Differential reduction of generalized hypergeometric functions in application to Feynman diagrams.

HyperDire project.

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Motivation

Oleg Tarasov (1996), and Davydychev-Delbourgo (1997), have suggested two elegant approaches for construction of hypergeometric representation of one-loop Feynman Diagrams. One of main achievement of these approaches are the essential reduction of independent variables. In accordance with Fleischer-Jegerlehner-Tarasov, 2003.

<table>
<thead>
<tr>
<th>Type of 1-loop diagram</th>
<th># 1</th>
<th># 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>N=2 (propagator)</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>N=3 (vertex)</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>N=4 (box)</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>N = 5 (pentagon)</td>
<td>15</td>
<td>4</td>
</tr>
<tr>
<td>N = k</td>
<td>k(k+1)/2</td>
<td>k-1</td>
</tr>
</tbody>
</table>

where
#1 the number of kinematic invariants (non-zero masses/momenta)
and
#2 the number of variables in hypergeometric representation.

In this approach, the one-loop $N$-point function is expressible in terms of hypergeometric functions of $N-1$ variables.

1-loop diagrams: Finite part

The program of constructing the analytical coefficients of the $\varepsilon$-expansion is a more complicated matter. The finite parts of one-loop diagrams in $d = 4$ dimension are expressible in terms of the Spence dilogarithm function
t’Hooft, Veltman, 1979;
Denner, Nierste, Scharf, 1991;
Ellis, Zanderighi, 2007;
Denner, Dittmaier, 2010

\[ \text{FeynmanDiagramm} = \frac{1}{\varepsilon^2} A + \frac{1}{\varepsilon} B + C + D\varepsilon + E\varepsilon^2 \ldots \]
The Task

to elaborate the algorithm and implementation for:

- manipulation with multiple hypergeometric (Horn-type) functions (express parameters of arbitrary values in terms of ones that differ from original by integers)
- construction of analytical coefficients of $\varepsilon$-expansion of multiple hypergeometric (Horn-type) functions

**Finally:**
Package for Numerical Evaluation of finite, $O(\varepsilon)$ and $O(\varepsilon^2)$ parts of one-loop Feynman Diagrams with an arbitrary set of kinematic invariants
Example: Sunset Diagram

\[ F_G = \int \frac{d^d k_1 d^d k_2}{[(k_1 - p)^2 - m_1^2][k_2^2 - m_2^2][(k_1 - k_2)^2 - m_3^2]} \]

\[ = \int_{-i\infty}^{i\infty} ds_1 ds_2 ds_3 \frac{m_1^{2s_1} m_2^{2s_2} m_3^{2s_3}}{(-p^2)^{s_1+s_2+s_3}} \Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3) \]

\[ \Gamma(3-d+s_1+s_2+s_3) \Gamma(d/2-1-s_1)\Gamma(d/2-1-s_2)\Gamma(d/2-1-s_3) \]

\[ \sim z_1^{d/2-1} z_2^{d/2-1} F_c^{(3)}(1, d/2, d/2, d/2, d/2, d/2; z_1, z_2, z_3) \]

\[ -z_1^{d/2-1} \Gamma^2(1-d/2) F_c^{(3)}(1, 2-d/2, d/2, 2-d/2, d/2, d/2, z_1, z_2, z_3) \]

\[ -z_2^{d/2-1} \Gamma^2(1-d/2) F_c^{(3)}(1, 2-d/2, d/2, 2-d/2, d/2, z_1, z_2, z_3) \]

\[ -\Gamma(d/2-1)\Gamma(1-d/2)\Gamma(3-d) F_c^{(3)}(3-d, 2-d/2, 2-d/2, 2-d/2, d/2, d/2, z_1, z_2, z_3) \]

in terms of the hypergeometric function (in the case \( n = 3 \))

\[ F_c^{(n)}(a, b; c_1, \cdots, c_n; z_1, \cdots z_n) = \sum_{k_1, \cdots k_n} \frac{(a)_{k_1+\cdots+k_n}(b)_{k_1+\cdots+k_n}}{(c_1)_{k_1} \cdots (c_n)_{k_n}} \frac{z_1^{k_1} \cdots z_n^{k_n}}{k_1! \cdots k_n!} \]

with arguments \( z_1 = m_1^2/m_3^2, \quad z_2 = m_2^2/m_3^2, \quad z_3 = p^2/m_3^2. \)
Horn-type Hypergeometric Functions

In accordance with Horn definition, a formal (Laurent) power series in \( r \) variables,
\[
\Phi(\vec{z}) = \sum C(\vec{m}) \vec{z}^\vec{m} \equiv \sum_{m_1, m_2, \ldots, m_r} C(m_1, m_2, \ldots, m_r) x_1^{m_1} \cdots x_r^{m_r},
\]
is called hypergeometric if for each \( i = 1, \ldots, r \) the ratio
\[
\frac{C(\vec{m} + e_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})} \Rightarrow C(\vec{m}) = \prod_{i=1}^{r} \lambda_i^{m_i} R(\vec{m}) \left( \frac{\prod_{j=1}^{N} \Gamma(\mu_j(\vec{m}) + \gamma_j)}{\prod_{k=1}^{M} \Gamma(\nu_k(\vec{m}) + \delta_k)} \right).
\]

\( P, Q, R \) are the rational functions in the index of summation: \( \vec{m} = (m_1, \ldots, m_r) \), and \( \vec{e}_j \) is unit vector with unity in the \( j^{\text{th}} \) place. The Horn hypergeometric function satisfies the following system of equation
\[
Q_j \left( \sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k} \right) \frac{1}{x_j} \Phi(\vec{z}) = P_j \left( \sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k} \right) \Phi(\vec{z}).
\]
**Current status**

The systematic algorithms for construction of analytical coefficients of $\varepsilon$-expansion for a large class of Horn-type hypergeometric functions around integer values of parameters was suggested by Moch-Uwer-Weinzierl, 2001.

<table>
<thead>
<tr>
<th>Hypergeometric function</th>
<th>Algorithm</th>
</tr>
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<tbody>
<tr>
<td>$pF_{p-1}$</td>
<td>A</td>
</tr>
<tr>
<td>$F_1$</td>
<td>A, B</td>
</tr>
<tr>
<td>$F_2, F_3$</td>
<td>A, C, D</td>
</tr>
<tr>
<td>$F_4$</td>
<td>does not work</td>
</tr>
</tbody>
</table>

Two of these algorithms $A \Rightarrow pF_{p-1}$ and $B \Rightarrow F_1$ was extended for zero-balance case: Weinzierl, 2004.
It was a few attempts to extend these approach to hypergeometric functions like $F_4$
Del Duca,Duhr,Glover,Smirnov, 2009
or to another (different from zero-balance case) set of parameters
Rottmann-Reina, 2011
Ablinger-Blümlein-Schneider, 2011
HyperDIRE project
(HYPERgeometric DIFFerential REduction)

HYPERDIRE is a set of Wolfram Mathematica based programs for differential reduction of Horn type hypergeometrical functions.

HYPERDIRE includes the following packages:

- **pfq** is relevant to manipulation with hypergeometrical functions $p+1F_p$
- **AppellF1F4** is relevant to manipulation with Appell`s hypergeometric functions of two variables $F_1$, $F_2,F_3,F_4$.
- Fd multiple variable function

The package is available at: https://sites.google.com/site/loopcalculations/home
Horn-type Hypergeometric Functions: reduction to the basis

Let us consider the series

$$
\Phi(\vec{\gamma}; \vec{\sigma}; \vec{z}) = \sum_{m_1, m_2, \ldots, m_r=0}^{\infty} \left( \frac{\prod_{j=1}^{K} \Gamma \left( \sum_{a=1}^{r} \mu_{ja} m_a + \gamma_j \right)}{\prod_{k=1}^{L} \Gamma \left( \sum_{b=1}^{r} \nu_{kb} m_b + \sigma_k \right)} \right) x_1^{m_1} \cdots x_r^{m_r},
$$

The sequences $\vec{\gamma} = (\gamma_1, \cdots, \gamma_K)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_L)$ are called upper and lower parameters of the hypergeometric function, respectively. Two functions with sets of parameters shifted by a unit, $\Phi(\vec{\gamma} + e^c; \vec{\sigma}; \vec{z})$ and $\Phi(\vec{\gamma}; \vec{\sigma}; \vec{z})$, are related by a linear differential operator:

$$
\Phi(\vec{\gamma} + e^c; \vec{\sigma}; \vec{z}) = \left( \sum_{a=1}^{r} \mu_{ca} x_a \frac{\partial}{\partial x_a} + \gamma_c \right) \Phi(\vec{\gamma}; \vec{\sigma}; \vec{z})
$$

$$
\Phi(\vec{\gamma}; \vec{\sigma} - e^c; \vec{z}) = \left( \sum_{b=1}^{r} \nu_{cb} x_b \frac{\partial}{\partial x_b} + \sigma_c \right) \Phi(\vec{\gamma}; \vec{\sigma}; \vec{z}) .
$$
Horn-type Hypergeometric Functions: Inverse Operators

Staring from homogeneous system of PDE and direct differential operators, the inverse differential operators can be constructed:

$$\Phi(\vec{\gamma} - \vec{e}_c; \vec{\sigma}; z) = \sum_{a} S_a(z, \vec{\partial}_x) \Phi(\vec{\gamma}; \vec{\sigma}; z)$$

$$\Phi(\vec{\gamma}; \vec{\sigma} + \vec{e}_c; z) = \sum_{b} L_b(z, \vec{\partial}_x) \Phi(\vec{\gamma}; \vec{\sigma}; z).$$

In this way, any Horn-type function can be written as follows:

$$P_0(z)\Phi(\vec{\gamma} + \vec{k}; \vec{\sigma} + \vec{l}; z) = \sum_{m_1, \ldots, m_p=0} P_{m_1, \ldots, m_r}(z) \left( \frac{\partial}{\partial z} \right)^{m} \Phi(\vec{\gamma}; \vec{\sigma}; z),$$

where $P_0(z)$ and $P_{m_1, \ldots, m_p}(z)$ are polynomials with respect to $\vec{\gamma}, \vec{\sigma}$ and $z$ and $\vec{k}, \vec{l}$ are lists of integers.

Simplify the procedure of Factorization.
Differential reduction algorithm for \( p+1 F_p \) hypergeometric funct.

- Differential identities:
  \[
  _p F_q(a_1 + 1, \bar{a}; \bar{b}; z) = B^+_{a_1} F_q(a_1, \bar{a}; \bar{b}; z) = \frac{1}{a_1} (\theta + a_1) \ F_q(a_1, \bar{a}; \bar{b}; z)
  \]
  \[
  _p F_q(\bar{a}; b_1 - 1, \bar{b}; z) = H^+_{b_1} F_q(\bar{a}; b_1, \bar{b}; z) = \frac{1}{b_1 - 1} (\theta + b_1 - 1) \ F_q(\bar{a}; b_1, \bar{b}; z)
  \]

- Inverse operators:
  \[
  p+1 F_p(\bar{a}; b_i + 1, \bar{b}; z) = H^+_{b_i} F_p(\bar{a}; b_i, \bar{b}; z) ,
  \]
  \[
  H^+_a = \frac{b_i - 1}{d_i} \left[ \frac{d}{dz} \prod_{j \neq i} (\theta + b_j - 1) - s_i(\theta) \right]_{b_i \to b_i + 1} ,
  \]
  \[
  d_i = \prod_{j=1}^{p+1} (1 + a_j - b_i) ,
  \]
  \[
  s_i(x) = \frac{\prod_{j=1}^{p+1} (x + a_j) - d_i}{x + b_i - 1} ,
  \]

Inverse operators:

- \( p+1 F_p(a_i - 1, \bar{a}; \bar{b}; z) = B^-_{a_i} F_p(a_i, \bar{a}; \bar{b}; z) , \)
  \[
  B^-_{a_i} = -\frac{a_i}{c_i} \left[ t_i(\theta) - z \prod_{j \neq i} (\theta + a_j) \right]_{a_i \to a_i - 1} ,
  \]
  \[
  c_i = -a_i \prod_{j=1}^{p} (b_j - 1 - a_i) ,
  \]
  \[
  t_i(x) = \frac{x \prod_{j=1}^{p} (x + b_j - 1) - c_i}{x + a_i} ,
  \]
Differential reduction algorithm for \( p+1F_p \) hypergeometric funct.

- Example of differential reduction:

\[
\begin{align*}
\begin{pmatrix} a_1-1, a_2, a_3 \mid z \end{pmatrix}_{b_1, b_2} (b_1-a_1)(b_2-a_1) &= \left\{ (1-z)\theta^2 \\
+ [(b_1+b_2-1-a_1)-z(a_2+a_3)] \theta + (b_1-a_1)(b_2-a_1)-za_2a_3 \right\} \begin{pmatrix} a_1, a_2, a_3 \mid z \end{pmatrix}_{b_1, b_2}
\end{align*}
\]

- In reduction on more units the structure of equality will be the same
Implementation of algorithm

• The package called HYPERDIRE (HYPERgeometric DIFFerential REduction), based on language of program “Mathematica”

• Key feature is that the product of non-commutative step-up and step-down operators of differential reduction turn into product of special 2-dimensional matrices and vectors which greatly simplify and reduce the time of calculation

• The functional programming style reduce the calculation time

```mathematica
MapAt[ReplacePart[#, Join[#[[1, 1]], Table[0, {i, 1, Length[#[[2, 1]]] - Length[#[[1, 1]]]]], {1, 1}] &, Map[ReplacePart[#, {Table[SymmetricPolynomial[-i, #[[1]]], {i, -Length[#[[1]]], 0}]}], 1] &, Nest[If[(numberOfAdBup = 1 + Sum[Length[#[[i, 1]]], {i, 1, Length[#]}]; changevar = AdBupvector[#[[-1, 2]], listOfAdownAndBupch[[numberOfAdBup]]]; Mod[Length[#[[-1, 1]]], Length[#[[-1, 2, 2]]]] == 0
), Append[#, {{changevar[[1]]}, changevar[[2]], 1/changevar[[1]], 0}], ReplacePart[ReplacePart[Insert[#, changevar[[1]], {-1, 1, 1}], changevar[[2]], {-1, 2} ], #[[[-1, 3]]/ changevar[[1]], {-1, 3}]] & , initialVector, Length[listOfAdownAndBupch]]
, -1]
```
Example of module PFQ

ToGroebnerBasis [ {[1+a_1,1+a_2, a_3,a_4],[1+b_1, b_2+1,b_3],x} ] ,

IntegerPart={1,1,0,0,1,1,0} changeVector={-1,-1,0,0,0,0,1}

\left\{1, \frac{1}{a_2} + \frac{1}{b_3} + \frac{1}{a_1}, \frac{a_1+a_2+b_3}{a_1a_2b_3}, \frac{1}{a_1a_2b_3}\right\}, \{\{a_1, a_2, a_3, a_4\}, \{b_1 + 1, b_2 + 1, b_3 + 1\}, x\}, 1

Hypergeometric function parameters transformation

\[
\begin{aligned}
_{4}F_{3}\left(\begin{array}{c}
1+a_1, 1+a_2, a_3, a_4 \\
1+b_1, 1+b_2, b_3
\end{array} \bigg| z \right) &= 
\left[1 + \left(\frac{1}{a_2} + \frac{1}{b_3} + \frac{1}{a_1}\right) \theta \right]
\left[1 + \frac{a_1 + a_2 + b_3}{a_1a_2b_3} \theta^2 + \frac{1}{a_1a_2b_3} \theta^3\right]
_{4}F_{3}\left(\begin{array}{c}
a_1, a_2, a_3, a_4 \\
b_1 + 1, b_2 + 1, b_3 + 1
\end{array} \bigg| x \right)
\end{aligned}
\]
From differential reduction formulas could be derived reducibility criteria:
under which conditions the hypergeometric function could be expressed in
terms of hyp. function of lower order (four criteria)
Example of module PFQ, reducibility

\[ \text{ToGroebnerBasis} \left[ \left\{ \left\{ 3+b_1, 1+a_2, 1+a_3 \right\}, \left\{ 2+b_1, 2+b_2 \right\}, x \right\} \right]; \]

\[ \text{IntegerPart} = \{3,1,1,2,2\} \quad \text{changeVector} = \{-1,-1,-1\} \]

\[ \left\{ \frac{b_2+1}{(x-1)(b_1+2)}, \frac{(a_2x+a_3x-b_1x-x+b_1-b_2+1)(b_2+1)}{(x-1)xa_2a_3(b_1+2)} \right\}, \left\{ \{a_2, a_3\}, \{b_2 + 1\}, x \right\}, 1 \]

Hypergeometric function parameters transformation

\[ _3F_2 \left( \begin{array}{c} 3 + b_1, 1 + a_2, 1 + a_3 \\ 2 + b_1, 2 + b_2 \end{array} \bigg| x \right) \\
= \left[ -\frac{b_2+1}{(x-1)(b_1+2)} - \frac{(a_2x+a_3x-b_1x-x+b_1-b_2+1)(b_2+1)}{(x-1)xa_2a_3(b_1+2)} \theta \right] _2F_1 \left( \begin{array}{c} a_2, a_3 \\ b_2+1 \end{array} \bigg| x \right) \]
\[ J^q_{22}(m^2, p^2, \alpha_1, \alpha_2, \sigma_1, \cdots, \sigma_{q-1}) = \left[ i^{1-n} \pi^{n/2} \right]^q \left( -m^2 \right)^{\frac{n}{2} q - \alpha_1,2 - \sigma} \frac{\prod_{k=1}^{q-1} \Gamma \left( \frac{n}{2} - \sigma_k \right)}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \left\{ \Gamma \left( \alpha_1 + \sigma - \frac{n}{2} (q-1) \right) \Gamma \left( \alpha_2 + \sigma - \frac{n}{2} (q-1) \right) \Gamma \left( \sigma - \frac{n}{2} (q-2) \right) \Gamma \left( \alpha_1,2 + \sigma - \frac{n}{2} q \right) \right\} \\
\times \frac{\Gamma \left( \alpha_1 + \sigma - \frac{n}{2} (q-1) \right) \Gamma \left( \alpha_2 + \sigma - \frac{n}{2} (q-1) \right) \Gamma \left( \sigma - \frac{n}{2} (q-2) \right) \Gamma \left( \alpha_1,2 + \sigma - \frac{n}{2} q \right)}{\Gamma(\alpha_1,2 + 2\sigma - n(q-1)) \Gamma \left( \frac{n}{2} \right)} \right. \\
\left. 4 F_3 \left( \begin{array}{c} \alpha_1 + \sigma - \frac{n}{2} (q-1), \alpha_2 + \sigma - \frac{n}{2} (q-1), \sigma - \frac{n}{2} (q-2), \alpha_1,2 + \sigma - \frac{n}{2} q \\ \frac{n}{2}, \frac{1}{2} (\alpha_1,2 - n(q-1)) + \sigma, \frac{1}{2} (1 + \alpha_1,2 - n(q-1)) + \sigma \end{array} \right| \frac{p^2}{4m^2} \right) \right]. \\

- **Criteria of reducibility:**

- **q=1**
  \[ 2 F_1 \left( \begin{array}{c} 1, I_1 - \frac{n}{2} \\ I_2 \end{array} \right| z \right) \quad \text{IBP gives 1 MI} \\

- **q=2**
  \[ (1, \theta) \times 3 F_2 \left( \begin{array}{c} 1, I_1 - \frac{n}{2}, I_2 - n \\ I_3 + \frac{n}{2}, I_4 + \frac{1}{2} - \frac{n}{2} \end{array} \right| z \right) \quad \text{IBP gives 2 MI} \\

- **q=3,4,5....**
  \[ (1, \theta, \theta^2) \times 3 F_2 \left( \begin{array}{c} I_1 - \frac{n}{2} (q-1), I_2 - \frac{n}{2} (q-2), I_3 - \frac{n}{2} q \\ \frac{n}{2}, I_4 + \frac{1}{2} - \frac{n}{2} (q-1) \end{array} \right| z \right) \quad \text{IBP gives ???} \]
Possible applications

pFq package could work even with $\textbf{11F}_{10}$ and reduce it to the function of type $\textbf{7F}_{6}$
Appell Function F1,F2,F3,F4  
the case of two variables

Let us consider the system of linear differential equations of the second order for the functions $\omega(\bar{z})$:

$$
\begin{align*}
\theta_{11}\omega(\bar{z}) &= \left\{ P_0(\bar{z})\theta_{12} + P_1(\bar{z})\theta_1 + P_2(\bar{z})\theta_2 + P_3(\bar{z}) \right\} \omega(\bar{z}), \\
\theta_{22}\omega(\bar{z}) &= \left\{ R_0(\bar{z})\theta_{12} + R_1(\bar{z})\theta_1 + R_2(\bar{z})\theta_2 + R_3(\bar{z}) \right\} \omega(\bar{z}), \\
\theta_j &= z_j \frac{\partial}{\partial z_j}.
\end{align*}
$$

The differential reduction algorithm in application to the Appell function could be done in similar way as for the case of one variable hypergeometrical function.

\begin{align*}
R(x, y) F_1(\bar{A} + \bar{m}; x, y) &= [P_0(x, y) + P_1(x, y)\theta_x + P_2(x, y)\theta_y] F_1(\bar{A}; x, y), \\
S(x, y) F_j(\bar{A} + \bar{m}; x, y) &= [Q_0(x, y) + Q_1(x, y)\theta_x + Q_2(x, y)\theta_y + Q_3(x, y)\theta_x\theta_y] F_j(\bar{A}; x, y).
\end{align*}
Differential reduction for $F_1$

the direct differential expressions reads:

\[
\begin{align*}
aF_1(a + 1, b_1, b_2, c; x, y) &= (\theta_x + \theta_y + a)F_1(a, b_1, b_2, c; x, y), \\
b_1F_1(a, b_1 + 1, b_2, c; x, y) &= (\theta_x + b_1)F_1(a, b_1, b_2, c; x, y), \\
(c - 1)F_1(a, b_1, b_2, c - 1; x, y) &= (\theta_x + \theta_y + c - 1)F_1(a, b_1, b_2, c; x, y).
\end{align*}
\]

Inverse differential relations:

\[
\begin{align*}
(c-a)F_1(a-1, b_1, b_2, c; x, y) &= \\
[c-a-b_1x-b_2y+(1-x)\theta_x+(1-y)\theta_y]F_1(a, b_1, b_2, c; x, y), \\
(c-b_1-b_2)F_1(a, b_1 - 1, b_2, c; x, y) &= \\
\left[c-b_1-b_2-ax+(1-x)\theta_x-x\left(1-\frac{1}{y}\right)\theta_y\right]F_1(a, b_1, b_2, c; x, y), \\
(c-a)(c-b_1-b_2)F_1(a, b_1, b_2, c + 1; x, y) &= \\
c\left[(c-a-b_1-b_2)-\left(1-\frac{1}{x}\right)\theta_x-\left(1-\frac{1}{y}\right)\theta_y\right]F_1(a, b_1, b_2, c; x, y).
\end{align*}
\]
Example of module AppellF1F4

In explicit form:

\[
F_1(a, b_1, b_2, c; z_1, z_2) =  
\left[  
\begin{array}{c}
-az_1 + a + b_1z_1 + b_2z_2 - c - z_1 + 1 \\
\frac{a(-z_1)+a+b_1z_1+b_2z_2-c-z_1+1}{a-c+1} - \frac{(z_1-1)(a-b_1+1)}{(b_1-1)(a-c+1)} \theta_1 + \frac{z_2-1}{a-c+1} \theta_2 \\
\end{array}
\right]  
\times F_1(a + 1, b_1 - 1, b_2, c; z_1, z_2).
\]

The similar procedures are implemented for Appell function \( F_2, F_3, F_4 \)
Application AppellF1F4

massive q-loop propagator could be expressed through the F4 hypergeometrical function.

\[ J_{023}^q(M_1^2, M_2^2, \alpha_1, \alpha_2, \sigma_1, \ldots, \sigma_{q-1}) = \frac{[i^{1-n} \pi^{q/2}] q(-M_1^2)^{n/2} q^{-a_{\alpha_1, \alpha_2, \sigma}}}{\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma\left(\frac{n}{2}\right)} \left\{ \prod_{i=1}^{q-1} \frac{\Gamma\left(\frac{n}{2} - \sigma_i\right)}{\Gamma(\sigma_i)} \right\} \]

\[
\left[ \Gamma\left(\frac{n}{2} - \alpha_2\right) \Gamma\left(a_{\alpha_1, \alpha_2, \sigma} - \frac{n}{2} q\right) \Gamma\left(a_{\alpha_2, \sigma} - \frac{n}{2} (q-1)\right) \right.
\times F_4\left(a_{\alpha_1, \alpha_2, \sigma} - \frac{n}{2} q, a_{\alpha_2, \sigma} - \frac{n}{2} (q-1), \frac{n}{2}, 1 + \alpha_2 - \frac{n}{2} \left| \frac{p^2}{M_1^2}, \frac{M_2^2}{M_1^2}\right.\right]
\left. + \left(\frac{M_2^2}{M_1^2}\right)^{\frac{n}{2} - \alpha_2} \Gamma\left(\alpha_2 - \frac{n}{2}\right) \Gamma\left(a_{\alpha_1, \sigma} - \frac{n}{2} (q-1)\right) \Gamma\left(\sigma - \frac{n}{2} (q-2)\right) \right.
\times F_4\left(a_{\alpha_1, \sigma} - \frac{n}{2} (q-1), \sigma - \frac{n}{2} (q-2), \frac{n}{2}, 1 - \alpha_2 + \frac{n}{2} \left| \frac{p^2}{M_1^2}, \frac{M_2^2}{M_1^2}\right.\right) \right].
\] (132)
The case of multiple variables

- Functions $F_A, F_B, F_C, F_D$ are the extensions of two variable functions $F_1, F_2, F_3, F_4$ to the mutivariable case.
- In HyperDIRE project now is implemented only $F_D$ differential reduction for any number of argument:

\[
F_D(a; b_1, b_2, b_3, b_4, b_5; c; z_1, z_2, z_3, z_4, z_5) \text{ is expressed in the terms of the function } \\
F_D(a - 1; b_1 + 1, b_2 - 1, b_3, b_4, b_5; c; z_1, z_2, z_3, z_4, z_5) \text{ and its five derivatives}
\]
thank You for an attention!