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HADRONIC EFFECTS IN LOW–ENERGY QCD: ADLER FUNCTION AND τ DECAY

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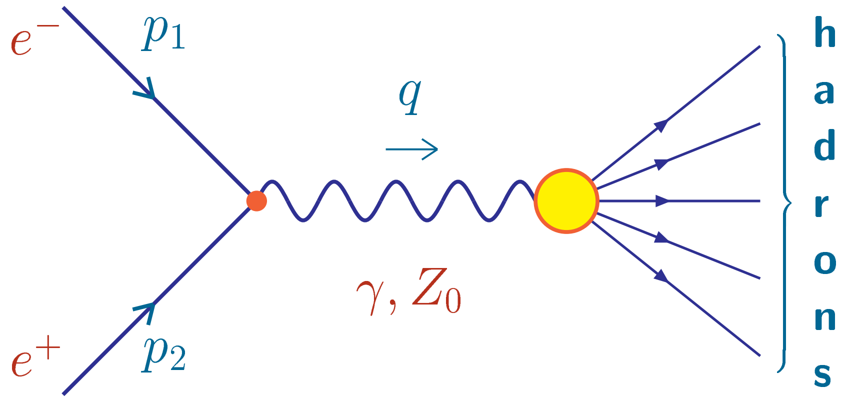
INTRODUCTION

Hadronic vacuum polarization function $\Pi(q^2)$ plays a crucial role in various issues of elementary particle physics. Indeed, the theoretical description of some strong interaction processes and of the hadronic contributions to electroweak observables is inherently based on $\Pi(q^2)$:

- **electron–positron annihilation into hadrons**
- **hadronic τ lepton decay**
- **muon anomalous magnetic moment**
- **running of the electromagnetic coupling**

The cross–section of e^+e^- annihilation into hadrons reads



$$\sigma = 4\pi^2 \frac{2\alpha^2}{s^3} L^{\mu\nu} \Delta_{\mu\nu},$$

where $s = q^2 = (p_1 + p_2)^2 > 0$,

$$L_{\mu\nu} = \frac{1}{2}\Big[q_\mu q_\nu - g_{\mu\nu} q^2 - (p_1 - p_2)_\mu (p_1 - p_2)_\nu\Big],$$

$$\Delta_{\mu\nu} = (2\pi)^4 \sum_\Gamma \delta(p_1 + p_2 - p_\Gamma) \langle 0|J_\mu(-q)|\Gamma\rangle \langle\Gamma|J_\nu(q)|0\rangle,$$

Γ denotes a final hadron state, and $J_\mu = \sum_f Q_f :\bar{q}\,\gamma_\mu\, q:$ stands for the electromagnetic quark current.

It is worth stressing that $\Delta_{\mu\nu}(q^2)$ exists only for $q^2 \geq 4m_\pi^2$, since otherwise no hadron state Γ could be excited:

- *R.P.Feynman (1972); S.L.Adler, PRD10 (1974).*

The hadronic tensor can be represented as $\Delta_{\mu\nu} = 2\,\mathrm{Im}\,\Pi_{\mu\nu}$,

$$\Pi_{\mu\nu}(q^2) = i\int e^{iqx}\langle 0|\,T\{J_\mu(x)\,J_\nu(0)\}\,|0\rangle\,d^4x = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2).$$

The hadronic vacuum polarization function $\Pi(q^2)$ **satisfies the once–subtracted dispersion relation (cut for** $q^2 \geq 4m_\pi^2$**)**

$$\Pi(q^2) = \Pi(q_0^2) - \left(q^2 - q_0^2\right)\int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s-q^2)(s-q_0^2)}\,ds,$$

where $m_\pi \simeq 135\,\mathrm{MeV}$ **is the mass of the** π **meson and** $R(s)$ **denotes the measurable ratio of two cross–sections:**

$$R(s) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0_+}\Big[\Pi(s-i\varepsilon) - \Pi(s+i\varepsilon)\Big] = \frac{\sigma\left(e^+e^- \to \mathrm{hadrons};\, s\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-;\, s\right)}.$$

It is worth noting here that $R(s) \equiv 0$ **for** $s < 4m_\pi^2$ **because of the kinematic restrictions mentioned above:**

- ***R.P.Feynman (1972).***

For practical purposes it proves to be convenient to deal with the so–called Adler function $D(Q^2)$ $(Q^2 = -q^2 \geq 0)$:

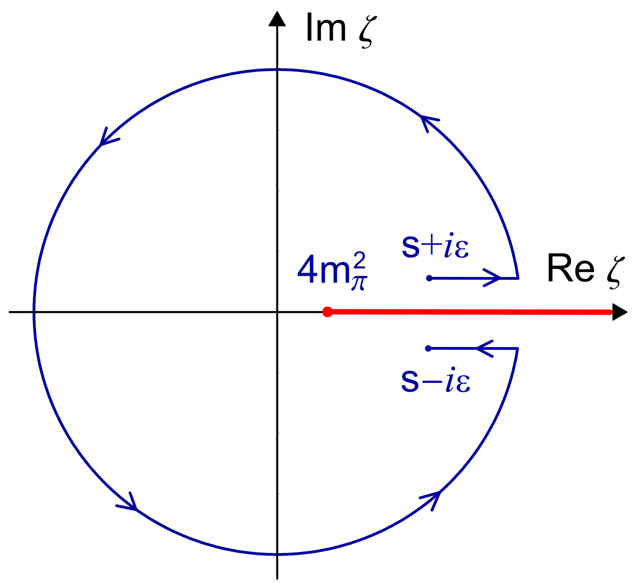
$$D(Q^2) = \frac{d\,\Pi(-Q^2)}{d\ln Q^2}, \qquad D(Q^2) = Q^2 \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\,ds,$$

which plays an indispensable role for the congruous analysis of the timelike and spacelike experimental data:

■ ***S.L.Adler PRD10 (1974); A.Rujula, H.Georgi PRD13 (1976); J.D.Bjorken (1989).***

The inverse relation between $D(Q^2)$ and $R(s)$ reads

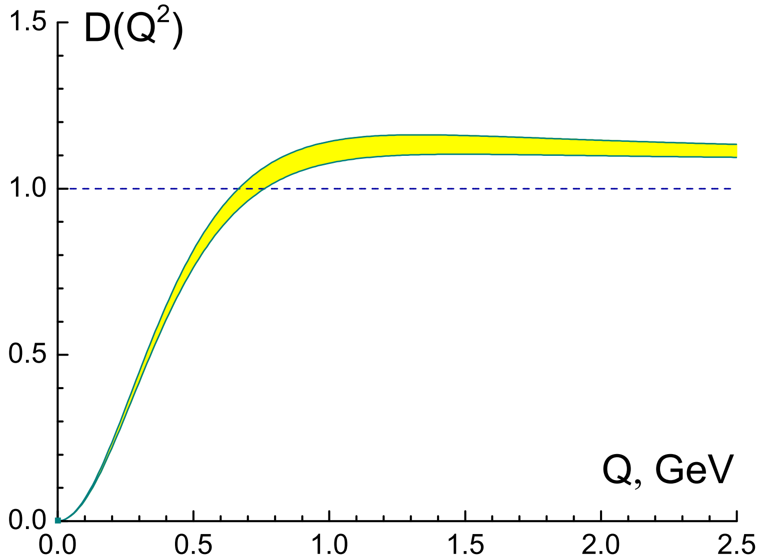
$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\,\frac{d\zeta}{\zeta}$$



■ ***A.V.Radyushkin (1982), hep-ph/9907228; N.V.Krasnikov, A.A.Pivovarov PLB116 (1982).***

Although there are no direct measurements of the Adler function, it can be restored by employing the experimental data on $R(s)$ (overall factor $N_{\mathbf{c}} \sum_f Q_f^2$ is omitted):

$$D_{\mathbf{exp}}(Q^2) = Q^2 \int_{4m_\pi^2}^{s_0} \frac{R_{\mathbf{exp}}(s)}{(s+Q^2)^2}\, ds + Q^2 \int_{s_0}^{\infty} \frac{R_{\mathbf{theor}}(s)}{(s+Q^2)^2}\, ds$$

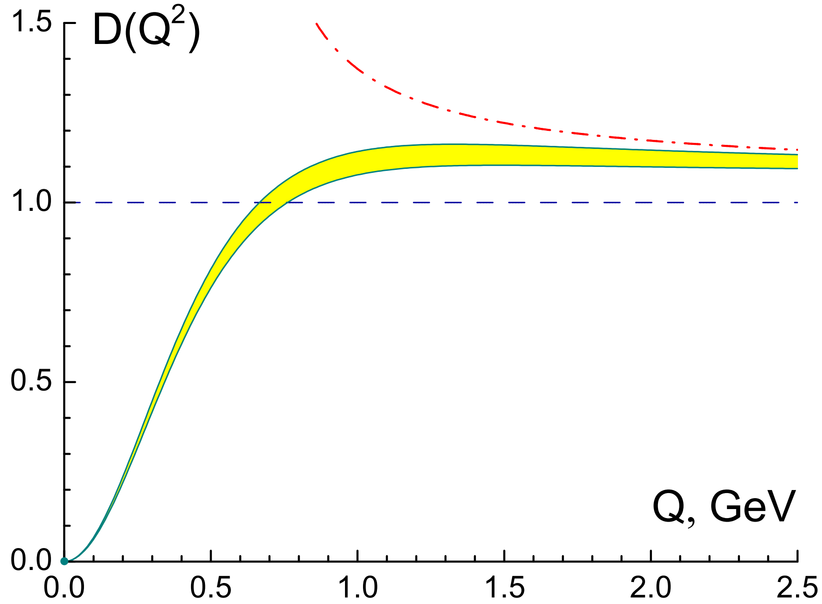


There is also a number of lattice simulations, which generally agree with the shown result:

- ***JLQCD and TWQCD Collabs., PRD79 (2009); QCDSF Collab., NPB688 (2004).***

On the one hand, perturbation theory provides an explicit expression for the Adler function valid at $Q^2 \to \infty$:

$$D^{(\ell)}_{\mathbf{pert}}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j \left[\alpha_{\mathbf{s}}^{(\ell)}(Q^2)\right]^j.$$



■ *S.G.Gorishny, A.L.Kataev, S.A.Larin PLB259 (1991); L.R.Surguladze, M.A.Samuel PRL66 (1991); P.A.Baikov, K.G.Chetyrkin, J.H.Kuhn, PRL101 (2008).*

On the other hand, this perturbative approximation is inconsistent with the dispersion relation for $D(Q^2)$ due to unphysical singularities of the strong running coupling $\alpha_{\mathbf{s}}(Q^2)$:

$$D^{(1)}_{\mathbf{pert}}(Q^2) = 1 + d_1\,\alpha_{\mathbf{s}}^{(1)}(Q^2), \qquad \alpha_{\mathbf{s}}^{(1)}(Q^2) = \frac{4\pi}{\beta_0}\frac{1}{\ln(Q^2/\Lambda^2)},$$

where $d_1 = 1/\pi$ and $\beta_0 = 11 - 2n_{\mathrm{f}}/3$.

Dispersion relation imposes stringent constraints on $D(Q^2)$:

$$D(Q^2) = Q^2 \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds$$

- **Since $R(s)$ assumes finite values and $R(s) \to \text{const}$ when $s \to \infty$, then $D(Q^2) \to 0$ at $Q^2 \to 0$ (holds for $m_\pi \neq 0$ only)**
- **Adler function possesses the only cut $Q^2 \leq -4m_\pi^2$ along the negative semiaxis of real Q^2**

PRIMARY OBJECTIVE: to merge these nonperturbative constraints with perturbative result for the Adler function.

Perturbation theory + Dispersion relations: the basic idea of the "Analytic" (or "Dispersive") approach to QFT.

QCD Analytic Perturbation Theory: [see A.Bakulev's talk]

- ***D.V. Shirkov, I.L. Solovtsov, PRL79 (1997); EPJC22 (2001); TMP150 (2007).***

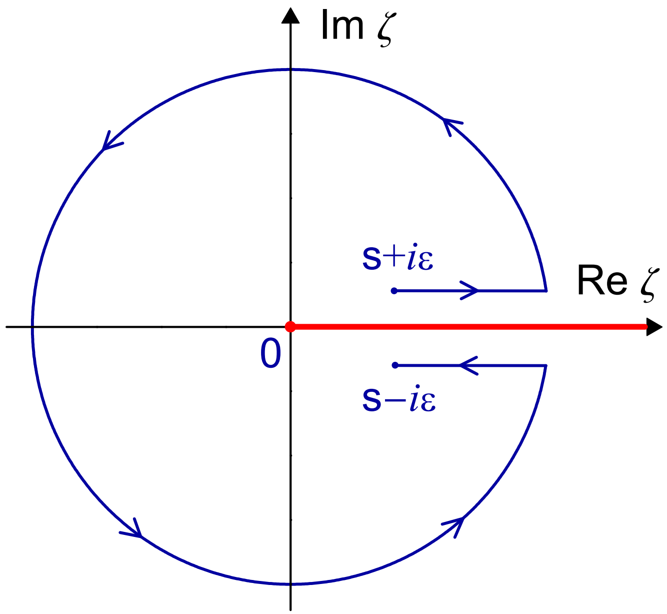
MASSLESS π MESON

This objective can be achieved by deriving the integral representations for the Adler function and $R(s)$–ratio, which involve the common spectral function.

$$D(Q^2) = Q^2 \int_0^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds$$

$$\updownarrow$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\, \frac{d\zeta}{\zeta}$$



Parton model predictions: $R_0(s) = 1 \quad \longleftrightarrow \quad D_0(Q^2) = 1.$

Strong corrections: $R(s) = 1 + r(s), \qquad D(Q^2) = 1 + d(Q^2).$

It is convenient to express $d(Q^2)$ and $r(s)$ in terms of the common spectral function $\rho_{\mathbf{D}}(\sigma)$:

$$D(Q^2) = 1 + d(Q^2)$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta}$$

$$R(s) = 1 + \int_s^{\infty} \rho_{\mathbf{D}}(\sigma) \frac{d\sigma}{\sigma}$$

$$D(Q^2) = Q^2 \int_0^{\infty} \frac{R(s)}{(s + Q^2)^2} \, ds$$

$$D(Q^2) = 1 + \int_0^{\infty} \frac{\rho_{\mathbf{D}}(\sigma)}{\sigma + Q^2} \, d\sigma$$

$$\rho_{\mathbf{D}}(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[D_{\mathbf{theor}}(-\sigma - i\varepsilon) - D_{\mathbf{theor}}(-\sigma + i\varepsilon) \Big] = - \frac{d\, R_{\mathbf{exp}}(\sigma)}{d \ln \sigma}.$$

Thus one arrives at the following integral representations:

$$D(Q^2) = 1 + \int_0^{\infty} \frac{\rho_{\mathbf{D}}(\sigma)}{\sigma + Q^2}\, d\sigma \quad \longleftrightarrow \quad R(s) = 1 + \int_s^{\infty} \rho_{\mathbf{D}}(\sigma) \frac{d\sigma}{\sigma}$$

$$\rho_{\mathbf{D}}(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[D_{\mathbf{theor}}(-\sigma - i\varepsilon) - D_{\mathbf{theor}}(-\sigma + i\varepsilon) \Big] = -\frac{d\, R_{\mathbf{exp}}(\sigma)}{d \ln \sigma}$$

- nonperturbative constraint on $D(Q^2)$ satisfied (cut $Q^2 \le 0$)
- congruent analysis of spacelike and timelike processes

In this study only perturbative contributions to the spectral function are retained; at the ℓ–loop level it reads

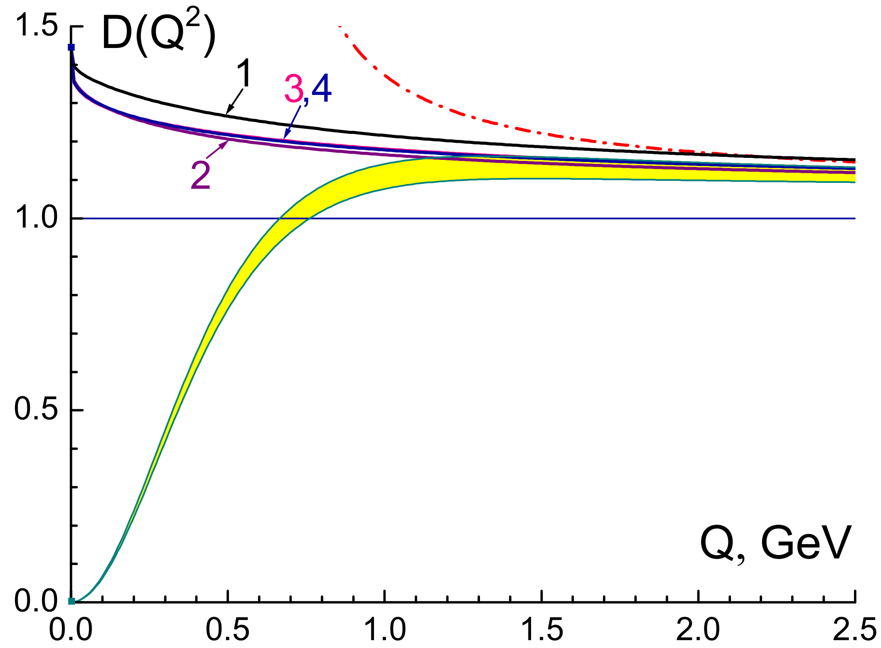
$$\rho^{(\ell)}_{\mathbf{pert}}(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \Big[D^{(\ell)}_{\mathbf{pert}}(-\sigma - i\varepsilon) - D^{(\ell)}_{\mathbf{pert}}(-\sigma + i\varepsilon) \Big].$$

In this case the obtained expressions for $D(Q^2)$ and $R(s)$ become identical to those of Shirkov–Solovtsov's APT:

■ *D.V. Shirkov, I.L. Solovtsov, PRL79 (1997); EPJC22 (2001); TMP150 (2007).*

$$D^{(\ell)}_{\mathbf{APT}}(Q^2) = 1 + \int_0^{\infty} \frac{\rho^{(\ell)}_{\mathbf{pert}}(\sigma)}{\sigma + Q^2}\, d\sigma$$

- no spurious singularities
- no free parameters
- good higher loop stability
- mild scheme dependence

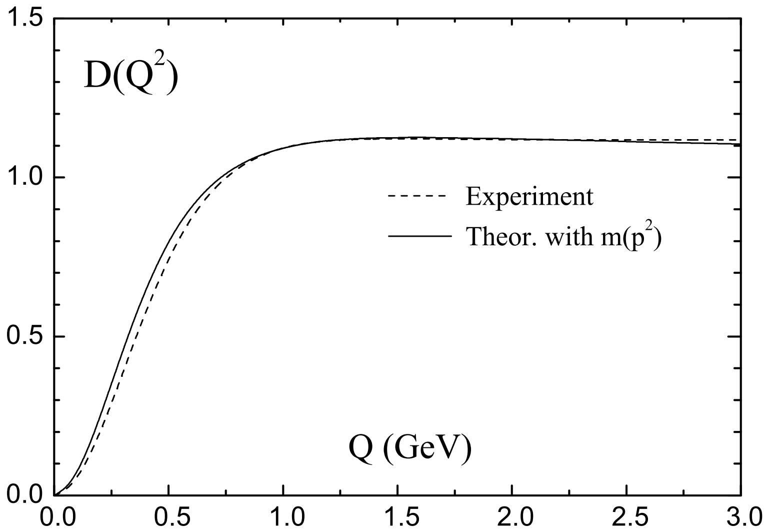


The APT method extends the range of applicability of perturbation theory, but fails to describe $D(Q^2)$ below 1 GeV.

This situation can be rescued at the expense of:

- **relativistic quark mass ($250\,\mathrm{MeV}$) threshold resummation:**
  K.A.Milton, I.L.Solovtsov, O.P.Solovtsova (2001)–(2006)
- **vector meson dominance assumption:**
  G.Cvetic, C.Valenzuela, I.Schmidt (2005)–(2007)
- **keep the pion mass m_π nonvanishing:**
  A.V.Nesterenko, J.Papavassiliou (2006)–(2009)

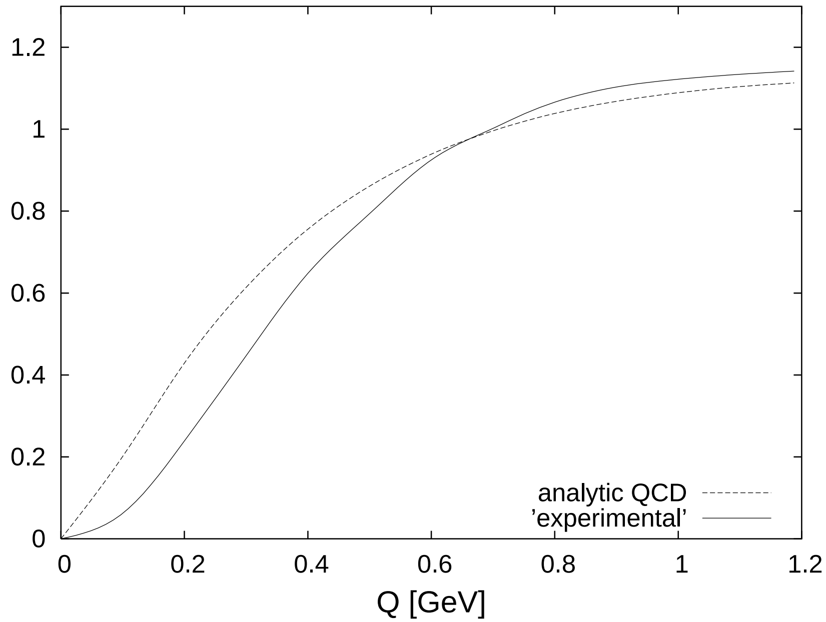
APT + relativistic quark mass threshold resummation:



[taken from MPLA21 (2006)]

K.A. Milton, I.L. Solovtsov, O.P. Solovtsova (2001)–(2006)

APT + vector meson dominance assumption:



[taken from NPBPS164 (2007)]

G. Cvetic, C. Valenzuela, I. Schmidt (2005)–(2007)

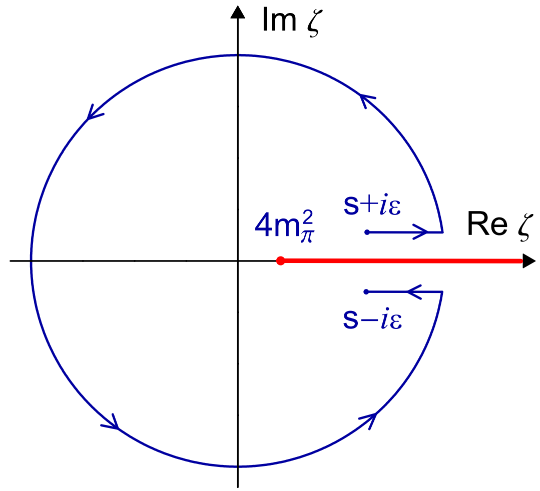
MASSIVE π MESON

The effects due to the masses of the light hadrons can be safely neglected only in the ultraviolet asymptotic. Such effects can be accounted for in a similar way:

$$D(Q^2) = Q^2 \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s+Q^2)^2}\, ds$$

$$\Updownarrow$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta)\, \frac{d\zeta}{\zeta}$$



Parton model prediction + kinematic restriction on $R(s)$:

$$R_0(s) = \theta(s - 4m_\pi^2) \quad \Longleftrightarrow \quad D_0(Q^2) = \frac{Q^2}{Q^2 + 4m_\pi^2}$$

- ***R.P. Feynman (1972).***

$$D(Q^2) = \frac{Q^2}{Q^2 + 4m_\pi^2} + d(Q^2)$$

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \, \frac{d\zeta}{\zeta}$$

$$R(s) = \theta(s - 4m_\pi^2) \left[1 + \int_s^\infty \rho_{\mathrm{D}}(\sigma) \, \frac{d\sigma}{\sigma} \right]$$

$$D(Q^2) = Q^2 \int_{4m_\pi^2}^\infty \frac{R(s)}{(s + Q^2)^2} \, ds$$

$$D(Q^2) = \frac{Q^2}{Q^2 + 4m_\pi^2} \left[1 + \int_{4m_\pi^2}^\infty \rho_{\mathrm{D}}(\sigma) \, \frac{\sigma - 4m_\pi^2}{\sigma + Q^2} \, \frac{d\sigma}{\sigma} \right]$$

$$\rho_{\mathrm{D}}(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \left[D_{\mathbf{theor}}(-\sigma - i\varepsilon) - D_{\mathbf{theor}}(-\sigma + i\varepsilon) \right] = - \frac{d \, R_{\mathbf{exp}}(\sigma)}{d \, \ln \sigma}$$

- ***A.V. Nesterenko, J. Papavassiliou, JPG32 (2006).***

Thus one arrives at the following integral representations:

$$D(Q^2) = \frac{Q^2}{Q^2 + 4m_\pi^2}\left[1 + \int_{4m_\pi^2}^{\infty} \rho_{\mathrm{D}}(\sigma)\, \frac{\sigma - 4m_\pi^2}{\sigma + Q^2}\, \frac{d\sigma}{\sigma}\right]$$

$$R(s) = \theta(s - 4m_\pi^2)\left[1 + \int_{s}^{\infty} \rho_{\mathrm{D}}(\sigma) \frac{d\sigma}{\sigma}\right]$$

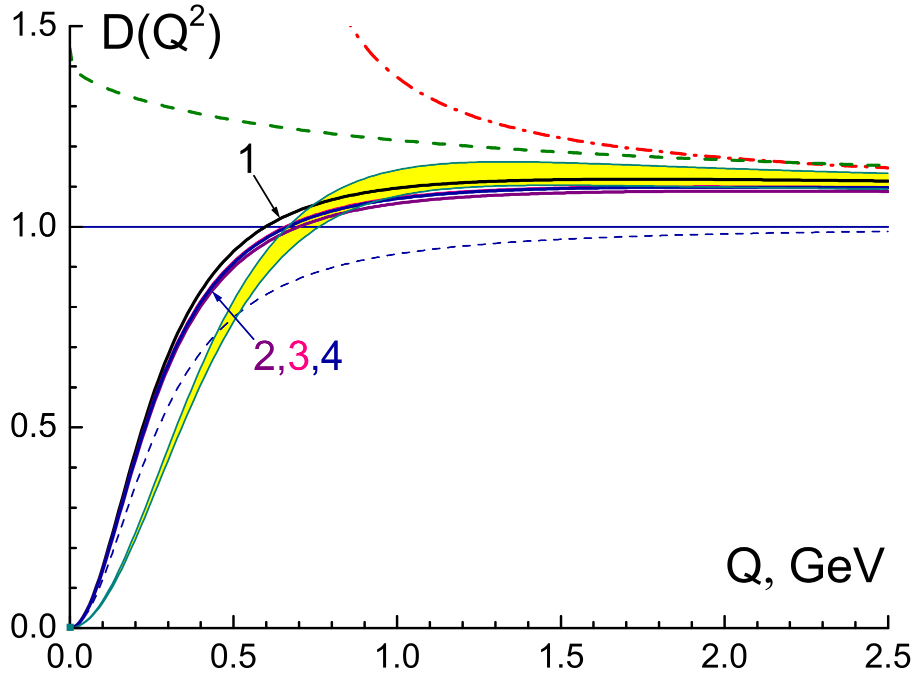
$$\rho_{\mathrm{D}}(\sigma) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0_+} \left[D_{\mathbf{theor}}(-\sigma - i\varepsilon) - D_{\mathbf{theor}}(-\sigma + i\varepsilon)\right] = -\frac{d\, R_{\mathbf{exp}}(\sigma)}{d\, \ln \sigma}$$

■ ***A.V. Nesterenko, J. Papavassiliou, JPG32 (2006).***

- **all nonperturbative constraints on $D(Q^2)$ are satisfied**
- **congruent analysis of spacelike and timelike processes**

In the limit $m_\pi = 0$ the obtained expressions become identical to those of the Analytic perturbation theory.

Only perturbative contributions to the spectral function $\rho_{\mathrm{D}}(\sigma)$ are retained here. The obtained results are in reasonable agreement with the "experimental" Adler function $D(Q^2)$ in the entire energy range $0 \leq Q^2 \leq \infty$.



Besides, this approach has the same advantages as APT:

- no unphysical singularities
- no free parameters
- mild scheme dependence
- enhanced loop stability
- congruent analysis of spacelike and timelike processes
- coincides with perturbation theory at high energies

One–loop Adler function within considered approaches:

- **PT:** $D^{(1)}_{\mathbf{pert}}(Q^2) = 1 + \dfrac{4}{\beta_0}\, a^{(1)}_{\mathbf{s}}(Q^2), \quad a^{(1)}_{\mathbf{s}}(Q^2) = \dfrac{1}{\ln z}, \quad z = \dfrac{Q^2}{\Lambda^2}$

  reliable for $Q \gtrsim 1.5\,\mathrm{GeV}$

- **APT:** $D^{(1)}_{\mathbf{APT}}(Q^2) = 1 + \dfrac{4}{\beta_0}\, a^{(1)}_{\mathbf{an}}(Q^2), \quad a^{(1)}_{\mathbf{an}}(Q^2) = \dfrac{1}{\ln z} + \dfrac{1}{1-z}$

  reliable for $Q \gtrsim 1\,\mathrm{GeV}$

- **MAPT:** $D^{(1)}_{\mathbf{MAPT}}(Q^2) = \dfrac{Q^2}{Q^2 + 4m_\pi^2} + \dfrac{4}{\beta_0}\, a^{(1)}_{\mathbf{eff}}(Q^2, 4m_\pi^2)$

  reliable in the entire energy range

The one–loop massive effective coupling takes the form

$$a^{(1)}_{\mathbf{eff}}(Q^2, m^2) = \frac{1}{\ln z} + \frac{z}{1-z}\,\frac{1+\chi}{z+\chi} - \frac{z}{z+\chi}\int_0^{\chi} \frac{1-\chi/\sigma}{\ln^2\sigma + \pi^2}\,\frac{d\sigma}{\sigma+z},$$

where $\chi = m^2/\Lambda^2$.

INCLUSIVE τ LEPTON DECAY

The inclusive semileptonic branching ratio:

$$R_\tau = \frac{\Gamma(\tau^- \to \mathrm{hadrons}^-\,\nu_\tau)}{\Gamma(\tau^- \to e^-\,\bar{\nu}_e\,\nu_\tau)} = R_{\tau,\mathrm{V}} + R_{\tau,\mathrm{A}} + R_{\tau,\mathrm{S}}.$$

Its nonstrange part associated with vector quark currents:

$$R_{\tau,\mathrm{V}} = \frac{N_{\mathbf{c}}}{2}\,|V_{\mathbf{ud}}|^2 S_{\mathrm{EW}}\left(\Delta_{\mathrm{QCD}} + \delta'_{\mathrm{EW}}\right) = 1.764 \pm 0.016$$

- *OPAL Collab., EPJC7 (1999); ALEPH Collab., EPJC4 (1998), RMP78 (2006).*

In this equation $N_{\mathbf{c}} = 3$**,** $|V_{\mathbf{ud}}| = 0.9738 \pm 0.0005$**,** $\delta'_{\mathrm{EW}} = 0.0010$**,** $S_{\mathrm{EW}} = 1.0194 \pm 0.0050$**,** $M_\tau = 1.777\,\mathrm{GeV}$**, and**

$$\Delta_{\mathrm{QCD}} = 2\int_0^{M_\tau^2}\left(1 - \frac{s}{M_\tau^2}\right)^2\left(1 + 2\,\frac{s}{M_\tau^2}\right) R(s)\,\frac{ds}{M_\tau^2}\,.$$

Perturbative approach :

$$\Delta_{\mathbf{QCD}} = 1 + d_1\, \alpha_{\mathbf{s}}^{(1)}(M_\tau^2) \quad \longrightarrow \quad \Lambda = (678 \pm 55)\,\mathrm{MeV}, \qquad n_{\mathrm{f}} = 2$$

■ *E. Braaten, S. Narison, A. Pich, NPB373 (1992).*

Current analysis :

$$\Delta_{\mathbf{QCD}} = 1 + d_1 \alpha_{\mathrm{TL}}^{(1)}(M_\tau^2) - \delta_\Gamma + \frac{4}{\beta_0}\int_\chi^1 f(\xi)\rho^{(1)}\big(\xi M_\tau^2\big)\,d\xi - d_1 \delta_\Gamma \alpha_{\mathrm{TL}}^{(1)}(m_\Gamma^2),$$

$$f(\xi) = \xi^3 - 2\xi^2 + 2, \qquad \chi = \frac{m_\Gamma^2}{M_\tau^2}, \qquad \delta_\Gamma = \chi\, f(\chi) \simeq 0.048, \qquad d_1 = \frac{1}{\pi},$$

$$\alpha_{\mathrm{TL}}^{(1)}(s) = \frac{4\pi}{\beta_0}\theta(s - m_\Gamma^2)\int_s^\infty \rho^{(1)}(\sigma)\,\frac{d\sigma}{\sigma}, \qquad m_\Gamma = m_{\pi^0} + m_{\pi^-}$$

- **massive case:** $\Lambda = (941 \pm 86)\,\mathrm{MeV}$
- **massless limit:** $\Lambda = (493 \pm 56)\,\mathrm{MeV}$

■ *A.V. Nesterenko, NPBPS186 (2009).*

SUMMARY

- **New integral representations for the Adler function and $R(s)$–ratio are derived**

- **These representations possess appealing features:**
  - **unphysical perturbative singularities are eliminated**
  - **additional parameters are not introduced**
  - **the π^2–terms are automatically taken into account**
  - **reasonable description of $D(Q^2)$ in entire energy range**

- **The effects due to the pion mass play a substantial role in processing the data on the inclusive τ lepton decay**