Group structure of the integration-by-part identities

Towards the general algorithm of IBP reduction

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Outline

Introduction

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3. Evaluate the master integrals with the help of methods available. Among them
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    2. Differential equations with respect to some external parameter
    3. Recurrence relation with respect to the power of some denominator
    4. Recurrence relation with respect to space-time dimension

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IBP equations are widely used in the above procedures.

Approaches to reduction

No universal approach to IBP reduction is available.

1. Laporta's method: Gauss elimination, starting from the simplest identities.(Laporta 2000)

Simple to implement and use, always works.

Accumulating database is time consuming, keeping it — memory consuming. The database can be insufficient.

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Except for the main topology, requires manual heuristic work.

3. Smirnovs' method: construct Gröbner-like bases in sectors.(Smirnov and Smirnov 2006)

No need to keep database, efficient, can be automatized.

Requires some manual heuristic work in choosing ordering. In some cases, it is inefficient.

Loop Integral

L-loop diagram with E external momenta $p_1, \ldots p_E$:

Loop integral

$$J(\mathbf{n}) = J(n_1, \ldots, n_N) = \int d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L j(\mathbf{n}) = \int \frac{d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L}{D_1^{n_1} \ldots D_N^{n_N}}$$

where $D_1, \ldots, D_M$ are denominators of the diagram, and $D_{M+1}, \ldots, D_N$ are some additionally chosen numerators.

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Prerequisites

All denominators and numerators linearly depend on $l_i \cdot q_j$. Any product $l_i \cdot q_j$ can be expressed via D_k.

Notation

$$q_i = \begin{cases} l_i, & i \leqslant L \\ p_{i-L}, & i > L \end{cases}$$

The total number of denominators and numerators

$$N = L(L+1)/2 + LE, \quad N \geqslant M$$

Integration-by-part identities

The integration-by-part identities arise due to the fact, that, in dimensional regularization the integral of the total derivative is zero (Tkachov 1981, Chetyrkin and Tkachov 1981)

IBP identities

$$\int d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L \, O_{ij} j(\mathbf{n}) = 0$$

IBP operators

$$O_{ij} = \frac{\partial}{\partial l_i} \cdot q_j$$

Explicitely differentiating, we obtain the relation between integrals with shifted indices.

Lorentz-invariance identities

The Lorentz-invariance identities (Gehrmann and Remiddi 2000) follow from the fact that the integral is a scalar function of p_i:

LI identities

$$\overbrace{\left(\sum_k p_{k[\nu} \frac{\partial}{\partial p_k^{\mu]}}\right)}^{\text{Lorentz generator}} J(n_1, n_2, \ldots, n_N) = 0$$

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$$p_i^{\mu} p_j^{\nu} \left(\sum_k p_{k[\nu} \frac{\partial}{\partial p_k^{\mu]}} \right) J(n_1, n_2, \ldots, n_N) = 0$$

Explicitly differentiating, we obtain LI identity. Different choice of $p_i^{\mu} p_j^{\nu}$ gives $E\left(E-1\right)/2$ identities.

Ordering of integrals

The goal of the reduction procedure

Any reduction procedure must have a goal, i.e., we have to know, what is simpler. Ordering of the integrals is required.

Common sense

Integrals with fewer denominators are simpler.

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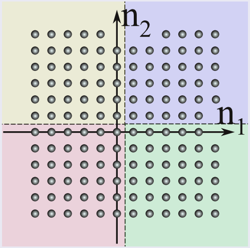
Sectors & Ordering

Integrals with the same set of denominators form a sector in $\mathbb{Z}^N$.



Example

$$J\left(n_1,n_2\right)=\int\frac{d^{\mathscr{D}}l}{\left[l^2-m^2\right]^{n_1}\left[(l-p)^2-m^2\right]^{n_2}}$$

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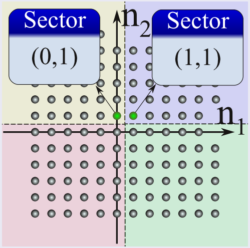

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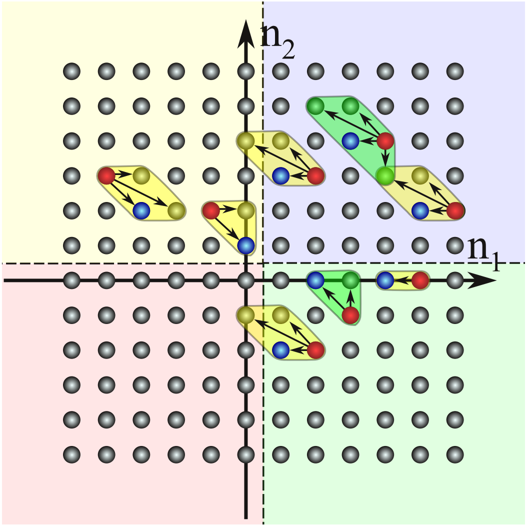
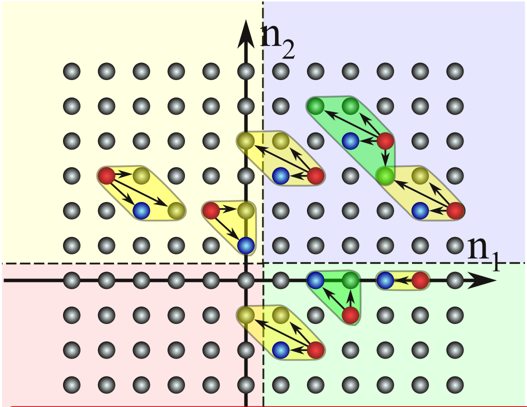
1. The number of denominators.
2. Total power of denominators and numerators .
3. Number of numerators .
4. $n_1,\ n_2,\ \ldots$

Reduction to simpler integrals



- Blue dot — the integrand differentiated to obtain the identity.
- Hightlighted region — the result of the differentiation. Different colors denote different differential operators.
- Red dot — the most complex integral of the identity.

Reduction to simpler integrals



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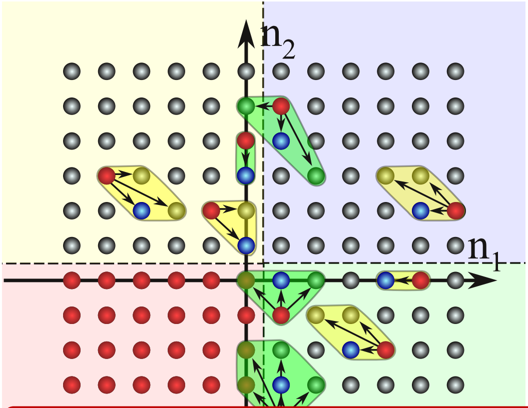
Huge redundancy

In each point we have $\overbrace{L(L+E)}^{\text{IBP}} + \overbrace{E(E-1)/2}^{\text{LI}}$ identities.

Reduction to simpler integrals



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Huge redundancy

In fact, we need only one identity per integral. Others can be reduced to $0 = 0$, which takes a lot of time. Which identities can be discarded?

Elimination of LI identities

LI identities can be discarded

All LI identities can be represented as linear combination of IBP identities.

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Observation

$$J(\mathbf{n}) = \int d^{\mathscr{D}} l_1 \dots d^{\mathscr{D}} l_L j(\mathbf{n}) = \int \frac{d^{\mathscr{D}} l_1 \dots d^{\mathscr{D}} l_L}{D_1^{n_1} \dots D_N^{n_N}}$$

The integral $J(\mathbf{n})$ is a scalar function of external momenta p_i, while the integrand $j(\mathbf{n})$ is a scalar function of all momenta q_i.Thus the operator

$$\sum_{k=1}^{L+E} q_{k[\nu} \frac{\partial}{\partial q_k^{\mu]}} \equiv \sum_{k=1}^{L} l_{k[\nu} \frac{\partial}{\partial l_k^{\mu]}} + \sum_{k=1}^{E} p_{k[\nu} \frac{\partial}{\partial p_k^{\mu]}},$$

being the generator of the Lorentz transformations on the scalar functions of q_i annihilates the integrand $j(\mathbf{n})$ identically.

Elimination of LI identities

Proof

$$p_i^{\mu} p_j^{\nu} \sum_{k=1}^{E} p_{k[\mu} \frac{\partial}{\partial p_k^{\nu]}} j(\mathbf{n}) = -p_i^{\mu} p_j^{\nu} \sum_{k=1}^{L} l_{k[\mu} \frac{\partial}{\partial l_k^{\nu]}} j(\mathbf{n}) + p_i^{\mu} p_j^{\nu} \sum_{k=1}^{L+E} q_{k[\mu} \frac{\partial}{\partial q_k^{\nu]}} j(\mathbf{n})$$

Elimination of LI identities

Proof

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$$= \sum_{k=1}^{L} \left[(p_j \cdot l_k) p_i \cdot \frac{\partial}{\partial l_k} - (p_i \cdot l_k) p_j \cdot \frac{\partial}{\partial l_k} \right] j(\mathbf{n})$$

Elimination of LI identities

Proof

$$
\begin{aligned}
p_i^{\mu} p_j^{\nu} \sum_{k=1}^{E} p_{k[\mu} \frac{\partial}{\partial p_k^{\nu]}} j(\mathbf{n}) &= -p_i^{\mu} p_j^{\nu} \sum_{k=1}^{L} l_{k[\mu} \frac{\partial}{\partial l_k^{\nu]}} j(\mathbf{n}) \\
&= \sum_{k=1}^{L} \left[(p_j \cdot l_k) p_i \cdot \frac{\partial}{\partial l_k} - (p_i \cdot l_k) p_j \cdot \frac{\partial}{\partial l_k} \right] j(\mathbf{n}) \\
&= \sum_{k=1}^{L} \left[\frac{\partial}{\partial l_k} \cdot p_i (p_j \cdot l_k) - \frac{\partial}{\partial l_k} \cdot p_j (p_i \cdot l_k) \right] j(\mathbf{n})
\end{aligned}
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&= \sum_{k=1}^{L} \left[\frac{\partial}{\partial l_k} \cdot p_i \, (p_j \cdot l_k) - \frac{\partial}{\partial l_k} \cdot p_j \, (p_i \cdot l_k) \right] j(\mathbf{n})
\end{aligned}
$$

Since the highlighted scalar products can be expressed via D_i, the last line is some linear combination of the IBP identities

Operator representation

Introduce the operators, acting on the functions on $\mathbb{Z}^N$ (similar to A, Y, Y^{-1} of(Smirnov and Smirnov 2006)):

Operators $A_1, \ldots, A_N, B_1, \ldots, B_N$

$$(A_\alpha f)(n_1, \ldots, n_N) = n_\alpha f(n_1, \ldots, n_\alpha + 1, \ldots, n_N),$$
$$(B_\alpha f)(n_1, \ldots, n_N) = f(n_1, \ldots, n_\alpha - 1, \ldots, n_N).$$

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Commutator

$$[A_\alpha, B_\beta] = \delta_{\alpha\beta}$$

A and B well suited to sectors

For any polynomial $P(A, B)$ the result of action

$$P(A, B)J(\mathbf{n}) = \sum C_i J(\mathbf{n}_i)$$

contains only integrals of the same and lower sectors as $J(\mathbf{n})$.

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Operator representation

IBP identity

$$-\int d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_N O_{ij} j(\mathbf{n}) = -\int d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_N \frac{\partial}{\partial l_i} \cdot q_j j(\mathbf{n}) = 0$$

can be rewritten in terms of operators $A_1, \ldots, A_N, B_1, \ldots, B_N$:

$$(\tilde{O}_{ij}(A, B) J)(\mathbf{n}) = 0$$

Operator representation

Example

When acting on the integrand in

$$J\left(n_1,n_2\right)=\int\frac{d^{\mathscr{D}}l}{\left[l^2-1\right]^{n_1}\left[\left(l-p\right)^2-1\right]^{n_2}}$$

by the operator $\frac{\partial}{\partial l}\cdot l$, we obtain the following identity

IBP identity

$$\left(\mathscr{D}-2n_1-n_2\right)J\left(n_1,n_2\right)-n_2J\left(n_1-1,n_2+1\right)$$
$$+n_2\left(p^2-2\right)J\left(n_1,n_2+1\right)-2n_1J\left(n_1+1,n_2\right)=0$$

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IBP identity

In terms of index shifting "operators" $\mathbf{I}^{\pm}$:

$$\left[\mathscr{D}-2n_1-n_2-n_2\mathbf{1}^{-}\mathbf{2}^{+}+n_2\left(p^2-2\right)\mathbf{2}^{+}-2n_1\mathbf{1}^{+}\right]J\left(n_1,n_2\right)=0$$

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In terms of the operators $A_1,\ldots,A_N,B_1,\ldots,B_N$:

$$\left(\mathscr{D}-2A_1B_1-A_2B_2-A_2B_1+\left(p^2-2\right)A_2-2A_1\right)J=0$$

Very similar notation, but not the same.

Operator representation

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IBP identity

Acting from the left is not allowed:

$$\mathbf{1}^+\left[\mathscr{D}-2n_1-n_2-n_2\mathbf{1}^-\mathbf{2}^+ + n_2\left(p^2-2\right)\mathbf{2}^+ - 2n_1\mathbf{1}^+\right] J(n_1,n_2) \neq 0$$

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Acting from the left is allowed:

$$A_1(\mathscr{D}-2A_1B_1-A_2B_2-A_2B_1+\left(p^2-2\right)A_2-2A_1)J = 0$$

Very similar notation, but not the same.

Operator representation

General IBP constraint in operator form

Shifting the indices

$$F(\mathbf{n}) = M(A,B)F(1,\ldots,1),$$

where $M(A,B)$ is some monomial.

Operator representation

General IBP constraint in operator form

Shifting the indices

In particular,

$$\tilde{O}_{ij} J\left(4, -1, 0, \ldots\right) = 0$$

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$$\frac{A_1^3}{3!}\tilde{O}_{ij}J\,(1,-1,0,\dots)=0$$

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$$B_2^2 \frac{A_1^3}{3!} \tilde{O}_{ij} J\left(1, \quad 1, 0, \ldots\right) = 0$$

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General IBP constraint

$$\left[\left(\sum_{i=1}^{L} \sum_{j=1}^{L+E} C_{ij}\left(A,B\right) \tilde{O}_{ij}\left(A,B\right)\right) J\right]\left(1,\ldots,1\right) = 0,$$

$C_{i,j}\left(A,B\right)$— some polynomials.

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$C_{i,j}\left(A,B\right)$— some polynomials.
Another way of saying the same:

$$LJ\left(1,\ldots,1\right) = 0,$$

where $L \in \mathscr{L}$ and $\mathscr{L}$ is the left ideal generated by $\tilde{O}_{ij}\left(A,B\right)$.

Operator representation

Basic idea of the reduction(Smirnov and Smirnov 2006)

Suppose we know how to reduce any monomial M modulo $\mathscr{L}$.

Division with the remainder by $\mathscr{L}$

$$M = L + r, \quad L \in \mathscr{L},$$

r is the remainder (simplest possible)

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$$M = L + r, \quad L \in \mathscr{L},$$

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Then we can reduce $J(n_1, \ldots, n_N)$:

Reduction of $J(\mathbf{n})$:

$$\begin{aligned} J(\mathbf{n}) &= MJ(1,\ldots,1) \\ &= LJ(1,\ldots,1) + rJ(1,\ldots,1) = rJ(1,\ldots,1) \end{aligned}$$

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The algorithm is known

Buchberger algorithm allows one to construct a Groebner basis, suitable for "division with the remainder".

Operator representation

Disappointment

Does not work appropriately (the reduction is not satisfactory).

The spoiler

For any function holds $B_i A_i f(1,\ldots,1) = 0$. In general, $Rf(1,\ldots,1) = 0$, where R belongs to the right ideal $\mathscr{R}$, generated by $B_1A_1,\ldots,B_NA_N$.

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Does not work appropriately (the reduction is not satisfactory).

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Need: Division with the remainder by $\mathscr{L} \oplus \mathscr{R}$

$$M = L + R + r, \quad L \in \mathscr{L}, R \in \mathscr{R},$$

The algorithm is not known

In practice, the algorithm useful for many cases suggested in (Smirnov and Smirnov 2006). In some cases, it is inefficient (Smirnov and Smirnov 2007).

Commutation relations

The differential operators $O_{ij} = \frac{\partial}{\partial l_i} \cdot q_j$ (and the corresponding operators $\tilde{O}_{ij}(A,B)$) form a closed Lie-algebra

Commutation relations

$$[O_{ij}, O_{kl}] = \delta_{il} O_{kj} - \delta_{kj} O_{il}$$

These properties of the IBP operators were not used so far

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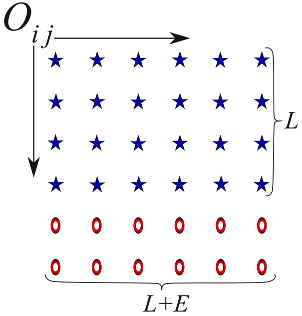
Commutation relations

$$[O_{ij}, O_{kl}] = \delta_{il} O_{kj} - \delta_{kj} O_{il}$$

These properties of the IBP operators were not used so far

The algebra is the same as if O_{ij} denotes a matrix with 1 in i-th row, j-th column, and zero everywhere else.



Commutation relations

The differential operators $O_{ij} = \frac{\partial}{\partial l_i} \cdot q_j$ (and the corresponding operators $\tilde{O}_{ij}(A,B)$) form a closed Lie-algebra

Commutation relations

$$[O_{ij}, O_{kl}] = \delta_{il} O_{kj} - \delta_{kj} O_{il}$$

These properties of the IBP operators were not used so far

Linear changes of variables (LCV)

$$l_i \to l_i' = M_{ij} q_j, \quad \text{where } M \text{ is } L \times (L+E) \text{ matrix.}$$

Representation

O_{ij} corresponds to the infinitesimal transformation $l_i \to l_i' = l_i + \varepsilon q_j$

$$f(l' \cdot q') d^{\mathscr{D}} l_1' \ldots d^{\mathscr{D}} l_L' = \left\{ f(l \cdot q) + \varepsilon \left[O_{ij} f(l \cdot q) \right] \right\} d^{\mathscr{D}} l_1 \ldots d^{\mathscr{D}} l_L .$$

Criterion of zero sectors

Definition

Scaleless integral is the integral, which gains non-unity factor under some LCV transformation(s). In dimensional regularization it is zero.

Criterion of zero sectors

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Example

The integral

$$J = \int \frac{d^{\mathscr{D}} l_1 \, d^{\mathscr{D}} l_2}{l_1^2 l_2^2 \left(l_1 - l_2\right)^2}$$

is scaleless since it transforms as

$$J \to \alpha^{2\mathscr{D}-6} J$$

under the transformation

$$l_{1,2} \to \alpha l_{1,2}$$

Criterion of zero sectors

Definition

Scaleless integral is the integral, which gains non-unity factor under some LCV transformation(s). In dimensional regularization it is zero.

Zero sector criterion

Solve IBP identities in the corner point $(\theta_1, \ldots, \theta_N)$ of the sector. Iff the identity

$$J(\theta) = J(\theta_1, \ldots, \theta_N) = 0$$

comes out, the sector is zero.

Proof.

By the condition, $J(\theta)$ can be represented as the action of some linear combination of O_{ik} on $J(\theta)$. These operators are generators of the LCV transformation $\Rightarrow J(\theta)$ is scaleless $\Rightarrow$ whole sector is zero. $\square$

Using the commutation relations

Demonstration

Commutator:

$$\mathscr{P}_1 = [\mathscr{P}_2, \mathscr{P}_3]$$

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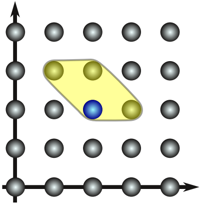

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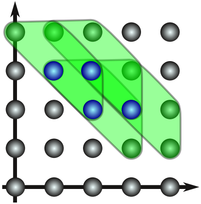

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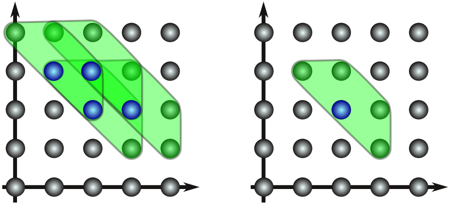
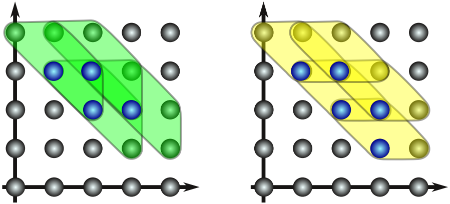


Using the commutation relations

Demonstration

Commutator:

$$\color{red}{\mathscr{P}_1} = [\color{green}{\mathscr{P}_2}, \color{yellow}{\mathscr{P}_3}]$$



Using the commutation relations

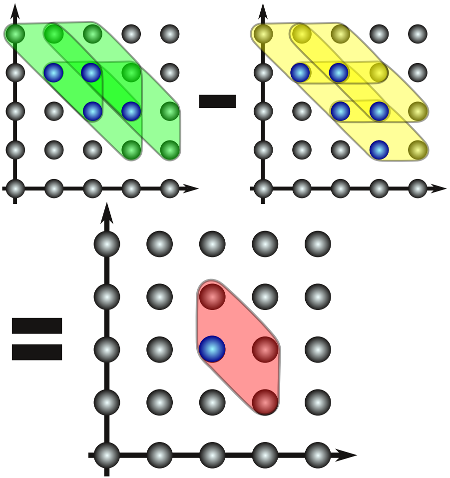
Demonstration

Commutator:

$$\mathscr{P}_1 = [\mathscr{P}_2, \mathscr{P}_3]$$

Therefore

Any identity, generated by $\mathscr{P}_1$ is a linear combination of the i identities, generated by $\mathscr{P}_2$ and $\mathscr{P}_3$.



Reduced set of IBPs

Simple application

One can use the smaller set of the IBP identities, generated by

Reduced set of IBP operators

$$
\begin{array}{c} L+E+1 \\ \text{identities} \end{array}
\left\{
\begin{array}{l}
\dfrac{\partial}{\partial l_i}\cdot l_{i+1}, \quad i=1,\ldots,L, \quad l_{L+1}\equiv l_1 \\
\displaystyle\sum_{i=1}^{L}\frac{\partial}{\partial l_i}\cdot l_i, \quad \frac{\partial}{\partial l_1}\cdot p_j, \quad j=1,\ldots,E
\end{array}
\right.
$$

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$$L+E+1 \text{ identities} \begin{cases} \dfrac{\partial}{\partial l_i} \cdot l_{i+1}, \quad i=1,\ldots,L, \quad l_{L+1} \equiv l_1 \\ \displaystyle\sum_{i=1}^{L} \frac{\partial}{\partial l_i} \cdot l_i, \quad \frac{\partial}{\partial l_1} \cdot p_j, \quad j=1,\ldots,E \end{cases}$$

Other identities are linear combinations of these. Reason: other IBP operators are commutators of the chosen ones.

Example

$$\frac{\partial}{\partial l_1} \cdot l_3 = \left[\frac{\partial}{\partial l_2} \cdot l_3, \frac{\partial}{\partial l_1} \cdot l_2 \right]$$

Criterion of redundancy I

Preconditions

- Identities, generated by the operators $\{\mathscr{P}_1, \ldots \mathscr{P}_k\}$, have been solved.
- The operator $\mathscr{P}$ "almost commutes" with $\{\mathscr{P}_1, \ldots \mathscr{P}_k\}$, i.e. its commutator with any $\mathscr{P}_i$ is a linear combination of $\mathscr{P}_i$:

$$[\mathscr{P}, \mathscr{P}_i] = \sum_{j=1}^{k} C_i^j \mathscr{P}_j$$

Criterion

If for some point $\mathbf{n} \in \mathbb{Z}^N$ the integral $J(\mathbf{n})$ can be reduced by $\{\mathscr{P}_1, \ldots \mathscr{P}_k\}$ to simpler integrals, then the identity $\mathscr{P} J(\mathbf{n})$ can also be reduced to simpler identities and thus can be discarded.

Criterion of redundancy I

Proof

Proof.

What does reducibility of $J\left(\mathbf{n}\right)$ mean? The following

$$J\left(\mathbf{n}\right)=o\left(\mathbf{n}\right)+Q_i\mathscr{P}_iJ\left(\mathbf{1}\right),$$

where $o\left(\mathbf{n}\right)$ contains integrals simpler than $J\left(\mathbf{n}\right)$, and Q are some polynomials of A,B.
Substituting $J\to\mathscr{P}J$ and using $[\mathscr{P},\mathscr{P}_i]=C_i^j\mathscr{P}_j$, we obtain

$$\mathscr{P}J\left(\mathbf{n}\right)=\mathscr{P}o\left(\mathbf{n}\right)+Q_i\mathscr{P}_i\mathscr{P}J\left(\mathbf{1}\right)=\mathscr{P}o\left(\mathbf{n}\right)+\left(Q_jC_j^i-Q_i\mathscr{P}\right)\mathscr{P}_iJ\left(\mathbf{1}\right).$$

$\square$

Criterion of redundancy I

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The first term in r.-h.s. contains the indentities (generated by $\mathscr{P}$) in simpler points $\square$

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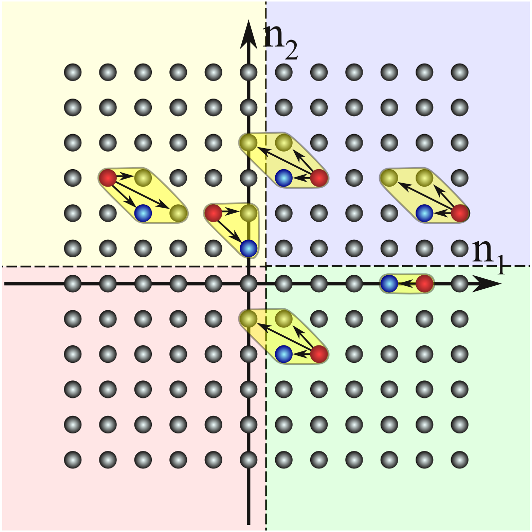
The first term in r.-h.s. contains the indentities (generated by $\mathscr{P}$) in simpler points, the second contains the identities already solved. $\square$

Criterion of redundancy I

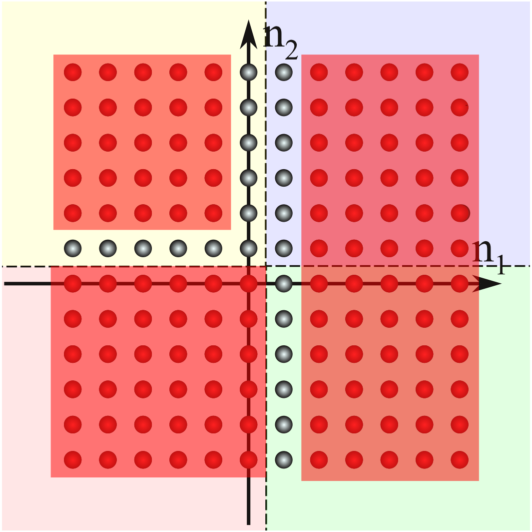
Illustration



1. Solve first identity.

Criterion of redundancy I

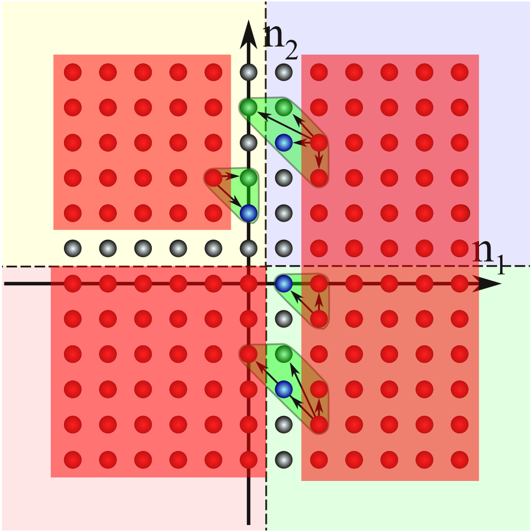
Illustration



1. Solve first identity.
2. The integrals on some hyperplanes are left unexpressed.

Criterion of redundancy I

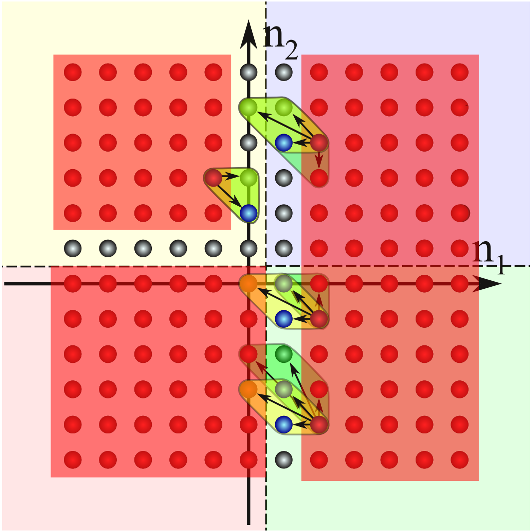
Illustration



1. Solve first identity.
2. The integrals on some hyperplanes are left unexpressed.
3. Consider next identity (if there is one satisfying preconditions) only on these hyperplanes.

Criterion of redundancy I

Illustration



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Criterion of redundancy I

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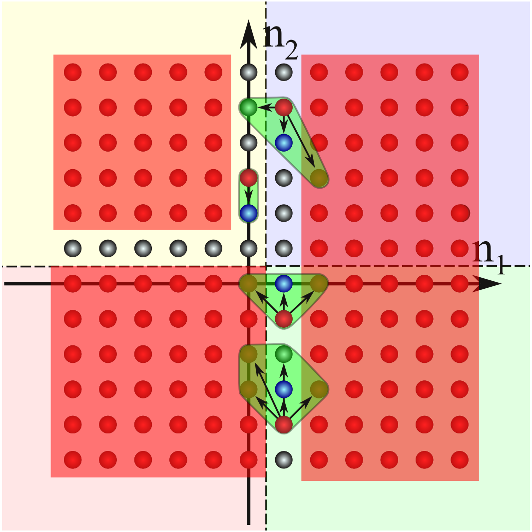

1. Solve first identity.
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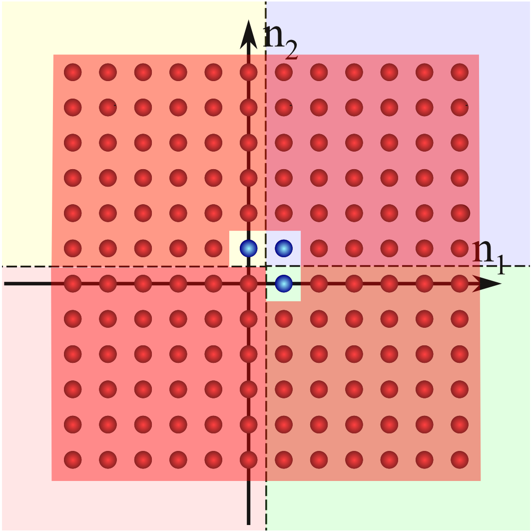
3. Consider next identity (if there is one satisfying preconditions) only on these hyperplanes.
4. Use already obtained rules.
5. Solve obtained IBP.

Criterion of redundancy I

Illustration



1. Solve first identity.
2. The integrals on some hyperplanes are left unexpressed.
3. Consider next identity (if there is one satisfying preconditions) only on these hyperplanes.
4. Use already obtained rules.
5. Solve obtained IBP.
6. Loop from 2.

Criterion of redundancy I

Idea of application

On each step of the above procedure the dimension of the hyperplanes decreases by one. Will we be able to continue the process to finish with the finite number of the master integrals (zero “dimension” hyperplane)?

Is it possible to find a suitable sequence of length $N = L(L+1)/2 + LE$?

The second operator should *almost commute* with the first one. The third operator should *almost commute* with the first two, etc.

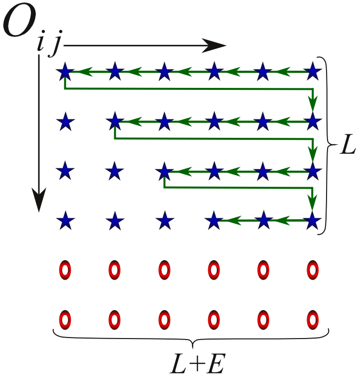
Criterion of redundancy I

Idea of application

Suitable sequence

$$\{\mathscr{P}_1,\dots \mathscr{P}_N\} = \{\tilde{O}_{1,L+E},\dots,\tilde{O}_{1,1},\tilde{O}_{2,L+E},\dots,\tilde{O}_{2,2},\dots,\tilde{O}_{L,L+E},\dots,\tilde{O}_{L,L}\}$$

For any k the operator $\mathscr{P}_{k+1}$ *almost commutes* with the set $\{\mathscr{P}_1,\dots \mathscr{P}_k\}$.



Sequence length= # of denominators

Total number of operators in the sequence is

$$\begin{aligned} N &= E+L\ + E+L-1\ + \ldots + E+1 \\ &= L(L+1)/2 + LE, \end{aligned}$$

i.e., equal to the number of denominators.

Criterion of redundancy II

Preconditions

Let for some IBP operator $\mathscr{P}_0$ and for all $\mathbf{n}$ in some sector holds

$$\mathscr{P}_0 J(\mathbf{n}) = J(\tilde{\mathbf{n}}) + o(\tilde{\mathbf{n}}), \quad \tilde{\mathbf{n}} \succ \mathbf{n}$$

Other IBP identities can be discarded in all points $\tilde{\mathbf{n}}$

Identity $\mathscr{P} J(\tilde{\mathbf{n}}) = 0$ is linear combination of some identities in simpler points and identities generated by $\mathscr{P}_0$.

Criterion of redundancy II

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Similar to the proof of Criterion I

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$\square$

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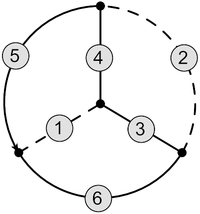
Criterion of redundancy II

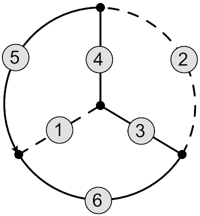
Application



(Broadhurst 1992)

Criterion of redundancy II

Application



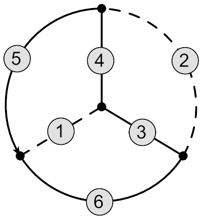
(Broadhurst 1992)

IBP operators

$$
\begin{aligned}
P_{11} &= -\mathscr{D} + 2A_3 + 2A_6 + A_1B_1 - A_6B_2 + A_1B_3 + 2A_3B_3 + A_6B_3 - A_1B_4 + A_6B_6\,,\\
P_{12} &= -A_3B_1 + A_3B_3 + A_3B_4 + 2A_3 + 2A_6 - A_1B_1 - A_6B_1 - A_6B_2 + A_1B_3 + A_6B_3 - A_1B_4 + A_6B_5\,,\\
P_{13} &= -A_3B_1 - A_3B_2 + A_3B_4 + A_3B_6 + 2A_3 + 2A_6 - A_1B_1 - A_6B_1 - A_1B_5 + A_6B_5 + A_1B_6 + A_6B_6\,,\\
P_{21} &= -A_4B_1 + A_4B_3 + A_4B_4 + 2A_4 - 2A_6 - A_1B_1 + A_2B_2 + A_6B_2 - A_1B_3 + A_2B_3 - A_6B_3 + A_1B_4 - A_2B_6 - A_6B_6\,,\\
P_{22} &= -\mathscr{D} + 2A_4 - 2A_6 + A_1B_1 + A_6B_1 + A_2B_2 + A_6B_2 - A_1B_3 - A_6B_3 + A_1B_4 + A_2B_4 + 2A_4B_4 - A_2B_5 - A_6B_5\,,\\
P_{23} &= -A_4B_2 + A_4B_4 + A_4B_5 + 2A_4 - 2A_6 + A_1B_1 + A_6B_1 - A_2B_2 + A_2B_4 + A_1B_5 - A_2B_5 - A_6B_5 - A_1B_6 - A_6B_6\,,\\
P_{31} &= -A_5B_1 - A_5B_2 + A_5B_4 + A_5B_6 + 2A_5 + 2A_6 - A_2B_2 - A_6B_2 - A_2B_3 + A_6B_3 + A_2B_6 + A_6B_6\,,\\
P_{32} &= -A_5B_2 + A_5B_4 + A_5B_5 + 2A_5 + 2A_6 - A_6B_1 - A_2B_2 - A_6B_2 + A_6B_3 - A_2B_4 + A_2B_5 + A_6B_5\,,\\
P_{33} &= -\mathscr{D} + 2A_5 + 2A_6 - A_6B_1 + A_2B_2 - A_2B_4 + A_2B_5 + 2A_5B_5 + A_6B_5 + A_6B_6
\end{aligned}
$$

Criterion of redundancy II

Application



(Broadhurst 1992)

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P_{13} &= -A_3B_1 - A_3B_2 + A_3B_4 + A_3B_6 + 2A_3 + 2A_6 - A_1B_1 - A_6B_1 - A_1B_5 + A_6B_5 + A_1B_6 + A_6B_6\,,\\
P_{21} &= -A_4B_1 + A_4B_3 + A_4B_4 + 2A_4 - 2A_6 - A_1B_1 + A_2B_2 + A_6B_2 - A_1B_3 + A_2B_3 - A_6B_3 + A_1B_4 - A_2B_6 - A_6B_6\,,\\
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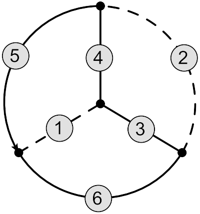
P_{33} &= -\mathscr{D} + 2A_5 + 2A_6 - A_6B_1 + A_2B_2 - A_2B_4 + A_2B_5 + 2A_5B_5 + A_6B_5 + A_6B_6
\end{aligned}
$$

Sharpening the system

"Sharpening" the system of IBP operators stands for the Gauss triangularization with respect to the most complex monomials.

Criterion of redundancy II

Application



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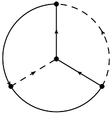
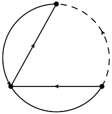
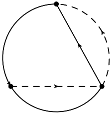
P_{13} &= -A_3B_1 - A_3B_2 + A_3B_4 + A_3B_6 + 2A_3 + 2A_6 - A_1B_1 - A_6B_1 - A_1B_5 + A_6B_5 + A_1B_6 + A_6B_6\,,\\
P_{21} &= -A_4B_1 + A_4B_3 + A_4B_4 + 2A_4 - 2A_6 - A_1B_1 + A_2B_2 + A_6B_2 - A_1B_3 + A_2B_3 - A_6B_3 + A_1B_4 - A_2B_6 - A_6B_6\,,\\
P_{22} &= -\mathscr{D} + 2A_4 - 2A_6 + A_1B_1 + A_6B_1 + A_2B_2 + A_6B_2 - A_1B_3 - A_6B_3 + A_1B_4 + A_2B_4 + 2A_4B_4 - A_2B_5 - A_6B_5\,,\\
P_{23} &= -A_4B_2 + A_4B_4 + A_4B_5 + 2A_4 - 2A_6 + A_1B_1 + A_6B_1 - A_2B_2 + A_2B_4 + A_1B_5 - A_2B_5 - A_6B_5 - A_1B_6 - A_6B_6\,,\\
P_{31} &= -A_5B_1 - A_5B_2 + A_5B_4 + A_5B_6 + 2A_5 + 2A_6 - A_2B_2 - A_6B_2 - A_2B_3 + A_6B_3 + A_2B_6 + A_6B_6\,,\\
P_{32} &= -A_5B_2 + A_5B_4 + A_5B_5 + 2A_5 + 2A_6 - A_6B_1 - A_2B_2 - A_6B_2 + A_6B_3 - A_2B_4 + A_2B_5 + A_6B_5\,,\\
P_{33} &= -\mathscr{D} + 2A_5 + 2A_6 - A_6B_1 + A_2B_2 - A_2B_4 + A_2B_5 + 2A_5B_5 + A_6B_5 + A_6B_6
\end{aligned}
$$



$P_{33} - P_{32}$ no masters



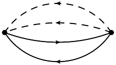
$P_{11} - P_{12}$ no masters



$P_{33} - P_{32}$ no masters

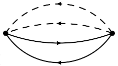
Criterion of redundancy II

Application

The most complex subtopology for the reduction: 

Criterion of redundancy II

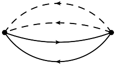
Application

The most complex subtopology for the reduction: 

IBP operators (sharpened)

$$\mathscr{P}_1 = A_4 + \dots, \qquad \mathscr{P}_2 = A_6B_5 + \dots, \qquad \mathscr{P}_3 = A_2B_3 + \dots,$$
$$\mathscr{P}_4 = A_2B_5 + \dots, \qquad \mathscr{P}_5 = A_1B_5 + \dots, \qquad \mathscr{P}_6 = A_6B_3 + \dots,$$
$$\mathscr{P}_7 = A_4B_3 + \dots, \qquad \mathscr{P}_8 = A_4B_5 + \dots, \qquad \mathscr{P}_9 = A_1B_3 + \dots$$

Criterion of redundancy II
Application

The most complex subtopology for the reduction: 

IBP operators (sharpened)

$$
\begin{array}{ccccc}
n_4 = 1 & \wedge & (n_6 = 1 \vee n_5 = 0) & \wedge & (n_2 = 1 \vee n_3 = 0) \\
\mathscr{P}_1 = A_4 + \ldots, & & \mathscr{P}_2 = A_6 B_5 + \ldots, & & \mathscr{P}_3 = A_2 B_3 + \ldots, \\
\mathscr{P}_4 = A_2 B_5 + \ldots, & & \mathscr{P}_5 = A_1 B_5 + \ldots, & & \mathscr{P}_6 = A_6 B_3 + \ldots, \\
\mathscr{P}_7 = A_4 B_3 + \ldots, & & \mathscr{P}_8 = A_4 B_5 + \ldots, & & \mathscr{P}_9 = A_1 B_3 + \ldots
\end{array}
$$

$\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$ give reduction everywhere, except the hyperplanes on which all conditions hold.

Simplifications due to Criterion II:

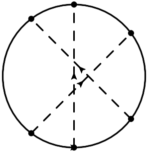
- Identities generated by $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$ need not be considered anymore. Rather, the corresponding rules need to be applied.
- Identities generated by $\mathscr{P}_{4-6}$ should be considered only on the above hyperplanes. For Laporta this means running over 3-parametric space rather than over original 6-parametric.

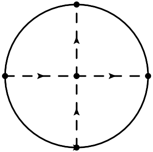
Criterion of redundancy II

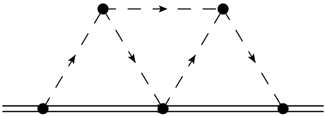
General case

In general case the situation is similar

We can find $[N/2]$ identities satisfying Criterion II. They reduce the number of free parameters by half and other identities should be considered only on the reduced set of points.







Conclusion& Outlook

- Huge redundancy of the IBP identities can be dramatically reduced using the group properties of the IBP reduction.
- Criteria of redundancy suggest an algorithm for the effective reduction procedure. In particular, Criterion II has been implemented in recent algorithm `FIRE` (Smirnov 2008).
- Lorentz-invarance identities can be completely discarded. All information contained in LIs is already contained in IBPs.
- The problem of the reduction can be reformulated as that of division with the remainder by the sum of the left and right ideal.
- The computer program partly based on the above ideas has been used in 4-loop and 3-loop calculations (Kirilin and Lee 2009, Grozin and Lee 2009)

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