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Infinite sum of four-point ladder diagrams

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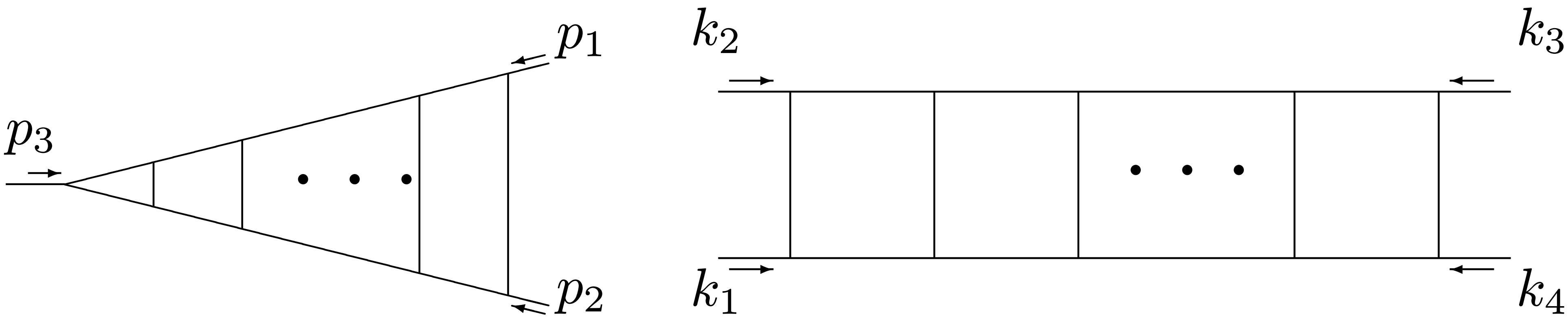
based on work with

D. J. Broadhurst and N. I. Ussyukina

Off-shell 3-point and 4-point ladder diagrams in ϕ^3 theory

For arbitrary off-shell values of the momenta and massless internal propagators, analytical results 3-point and 4-point ladder diagrams were found for an arbitrary number of loops, L.



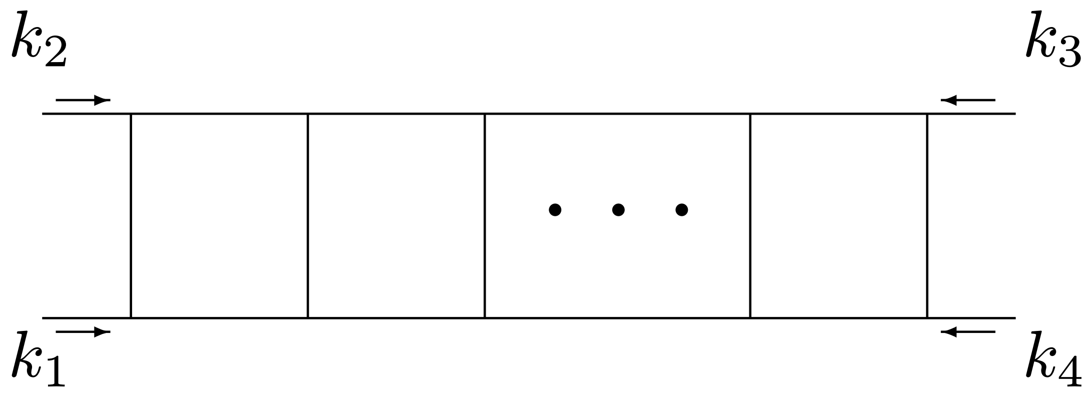
3-point and 4-point L-loop diagrams in ϕ^3 theory

[UD1] = [N.I. Ussyukina & A.I.D., Phys. Lett. B298 (1993) 363] – two-loop 3- and 4-point ladder diagrams

[UD2] = [N.I. Ussyukina & A.I.D., Phys. Lett. B305 (1993) 136] – L-loop 3- and 4-point ladder diagrams

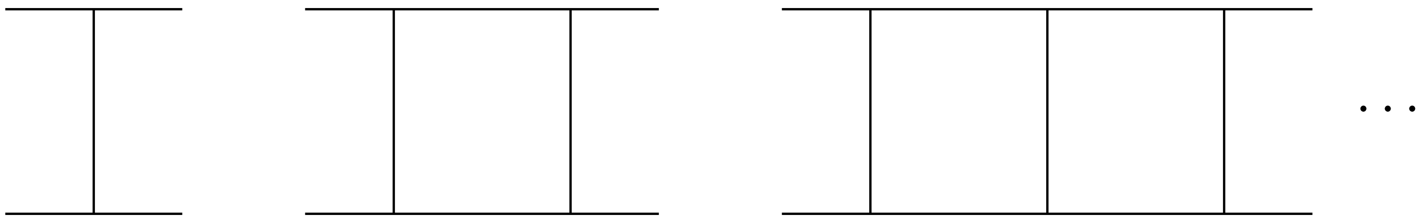
[Bro] = [D.J. Broadhurst, Phys. Lett. B307 (1993) 132] – using Gegenbauer-polynomial methods

Four-point ladder diagrams



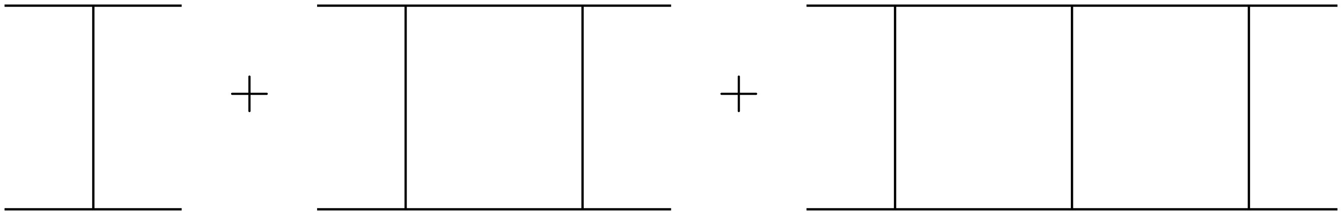
The off-shell results are finite and depend on 6 kinematic invariants

$$k_1^2,\ k_2^2,\ k_3^2,\ k_4^2,\ s = (k_1 + k_2)^2,\ t = (k_2 + k_3)^2.$$



zero-loop $\mathcal{T}$ one-loop two-loop

Infinite sum of 4-point ladder diagrams

$\mathcal{D} =$  $+ \ldots$

The infinite sum should satisfy a Dyson–Schwinger equation, of the schematic form

$$\mathcal{D} = \mathcal{T} + g^2 \int \mathrm{d}^4 k \; \mathcal{T} \cdot \mathcal{D}.$$

- $\mathcal{T}$ is the t-channel tree-diagram (normalized to $1/t$)
- the dot indicates convolution under the 4-dim integration that adds another loop
- $\mathcal{D}$ can be understood as the Bethe–Salpeter kernel in ladder approximation

[B.A. Arbuzov and V.E. Rochev], [K.G. Klimenko and V.E. Rochev]

Analogy with the set of non-negative integers, $\mathcal{N} = \{0, 1, 2, 3, \ldots\}$:

$$\mathbf{1}^+\mathcal{N} = \{1, 2, 3, 4, \ldots\}, \qquad \mathcal{N} = \{0, \mathbf{1}^+\mathcal{N}\}$$

Motivations

- To study the infinite coupling limit – in particular, to check our conjecture (back in 1993) that by including the tree-diagram $\mathcal{T}$ in $\mathcal{D}$ we would obtain zero for the sum of 4-point ladder diagrams at infinite coupling.

- Ladder approximations are of interest to $\mathcal{N} = 4$ super Yang–Mills theory whose strong coupling limit may be governed by an AdS/CFT correspondence.
  [B. Eden, P.S. Howe, C. Schubert, E. Sokatchev, P.C. West, M. Bianchi, S. Kovacs, G. Rossi, Y.S. Stanev, F.A. Dolan, H. Osborn, N. Beisert, C. Kristjansen, J. Plefka, G.W. Semenoff, M. Staudacher, J.M. Drummond, G.P. Korchemsky, J. Henn, V.A. Smirnov, D. Nguyen, M. Spradlin, A. Volovich, L.F. Alday, R. Roiban, B. Basso, L.V. Bork, D.I. Kazakov, G.S. Vartanov]
  Another interesting application is the conformal quantum mechanics [A.P. Isaev]

- Recent interest to studying properties of the functions occurring in such ladder diagrams [I. Kondrashuk, A. Vergara, A.V. Kotikov, I. Gonzalez, e.a.]

In any case, we hope that it may be of interest to see the explicit form of a 4-point ladder sum, as a function of the 6 kinematic invariants and the coupling g^2, which also has the dimensions of $(\text{mass})^2$ in ϕ^3 theory.

[BrD] = [D.J. Broadhurst and A.I Davydychev, Nucl. Phys. B (Proc. Suppl.), 205–206 (2010) 326]

The L-loop term

We write the perturbation series of ladder diagrams as

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{1}{t} \left\{ 1 + \sum_{L=1}^{\infty} \left(-\frac{\kappa^2}{4} \right)^L \Phi^{(L)}(X, Y) \right\}$$

with dimensionless ratios

$$X \equiv \frac{k_1^2 k_3^2}{st}, \quad Y \equiv \frac{k_2^2 k_4^2}{st}, \quad \kappa^2 \equiv \frac{g^2}{4\pi^2 s}$$

that we assume to be positive.

Here the dimensionless function $\Phi^{(L)}(X, Y)$, accompanied by the factor $\frac{1}{t} \left(-\frac{\kappa^2}{4} \right)^L$, represents the contribution of the L-loop term.

The L-loop term (continued)

As shown in [UD2], the L-loop term

$$\Phi^{(L)}(X,Y) = -\frac{1}{L!\,(L-1)!}\int_0^1 \frac{\mathrm{d}\xi}{Y\xi^2+(1-X-Y)\xi+X} \times \left[\ln\xi\left(\ln\frac{Y}{X}+\ln\xi\right)\right]^{L-1}\left(\ln\frac{Y}{X}+2\ln\xi\right)$$

depends only on the cross ratios X and Y and **is described by the same function as the ladder 3-point function**. When scaled by an appropriate power of p_3^2, the latter depends only on the ratios $x = p_1^2/p_3^2$ and $y = p_2^2/p_3^2$ and is given by $\Phi^{(L)}(x,y)$.

The origin of this simplification was explained in [UD2] and [Bro] by applying the conformal transformation that relates 4-point ladder diagram to the 3-point one.

The L-loop term (continued)

This integral may be evaluated in terms of polylogarithms Li_j [UD2].
Let us consider the case where the Källen function

$$\mu = \sqrt{4XY - (X+Y-1)^2}$$

is real and positive. Then we are outside the region that contains Landau singularities and hence may define the geometrical angle ϕ (with $0 < \phi < \pi$):

$$\phi = \arccos\left(\frac{X+Y-1}{2\sqrt{XY}}\right), \quad \text{so that} \quad \mu = 2\sqrt{XY}\,\sin\phi.$$

In this region, the L-loop term [UD2] can be presented in terms of Li_j as

$$\Phi^{(L)}(X,Y) = \frac{2}{\mu L!}\sum_{j=L}^{2L}\frac{j!}{(j-L)!\,(2L-j)!}\left(\ln\frac{X}{Y}\right)^{2L-j}\,\mathrm{Im}\,\mathrm{Li_j}\left(\sqrt{\frac{\mathrm{Y}}{\mathrm{X}}}\,\mathrm{e}^{\mathrm{i}\phi}\right)$$

involving powers of $\ell \equiv \ln(X/Y)$ and $\mathrm{Im}\,\mathrm{Li_j}$ ($L \le j \le 2L$). The symmetry $\Phi^{(L)}(X,Y) = \Phi^{(L)}(Y,X)$ is ensured by the inversion formula for Li_j, see [Lewin].

Infinite sum: an integral with a Bessel function

Let us omit the tree term $1/t$ and use the integral representation for the L-loop term to sum the series

$$\sum_{L=1}^{\infty} \left(-\frac{\kappa^2}{4}\right)^L \Phi^{(L)}(X,Y) = \frac{\kappa}{2}\int\limits_0^1 \frac{\mathrm{d}\xi}{X+(1-X-Y)\xi+Y\xi^2} \left(\ln\frac{Y}{X}+2\ln\xi\right)$$

$$\times \frac{1}{\sqrt{\ln\xi\,\left(\ln\frac{Y}{X}+\ln\xi\right)}}\, J_1\left(\kappa\sqrt{\ln\xi\,\left(\ln\frac{Y}{X}+\ln\xi\right)}\right)$$

where J_1 is a Bessel function,

$$J_1(z) = -\frac{2}{z}\sum_{L=1}^{\infty}\frac{1}{L!\,(L-1)!}\left(-\frac{z^2}{4}\right)^L$$

To remind,

$$X \equiv \frac{k_1^2 k_3^2}{st}, \quad Y \equiv \frac{k_2^2 k_4^2}{st}, \quad \kappa^2 \equiv \frac{g^2}{4\pi^2 s}$$

Infinite sum: an integral with a Bessel function (continued)

Substituting $\xi = e^{-\eta}$ and denoting $\ell \equiv \ln \frac{X}{Y}$, we obtain

$$\sum_{L=1}^{\infty} \left(-\frac{\kappa^2}{4}\right)^L \Phi^{(L)}(X,Y) = -\frac{\kappa}{2} \int\limits_0^\infty \frac{e^{-\eta}\, \mathrm{d}\eta}{X + (1-X-Y)e^{-\eta} + Ye^{-2\eta}} \times \frac{2\eta+\ell}{\sqrt{\eta(\ell+\eta)}} J_1\left(\kappa\sqrt{\eta(\ell+\eta)}\right).$$

The denominator may be re-written as

$$X + (1-X-Y)e^{-\eta} + Ye^{-2\eta} = e^{-\eta}\left[1 - X - Y + 2\sqrt{XY}\cosh\left(\eta + \frac{\ell}{2}\right)\right]$$
$$= -2\sqrt{XY}e^{-\eta}\left[\cos\phi - \cosh\left(\eta + \frac{\ell}{2}\right)\right].$$

Infinite sum: an integral with a Bessel function (continued)

In this way, we arrived at

$$\sum_{L=1}^{\infty} \left(-\frac{\kappa^2}{4}\right)^L \Phi^{(L)}(X,Y) \;=\; \frac{\kappa}{4\sqrt{XY}} \int\limits_0^\infty \frac{\mathrm{d}\eta}{\cos\phi - \cosh\left(\eta + \frac{\ell}{2}\right)} \times \frac{2\eta+\ell}{\sqrt{\eta(\ell+\eta)}} J_1\left(\kappa\sqrt{\eta(\ell+\eta)}\right)$$

and obtained, in 1999–2000, an explicit summation of all 4-point ladder diagrams with loop numbers $L > 0$.

Yet we could find no way of investigating our hunch that inclusion of the tree diagram, with $L = 0$, might give an exponentially vanishing result at infinitely strong coupling.

Infinite sum: an integral with a Bessel function (continued)

The first break-through came from noticing that

$$\frac{2\eta+\ell}{\sqrt{\eta(\ell+\eta)}}\, J_1\left(\kappa\sqrt{\eta(\ell+\eta)}\right) = -\frac{2}{\kappa}\,\frac{\mathrm{d}}{\mathrm{d}\eta} J_0\left(\kappa\sqrt{\eta(\ell+\eta)}\right).$$

Then, integrating by parts, we found that the Dyson–Schwinger solution is

$$\begin{aligned}\mathcal{D}(k_1^2,k_2^2,k_3^2,k_4^2,s,t) &= \frac{1}{t} + \frac{1}{t}\sum_{L=1}^{\infty}\left(-\frac{\kappa^2}{4}\right)^L \Phi^{(L)}(X,Y)\\ &= \frac{1}{2t\sqrt{XY}}\int\limits_0^\infty \mathrm{d}\eta\, \frac{\sinh\left(\eta+\frac{\ell}{2}\right) J_0\left(\kappa\sqrt{\eta(\ell+\eta)}\right)}{\left[\cosh\left(\eta+\frac{\ell}{2}\right)-\cos\phi\right]^2}\end{aligned}$$

where the tree-term $1/t$ is precisely included by the surface term of the partial integration.

Infinite sum: an integral with a Bessel function (continued)

Next, we shift the integration variable η and obtain

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{1}{2t\sqrt{XY}} \int\limits_{\ell/2}^{\infty} \mathrm{d}\eta \; \frac{\sinh\eta \; J_0\left(\kappa\sqrt{\eta^2 - \frac{1}{4}\ell^2}\right)}{(\cosh\eta - \cos\phi)^2}.$$

The $X \longleftrightarrow Y$ symmetry of the result is now quite easy to understand:

- If we were to interchange X and Y, then the only thing that would change is the lower limit of integration: $\ell/2 \to -\ell/2$, since $\phi \equiv \arccos((X+Y-1)/(2\sqrt{XY}))$ is symmetric in (X, Y).

- The integral between $-\ell/2$ and $\ell/2$ is zero, since the integrand is an odd function of η and an even function of $\ell \equiv \ln(X/Y)$.

$\Rightarrow$ We may take $\frac{1}{2}|\ell| = \frac{1}{2}|\ln X - \ln Y|$ as the lower limit of integration.

Infinite sum: getting rid of the Bessel function

We re-write the result as

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) \;=\; \frac{1}{2t\sqrt{XY}} \int\limits_0^\infty \frac{\mathrm{d}\eta\;\sinh\eta}{(\cosh\eta - \cos\phi)^2}\, J_0\!\left(\kappa\sqrt{\eta^2 - \tfrac{1}{4}\ell^2}\right) \vartheta\left(\eta^2 - \tfrac{1}{4}\ell^2\right),$$

where $\vartheta(x)$ is the Heaviside function: $\vartheta(x)\!=\!1$, for $x > 0$, and $\vartheta(x)\!=\!0$, otherwise.
Now, let us use the integral representation

$$\int\limits_0^\infty \mathrm{d}\tau\;\sin\left(\kappa\eta\cosh\tau\right)\;\cos\left(\tfrac{1}{2}\ell\kappa\sinh\tau\right) = \frac{\pi}{2}\, J_0\!\left(\kappa\sqrt{\eta^2 - \tfrac{1}{4}\ell^2}\right)\;\vartheta\left(\eta^2 - \tfrac{1}{4}\ell^2\right)$$

which may be obtained from [PBM1] (Equation (2.5.25.9), with the substitutions $x = \kappa\sinh\tau$, $y = \kappa$, $c = \eta$, and $b = \frac{1}{2}\ell$).
The key point is that we are rid of the integration limit $\ell/2$.

Infinite sum: getting rid of the Bessel function (continued)

By this device, we obtain a double integral

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{1}{\pi t \sqrt{XY}} \int_0^\infty \frac{\mathrm{d}\eta \;\sinh\eta}{(\cosh\eta - \cos\phi)^2} \times \int_0^\infty \mathrm{d}\tau \sin(\kappa\eta\cosh\tau) \cos\left(\tfrac{1}{2}\ell\kappa\sinh\tau\right)$$

Next, substitution $z = \kappa\cosh\tau$ gives $\kappa\sinh\tau = \sqrt{z^2-\kappa^2}$ and $\mathrm{d}\tau = \mathrm{d}z/\sqrt{z^2-\kappa^2}$. Hence we obtain

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{1}{\pi t \sqrt{XY}} \int_0^\infty \frac{\mathrm{d}\eta \;\sinh\eta}{(\cosh\eta - \cos\phi)^2} \times \int_\kappa^\infty \frac{\mathrm{d}z \;\sin(\eta z)}{\sqrt{z^2-\kappa^2}} \cos\left(\tfrac{1}{2}\ell\sqrt{z^2-\kappa^2}\right).$$

Infinite sum: getting rid of the Bessel function (continued)

Now we reverse the order of the integrations, obtaining

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{1}{\pi t \sqrt{XY}} \int\limits_{\kappa}^{\infty} \frac{\mathrm{d}z}{\sqrt{z^2 - \kappa^2}} \cos\left(\tfrac{1}{2}\ell\sqrt{z^2-\kappa^2}\right) \times \int\limits_0^{\infty} \frac{\mathrm{d}\eta \; \sinh\eta \; \sin(\eta z)}{(\cosh\eta - \cos\phi)^2} \, .$$

From Equation (2.5.48.18) of [PBM1] (with $t = \pi - \phi$, $c = 1$, $b = z$), we obtain

$$\int\limits_0^{\infty} \frac{\mathrm{d}\eta \; \sinh\eta \; \sin(\eta z)}{(\cosh\eta - \cos\phi)^2} = \frac{\pi z}{\sin\phi} \, \frac{\sinh\left[(\pi - \phi) z\right]}{\sinh(\pi z)} .$$

Infinite sum of ladder diagrams: the final solution

Recalling that $\mu = 2\sqrt{XY}\,\sin\phi$, we obtain

$$\mathcal{D}(k_1^2, k_2^2, k_3^2, k_4^2, s, t) = \frac{2}{t\mu}\int\limits_{\kappa}^{\infty} \frac{z\,\mathrm{d}z}{\sqrt{z^2-\kappa^2}}\,\frac{\sinh\left[(\pi-\phi)z\right]}{\sinh(\pi z)}\,\cos\left(\tfrac{1}{2}\ell\sqrt{z^2-\kappa^2}\right)\,.$$

This is our final solution to the Dyson–Schwinger equation that sums all L-loop 4-point ladder diagrams, including (most crucially) the tree-diagram, with $L=0$ loops. The sum manifestly vanishes, exponentially fast, as the dimensionless coupling $\kappa = g/(2\pi\sqrt{s})$ tends to infinity, since the ratio of sinh functions in the integrand satisfies

$$\frac{\sinh\left[(\pi-\phi)z\right]}{\sinh(\pi z)} \le \frac{\sinh\left[(\pi-\phi)\kappa\right]}{\sinh(\pi\kappa)} = \mathcal{O}(e^{-\kappa\phi})$$

with $0 < \phi < \pi$.

So we are done, 17 years after conjecturing such an exponential suppression.