About all order ε-expansion of the hypergeometric function

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Introduction

Forty-five years ago, Regge [1967] proposed that any Feynman diagram can be understood in terms of a special class of hypergeometric functions satisfying some system of differential equations so that the singularity surface of the relevant hypergeometric function coincides with the surface of the Landau singularities of the original Feynman diagram. Based on Regge's conjecture, explicit systems of differential equations for particular types of diagrams have been constructed:

the hypergeometric representation for N-point one-loop diagrams via a series representation (Appell functions and Lauricella functions appear here)

(Kershaw, 1973; Wu, 1974; Mano, 1975);

the system of differential equations and its solution in terms of Lappo-Danilevsky functions (Lappo-Danilevsky, 1934) has been constructed by (Barucchi,Ponzano, 1973) and the monodromy structure of some Feynman diagrams has been studied by (Ponzano, Regge, Speer, Westwater, 1969)

It was known at mid-1970's that each Feynman diagram is a function of the "Nilsson class." This means that the Feynman diagram is a multivalued analytical function in complex projective space $\mathbb{CP}^n$. The singularities of this function are described by Landau's equation. Later, Kashiwara, Kawai, 1977, showed that any regularized Feynman integral satisfies some holonomic system of linear differential equations whose characteristic variety is confined to the extended Landau variety.

Hypergeometric Functions: I

Let us recall that there are several different ways to describe special functions:

- as an integral of the Euler type;
- by a series whose coefficients satisfy certain recurrence relations;
- as a solution of a system of differential and/or difference equations (holonomic approach).

These approaches and interrelations between them have been discussed in series of a papers by
I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky,
Adv. Math. **84** (1990) 255;
I.M. Gel'fand, M.I. Graev, V.S. Retakh,
Russian Math. Surveys **47** (1992) 1;
I.M. Gelfand, M.I. Graev,
Russian Math. Surveys **52** (1997) 639;
Russian Math. Surveys **56** (2001) 615.

Integral represenation

An Euler integral has the form

$$\Phi(\vec{\alpha},\vec{\beta},P) = \int_\Sigma \Pi_i P_i(x_1,\cdots,x_k)^{\beta_i} x_1^{\alpha_1}\cdots x_k^{\alpha_k} dx_1\cdots dx_k \ ,$$

where P_i is some Laurent polynomial with respect to variables $x_1,\cdots,x_k$:
$P_i(x_1,\cdots,x_k) = \sum c_{\omega_1\cdots\omega_k} x_1^{\omega_1}\ldots x_k^{\omega_k}$,

with $\omega_j \in \mathbb{Z}$, and $\alpha_i,\beta_j \in \mathbb{C}$.

We assume that the region Σ is chosen such that the integral exists.

Holonomic representations

A combination of differential and difference equations can be found to describe functions of the form

$$\Phi(\vec{z},\vec{x},W) = \sum_{k_1,\cdots,k_r=0}^{\infty} \left(\Pi_{a=1}^{m} \frac{1}{z_a + \sum_{b=1}^{r} W_{ab} k_j} \right) \Pi_{j=1}^{r} \frac{x_j^{k_j}}{k_j!} \; ,$$

where W is an $r \times m$ matrix. In particular, this function satisfies the equations

$$\frac{\partial \Phi(\vec{z},\vec{x},W)}{\partial x_j} = \Phi(\vec{z}+\omega_j, x, W) \; , \quad j = 1, \cdots, r,$$

$$\frac{\partial}{\partial z_i} \left(z_i \Phi + \sum_{j=1}^{r} W_i x_j \frac{\partial \Phi}{\partial x_j} \right) = 0 \; , \quad i = 1, \cdots, m,$$

where ω_j is the $j^{\rm th}$ column of the matrix W.

Series representation

We will take the Horn definition of the series representation. In accordance with this definition, a formal (Laurent) power series in r variables,

$$
\begin{aligned}
\Phi(\vec{x}) &= \sum C(\vec{m})\vec{x}^m \\
&\equiv \sum_{m_1,m_2,\cdots,m_r} C(m_1,m_2,\cdots,m_r) x_1^{m_1}\cdots x_r^{m_r},
\end{aligned}
$$

is called hypergeometric if for each $i = 1,\cdots,r$ the ratio

$$
C(\vec{m}+\vec{e}_i)/C(\vec{m})
$$

is a rational function in the index of summation: $\vec{m} = (m_1,\cdots,m_r)$, where $\vec{e}_j = (0,\cdots,0,1,0,\cdots,0)$, is unit vector with unity in the $j^{\rm th}$ place. Ore[1930], Sato[1990] found that the coefficients of such a series have the general form

$$
C(\vec{m}) = \Pi_{i=1}^r \lambda_i^{m_i} R(\vec{m}) \left(\Pi_{j=1}^N \Gamma(\mu_j(\vec{m}) + \gamma_j + 1) \right)^{-1},
$$

where $N \geq 0$, $\lambda_j, \gamma_j \in \mathbb{C}$ are arbitrary complex numbers, $\mu_j : \mathbb{Z}^{\smallsetminus} \to \mathbb{Z}$ are arbitrary linear maps, and R is an arbitrary rational function.

A series of this type is called a "Horn-type" hypergeometric series.

Example

To illustrate difference between series representation and combination of differential/difference equation, let us consider the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ which we introduce via series representation

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \, , $$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol. The differential equation is

$$ \frac{d}{dz}\left(z\frac{d}{dz} + c - 1 \right) {}_2F_1(a,b;c;z) = \left(z\frac{d}{dz} + a \right) \left(z\frac{d}{dz} + b \right) {}_2F_1(a,b;c;z) \, . $$

Holonomic definition of Gauss hypergeometric function:

$$ \begin{aligned} & \frac{d}{dz} {}_2F_1(a,b;c;z) = {}_2F_1(a+1,b+1;c+1;z) \, , \\ & {}_2F_1(a+1,b;c;z) - z {}_2F_1(a+1,b+1;c+1;z) = \\ & \qquad a {}_2F_1(a,b;c;z) \, , \\ & {}_2F_1(a,b+1;c;z) - z {}_2F_1(a+1,b+1;c+1;z) = \\ & \qquad b {}_2F_1(a,b;c;z) \, , \\ & {}_2F_1(a,b;c-1;z) - z {}_2F_1(a+1,b+1;c+1;z) = \\ & \qquad (c-1) {}_2F_1(a,b;c;z) \, . \end{aligned} $$

Construction of ε-expansion: Formulation of problem

The elaboration of the algorithm for analytical evaluation of the higher order terms of the ε-expansion of any hypergeometric functions of several variables with arbitrary set of parameters.

There is not universal agreement on what it means to express a solution in terms of known special functions. One reasonable answer has been presented by Kitaev, when he quotes R. Askye's Forward to the book *Symmetries and Separation of Variables* by W. Miller, Jr., which says "One term which has not been defined so far is 'special function'. My definition is simple, but not time invariant. A function is a special function if it occurs often enough so that it gets a name".
Kitaev adds, "... most of the people who apply them . . . understand, under the notion of special functions, a set of functions which can be found in one of the well-known reference books. . . ." To this, we may add "functions which can be found in one of the well-known computer algebra systems."

Multiple polylogarithms (Hyperlogarithms)

The starting point of our consideration is integral

$$
\begin{aligned}
& I(z; a_k, a_{k-1} \cdots, a_1) \\
&= \int_0^z \frac{dt_k}{t_k - a_k} \int_0^{t_k} \frac{dt_{k-1}}{t_{k-1} - a_{k-1}} \cdots \int_0^{t_2} \frac{dt_1}{t_1 - a_1} \\
&= \int_0^z \frac{dt}{t - a_k} I(a_{k-1} \cdots, a_1; t) \; ,
\end{aligned}
$$

where we put that all $a_k \neq 0$. In early consideration by Kummer, Poincare, Lappo-Danilevky this integral was called as hyperlogarithms It was treated as analytical functions of one variable z, the upper limit of integration. Goncharov has analysed it as multivalued analytical functions on $a_1, \cdots, a_k, z$. One of the property of hyperlogarithms is the scaling invariance:

$$
I(z; a_1, \cdots, a_k) = I\left(1; \frac{a_1}{z}, \cdots, \frac{a_k}{z}\right) \; .
$$

By definition, the multiple polylogarithm

$$
\mathrm{Li}_{k_1,k_2,\cdots,k_n}(x_1, x_2, \cdots, x_n) = \sum_{m_n > \cdots > m_1 > 0}^{\infty} \frac{x_1^{m_1}}{m_1^{k_1}} \frac{x_2^{m_2}}{m_2^{k_2}} \cdots \frac{x_n^{m_n}}{m_n^{k_n}} \; ,
$$

where **weight** $k = k_1 + k_2 + \cdots + k_n$ and **depth** is equal to n.

The multiple polylogarithm is a special case of iterated integral

$$
\begin{aligned}
& G_{m_n,m_{n-1},\cdots,m_1}(z;x_n,\cdots,x_1) \\
& \equiv I(z;\underbrace{0,\cdots,0}_{m_n-1 \text{ times}},x_n,\cdots,\underbrace{0,\cdots,0}_{m_1-1 \text{ times}},x_1) \\
& = (-1)^n \mathrm{Li}_{m_1,m_2,\cdots,m_n}\left(\frac{x_2}{x_1},\frac{x_3}{x_2},\cdots,\frac{z}{x_n}\right) \; .
\end{aligned}
$$

The inverse relation is

$$
\begin{aligned}
& \mathrm{Li}_{k_1,k_2,\cdots,k_n}(y_1,y_2,\cdots,y_n) \\
& = (-1)^n G_{1;k_n,k_{n-1},\cdots,k_2,k_1}\left(\frac{1}{y_n},\frac{1}{y_n y_{n-1}},\cdots,\frac{1}{y_1\cdots y_n}\right) \; .
\end{aligned}
$$

The multiple polylogarithms satisfy two Hopf algebras, so called shuffle and stuffle ones. The first is related with integral representation, the second one with series.

Multiple polylogarithms: Particular cases

A particular case of the multiple polylogarithm is the "generalized polylogarithm" defined by

$$\mathrm{Li}_{k_1,k_2,\ldots,k_n}(z) = \sum_{m_n > m_{n-1} > \cdots > m_1 > 0}^{\infty} \frac{z^{m_n}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}$$

where $|z| < 1$ when all $k_i \geq 1$, or $|z| \leq 1$ when $k_n \leq 2$.

Another particular case is a "multiple polylogarithm of a square root of unity," defined as

$$\mathrm{Li}_{\left(\substack{\sigma_1,\sigma_2,\cdots,\sigma_n \\ s_1,s_2,\cdots,s_n}\right)}(z) = \sum_{m_n > m_{n-1} > \cdots m_1 > 0} z^{m_n} \frac{\sigma_n^{m_n} \cdots \sigma_1^{m_1}}{m_n^{s_n} \cdots m_1^{s_1}} \,.$$

where $\vec{s} = (s_1, \cdots s_n)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_n)$ are multi-indices and σ_k belongs to the set of the square roots of unity, $\sigma_k = \pm 1$. This particular case of multiple polylogarithms has been analyzed in detail by Remiddi and Vermaseren, 2000

For the numerical evaluation of multiple polylogarithms: Vollinga & Weinzierl, 2005;

Integral & Series representation

In the Euler integral representation, the most important results are related to the construction of the all-order ε expansion of Gauss hypergeometric function with special values of parameters in terms of Nielsen polylogarithms
A.I.Davydychev , Phys.Rev.(1999);
A.I.Davydychev & M.K., Nucl.Phys.Proc.Suppl.89 (2000);
A.I.Davydychev & M.K., Nucl.Phys.B605 (2001)

The series representation is an intensively studied approach. Particularly impressive results were derived in the framework of the nested-sum approach for hypergeometric functions with a balanced set of parameters by
Moch,Uwer,Weinzierl,2002; Weinzierl,2004;
Computer realizations of nested sums approach to expansion of hypergeometric functions are given in
Weinzierl, 2002; Moch & Uwer, 2006;
Huber & Maître, 2206, 2008

Generating-function approach have been applied to construction of ε-expansion for hypergeometric functions with one unbalanced set of parameters
M.K.,Davydychev,2004; M.K.,Ward,Yost,2007; M.K.,Kniehl,2008

Series representation

The pioneering systematic activity in studying the Laurent series expansion of hypergeometric functions at particular values of the argument $(z = 1)$ was started by David Broadhurst in the context of Euler-Zagier sums (or multidimensional zeta values). This activity has received further consideration for another, physically interesting point, $z = 1/4$ and also for the "primitive sixth roots of unity"
Broadhurst, 1999; Borwein, et.al., 1999; Fleischer, M.K., 1999; M.K., Veretin, 2000; Davydychev, M.K., 2000;
Over time, other types of sums have been analysed in a several publications:

- *harmonic* sums
- *generalized harmonic sums*
- *binomial* sums
- *inverse binomial* sums

How to calculate this sums analytically?

Generating function approach

H.S. Wilf, *Generatingfunctionology*, Academic Press, London, 1994.

Let us rewrite an arbitrary series as

$$\Sigma_{\vec{A}}(\vec{z}) = \sum_{j=1}^{\infty} \vec{z}^j \eta_{\vec{A}}(j) \; ,$$

where $\vec{A}$ denote the collective sets of indices, whereas $\eta_{\vec{A}}(j)$ is the coefficient of $\vec{z}^j$.

The idea is to find a recurrence relation with respect to j, for the coefficients $\eta_{vecA}(j)$, and then transform it into a differential equation for the *generating* function $\Sigma_{\vec{A}}(z)$. In this way, the problem of summing the series would be reduced to solving a differential equation.

Generating functions approach (II)

M.K.,Ward,Yost, 07 M.K.,Kniehl, 08

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=-\frac{(1-y)^2}{y}}$$

$$= \frac{1-y}{1+y} \sum_{p,\vec{s}} c_{p,\vec{s}} \ln^p y \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y)$$

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j^c} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=-\frac{(1-y)^2}{y}}$$

$$= \sum_{p,\vec{s}} \tilde{c}_{p,\vec{s}} \ln^p y \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y) \ , \quad c \geq 2$$

$$\sum_{j=1}^{\infty} \binom{2j}{j} u^j S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=\frac{\chi}{(1+\chi)^2}}$$

$$= \sum_{p,\vec{s}} \left[\frac{c_{p,\vec{s}}}{1-\chi} + d_{p,\vec{s}} \right] \ln^p \chi \ \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(\chi) \ ,$$

$$\sum_{j=1}^{\infty} \binom{2j}{j} \frac{u^j}{j^c} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=\frac{\chi}{(1+\chi)^2}}$$

$$= \sum_{p,\vec{s}} \tilde{c}_{p,\vec{s}} \ln^p \chi \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(\chi) \ , \quad c \geq 1$$

where c is a positive integer, $c_{p,\vec{s}}$, $\tilde{c}_{p,\vec{s}}$ and $d_{p,\vec{s}}$ are rational coefficients, $\mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(z)$ is the multiple polylogarithm of a square root of unity and

$$S_a(j-1) = \sum_{i=1}^{j-1} \frac{1}{i^a} \ ,$$

An approach using the iterated solution of differential equations has been explored by
Shu Oi, 2004, M.K.,Ward,Yost,2007, M.K.,Kniehl,2008

One of the advantages of the iterated-solution approach over the series approach is that it provides a more efficient way to calculate each order of the ε expansion, since it relates each new term to the previously derived terms, rather than having to work with an increasingly large collection of independent sums at each order. This technique includes two steps: (i) the differential-reduction algorithm (to reduce a generalized hypergeometric function to basic functions); (ii) iterative solution of the proper differential equation for the basic functions (equivalent to iterative algorithms for calculating the analytical coefficients of the ε expansion).

A system of homogeneous linear differential equations,

$$\frac{d}{dt}\vec{u}(t) = A(t, \vec{u}(t)) \ ,$$

where A is an $n \times n$ matrix and $\vec{u}$ is an n-dimensional vector, can be formally written via Picard's method of approximation as

$$\begin{aligned}
\vec{u}_1 &= \vec{u}_0 + \int_{u_0}^{t} dt A(t, \vec{u}_0) \ , \\
\vec{u}_2 &= \vec{u}_0 + \int_{u_0}^{t} dt A(t, \vec{u}_1(t)) \ , \\
&\cdots \\
\vec{u}_n &= \vec{u}_0 + \int_{u_0}^{t} dt A(t, \vec{u}_{n-1}(t)) \ ,
\end{aligned}$$

where $\vec{u}_0 = \vec{u}(t_0)$ is an initial condition. It can be proven that, in the region where this integral exists, the following properties are satisfied: (i) as n increases indefinitely, the sequence of functions $\vec{u}_n$ tends to a limit which is a continuous function of t; (ii) this limiting function satisfies the differential equation; (iii) the solution thus defined assumes the value $\vec{u}_0$ when $t = t_0$ and is the only continuous solution.

Let us introduce the set of functions,

$$F_p(t) = \int_{t_0}^{t} d\tau A(\tau) F_{p-1}(\tau) \ ,$$

and write the relations (**??**) as

$$\vec{u}(t) = F_0(t)\vec{u}_0 + \cdots + F_p(t)\vec{u}_0 + \cdots \ ,$$

where F_0 is the identical transformation, $F_0 = I$. For $A(t) = \sum_j \frac{U_j}{t-\alpha_j}$, the iterative solution coincides with hyperlogarithms of configuration α_j.

Differential Equation approach for construction of ε-expansion

Let us consider as basis the Gauss hypergeometric function with the following set of parameters:

$$\omega(z) = {}_2F_1\left(\frac{p_1}{q_1}+a_1\varepsilon, \frac{p_2}{q_2}+a_2\varepsilon; 1-\frac{p_3}{q_3}+c\varepsilon; z\right);.$$

It is the solution of the differential equation

$$\left[z\frac{d}{dz}+\frac{p_1}{q_1}+a_1\varepsilon\right]\left[z\frac{d}{dz}+\frac{p_2}{q_2}+a_2\varepsilon\right]\omega(z) = \frac{d}{dz}\left[z\frac{d}{dz}-\frac{p_3}{q_3}+c\varepsilon\right]\omega(z)\ .$$

with boundary conditions $w(0)=1$ and $z\frac{d}{dz}w(z)\big|_{z=0}=0$. Due to analyticity of Gauss hypergeometric function with respect to parameters, this equation is valid in each order of ε,

$$w(z) = \sum_{k=0}^{\infty} w_k(z)\varepsilon^k.$$

In terms of coefficients functions $\omega_k(z)$, we have

$$\left[(1-z)\frac{d}{dz}-\left(\frac{p_1}{q_1}+\frac{p_2}{q_2}\right)-\frac{1}{z}\frac{p_3}{q_3}\right]\left(z\frac{d}{dz}\right)\omega_k-\frac{p_1p_2}{q_1q_2}\omega_k = \left(a_1+a_2-\frac{c}{z}\right)\left(z\frac{d}{dz}\right)\omega_{k-1} + \left(a_1\frac{p_2}{q_2}+a_2\frac{p_1}{q_1}\right)\omega_{k-1}+a_1a_2\omega_{k-2}\ .$$

Differential Equation approach for construction of ε-expansion (II)

The main idea is to find a new parametrization (change of variable) $z \to \xi(z)$, and to define a new functions $\rho_k(\xi)$, related with a first derivative of original functions $\omega_k(\xi)$,

$$\rho_k(\xi) = \sum_j \Gamma_{kj}(\xi) \frac{d}{d\xi} \omega_j(\xi) \ ,$$

so that original equation can be rewritten as system of linear differential equations of the first order with an rational coefficients:

$$\begin{aligned}
\frac{d}{d\xi} w_k(\xi) &= \rho_k(\xi) \sum_j \frac{A_j}{\xi - \alpha_j} \ , \\
\frac{d}{d\xi} \rho_k(\xi) &= \rho_{k-1}(\xi) \sum_j \frac{B_j}{\xi - \beta_j} \\
& \quad + \omega_{k-1}(\xi) \sum_j \frac{C_j}{\xi - \gamma_j} + \omega_{k-2}(\xi) \sum_j \frac{D_j}{\xi - \sigma_j} \ ,
\end{aligned}$$

where $A_j, B_j, C_j, D_j, \alpha_j, \beta_j, \gamma_j, \sigma_j$ Then the iterative solution of this system can be constructed. Under condition, $\omega_0(z) = 1 (\rho_0 = 0)$, this solution are expressible in terms of hyperlogarithms depending on parameters $\alpha_j, \beta_j, \gamma_j, \sigma_j$, (possible) times on powers of logarithms.

Iterative solution of Gauss hypergeometric function

M.K.,Ward,Yost, JHEP, 07.

$$ {}_2F_1(a_1\varepsilon, a_2\varepsilon; 1+c\varepsilon; z) $$

Starting equation is

$$ \begin{aligned} & (1-z)\frac{d}{dz}\left(z\frac{d}{dz}\right)w_k(z) \\ & = \left(a_1+a_2-\frac{c}{z}\right)\left(z\frac{d}{dz}\right)w_{k-1}(z)+a_1a_2w_{k-2}(z) \; . \end{aligned} $$

Let us introduce a new function: and rewrite original equation as

$$ \begin{aligned} (1-z)\frac{d}{dz}\rho_i(z) &= \left(a_1+a_2-\frac{c}{z}\right)\rho_{i-1}(z)+a_1a_2w_{i-2}(z) \; , \\ z\frac{d}{dz}w_i(z) &= \rho_i(z) \; . \end{aligned} $$

The solution of this system can be presented in an iterated form:

$$ \begin{aligned} \rho_i(z) &= (a_1+a_2-c)\int_0^z \frac{dt}{1-t}\rho_{i-1}(t) \\ & \qquad +a_1a_2\int_0^z \frac{dt}{1-t}w_{i-2}(t)-cw_{i-1}(z) \; , \quad i \geq 1 \; , \\ w_i(z) &= \int_0^z \frac{dt}{t}\rho_i(t) \; , \quad i \geq 1 \; . \end{aligned} $$

Iterative solution of Gauss hypergeometric function: II

$$
{}_2F_1\left(\begin{array}{c} 1+a_1\varepsilon, 1+a_2\varepsilon \\ 2+c\varepsilon \end{array} \middle| z \right) = \frac{1+c\varepsilon}{z} \Biggl(-\ln(1-z)
$$

$$
-\varepsilon \left\{ \frac{c-a_1-a_2}{2} \ln^2(1-z) + c\mathrm{Li}_2(z) \right\}
$$

$$
+\varepsilon^2 \left\{ \left[(a_1+a_2)c - c^2 - 2a_1a_2 \right] \mathrm{S}_{1,2}(z) \right.
$$

$$
+ \left[(a_1+a_2)c - c^2 - a_1a_2 \right] \ln(1-z)\mathrm{Li}_2(z)
$$

$$
\left. + c^2\mathrm{Li}_3(z) - \frac{1}{6}(c-a_1-a_2)^2 \ln^3(1-z) \right\}
$$

$$
-\varepsilon^3 \left\{ c \left[(a_1+a_2)c - c^2 - 2a_1a_2 \right] \mathrm{S}_{2,2}(z) \right.
$$

$$
+ c \left[(a_1+a_2)c - c^2 - a_1a_2 \right] \ln(1-z)\mathrm{Li}_3(z)
$$

$$
+ (c-a_1)(c-a_2)(c-a_1-a_2)
$$

$$
\times \left[\ln(1-z)\mathrm{S}_{1,2}(z) + \frac{1}{2} \ln^2(1-z)\mathrm{Li}_2(z) \right]
$$

$$
+ \frac{1}{24}(c-a_1-a_2)^3 \ln^4(1-z)
$$

$$
\left. + c(c-a_1-a_2)^2\mathrm{S}_{1,3}(z) + c^3\mathrm{Li}_4(z) \right\} + \mathcal{O}(\varepsilon^4) \Biggr) ,
$$

Coincides with
Fleischer,Kotikov,Veretin, 1999

Iterative solution of Gauss hypergeometric function: III

$$
{}_2F_1\left(\begin{array}{c} a_1\varepsilon, a_2\varepsilon \\ \frac{1}{2}+f\varepsilon \end{array} \middle| z \right) .
$$

Original equation is

$$
\begin{aligned}
&\left[(1-z)\frac{d}{dz} - \frac{1}{2z}\right]\left(z\frac{d}{dz}\right) w_i(z) \\
&= \left[(a_1+a_2) - \frac{f}{z}\right]\left(z\frac{d}{dz}\right) w_{i-1}(z) + a_1 a_2 w_{i-2}(z) .
\end{aligned}
$$

Let us introduce the new variable y

$$
y = \frac{1-\sqrt{\frac{z}{z-1}}}{1+\sqrt{\frac{z}{z-1}}} , \quad z = -\frac{(1-y)^2}{4y} ,
$$

and define a set of a new functions $\rho_i(y)$

$$
\begin{aligned}
y\frac{d}{dy}\rho_i(y) &= (a_1+a_2)\frac{1-y}{1+y}\rho_{i-1}(y) \\
&\quad + 2f\left(\frac{1}{1-y} - \frac{1}{1+y}\right)\rho_{i-1}(y) + a_1 a_2 w_{i-2}(y) , \\
y\frac{d}{dy} w_k(y) &= -\rho_k(y) .
\end{aligned}
$$

The solution of these differential equations has the form

$$
\begin{aligned}
\rho_i(y) &= \int_1^y dt \left[2f\frac{1}{1-t} - 2(a_1+a_2-f)\frac{1}{1+t}\right]\rho_{i-1}(t) \\
&\quad - (a_1+a_2)\left[w_{i-1}(y) - w_{i-1}(1)\right] \\
&\quad + a_1 a_2 \int_1^y \frac{dt}{t} w_{i-2}(t) , \quad i \geq 1 , \\
w_i(y) &= -\int_1^y \frac{dt}{t}\rho_i(t) , \quad i \geq 1 .
\end{aligned}
$$

Iterative solution of Gauss hypergeometric function: IV

Davydychev,M.K.2004; M.K.,06; M.K.,Ward,Yost,06

$$
{}_2F_1\left(\begin{array}{c} 1+a_1\varepsilon, 1+a_2\varepsilon \\ \frac{3}{2}+f\varepsilon \end{array} \middle| z \right) = \frac{(1+2f\varepsilon)}{2z}\frac{1-y}{1+y}\Bigg(\ln y
$$

$$
+\varepsilon \Bigg\{ 2(f-a_1-a_2)\left[\mathrm{Li}_2(-y)+\ln y \ln(1+y)\right]
$$

$$
-2f\left[\mathrm{Li}_2(y)+\ln y \ln(1-y)\right]
$$

$$
+\frac{1}{2}(a_1+a_2)\ln^2 y + \zeta_2(3f-a_1-a_2) \Bigg\}
$$

$$
+\mathcal{O}(\varepsilon^2)\Bigg) \,, \qquad (1)
$$

M.K., JHEP 06.

Let us consider a Gauss hypergeometric functions with integer or half-integer values of ε-independent parameters. We will call these basis functions as functions of type **A, B, C, D, E, F**. For each type the values of a, b, c, parameters of our basis, are presented in Table I:

$$ {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| z \right) $$

Table I						
	A	**B**	**C**	**D**	**E**	**F**
a	$a_1\varepsilon$	$a\varepsilon$	$a\varepsilon$	$\frac{1}{2}+b_1\varepsilon$	$a_1\varepsilon$	$\frac{1}{2}+b_1\varepsilon$
b	$a_2\varepsilon$	$\frac{1}{2}+b\varepsilon$	$\frac{1}{2}+b\varepsilon$	$\frac{1}{2}+b_2\varepsilon$	$a_2\varepsilon$	$\frac{1}{2}+b_2\varepsilon$
c	$\frac{1}{2}+f\varepsilon$	$1+c\varepsilon$	$\frac{1}{2}+f\varepsilon$	$\frac{1}{2}+f\varepsilon$	$1+c\varepsilon$	$1+c\varepsilon$

The number of independent basis hypergeometric functions, enumerated in Table I, can be reduced by help of the Kummer transformations of variable z.

$$ z \to \frac{1}{z}, 1-z, \frac{1}{1-z}, \frac{-z}{1-z}, 1-\frac{1}{z} $$

With respect to this transformations the functions of type **A, B, C, D** are transformed into each other. This allows us to reduce the number of independent hypergeometric functions. The functions of type **E, F** transform into functions of the same type.

M.K., JHEP 06.

Let us illustrate how functions of type **B, C, D** can be expressed in terms of functions of type **A**:

$\mathbf{D}$-type:

$$
{}_2F_1\left(\begin{array}{c} \frac{1}{2}+b_1\varepsilon, \frac{1}{2}+b_2\varepsilon \\ \frac{1}{2}+f\varepsilon \end{array} \middle| z \right)
= \frac{(1-z)^{(f-b_1-b_2)\varepsilon}}{(1-z)^{1/2}} \, {}_2F_1\left(\begin{array}{c} (f-b_1)\varepsilon, (f-b_2)\varepsilon \\ \frac{1}{2}+f\varepsilon \end{array} \middle| z \right) ,
$$

C-type:

$$
{}_2F_1\left(\begin{array}{c} \frac{1}{2}+b, a\varepsilon \\ \frac{1}{2}+f\varepsilon \end{array} \middle| z \right)
= \frac{1}{(1-z)^{a\varepsilon}} \, {}_2F_1\left(\begin{array}{c} a\varepsilon, (f-b)\varepsilon \\ \frac{1}{2}+f\varepsilon \end{array} \middle| -\frac{z}{1-z} \right)
$$

B-type:

$$
{}_2F_1\left(\begin{array}{c} \frac{1}{2}+b\varepsilon, a\varepsilon \\ 1+c\varepsilon \end{array} \middle| z \right) =
\frac{\Gamma(1+c\varepsilon)\Gamma\left(-\frac{1}{2}-(c-a-b)\varepsilon\right)}{\Gamma(a\varepsilon)\Gamma\left(\frac{1}{2}+b\varepsilon\right)} \frac{(1-z)^{1/2+(c-a-b)\varepsilon}}{z^{1-(a-c)\varepsilon}}
{}_2F_1\left(\begin{array}{c} 1+(c-a)\varepsilon, 1-a\varepsilon \\ \frac{3}{2}+(c-a-b)\varepsilon \end{array} \middle| 1-\frac{1}{z} \right)
+\frac{1}{z^{a\varepsilon}} \frac{\Gamma(1+c\varepsilon)\Gamma\left(\frac{1}{2}+(c-a-b)\varepsilon\right)}{\Gamma(1+(c-a)\varepsilon)\Gamma\left(\frac{1}{2}+(c-b)\varepsilon\right)}
{}_2F_1\left(\begin{array}{c} a\varepsilon, (a-c)\varepsilon \\ \frac{1}{2}+(a+b-c)\varepsilon \end{array} \middle| 1-\frac{1}{z} \right)
$$

As a result, we get the following statement:

Any functions of type A, B, C, D can be expressed in an algebraic way in terms of just one of these types.

Generalized hypergeometric functions

The generalized hypergeometric function can be written as series

$$
{}_PF_Q\left(\begin{array}{c} \{A_1+a_1\varepsilon\}, \{A_2+a_2\varepsilon\}, \cdots \{A_P+a_P\varepsilon\} \\ \{B_1+b_1\varepsilon\}, \{B_2+b_2\varepsilon\}, \cdots \{B_Q+b_q\varepsilon\} \end{array} \middle| z \right)
= \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{\Pi_{s=1}^P (A_s+a_s\varepsilon)_j}{\Pi_{r=1}^Q (B_r+b_r\varepsilon)_j} \, ,
$$

where $(\alpha)_j \equiv \Gamma(\alpha+j)/\Gamma(\alpha)$ is the Pochhammer symbol.

We want to construct the ε-expansion of this series.

$$
{}_PF_Q = \begin{cases} P \leq Q & \text{converges for all finite z} \\ P = Q+1 & \text{converges for all } |z|<1 \\ P > Q+1 & \text{diverges for all } z \neq 0 \end{cases}
$$

Reduction of hypergeometric function

It is well known that any function

$$
{}_pF_{p-1}(\vec{a}+\vec{m}; \vec{b}+\vec{k}; z)
$$

is expressible in terms of p other functions of the same type:

$$
R_{p+1}(\vec{a},\vec{b},z)\,{}_pF_{p-1}(\vec{a}+\vec{m}; \vec{b}+\vec{k}; z) = \sum_{k=1}^{p} R_k(\vec{a},\vec{b},z)\,{}_pF_{p-1}(\vec{a}+\vec{e}_k; \vec{b}+\vec{E}_k; z) \, ,
$$

where $\vec{m}$, $\vec{k}$, $\vec{e}_k$, and $\vec{E}_k$ are lists of integers and R_k are polynomials in parameters $\vec{a}$, $\vec{b}$, and z.

Construction of all-order ε-expansion via Differential equation

M.K.,Ward,Yost, JHEP, 07.

$$\omega(z) = {}_pF_{p-1}(\vec{a}\varepsilon; \vec{1} + \vec{b}\varepsilon; z)$$

Defining the coefficients functions $w_k(z)$ at each order by

$$\omega(z) = \sum_{k=0}^{\infty} w_k(z)\varepsilon^k,$$

The differential equation is

$$\left[(1-z)\frac{d}{dz}\right]\left(z\frac{d}{dz}\right)^{p-1} w_k(z)$$
$$= \sum_{i=1}^{p-1}\left[P_i(\vec{a}) - \frac{1}{z}Q_i(\vec{b})\right]\left(z\frac{d}{dz}\right)^{p-i} w_{k-i}(z) + P_p(\vec{a})w_{k-p}(z) ,$$

where $P_j(\vec{a})$ and $Q_j(\vec{b})$ are polynomials of order j depending on vectors $\vec{a}$ and $\vec{b}$, respectively.

$$\begin{aligned} z\frac{d}{dz}\rho_k^{(j)}(z) &= \rho_k^{(j+1)}(z) , \quad j = 0, 1, \cdots, p-1 \\ (1-z)\frac{d}{dz}\rho_k^{(p-1)}(z) &= \sum_{i=1}^{p}\left[P_i(\vec{a}) - \frac{1}{z}Q_i(\vec{b})\right]\rho_{k-i}^{(p-i)}(z) , \end{aligned}$$

The solution is iterated integral:

$$\begin{aligned} \rho_k^{(p-1)}(z) &= \sum_{i=1}^{p}\left[P_i(\vec{a}) - Q_i(\vec{b})\right]\int_0^z \frac{dt}{1-t}\rho_{k-i}^{(p-i)}(t) \\ &\quad - \sum_{i=1}^{p-2} Q_i(\vec{b})\rho_{k-i}^{(p-i-1)}(z) \\ &\quad - Q_{p-1}(\vec{b})[w_{k-p+1}(z) - \delta_{0,k-p+1}] , \\ \rho_k^{(j-1)}(z) &= \int_0^z \frac{dt}{t}\rho_k^{(j)}(t) , \quad k \geq 1 , \quad j = 1, 2, \cdots, p-1 , \end{aligned}$$

Results

Here, we will mention some of the existing results.

- If I_1, I_2, I_3 are arbitrary integers, the Laurent expansions of the Gauss hypergeometric functions

$$
\begin{aligned}
&{}_2F_1(I_1 + a\varepsilon, I_2 + b\varepsilon; I_3 + \tfrac{p}{q} + c\varepsilon; z) \ , \\
&{}_2F_1(I_1 + \tfrac{p}{q} + a\varepsilon, I_2 + \tfrac{p}{q} + b\varepsilon; I_3 + \tfrac{p}{q} + c\varepsilon; z) \ , \\
&{}_2F_1(I_1 + \tfrac{p}{q} + a\varepsilon, I_2 + b\varepsilon; I_3 + c\varepsilon; z) \ , \\
&{}_2F_1(I_1 + \tfrac{p}{q} + a\varepsilon, I_2 + b\varepsilon; I_3 + \tfrac{p}{q} + c\varepsilon; z)
\end{aligned}
$$

  are expressible in terms of multiple polylogarithms of arguments being powers of q-roots of unity and a new variable, that is an algebraic function of z, with coefficients that are ratios of polynomials.

- If $\vec{A}, \vec{B}$ are lists of integers and I, p, q are integers, the Laurent expansions of the generalized hypergeometric functions

$$
\begin{aligned}
&{}_pF_{p-1}(\vec{A} + \vec{a}\varepsilon, \tfrac{p}{q} + I; \vec{B} + \vec{b}\varepsilon; z) \ , \\
&{}_pF_{p-1}(\vec{A} + \vec{a}\varepsilon; \vec{B} + \vec{b}\varepsilon, \tfrac{p}{q} + I; z)
\end{aligned}
$$

  are expressible in terms of multiple polylogarithms of arguments that are powers of q-roots of unity and a new variable that is an algebraic function of z, with coefficients that are ratios of polynomials.

Results: II

- If $\vec{A}, \vec{B}$ are lists of integers, the Laurent expansion of the generalized hypergeometric function

$$ {}_pF_{p-1}(\vec{A}+\vec{a}\varepsilon; \vec{B}+\vec{b}\varepsilon; z) $$

are expressible via generalized polylogarithms.

- If p, q, I_k are any integers and $\vec{A}, \vec{B}$ are lists of integers, the generalized hypergeometric function

$$ {}_pF_{p-1}(\{\tfrac{p}{q}+\vec{A}+\vec{a}\varepsilon\}_r, \vec{I_1}+\vec{c}\varepsilon; \{\tfrac{p}{q}+\vec{B}+\vec{b}\varepsilon\}_r, \vec{I_2}+\vec{d}\varepsilon; z) $$

is expressible in terms of multiple polylogarithms of arguments that are powers of q-roots of unity and the new variable $z^{1/q}$, with coefficients that are ratios of polynomials.

- the coefficients of the ε expansion of the hypergeometric functions

$$ {}_{p+1}F_p\left(\left.\begin{array}{c}\vec{A}+\frac{r}{q}+\vec{a}\varepsilon\\ \vec{B}+\frac{r}{q}+\vec{b}\varepsilon\end{array}\right| z\right), \quad {}_{p+1}F_p\left(\left.\begin{array}{c}\vec{A}+\vec{a}\varepsilon\\ \vec{B}+\vec{b}\varepsilon, I+\frac{r}{q}+c\varepsilon\end{array}\right| z\right), $$
$$ {}_{p+1}F_p\left(\left.\begin{array}{c}I+\frac{r}{q}+c\varepsilon, \vec{A}+\vec{a}\varepsilon\\ \vec{B}+\vec{b}\varepsilon\end{array}\right| z\right), $$

where $\vec{A}, \vec{B}, \vec{a}, \vec{b}, c$ and I are all integers, are related to each other.

Conclusion

At the present moment it is unclear is there some limitation on the type of functions generated by Feynman diagrams or a zoo of a new functions is artifact of using technique? In particular, the statement that results of calculations can be written in terms of restricted set of special functions will allow to use the restricted set of programs for numerical evaluation of physical results. Another application is related with evaluation of so called single scale diagrams, where an explicit prediction of possible transcendental constants can be done.

The strategy of such kind analysis is well know in theory of special functions and analytical theory of differential equations. As well know, any Feynman diagram satisfy to the system of linear differential/difference equations with the polynomial equation. At the modern mathematical language, such system can be associated with the Gelfand-Karpanov-Zelevinskii functions or D-modules. So, any question about "zoo" of special functions generated by ε-expansion of Feynman diagrams could be reduced to the problem of construction of Laurent expansion of D-modules around some values of their parameters.

Since the power of propagator is an integer number in a covariant gauge and any (irreducible) numerator are expressible in terms of integral of the same topology with a shifted powers which is again integer number. it is enough to consider an hypergeometric functions of several variables (in general, the number of variables of equal to number of kinematic invariants minus one) with integer values of parameters only. Fortunately, when some of the kinematic invariants are proportional (or equal) to each other, the number of variables in the proper hypergeometric series could be reduced. But price of this reduction is rational values of parameters. All known exactly solvable cases. has confirmed this statement. Typically, only integer and half-integer values of parameters are generated and only recently the inverse cubic values have been detected.

M.K. & Davydychev 99/05.
It is evident that some (or all, if the basis is complete) of the alternating or non-alternating multiple Euler-Zagier sums (or multiple zeta values) can be written in terms of multiple (inverse) binomial sums of special values of arguments. Two arguments where such a representation is possible are trivially obtained by setting the arguments of the harmonic polylogarithms y, χ to ± 1:

$$\begin{aligned} u &= 4 \;, \quad y = -1 \;, \\ u &= \frac{1}{4} \;, \quad \chi = 1 \;. \end{aligned}$$

Another such point is "golden ratio",

$$u = -1 \;, \quad y = \frac{3 - \sqrt{5}}{2}$$

has been discussed intensively in the context of Apéry-like expressions for Riemann zeta functions. For two other points

$$\begin{aligned} u &= 1 \;, \quad y = \exp\left(i\frac{\pi}{3}\right) \;, \\ u &= 2 \;, \quad y = i \;, \end{aligned}$$

the relation between multiple inverse binomial sums and multiple zeta values was analysed mainly by the method of experimental mathematics.

Let us make a few comments about harmonic polylogarithms of a complex argument. For the case $0 \leq u \leq 4$, the variable y belongs to a complex unit circle, $y = \exp(i\theta)$. In this case, the colored polylogarithms of a square root of unity can be split into real and imaginary parts and generalized log-sine functions are generated.

The case of one-loop Feynman diagrams has been studied the most. The hypergeometric representations for N-point one-loop diagrams with arbitrary powers of propagators and an arbitrary space-time dimension have been derived for non-exceptional kinematics by
Davydychev & Boos, 1991; Davydychev 1991
His approach is based on the Mellin-Barnes technique.

An alternative hypergeometric representation for one-loop diagrams has been derived recently by Fleischer, Jegerlehner, Tarasov, 2003 using a difference equation in the space-time dimension. In this approach, the one-loop N-point function was shown to be expressible in terms of hypergeometric functions of $N-1$ variables. One remarkable feature of the derived results is a one-to-one correspondence between arguments of the hypergeometric functions and Gram and Cayley determinants, which are two of the main characteristics of diagrams.

Application to Feynman diagrams: construction ε-expansion

The program of constructing the analytical coefficients of the ε-expansion is a more complicated matter. The finite parts of one-loop diagrams in $d = 4$ dimension are expressible in terms of the Spence dilogarithm function
Hooft, Veltman, 1979; Denner, Nierste, Scharf, 1991

Only partial results for higher-order terms in the ε-expansion are known at one loop. The all-order ε-expansion of the one-loop propagator with an arbitrary values of masses and external momentum has been constructed in terms of Nielsen polylogarithms:
A.I.Davydychev, 1999; A.I.Davydychev & M.K., 2000, 2001;

The term linear in ε for the one-loop vertex diagram with non-exceptional kinematics has also been constructed in terms of Nielsen polylogarithms
Nierste, Müller, Böhm, 1993

The all-order ε expansion for the one-loop vertex with non-exceptional kinematics is expressible in terms of multiple polylogarithms of two variables:
Davydychev, 2006; Tarasov, 2008

Beyond these examples, the situation is less complete. The term linear in ε for the box diagram is still under construction. Some cases for particular masses have been analyzed

Fleischer, Riemann, Tarasov, 2003; Körner, Merebashvili, Rogal, 2005,2006; Tarasov, Kniehl, 2009;

Many physically interesting particular cases have been considered beyond one loop.