$$

- change of variables

$$\Psi(y_1) = \eta(y_1) e^{-\Omega_0 y_1/2} \qquad w = e^{-y_1}, \text{ and } x = e^{i\phi}$$

$$\partial_w w \partial_w \eta = -\Omega_0(x)\partial_w \eta + \hat{\kappa}\left[\frac{1}{w+x^{-1}} - \frac{1}{w+x}\right]\eta\,, \qquad \hat{\kappa} = \frac{\hat{\lambda}\,x}{4\pi^2(1-x^2)}$$

- Ω_0 from boundary condition $\partial_{y_1}\Psi(y_1)|_{y_1=0} = 0$
- iterative solution

$$\Omega_0 = \hat{\kappa}\Omega_0^{(1)} + \hat{\kappa}^2\Omega_0^{(2)} + \ldots \qquad \eta = 1 + \hat{\kappa}\eta^{(1)} + \ldots$$

all-loop solution in terms of HPLs

JMH, Huber

- solution for $\eta(w,x)$ in terms of iterated integrals
- special case of Goncharov polylogarithms $G(a_i;z)$

  here $a_i \in \{0, -x, -1/x\}\,, \quad z = w$

- compute differential

$$d\eta^{(L)} = f_1\, d\log x + f_2\, d\log(1+x) + f_3\, d\log(1-x) + f_4\, d\log(w+x) + f_5\, d\log(w+1/x)\,,$$

and hence, at $w=1$

$$d\Omega^{(L)} = g_1\, d\log x + g_2\, d\log(1+x) + g_3\, d\log(1-x)$$

- $\Omega^{(L)}$ is given by HPLs of weight (2L-1)
- can be found algorithmically in principle at any loop order

explicit results, and surprises

Correa, JMH, Maldacena, Sever
JMH, Huber

Surprise #1:
only indices 0,1 are needed if argument x^2 is chosen! (at least to 5 loops)

- for example, to three loops (argument x^2 is implicit)

$$\Omega_0^{(1)}(x) = -H_0 \, , \tag{3.34}$$

$$\Omega_0^{(2)}(x) = 4\,\zeta_3 + 2\,\zeta_2\,H_0 + 2\,H_{2,0} + H_{0,0,0} \, , \tag{3.35}$$

$$\begin{aligned}\Omega_0^{(3)}(x) = & -8\,\zeta_2\,\zeta_3 - 12\,\zeta_5 - 12\,\zeta_4\,H_0 - 16\,\zeta_3\,H_2 - 8\,\zeta_2\,H_3 - 4\,\zeta_3\,H_{0,0} - 8\,\zeta_2\,H_{2,0} \\ & - 8\,H_{4,0} - 8\,\zeta_2\,H_{0,0,0} - 8\,H_{2,2,0} - 4\,H_{3,0,0} - 8\,H_{3,1,0} - 4\,H_{2,0,0,0} - 6\,H_{0,0,0,0,0} \, .\end{aligned} \tag{3.36}$$

- at four loops

$$\begin{aligned}\Omega_0^{(4)}(x) = {} & 48\,\zeta_3\,\zeta_4 + 24\,\zeta_2\,\zeta_5 + 36\,\zeta_7 + 8\,\zeta_3^2\,H_0 + 51\,\zeta_6\,H_0 + 48\,\zeta_2\,\zeta_3\,H_2 + 72\,\zeta_5\,H_2 \\ & + 96\,\zeta_4\,H_3 + 88\,\zeta_3\,H_4 + 80\,\zeta_2\,H_5 + 32\,\zeta_2\,\zeta_3\,H_{0,0} + 20\,\zeta_5\,H_{0,0} + 72\,\zeta_4\,H_{2,0} \\ & + 96\,\zeta_3\,H_{2,2} + 48\,\zeta_2\,H_{2,3} + 32\,\zeta_3\,H_{3,0} + 128\,\zeta_3\,H_{3,1} + 64\,\zeta_2\,H_{3,2} + 80\,\zeta_2\,H_{4,0} \\ & + 48\,\zeta_2\,H_{4,1} + 92\,H_{6,0} + 114\,\zeta_4\,H_{0,0,0} + 24\,\zeta_3\,H_{2,0,0} + 48\,\zeta_2\,H_{2,2,0} + 48\,H_{2,4,0} \\ & + 64\,\zeta_2\,H_{3,0,0} + 64\,\zeta_2\,H_{3,1,0} + 64\,H_{3,3,0} + 80\,H_{4,2,0} + 80\,H_{5,0,0} + 80\,H_{5,1,0} \\ & + 24\,\zeta_3\,H_{0,0,0,0} + 48\,\zeta_2\,H_{2,0,0,0} + 48\,H_{2,2,2,0} + 24\,H_{2,3,0,0} + 48\,H_{2,3,1,0} + 64\,H_{3,1,2,0} \\ & + 32\,H_{3,2,0,0} + 64\,H_{3,2,1,0} + 64\,H_{4,0,0,0} + 24\,H_{4,1,0,0} + 48\,H_{4,1,1,0} + 92\,\zeta_2\,H_{0,0,0,0,0} \\ & + 24\,H_{2,2,0,0,0} + 48\,H_{3,0,0,0,0} + 32\,H_{3,1,0,0,0} + 36\,H_{2,0,0,0,0,0} + 92\,H_{0,0,0,0,0,0,0} \, .\end{aligned} \tag{3.37}$$

curious relative signs!

explicit results, and surprises

JMH, Huber

Surprise #2:
in small x limit, only single zeta values appear, at least to six loops

- e.g. at six loops:

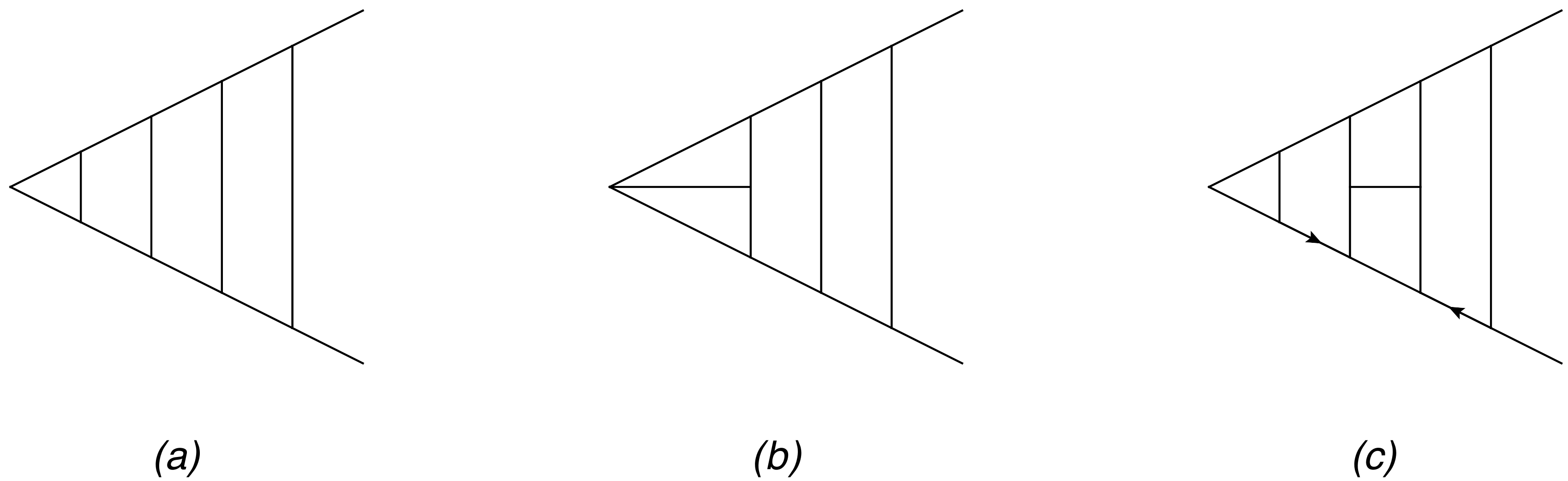
$$\begin{aligned}\Omega_0^{(6)}(x) \overset{x\to 0}{=} & \frac{339008}{51975}\log^{11}x + \frac{339008}{2835}\zeta_2\log^9 x + \frac{4288}{63}\zeta_3\log^8 x \\ & + \frac{12800}{7}\zeta_4\log^7 x + \left(\frac{34304}{45}\zeta_2\zeta_3 + \frac{10688}{45}\zeta_5\right)\log^6 x \\ & + \left(\frac{2944}{15}\zeta_3^2 + \frac{110944}{15}\zeta_6\right)\log^5 x + (5792\,\zeta_3\zeta_4 + 1376\,\zeta_2\zeta_5 + 528\,\zeta_7)\log^4 x \\ & + \left(\frac{2944}{3}\zeta_2\zeta_3^2 + \frac{2432}{3}\zeta_3\zeta_5 + \frac{80048}{9}\zeta_8\right)\log^3 x \\ & + (128\,\zeta_3^3 + 3792\,\zeta_4\zeta_5 + 7584\,\zeta_3\zeta_6 + 1152\,\zeta_2\zeta_7 + 664\,\zeta_9)\log^2 x \\ & + \left(1824\,\zeta_3^2\zeta_4 + 1152\,\zeta_2\zeta_3\zeta_5 + 336\,\zeta_5^2 + 672\,\zeta_3\zeta_7 + \frac{8292}{5}\zeta_{10}\right)\log x \\ & + \frac{256}{3}\zeta_2\zeta_3^3 + 160\,\zeta_3^2\zeta_5 + 612\,\zeta_5\zeta_6 + 432\,\zeta_4\zeta_7 + \frac{2480}{3}\zeta_3\zeta_8 \\ & + \frac{680}{3}\zeta_2\zeta_9 + 372\,\zeta_{11} + \mathcal{O}(x)\,. \end{aligned} \qquad (3.44)$$

- no multiple zetas! E.g. $\zeta_{5,3}$ could have appeared at weight 8
- BES equation has same property - can one prove it here from field theory?

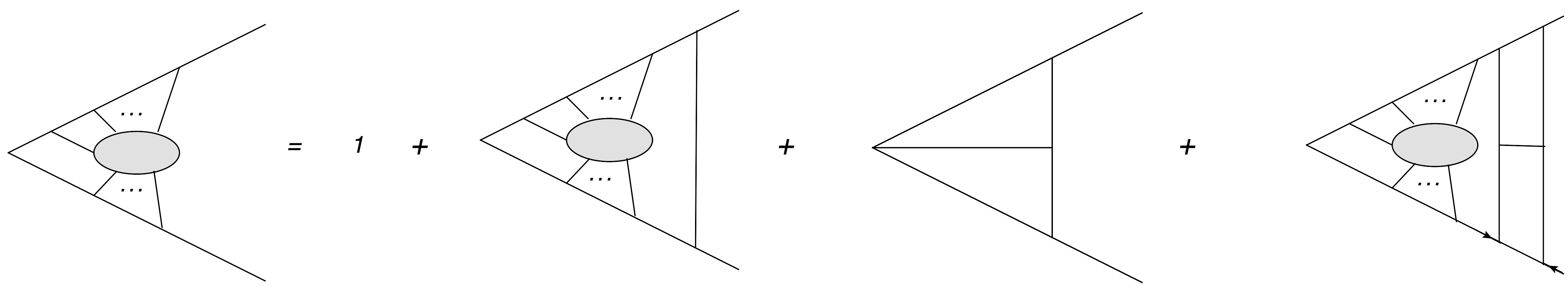
scaling limit beyond the LO

JMH, Huber

- only three integral classes at LO + NLO:



- modified Bethe-Salpeter equation



- leads to Schroedinger equation with inhomogeneous term!

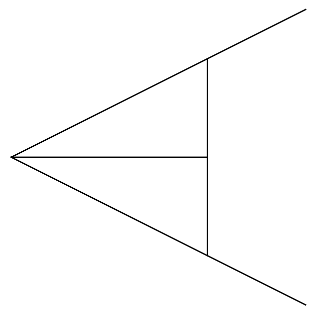
perturbative solution at NLO

JMH, Huber

- integral class (b)

$$\left[-\partial_{y_1}^2 - \frac{\hat\lambda}{8\pi^2} \frac{1}{(\cosh y_1 + \cos\phi)} + \frac{\Omega^2(\phi)}{4} \right] \Psi(y_1,\phi) = $$
$$= c \, \frac{\lambda \, \hat\lambda}{(\cosh y_1 + \cos\phi)} \, \Phi^{(1)} \left(\frac{e^{y_1}/2}{\cosh y_1 + \cos\phi}, \frac{e^{-y_1}/2}{\cosh y_1 + \cos\phi} \right) .$$

- ``seed'' of iteration is simple function:



$$\Phi^{(n)}(x,y) = \frac{1}{\sqrt{(1-x-y)^2 - 4xy}} \, \tilde\Phi^{(n)}(x,y)$$

$$\tilde\Phi^{(L)}(x,y) = \sum_{f=0}^{L} \frac{(-1)^f (2L-f)!}{L! f! (L-f)!} \log^f(z_1 z_2) \left[\mathrm{Li}_{2L-f}(z_1) - \mathrm{Li}_{2L-f}(z_2) \right]$$

Isaev; Davydychev, Usyukina

$$x = z_1 z_2 \, , \qquad y = (1-z_1)(1-z_2)$$

- can show: gives rise to same function class as at LO!

JMH, Huber

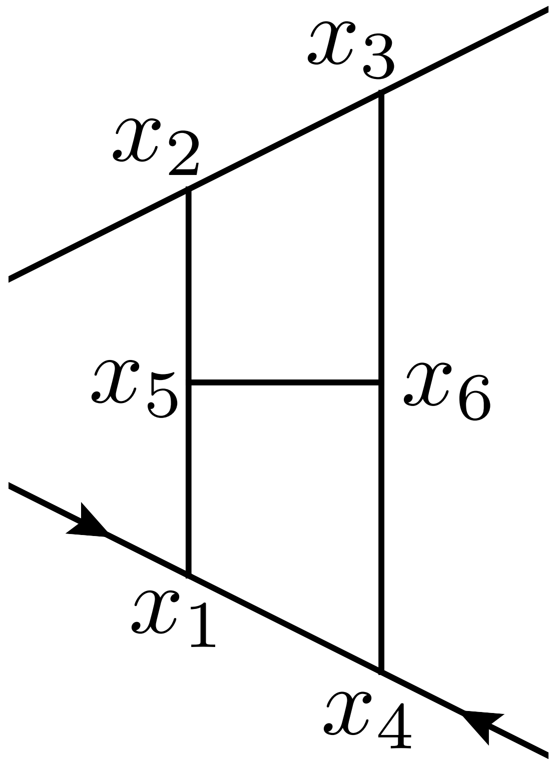
- can be solved at any loop order in terms of HPLs!

perturbative solution at NLO

JMH, Huber

- integral class (c)

- H-exchange kernel involves same function!



$$f(x_1,x_2,x_3,x_4) = (\partial_1+\partial_4)^2\, h(x_1,x_2;x_3,x_4)\,,$$

$$h(x_1,x_2;x_3,x_4) = \int \frac{d^4x_5 d^4x_6}{(i\pi^2)^2}\, \frac{1}{x_{15}^2 x_{25}^2 x_{36}^2 x_{46}^2 x_{56}^2}$$

diff. eq. : Beisert et al.; Sokatchev et al;

$$\begin{aligned}\tilde{f} \;=&\; x_{24}^2(x_{12}^2+x_{23}^2-x_{31}^2)\,\Phi^{(1)}\left(\frac{x_{12}^2}{x_{13}^2},\frac{x_{23}^2}{x_{13}^2}\right)+x_{13}^2(x_{12}^2+x_{14}^2-x_{24}^2)\,\Phi^{(1)}\left(\frac{x_{12}^2}{x_{24}^2},\frac{x_{14}^2}{x_{24}^2}\right)\\ &+x_{24}^2(x_{14}^2+x_{34}^2-x_{13}^2)\,\Phi^{(1)}\left(\frac{x_{34}^2}{x_{13}^2},\frac{x_{14}^2}{x_{13}^2}\right)+x_{13}^2(x_{23}^2+x_{34}^2-x_{24}^2)\,\Phi^{(1)}\left(\frac{x_{34}^2}{x_{24}^2},\frac{x_{23}^2}{x_{24}^2}\right)\\ &+(x_{13}^2x_{24}^2-x_{14}^2x_{23}^2-x_{12}^2x_{34}^2)\,\Phi^{(1)}\left(\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\right)\,,\end{aligned} \qquad (4.21)$$

$$\tilde{f} \;=\; (x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2) f$$

- needs to be integrated over line parameters
- will the result be given in terms of HPLs? Algorithm for all loops?

scaling limit at strong coupling to NLO

- expand string theory result of Drukker, Forini in limit

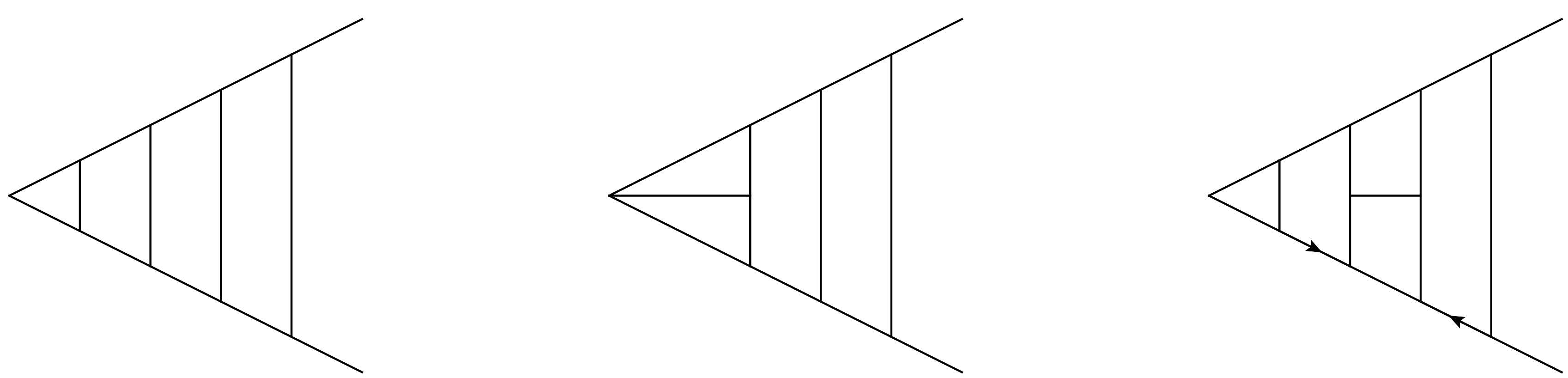
$$\Gamma = -\frac{\sqrt{\hat{\lambda}}}{2\pi \cos\frac{\phi}{2}} \left[1 - \frac{1}{2}\frac{\lambda}{\hat{\lambda}} \log\frac{\hat{\lambda}}{\lambda} + \mathcal{O}\left(\frac{\lambda}{\hat{\lambda}}\right)\right] .$$

- agreement with field theory at LO (ladders) Correa, JMH, Maldacena, Sever
- at NLO:

$$\Gamma^{(a)+(c)} = -\Omega_0 = -\frac{\sqrt{\hat{\lambda}}}{2\pi \cos\frac{\phi}{2}} \left[1 - \frac{1}{2}\frac{\lambda}{\hat{\lambda}} \log\frac{\hat{\lambda}}{\lambda} + \mathcal{O}\left(\frac{\lambda}{\hat{\lambda}}\right)\right]$$

JMH, Huber
Bykov, Zarembo for $\phi \to \pi$

agrees if integral class (b) is subleading at strong coupling



(a) (b) (c)

- NB: in principle there could be an order of limits ambiguity between scaling and strong coupling limit

Summary and discussion

$\Gamma_{\mathrm{cusp}}(\phi, \theta, \lambda, N)$ is interesting physical quantity

- exact result for small angles

  agrees with string theory result

- relation to Regge limit of massive amplitudes

  planar integrand for cusped Wilson loop known!

  full three-loop result

- new scaling limit; Schrödinger problem

  systematic solution to all loop orders

  surprises in structure of results:
  only certain HPLs, zeta values

Outlook

- prove more exact properties: HPLs, zeta values,...

  proofs are often constructive,
  - i.e. also solve the computational problem

- TBA equations from integrability

  Correa, Maldacena, Sever; Drukker

  simplify them in exactly known cases?
  - e.g. scaling limit; small angle limit (Bremsstrahlung)

  $\Gamma_{\rm cusp}$ is special case of massive amplitude
  - does integrability apply there too?

- non-planar corrections

  appear first at four-loops
  earlier in other Wilson loops with more external lines

- apply new ideas to QCD - three loops?